An adaptive high-order unfitted finite element method for elliptic interface problems

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Abstract We design an adaptive unfitted finite element method on the Cartesian mesh with hanging nodes. We derive an hp-reliable and efficient residual type a posteriori error estimate on K-meshes. A key ingredient is a novel hp-domain inverse estimate which allows us to prove the stability of the finite element method under practical interface resolving mesh conditions and also prove the lower bound of the hp a posteriori error estimate. Numerical examples are included.

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1 Introduction

We consider the following model elliptic interface problem

$$-\mathrm{div}(a\nabla u) = f \ \text{ in } \Omega, \tag{1.1}$$

$$[\![u]\!]_\Gamma = 0, \quad [\![a\nabla u \cdot \nu]\!]_\Gamma = 0 \ \text{ on } \Gamma, \tag{1.2}$$

$$u = g \ \text{ on } \partial\Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain, $f \in L^2(\Omega)$, $g \in H^{1/2}(\partial\Omega)$, Γ is a Lipschitz and piecewise C^2-smooth interface which divides Ω into two nonintersecting subdomains

$$\Omega_1 \subset \bar\Omega_1 \subset \Omega, \quad \Omega_2 = \Omega\backslash\bar\Omega_1, \quad \Gamma = \partial\Omega_1 \cap \partial\Omega_2.$$

For simplicity, we assume that the coefficient $a(x)$ is positive and piecewise constant, namely,

$$a = a_1\chi_{\Omega_1} + a_2\chi_{\Omega_2}, \ \ a_1, a_2 > 0,$$

where χ_{Ω_i} denotes the characteristic function of Ω_i, $i = 1, 2$. Here ν is the unit outer normal to Ω_1, and $[\![v]\!]_\Gamma := v|_{\Omega_1} - v|_{\Omega_2}$ stands for the jump of a function v across the interface Γ. In this paper we will assume Ω is a union of bounded rectangles so that it can be partitioned by Cartesian meshes. For general Lipschitz domains we can extend the ideas developed in this paper in the framework of fictitious domain finite element methods, which will be studied in a future work.

There are extensive studies in the literature for immersed or unfitted mesh methods which allow the interface intersecting elements in an arbitrary manner and thus are able to avoid expensive work in the mesh generation when using body-fitted methods [3, 18, 53]. For low order approximations, we refer to the immersed boundary method [43], the immersed interface method [35], the ghost fluid method [37], the immersed finite element method [17, 36], and the extended Nitsche's method or the cut finite element method [11, 27]. The seminal idea of "doubling of unknowns" in the interface element in [27] has motivated studies of unfitted high order h-methods in [10, 30, 31, 52] and hp-methods in [38, 51]. We also refer to [33] for the unfitted isoparametric finite element method and the recent review paper [8] for further references on the theory and application of unfitted finite element methods. We remark that a crucial ingredient in the design and analysis of unfitted high order finite element methods is the inverse trace inequality on curved domains for which various interface resolving mesh conditions are introduced.

A posteriori error estimates are computable quantities in terms of the discrete solution and the input data, which provide the estimation of the discrete error and are decisive in designing efficient adaptive methods [4]. There exists an extensive literature on hp-residual type a posteriori finite element error estimates, see [39, 40] for conforming finite element methods and [29] for discontinuous Galerkin methods. The recent work [21] proves that the equilibrated flux

a posteriori error estimate on conforming meshes is also polynomial degree robust. The convergence and quasi-optimality of h-adaptive methods based on a posteriori error estimates for discontinuous Galerkin methods have been studied in [32], [7] and the references therein. For the reliable and efficient residual type a posteriori error estimation for other unfitted finite element methods we refer to the recent work [28] for immersed finite element methods and [13] for the cut finite element method.

The purpose of this paper is two folds. We first introduce the concept of interface deviation and prove the domain inverse estimate, which allows us to show the hp-stability of an unfitted finite element method under new interface resolving mesh conditions that can be easily implemented in practical computations. The unfitted finite element method is based on the idea of doubling of unknowns in [27] and the idea of merging small elements with neighboring large elements in [31] in the framework of the local discontinuous Galerkin (LDG) method [20]. Secondly, we derive a residual type hp-a posteriori error estimate for the unfitted finite element method on the so called K-meshes with possible hanging nodes [4]. Here we extend the hp-quasi-interpolation operator in [40] and the hp-local smoothing operator in [29, 55] to K-meshes. We also show the hp approximation error of unfitted finite element functions by H^1 functions by using the $H^{1/2}$-norm localization lemma in [24]. The local lower bound of our a posteriori error estimate is established by using the domain inverse estimate. This argument is different from the classical argument in [39] to derive the lower bound and the result is slightly better (see the remark below Theorem 4.1). We remark that for simplicity, a uniform polynomial degree is used in this paper, but the change to a variable polynomial degree over the mesh can also be considered by the method in this paper.

The paper is organized as follows. In section 2 we introduce the unfitted finite element method and prove the domain inverse estimate. In section 3 we show the upper bound of the residual type a posteriori error estimate. In section 4 we prove the efficiency of our a posteriori error estimator. In section 5 we report several numerical examples to show the effectiveness of our adaptive unfitted finite element method.

2 The unfitted finite element method

We first introduce the notation and the unfitted finite element method in the first subsection. Then we prove the domain inverse estimate which plays a key role in this paper. In the third subsection we prove the stability of our finite element method.

2.1 Notation and the finite element method

Let $\mathcal{T}$ be a Cartesian finite element mesh with possible local refinements and hanging nodes. The elements of the mesh are (open) rectangles whose sides are

Fig. 2.1 Examples of a large element K with respect to Ω_i with (a) one, (b) two, and (c) three vertices in Ω_i. The element in (a) is an irregular large element with respect to Ω_i.

parallel to the coordinate axes. For any $K \in \mathcal{T}$, let h_K stand for its diameter. Denote $\mathcal{T}^\Gamma = \{K \in \mathcal{T} : K \cap \Gamma \neq \emptyset\}$ the set of interface elements. We assume the interface Γ intersects each element $K \in \mathcal{T}^\Gamma$ at most twice at different (open) sides and each element $K \in \mathcal{T}^\Gamma$ includes at most one singular point of Γ where Γ is not C^2-smooth.

Definition 2.1 (Large element) For $i = 1, 2$, an element $K \in \mathcal{T}$ is called a large element with respect to Ω_i if $K \subset \Omega_i$ or $K \in \mathcal{T}^\Gamma$ for which there exists a constant $\delta_0 \in (0, 1/2)$ such that $|e \cap \Omega_i| \geq \delta_0 |e|$ for each side e of K having nonempty intersection with Ω_i and, if K has only one vertex A_K^i in Ω_i and includes a singular point Q_K of Γ, $\mathrm{dist}(Q_K, e_j) \geq \frac{1}{2}\delta_0 \min(|e_1|, |e_2|)$, where e_j is the side of K having A_K^i as one of its end points and $\mathrm{dist}(Q_K, e_j)$ is the distance of Q_K to the side e_j, $j = 1, 2$, see Figure 2.1.

The large elements with respect to Ω_i which have only one vertex in Ω_i and include a singular point of Γ will be called *irregular* large elements with respect to Ω_i. The other kinds of large elements with respect to Ω_i will be called *regular* large elements with respect to Ω_i, $i = 1, 2$. We notice that if K is an irregular large element, then the triangle with vertices A_K^i, Q_K, and one of the intersection points of $\Gamma \cap \partial K$ is shape regular with the ratio of the radius of the maximal inscribed circle to the diameter of the triangle depending on δ_0.

One difficulty in the study of unfitted finite element methods is the possibility that K may not be large with respect to both Ω_1 and Ω_2. We make the following assumption on the finite element mesh which is inspired by Johansson and Larson [31] in which a fictitious boundary discontinuous Galerkin method for elliptic equations is developed.

Assumption (H1): For each $K \in \mathcal{T}^\Gamma$, there exists a rectangular macro-element $N(K)$ which is a union of K and its neighboring element (or elements) such that $N(K)$ is large with respect to both Ω_1 and Ω_2, see Figure 2.2. We assume $h_{N(K)} \leq C_0 h_K$ for some fixed constant C_0.

Fig. 2.2 The small element K and its macro-element $N(K)$ (shadow region).

One way to satisfy the assumption (H1) is to locally refine the neighboring elements K' of $K \in \mathcal{T}^\Gamma$ which is not large with respect to both Ω_1, Ω_2 so that the elements K' are of the same size as K and K' are completely included in Ω_1 or Ω_2. In this case, we can define $N(K)$ as the union of K and those neighboring elements K' (see Figure 2.2).

In the following, we will always set $N(K) = K$ if $K \in \mathcal{T}^\Gamma$ and K is large with respect to both Ω_1, Ω_2. Thus $\widetilde{\mathcal{T}} = \{N(K) : K \in \mathcal{T}^\Gamma\} \cup \{K \in \mathcal{T} : K \subset \Omega_i, i = 1, 2, K \not\subset N(K')$ for some $K' \in \mathcal{T}^\Gamma\}$ is also a Cartesian mesh of Ω. The elements in $\widetilde{\mathcal{T}}$ are large with respect to both domains Ω_1, Ω_2 and the interface intersects the boundary of each element $K \in \widetilde{\mathcal{T}}$ also twice at different sides. We will call $\widetilde{\mathcal{T}}$ the induced mesh of $\mathcal{T}$ and write $\widetilde{\mathcal{T}} = \text{Induced}\,(\mathcal{T})$.

For any rectangular element K, $K \cap \Gamma \neq \emptyset$, we denote $\Gamma_K = \Gamma \cap K$ and Γ_K^h the (open) straight segment connecting the two intersection points of Γ and ∂K. If K includes a singular point Q_K, then Γ_K is the union of two C^2-smooth curves $\Gamma_{1K} \cup \Gamma_{2K}$. We denote Γ_{jK}^h the (open) straight segment connecting Q_K and the intersecting point of $\Gamma_{jK} \cap \partial K$, $j = 1, 2$.

The concept of interface deviation which measures how far Γ_K deviates from Γ_K^h or $\Gamma_{1K}^h, \Gamma_{2K}^h$ plays an important role in our subsequent analysis.

Definition 2.2 For any rectangular element K, $K \cap \Gamma \neq \emptyset$, the interface deviation η_K is defined as $\eta_K = \max(\eta_K^1, \eta_K^2)$, where for $i = 1, 2$, if K is a regular large element with respect to Ω_i with $A_K^i \in \Omega_i$ being the vertex of K which has the maximum distance to Γ_K^h,

$$\eta_K^i = \frac{\text{dist}_\text{H}(\Gamma_K, \Gamma_K^h)}{\text{dist}(A_K^i, \Gamma_K^h)},$$

and if K is an irregular large element with respect to Ω_i with vertex $A_K^i \in \Omega_i$,

$$\eta_K^i = \max\left(\frac{\text{dist}_\text{H}(\Gamma_{1K}, \Gamma_{1K}^h)}{\text{dist}(A_K^i, \Gamma_{1K}^h)}, \frac{\text{dist}_\text{H}(\Gamma_{2K}, \Gamma_{2K}^h)}{\text{dist}(A_K^i, \Gamma_{2K}^h)}\right).$$

Here $\text{dist}_\text{H}(\Gamma_1, \Gamma_2) = \max_{x \in \Gamma_1}(\min_{y \in \Gamma_2} |x - y|)$ is the Hausdorff distance between two sets Γ_1, Γ_2 and $\text{dist}(A, \Gamma_1)$ is the distance of a point A to the set Γ_1.

Fig. 2.3 The element K and its macro-element $N(K)$ when K includes a singular point of Γ. (a) K has two vertices in Ω_i. (b) K has three vertices in Ω_i.

Lemma 2.1 *Let $K \in \widetilde{\mathcal{T}}^\Gamma$ which is large with respect to both Ω_1, Ω_2 and $N(K)$ be the macro-element which is the union of K and its two or three neighboring elements included in Ω_i depending on K having two or three vertices in Ω_i, $i = 1, 2$, see Figure 2.3. The neighboring elements are assumed to be of the same size as K. Then $\eta^i_{N(K)} \leq \max(1/2, (1 - \delta_0)/(1 + \delta_0))$.*

Proof We first prove the case when K has three vertices in Ω_i. Let Γ^h_K be the segment CD, $A^i_K \in \Omega_i$ be the vertex of K having the maximal distance to CD, and $A^i_{N(K)} \in \Omega_i$ be the vertex of $N(K)$ having the maximal distance to CD. We extend DC to intersect the extended segment $A^i_K B_K$ at B'_K and $A^i_{N(K)} B_{N(K)}$ at $B'_{N(K)}$, see Figure 2.3(b). Denote h_j the length of the side of K parallel to the jth coordinate axis, $j = 1, 2$. By elementary geometry, $\frac{|B_{N(K)} B'_{N(K)}|}{|B_K B'_K|} = \frac{|B_K C| + h_2}{|B_K C|} \geq 2$. Thus, since $\operatorname{dist}_H(\Gamma_K, \Gamma^h_K) \leq \operatorname{dist}(A^i_K, \Gamma^h_K)$,

$$\eta^i_{N(K)} \leq \frac{|B_K B'_K| + h_1}{|B_{N(K)} B'_{N(K)}| + 2h_1} \leq \frac{|B_K B'_K| + h_1}{2|B_K B'_K| + 2h_1} = \frac{1}{2}.$$

When K has two vertices in Ω_i, we use the notation in Figure 2.3(a). Since K is large with respect to both Ω_1, Ω_2, we have $\delta_0 h_1 \leq |A^i_K C| \leq (1 - \delta_0) h_1, \delta_0 h_1 \leq |B^i_K D| \leq (1 - \delta_0) h_1$. Thus it follows from $\operatorname{dist}_H(\Gamma_K, \Gamma^h_K) \leq \max(\operatorname{dist}(A^i_K, \Gamma^h_K), \operatorname{dist}(B^i_K, \Gamma^h_K))$ that

$$\eta^i_{N(K)} \leq \max\left(\frac{|A^i_K C|}{|A^i_{N(K)} C|}, \frac{|B^i_K D|}{|A^i_{N(K)} C|}\right) \leq \frac{(1 - \delta_0) h_1}{h_1 + \delta_0 h_1} = \frac{1 - \delta_0}{1 + \delta_0}.$$

This completes the proof. $\square$

We make the following assumption which can be viewed as a variant of interface resolving mesh conditions.

Assumption (H2): For any $K \in \widetilde{\mathcal{T}}^\Gamma$, there exists a rectangular macro-element $N(K)$ which is a union of K and its neighboring element (or elements)

such that $\eta_{N(K)} \leq \max(1/2, (1-\delta_0)/(1+\delta_0))$.

If Γ_K is C^2-smooth in K, it is easy to see that $\mathrm{dist}_H(\Gamma_K, \Gamma_K^h) \leq Ch_K^2$ (see, e.g., Feistauer [25, §3.3.2]) and thus $\eta_K \leq Ch_K$ for some constant C independent of h_K. When K is an irregular large element with respect to Ω_i, we still have $\mathrm{dist}_H(\Gamma_{jK}, \Gamma_{jK}^h) \leq Ch_K^2$, $j = 1, 2$, and thus $\eta_K^i \leq Ch_K$. Therefore, in these cases, Assumption (H2) can be satisfied with $N(K) = K$ if h_K is sufficiently small. When $K \in \widetilde{\mathcal{T}}^\Gamma$ includes a singular point of Γ and has two or three vertices in Ω_i, by Lemma 2.1, if h_K is sufficiently small, we may merge K with its neighboring elements in Ω_i to obtain a macro-element $N(K)$ so that $\eta_{N(K)}^i \leq \max(1/2, (1-\delta_0)/(1+\delta_0))$. Therefore, when the interface elements are sufficiently refined, Assumption (H2) can always be satisfied.

In the following, we denote $\mathcal{M}$ the induced mesh from $\widetilde{\mathcal{T}}$ by possibly merging elements in $\widetilde{\mathcal{T}}^\Gamma$ with their neighboring elements such that

$$\eta_K \leq \max(1/2, (1-\delta_0)/(1+\delta_0)) \quad \forall K \in \mathcal{M}. \tag{2.1}$$

Obviously, each element in $\mathcal{M}$ is large with respect to both Ω_1, Ω_2.

Now we introduce the finite element space using the idea of "doubling of unknowns" in Hansbo and Hansbo [27]. For any integer $p \geq 1$ and $K \in \mathcal{M}$, denote $Q_p(K)$ the set of all polynomials in K which is of order p in each variable. We define the unfitted finite element space as

$$\mathbb{X}_p(\mathcal{M}) = \{v_1 \chi_{\Omega_1} + v_2 \chi_{\Omega_2} : v_i|_K \in Q_p(K), i = 1, 2\}.$$

We also define the broken Sobolev space

$$H^1(\mathcal{M}) = \{v_1 \chi_{\Omega_1} + v_2 \chi_{\Omega_2} : v_i|_K \in H^1(K), i = 1, 2\}.$$

For any $v \in H^1(\mathcal{M})$, $v|_K = v_1 \chi_{K_1} + v_2 \chi_{K_2}$ $\forall K \in \mathcal{M}$, we denote $\nabla_h v|_K := \nabla v_1 \chi_{K_1} + \nabla v_2 \chi_{K_2}$, where $K_i = K \cap \Omega_i$, χ_{K_i} is the characteristic function of K_i, $i = 1, 2$.

Let $\mathcal{E} = \mathcal{E}^{\mathrm{side}} \cup \mathcal{E}^\Gamma \cup \mathcal{E}^{\mathrm{bdy}}$, where $\mathcal{E}^{\mathrm{side}} = \{e = \partial K \cap \partial K' : K, K' \in \mathcal{M}\}$, $\mathcal{E}^\Gamma = \{\Gamma_K : K \in \mathcal{M}\}$ and $\mathcal{E}^{\mathrm{bdy}} = \{e = \partial K \cap \partial \Omega : K \in \mathcal{M}\}$. Since hanging nodes are allowed, $e \in \mathcal{E}^{\mathrm{side}}$ can be part of a side of an adjacent element. For $i = 1, 2$, denote by $\mathcal{M}_i = \{K \in \mathcal{M} : K \cap \Omega_i \neq \emptyset\}$. Then $\Omega_i \subset \Omega_i^h = \cup\{K : K \in \mathcal{M}_i\}$. We denote $\mathcal{E}_i^{\mathrm{side}}$ the set of all sides of $\mathcal{M}_i$ interior to Ω_i^h, that is, not on the boundary $\partial \Omega_i^h$. Finally, we set $\bar{\mathcal{E}} = \mathcal{E}_1^{\mathrm{side}} \cup \mathcal{E}_2^{\mathrm{side}} \cup \mathcal{E}^\Gamma \cup \mathcal{E}^{\mathrm{bdy}}$.

For any subset $\hat{\mathcal{M}} \subset \mathcal{M}$ and $\hat{\mathcal{E}} \subset \bar{\mathcal{E}}$, we use the notation

$$(u, v)_{\hat{\mathcal{M}}} := \sum_{K \in \hat{\mathcal{M}}} (u, v)_K, \quad \langle u, v \rangle_{\hat{\mathcal{E}}} := \sum_{e \subset \hat{\mathcal{E}}} \langle u, v \rangle_e,$$

where $(u, v)_K$ is the inner product of $L^2(K)$ and $\langle u, v \rangle_e$ is the inner product of $L^2(e)$.

For any $e \in \mathcal{E}$, we fix a unit normal vector n_e of e with the convention that n_e is the unit outer normal to $\partial \Omega$ if $e \in \mathcal{E}^{\mathrm{bdy}}$ and n_e is the unit outer normal to $\partial \Omega_1$ if $e \in \mathcal{E}^\Gamma$. For any $v \in H^1(\mathcal{M})$, we define the jump of v across e as

$$[\![v]\!]_e := v_- - v_+ \quad \forall e \in \mathcal{E}^{\mathrm{side}} \cup \mathcal{E}^\Gamma, \quad [\![v]\!]_e := v_- \quad \forall e \in \mathcal{E}^{\mathrm{bdy}},$$

where $v_\pm$ is the trace of v on e in the $\pm n_e$ direction. We define the piecewise constant normal vector function $n \in L^\infty(\mathcal{E}) = \Pi_{e \in \mathcal{E}} L^\infty(e)$ by $n|_e = n_e \ \forall e \in \mathcal{E}$.

Now we introduce our unfitted finite element method in the framework of LDG method. We focus on the primal formulation by following Arnold, Brezzi, Cockburn and Marini [2], Perugia and Schötzau [42]. For any $v \in H^1(\mathcal{M}), g \in L^2(\partial\Omega)$, we define the liftings $\mathsf{L}(v) \in [\mathbb{X}_p(\mathcal{M})]^2, \mathsf{L}_1(g) \in [\mathbb{X}_p(\mathcal{M})]^2$ such that for any $r \in [\mathbb{X}_p(\mathcal{M})]^2$,

$$(r, \mathsf{L}(v))_\mathcal{M} = \langle \hat{r} \cdot n, [\![v]\!] \rangle_\mathcal{E}, \qquad (r, \mathsf{L}_1(g))_\mathcal{M} = \langle r \cdot n, g \rangle_{\mathcal{E}^{\mathrm{bdy}}}, \qquad (2.2)$$

where the numerical flux $\hat{r}|_e = \beta_e r_- + (1 - \beta_e) r_+ \ \forall e \in \mathcal{E}$. Here $\beta_e = 0$ or $\beta_e = 1$ for $e \in \mathcal{E}^{\mathrm{side}} \cup \mathcal{E}^\Gamma$ and $\beta_e = 1$ for $e \in \mathcal{E}^{\mathrm{bdy}}$ as suggested in [20] to enhance the sparsity of the stiffness matrix.

Our unfitted finite element method is to find $U \in \mathbb{X}_p(\mathcal{M})$ such that

$$a_h(U, v) = F_h(v) \qquad \forall v \in \mathbb{X}_p(\mathcal{M}), \qquad (2.3)$$

where the bilinear form $a_h : H^1(\mathcal{M}) \times H^1(\mathcal{M}) \to \mathbb{R}$ and the functional $F_h : H^1(\mathcal{M}) \to \mathbb{R}$ are given by

$$a_h(v, w) = (a(\nabla_h v - \mathsf{L}(v)), \nabla_h w - \mathsf{L}(w))_\mathcal{M} + \langle \alpha[\![v]\!], [\![w]\!] \rangle_{\bar{\mathcal{E}}},$$
$$F_h(v) = (f, v)_\mathcal{M} - (a\mathsf{L}_1(g), \nabla_h v - \mathsf{L}(v))_\mathcal{M} + \langle \alpha g, v \rangle_{\mathcal{E}^{\mathrm{bdy}}}.$$

Here for any $v = v_1 \chi_{\Omega_1} + v_2 \chi_{\Omega_2}, w = w_1 \chi_{\Omega_1} + w_2 \chi_{\Omega_2} \in H^1(\mathcal{M})$,

$$\langle \alpha[\![v]\!], [\![w]\!] \rangle_{\bar{\mathcal{E}}} := \sum_{i=1}^2 \langle \alpha[\![v_i]\!], [\![w_i]\!] \rangle_{\mathcal{E}_i^{\mathrm{side}}} + \langle \alpha[\![v]\!], [\![w]\!] \rangle_{\mathcal{E}^\Gamma \cup \mathcal{E}^{\mathrm{bdy}}}. \qquad (2.4)$$

We notice that the penalty is added on $\bar{\mathcal{E}} = \mathcal{E}_1^{\mathrm{side}} \cup \mathcal{E}_2^{\mathrm{side}} \cup \mathcal{E}^\Gamma \cup \mathcal{E}^{\mathrm{bdy}}$ instead of $\mathcal{E} = \mathcal{E}^{\mathrm{side}} \cup \mathcal{E}^\Gamma \cup \mathcal{E}^{\mathrm{bdy}}$. The interface penalty function $\alpha \in L^\infty(\mathcal{E})$ will be specified in §2.3 after we prove the inverse trace inequality on the curved domain in the next subsection. We remark that the stabilization term $\langle \alpha[\![v]\!], [\![w]\!] \rangle_{\mathcal{E}^\Gamma}$ plays the key role in weakly capturing the jump behavior of the finite element solution at the interface in the weak formulation (2.3).

To conclude this section, we remark that the unfitted finite element methods in the literature are mostly based on the interior penalty discontinuous Galerkin (IPDG) method. The LDG formulation allows us to prove the stability of the method without assuming the interface penalty constant α_0 being sufficiently large (see §2.3 below).

2.2 Domain inverse estimates

Let $I = (-1, 1)$ and $\{L_n\}_{n \geq 0}$ be the Legendre polynomials which are orthogonal in $L^2(I)$ and satisfy $L_n(1) = 1$, $n \geq 0$. We start by recalling the first integral of Laplace for the Legendre polynomials (see, e.g., Szegö [48, P.97]).

Lemma 2.2 *For $n \geq 0$, we have*

$$L_n(t) = \frac{1}{\pi} \int_0^\pi \left[t + (t^2 - 1)^{1/2} \cos \phi \right]^n d\phi \quad \forall t \in \mathbb{R}.$$

We remark that the integral on the right hand side of above identity is actually real if $|t| < 1$ since $\int_0^\pi (\cos \phi)^{2k+1} d\phi = 0$ for any integer $k \geq 0$.

Proof For the sake of completeness, we sketch the proof here. By Rodrigues' formula (cf., e.g., Bernardi and Maday [6]), we know that

$$L_n(t) = \frac{(-1)^n}{2^n n!} \left(\frac{d}{dt} \right)^n \left[(1 - t^2)^n \right] \quad \forall t \in \mathbb{R}.$$

By Cauchy's integration formula,

$$L_n(t) = \frac{1}{2\pi \mathbf{i}} \int_\Sigma \frac{L_n(z)}{z - t} dz = \frac{1}{2\pi \mathbf{i}} \frac{(-1)^n}{2^n n!} \int_\Sigma \left(\frac{d}{dz} \right)^n \left[(1 - z^2)^n \right] \frac{1}{z - t} dz$$

for any closed contour enclosing the point $z = t$. Integrating by parts we obtain

$$L_n(t) = \frac{1}{2\pi \mathbf{i}} \int_\Sigma \left(\frac{1}{2} \frac{z^2 - 1}{z - t} \right)^n \frac{dz}{z - t}.$$

The lemma is obvious if $t = \pm 1$. For $t \neq \pm 1$, we choose the circle $|z - t| = |t^2 - 1|^{1/2}$ as the contour of the integration. By writing $z = t + (t^2 - 1)^{1/2} e^{\mathbf{i}\phi}$, we obtain easily the formula of Laplace. $\square$

It follows from Lemma 2.2 that $|L_n(t)| \leq 1 \ \forall t \in [-1, 1]$, and

$$|L_n(t)| \leq \left(|t| + \sqrt{t^2 - 1} \right)^n \quad \forall |t| > 1, \ n \geq 0. \tag{2.5}$$

We now prove the one dimensional domain inverse estimate.

Lemma 2.3 *Let $I_\lambda = (-\lambda, \lambda), \lambda > 1$, we have*

$$\|g\|_{L^2(I_\lambda \setminus \bar{I})}^2 \leq \frac{1}{2} \left[(\lambda + \sqrt{\lambda^2 - 1})^{2p+1} - 1 \right] \|g\|_{L^2(I)}^2 \quad \forall g \in Q_p(I_\lambda),$$

where $Q_p(I_\lambda)$ is the set of polynomials of order p in I_λ.

Proof It is well known that $\|L_n\|_{L^2(I)} = (n + 1/2)^{-1/2}$ for $n \geq 0$. Thus, for any $g \in Q_p(I_\lambda)$, $g(t) = \sum_{n=0}^p a_n L_n(t)$ and $\|g\|_{L^2(I)}^2 = \sum_{n=0}^p a_n^2 (n + 1/2)^{-1}$. Now by Cauchy-Schwarz's inequality

$$\|g\|_{L^2(I_\lambda \setminus \bar{I})}^2 \leq \|g\|_{L^2(I)}^2 \cdot \sum_{n=0}^p (n + 1/2) \|L_n\|_{L^2(I_\lambda \setminus \bar{I})}^2. \tag{2.6}$$

By using (2.5) and taking the transform $s = t + \sqrt{t^2 - 1}$,

$$\sum_{n=0}^{p} (n + 1/2) \|L_n\|_{L^2(I_\lambda \setminus \bar{I})}^2 \leq 2 \sum_{n=0}^{p} (n + 1/2) \int_1^\lambda (t + \sqrt{t^2 - 1})^{2n} dt$$

$$= \sum_{n=0}^{p} (n + 1/2) \int_1^{\lambda + \sqrt{\lambda^2 - 1}} (s^{2n} - s^{2n-2}) ds.$$

Now by using the summation by parts, we have

$$\sum_{n=0}^{p} (n + 1/2) \|L_n\|_{L^2(I_\lambda \setminus \bar{I})}^2 \leq (p + 1/2) \int_1^{\lambda + \sqrt{\lambda^2 - 1}} s^{2p} ds$$

$$= \frac{1}{2} \left[(\lambda + \sqrt{\lambda^2 - 1})^{2p+1} - 1 \right].$$

This completes the proof by using (2.6). $\square$

It follows from Lemma 2.3 that for any $(a, b) \subset (a, c)$, we have

$$\int_b^c |g|^2 dt \leq \frac{1}{2} \left[(\lambda + \sqrt{\lambda^2 - 1})^{2p+1} - 1 \right] \int_a^b |g|^2 dt \quad \forall g \in Q_p(a, c), \quad (2.7)$$

where $\lambda = (c - t_0)/(b - t_0)$, $t_0 = (a + b)/2$ is the midpoint of the interval (a, b).

The following two dimensional domain inverse estimate plays a key role in the next subsection to study the stability of our unfitted finite element method.

Lemma 2.4 *Let Δ be a triangle with vertices $A = (a_1, a_2)^T, B = (0, 0)^T, C = (c_1, 0)^T$, where $a_2, c_1 > 0$. Let $\delta \in (0, a_2)$ and $\Delta_\delta = \{x \in \Delta : \text{dist}(x, BC) > \delta\}$, where $\text{dist}(x, BC) = \min\{|x - y| : y \in BC\}$. Then, we have*

$$\|v\|_{L^2(\Delta)} \leq \mathsf{T} \left(\frac{1 + \delta a_2^{-1}}{1 - \delta a_2^{-1}} \right)^{2p+3/2} \|v\|_{L^2(\Delta_\delta)} \quad \forall v \in Q_p(\Delta).$$

where $\mathsf{T}(t) = t + \sqrt{t^2 - 1} \ \ \forall t \geq 1$.

Proof The triangle Δ can be parametrized as $x = t(s, 0)^T + (1 - t)(a_1, a_2)^T$, $s \in (0, c_1), t \in (0, 1)$. The Jacobi determinant of the parametrization is $a_2 t$. Obviously,

$$\int_{\Delta_\delta} |v|^2 dx = \int_0^{c_1} \int_0^{1 - \delta a_2^{-1}} |v(ts + (1 - t)a_1, (1 - t)a_2)|^2 a_2 t \, dt \, ds.$$

Since for a fixed s, $\tilde{v}(t) = v(ts + (1-t)a_1, (1-t)a_2)t \in Q_{2p+1}(0,1)$, we use (2.7) to obtain

$$\int_{1-\delta a_2^{-1}}^{1} |v(ts + (1-t)a_1, (1-t)a_2)|^2 t \, dt$$

$$\leq \frac{1}{1-\delta a_2^{-1}} \int_{1-\delta a_2^{-1}}^{1} |tv(ts + (1-t)a_1, (1-t)a_2)|^2 dt$$

$$\leq \frac{1}{2} \left[\mathsf{T}\left(\frac{1+\delta a_2^{-1}}{1-\delta a_2^{-1}}\right)^{2(2p+1)+1} - 1 \right] \int_{0}^{1-\delta a_2^{-1}} |v(ts + (1-t)a_1, (1-t)a_2)|^2 t \, dt.$$

This completes the proof. $\square$

The following lemma will be used in section 4 to prove the efficiency of the a posteriori error estimators.

Lemma 2.5 *Let $\Delta \subset \mathbb{R}^2$ be a triangle and ρ_Δ the radius of its maximal inscribed circle. For any $\delta \in (0, \rho_\Delta/2)$, denote $\Delta_\delta = \{x \in \Delta : \mathrm{dist}(x, \partial\Delta) > \delta\}$. Then for any $v \in Q_p(\Delta)$, we have*

$$\|v\|_{L^2(\Delta)} \leq (1 + 7\sqrt{\delta/\rho_\Delta})^{2p+3/2} \|v\|_{L^2(\Delta_\delta)}.$$

Proof Let O be the center of the maximal inscribed circle of Δ. The triangle Δ is divided into three sub-triangles by connecting O and three vertices of Δ. We use Lemma 2.4 in each of the three triangles to obtain

$$\|v\|_{L^2(\Delta)} \leq \mathsf{T}(\lambda)^{2p+3/2} \|v\|_{L^2(\Delta_\delta)}, \quad \lambda = \frac{1 + \delta/\rho_\Delta}{1 - \delta/\rho_\Delta}.$$

Since $\mathsf{T}(\lambda) = 1 + \sqrt{\lambda - 1}(\sqrt{\lambda - 1} + \sqrt{\lambda + 1})$ and $\lambda < 3$ by the assumption $\delta \in (0, \rho_\Delta/2)$, we have

$$\|v\|_{L^2(\Delta)} \leq (1 + 2(2 + \sqrt{2})\sqrt{\delta/\rho_\Delta})^{2p+3/2} \|v\|_{L^2(\Delta_\delta)}.$$

This completes the proof. $\square$

2.3 Stability and a priori error analysis

We first recall the standard multiplicative trace inequality (cf., e.g., Burman and Ern [9]), for any $K \in \mathcal{M}$ and $v \in H^1(K)$,

$$\|v\|_{L^2(\partial K)} \leq C h_K^{-1/2} \|v\|_{L^2(K)} + C \|v\|_{L^2(K)}^{1/2} \|\nabla v\|_{L^2(K)}^{1/2}. \tag{2.8}$$

The following lemma is proved in Xiao, Xu and Wang [52] when the interface Γ is C^2-smooth. It can be extended to cover the case when Γ is Lipschitz and piecewise C^2 as assumed in this paper.

Lemma 2.6 *For any $K \in \mathcal{M}$, denote $K_i = K \cap \Omega_i$, $i = 1, 2$. Then there exists a constant C independent of h_K such that for $i = 1, 2$,*

$$\|v\|_{L^2(\Gamma_K)} \le C\|v\|_{L^2(K_i)}^{1/2}\|v\|_{H^1(K_i)}^{1/2} + \|v\|_{L^2(\partial K_i \setminus \bar{\Gamma}_K)} \qquad \forall v \in H^1(K_i).$$

Proof Since Γ is Lipschitz continuous and piecewise C^2, there is a set of sub-domains $\{U_j\}_{j=1}^r$ that covers Γ and a partition of unity $\{\phi_j\}_{j=1}^r$ subordinated to $\{U_j\}_{j=1}^r$, that is, $\phi_j \in C_0^\infty(U_j), 0 \le \phi_j \le 1, \sum_{j=1}^r \phi_j = 1$ in $\cup_{j=1}^r U_j$. More-over, let $\nu = (\nu_1, \nu_2)^T$ be the unit outer normal vector to $\partial\Omega_1$, we may assume in each U_j, there exists an index $k(j) = 1$ or 2, such that $|\nu_{k(j)}| \ge 1/2$ in U_j, $j = 1, \cdots, r$. Here for the points on Γ where ν is discontinuous, we define $\nu = (1/\sqrt{2}, 1/\sqrt{2})^T$. Since $\nu_{k(j)}$ does not change sign in each U_j, we have

$$\frac{1}{2}\int_{\Gamma_K} |v|^2 ds = \frac{1}{2}\sum_{j=1}^r \int_{\Gamma_K} |v|^2 \phi_j ds \le \sum_{j=1}^r \int_{\Gamma_K} |v|^2 \phi_j |\nu_{k(j)}| ds$$
$$\le \sum_{j=1}^r \left| \int_{\Gamma_K} |v|^2 \phi_j \nu_{k(j)} ds \right|.$$

Now by integration by parts, we obtain

$$\int_{\Gamma_K} |v|^2 \phi_j \nu_{k(j)} ds = \int_{\partial K_i} |v|^2 \phi_j \nu_{k(j)} ds - \int_{\partial K_i \setminus \bar{\Gamma}_K} |v|^2 \phi_j \nu_{k(j)} ds$$
$$= \int_{K_i} \frac{\partial}{\partial x_{k(j)}} \left[\phi_j |v|^2 \right] dx - \int_{\partial K_i \setminus \bar{\Gamma}_K} |v|^2 \phi_j \nu_{k(j)} ds$$
$$\le C\|v\|_{L^2(K_i)}^2 + 2\|v\|_{L^2(K_i)}\|\nabla v\|_{L^2(K_i)} + \|v\|_{L^2(\partial K_i \setminus \bar{\Gamma}_K)}^2,$$

where $C = \max_{1 \le j \le r} \|\nabla\phi_j\|_{L^\infty(U_j)}$. This completes the proof. $\square$

We will use the following inverse trace inequality in Warburton and Hes-thaven [50].

Lemma 2.7 *Let Δ be a triangle. For any $v \in P_p(\Delta)$, the set of all polynomials of order p in Δ, we have*

$$\|v\|_{L^2(\partial\Delta)} \le \sqrt{\frac{(p+1)(p+2)}{2} \frac{|\partial\Delta|}{|\Delta|}} \|v\|_{L^2(\Delta)}.$$

The following inverse trace inequality on curved domains plays a key role in our analysis.

Lemma 2.8 *Let $K \in \mathcal{M}^\Gamma := \{K \in \mathcal{M} : K \cap \Gamma \ne \emptyset\}$. Then for $i = 1, 2$,*

$$\|v\|_{L^2(\partial K_i)} \le Cph_K^{-1/2}\mathsf{T}\left(\frac{1 + 3\eta_K}{1 - \eta_K}\right)^{2p} \|v\|_{L^2(K_i)} \qquad \forall v \in Q_p(K),$$

where the constant C is independent of h_K, p, and η_K.

Fig. 2.4 The figure used in the proof of Lemma 2.8 and Lemma 4.1.

Proof We only prove the case when $K_i = K \cap \Omega_i$ is a curved trapezoid (see Figure 2.4). The other cases can be proved similarly. Let K_i^h be the trapezoid $A_i B C D$ which replaces Γ_K by the straight segment Γ_K^h, where A_i is the vertex of K in Ω_i having the maximum distance to Γ_K^h, B, C are the end points of Γ_K^h with C on the side of K opposite to A_i, and D the other vertex of K in Ω_i (see Figure 2.4). As K is large with respect to Ω_i, the triangles $\Delta A_i B C, \Delta A_i C D$ are shape regular with the shape regular constant depending possibly on δ_0 in Definition 2.1. By Lemma 2.6 and using Lemma 2.7 in each triangle $\Delta A_i B C, \Delta A_i C D$ we obtain

$$\|v\|_{L^2(\partial K_i)} \le C\|v\|_{L^2(K_i)}^{1/2}\|v\|_{H^1(K_i)}^{1/2} + \|v\|_{L^2(\partial K_i^h)}$$
$$\le C\|v\|_{L^2(K_i)}^{1/2}\|v\|_{H^1(K_i)}^{1/2} + Cph_K^{-1/2}\|v\|_{L^2(K_i^h)}. \tag{2.9}$$

Let $\delta = \mathrm{dist}(\Gamma_K, \Gamma_K^h)$ and $d_i = \mathrm{dist}(A_i, \Gamma_K^h)$. Then the interface deviation $\eta_K \ge \delta/d_i$ by Definition 2.2. Let $\Delta A_i B' C' \subset \Delta A B C \subset \Delta A_i B'' C''$ such that $B'C', B''C''$ are parallel to Γ_K^h and the distances of $B'C', B''C''$ to Γ_K^h are δ. B', C' are respectively on the segments $A_i B, A_i C$ and B'', C'' are respectively on the extended lines of $A_i B, A_i C$. Let D' on AD such that $D'C'$ is parallel to DC, see Figure 2.4. It is clear that $\Delta A_i C' D' \subset K_i$ and $\frac{|C'D'|}{|CD|} = \frac{|A_i C'|}{|A_i C|} = \frac{d_i - \delta}{d_i}$. Thus $\frac{|DD'|}{|A_i D|} = \frac{\delta}{d_i} \le \eta_K$.

Since $K_i^h = (\Delta A_i C D) \cup (\Delta A_i B C)$ and $\Delta A_i C' D', \Delta A_i B' C' \subset K_i$, we obtain by using Lemma 2.4 that

$$\|v\|_{L^2(K_i^h)} \le \|v\|_{L^2(\Delta A_i C D)} + \|v\|_{L^2(\Delta A_i B C)}$$
$$\le \mathsf{T}\left(\frac{1+\eta_K}{1-\eta_K}\right)^{2p+3/2}(\|v\|_{L^2(\Delta A_i C' D')} + \|v\|_{L^2(\Delta A_i B' C')})$$
$$\le C\mathsf{T}\left(\frac{1+\eta_K}{1-\eta_K}\right)^{2p+3/2}\|v\|_{L^2(K_i)}. \tag{2.10}$$

Since $K_i \subset (\Delta A_i CD) \cup (\Delta A_i B'' C'')$, by the inverse estimate for hp finite element method (cf., e.g., Schwab [46, Theorem 4.76]), we have

$$
\begin{aligned}
\|\nabla v\|_{L^2(K_i)} &\leq \|\nabla v\|_{L^2(\Delta A_i CD)} + \|\nabla v\|_{L^2(\Delta A_i B'' C'')} \\
&\leq C p^2 h_K^{-1} \|v\|_{L^2(\Delta A_i CD)} + C p^2 h_K^{-1} \|v\|_{L^2(\Delta A_i B'' C'')}. \quad (2.11)
\end{aligned}
$$

On the other hand, by using Lemma 2.4 again,

$$
\|v\|_{L^2(\Delta A_i CD)} \leq \mathsf{T} \left(\frac{1+\eta_K}{1-\eta_K} \right)^{2p+3/2} \|v\|_{L^2(\Delta A_i C' D')},
$$

$$
\begin{aligned}
\|v\|_{L^2(\Delta A_i B'' C'')} &\leq \mathsf{T} \left(\frac{1+2\delta(d_i+\delta)^{-1}}{1-2\delta(d_i+\delta)^{-1}} \right)^{2p+3/2} \|v\|_{L^2(\Delta A_i BC)} \\
&\leq \mathsf{T} \left(\frac{1+3\eta_K}{1-\eta_K} \right)^{2p+3/2} \|v\|_{L^2(K_i)}.
\end{aligned}
$$

Inserting these two estimates to (2.11), we obtain

$$
\|\nabla v\|_{L^2(K_i)} \leq C p^2 h_K^{-1} \mathsf{T} \left(\frac{1+3\eta_K}{1-\eta_K} \right)^{2p+3/2} \|v\|_{L^2(K_i)}. \quad (2.12)
$$

This, together with (2.9)-(2.10), completes the proof. $\square$

We remark that various interface resolving mesh conditions have been made in the literature to obtain the inverse trace inequality in Lemma 2.8, which is crucial in establishing the stability of unfitted finite element methods. For example, it is assumed in Massjung [38], Wu and Xiao [51] that each local interface Γ_K, $K \in \mathcal{M}$, is star shaped with respect to some point in Ω_i, which allows for the use of a local polar coordinate system.

To proceed, we define the interface penalty function $\alpha \in L^\infty(\mathcal{E})$:

$$
\alpha|_e = \alpha_0 \hat{a}_e \hat{\Theta}_e h_e^{-1} p^2 \quad \forall e \in \mathcal{E}, \quad (2.13)
$$

where $\alpha_0 > 0$ is some fixed constant which is taken to be 1 in all our numerical examples, and

$$
\hat{a}_e = \max\{a_K : e \cap \bar{K} \neq \emptyset\}, \quad \hat{\Theta}_e = \max\{\Theta_K : e \cap \bar{K} \neq \emptyset\},
$$

with

$$
a_K = \begin{cases} \frac{a_1+a_2}{2} & \text{if } K \in \mathcal{M}^\Gamma, \\ a_i & \text{if } K \subset \Omega_i. \end{cases}, \quad \Theta_K = \begin{cases} \mathsf{T}\left(\frac{1+3\eta_K}{1-\eta_K}\right)^{4p} & \text{if } K \in \mathcal{M}^\Gamma, \\ 1 & \text{otherwise.} \end{cases} \quad (2.14)
$$

Here $\mathsf{T}(t) = t + \sqrt{t^2-1}$, $\forall t \geq 1$. We remark that η_K is the interface deviation of the interface in $K \in \mathcal{M}$ defined in Definition 2.2, which is the only place that the geometry of the interface comes into our method. The mesh function $h|_e = (h_K + h_{K'})/2$ if $e = \partial K \cap \partial K' \in \mathcal{E}^{\text{side}}$ and $h|_e = h_K$ if $e = K \cap \Gamma \in \mathcal{E}^\Gamma$ or $e = \partial K \cap \partial \Omega \in \mathcal{E}^{\text{bdy}}$ for some $K \in \mathcal{M}$.

Lemma 2.9 *We have $\|a^{1/2}\mathsf{L}(v)\|_{\mathcal{M}} \le c_{\mathsf{L}}\|\alpha^{1/2}[\![v]\!]\|_{\mathcal{E}}$ $\forall v \in \mathbb{X}_p(\mathcal{M})$ for some constant $c_{\mathsf{L}} > 0$ independent of p, the mesh $\mathcal{M}$, and the coefficient a.*

Proof By taking $r = a\mathsf{L}(v)$ in (2.2), we have

$$\|a^{1/2}\mathsf{L}(v)\|_{\mathcal{M}}^2 \le \|\alpha^{-1/2}\widehat{a\mathsf{L}(v)}\|_{\mathcal{E}}\|\alpha^{1/2}[\![v]\!]\|_{\mathcal{E}}$$

$$\le C\left(\sum_{e\in\mathcal{E}}\|\hat{\Theta}_e^{-1/2}h_e^{1/2}p^{-1}\widehat{a^{1/2}\mathsf{L}(v)}\|_{L^2(e)}^2\right)^{1/2}\|\alpha^{1/2}[\![v]\!]\|_{\mathcal{E}}$$

$$\le C\left(\sum_{K\in\mathcal{M}}\sum_{i=1}^2\|\Theta_K^{-1/2}h_K^{1/2}p^{-1}(a^{1/2}\mathsf{L}(v))\|_{L^2(\partial K_i)}^2\right)^{1/2}\|\alpha^{1/2}[\![v]\!]\|_{\mathcal{E}}$$

$$\le C\|a^{1/2}\mathsf{L}(v)\|_{\mathcal{M}}\|\alpha^{1/2}[\![v]\!]\|_{\mathcal{E}},$$

where we have used Lemma 2.8 in the interface elements and a scaled version of Lemma 2.7 for the elements not intersecting the interface. This completes the proof. □

For any $v \in H^1(\mathcal{M})$, we define the DG norm

$$\|v\|_{\mathrm{DG}}^2 = \|a^{1/2}\nabla_h v\|_{\mathcal{M}}^2 + \|\alpha^{1/2}[\![v]\!]\|_{\bar{\mathcal{E}}}^2,$$

where $\|\alpha^{1/2}[\![v]\!]\|_{\bar{\mathcal{E}}}^2 := \langle\alpha[\![v]\!],[\![v]\!]\rangle_{\bar{\mathcal{E}}}$. By (2.4), we know that

$$\|\alpha^{1/2}[\![v]\!]\|_{\bar{\mathcal{E}}}^2 = \sum_{i=1}^2\sum_{e\in\mathcal{E}_i^{\mathrm{side}}}\|\alpha^{1/2}[\![v_i]\!]\|_{L^2(e)}^2 + \|\alpha^{1/2}[\![v]\!]\|_{\mathcal{E}^\Gamma\cup\mathcal{E}^{\mathrm{bdy}}}^2$$

$$\ge \|\alpha^{1/2}[\![v]\!]\|_{\mathcal{E}}^2. \tag{2.15}$$

Theorem 2.1 *We have $a_h(v,v) \ge (4 + c_{\mathsf{L}}^2)^{-1}\|v\|_{\mathrm{DG}}^2$ $\forall v \in \mathbb{X}_p(\mathcal{M})$, where $c_{\mathsf{L}} > 0$ is the constant in Lemma 2.9.*

Proof The argument is standard. For any $\delta_1 \in (0,1)$, by Lemma 2.9 and (2.15) we have

$$a_h(v,v) = \|a^{1/2}\nabla_h v\|_{\mathcal{M}}^2 + \|a^{1/2}\mathsf{L}(v)\|_{\mathcal{M}}^2 - 2(a\nabla_h v,\mathsf{L}(v))_{\mathcal{M}} + \|\alpha^{1/2}[\![v]\!]\|_{\bar{\mathcal{E}}}^2$$

$$\ge \|a^{1/2}\nabla_h v\|_{\mathcal{M}}^2 + (1 + (1-\delta_1)c_{\mathsf{L}}^{-2})\|a^{1/2}\mathsf{L}(v)\|_{\mathcal{M}}^2 - 2(a\nabla_h v,\mathsf{L}(v))_{\mathcal{M}}$$

$$+ \delta_1\|\alpha^{1/2}[\![v]\!]\|_{\bar{\mathcal{E}}}^2.$$

By the elementary inequality $a^2 - 2ab + (1+\epsilon)b^2 \ge \frac{\epsilon}{1+\epsilon}a^2$ $\forall a,b > 0, \epsilon > 0$, we obtain

$$a_h(v,v) \ge \frac{(1-\delta_1)c_{\mathsf{L}}^{-2}}{1 + (1-\delta_1)c_{\mathsf{L}}^{-2}}\|a^{1/2}\nabla_h v\|_{\mathcal{M}}^2 + \delta_1\|\alpha^{1/2}[\![v]\!]\|_{\bar{\mathcal{E}}}^2.$$

This completes the proof by choosing $\delta_1 = \frac{\sqrt{1+4c_{\mathsf{L}}^{-2}}-1}{\sqrt{1+4c_{\mathsf{L}}^{-2}}+1}$ to make the coefficients in the above inequality equal and noticing that $\delta_1 \ge (4 + c_{\mathsf{L}}^2)^{-1}$. □

The following a priori error estimate can be proved by using Theorem 2.1, the classical hp-interpolation error estimate in Babuška and Suri [5, Lemma 4.5], and the argument in [42], [51]. Here we omit the details.

Theorem 2.2 *Let the solution of the problem* (1.1)-(1.3) $u \in H^k(\Omega_1 \cup \Omega_2)$, $k \geq 2$. *Let* $U \in \mathbb{X}_p(\mathcal{M})$ *be the solution of* (2.3). *Then there exists a constant* C *independent of* p, *the mesh* $\mathcal{M}$, *and the coefficient* a *such that*

$$\|u - U\|_{\mathrm{DG}} \leq C \max_{e \in \mathcal{E}} |\alpha|_e|^{1/2} \frac{h^{\min(p+1,k)-1}}{p^{k-3/2}} \sum_{i=1}^{2} \|a_i^{1/2} \tilde{u}_i\|_{H^k(\Omega)}.$$

Here $h = \max_{K \in \mathcal{M}} h_K$ *and* $\tilde{u}_i \in H^k(\Omega)$ *is the Stein extension [1, P.154] of* $u_i \in H^k(\Omega_i)$ *for Lipschitz domains satisfying* $\|\tilde{u}_i\|_{H^k(\Omega)} \leq C \|u_i\|_{H^k(\Omega_i)}$, $i = 1, 2$.

We remark that the error estimate is slightly sub-optimal in p which is typical for discontinuous Galerkin methods (see e.g., Georgoulis, Hall and Melenk [26]). However, hp-optimal error estimates can be proved in some special cases for discontinuous Galerkin methods for Possion problem on 1-irregular meshes (each side containing at most 1 hanging node), see Stamm and Wihler [47].

3 A posteriori error estimation: reliability

We start by introducing some further notation. We assume the elements in $\mathcal{T}$ are obtained by local successive quad-refinements of some conforming initial mesh $\mathcal{T}_0$. A quad-refinement of an element consists of subdividing the element into four congruent rectangles.

Let $\mathcal{N}^0$ be the set of conforming nodes of the induced mesh $\mathcal{M}$ from $\mathcal{T}$ such that each element $K \in \mathcal{M}$ is large with respect to both Ω_1, Ω_2 and satisfies (2.1). A node is called conforming if it either locates on the boundary or is shared by the four elements to which it belongs. For each conforming node P, we define $\psi_P \in \mathbb{X}_1(\mathcal{M}) \cap H^1(\Omega)$, which is bilinear in each element and satisfies $\psi_P(Q) = \delta_{PQ}$ for any $Q \in \mathcal{N}^0$. Here δ_{PQ} is the Kronecker delta. It is proved in Babuška and Miller [4] that $\{\psi_P : P \in \mathcal{N}^0\}$ consists of a basis of $\mathbb{X}_1(\mathcal{M}) \cap H^1(\Omega)$ and satisfies the property of the partition of unity

$$\sum_{P \in \mathcal{N}^0} \psi_P = 1.$$

We impose the following assumption on the finite element mesh which is first introduced in Babuška and Miller [4] as the K-mesh (see Figure 3.1).

Assumption (H3) There exists a constant $C > 0$ uniform on the level of discretization of $\mathcal{M}$ such that for any conforming node $P \in \mathcal{N}^0$,

$$\mathrm{diam}(\mathrm{supp}(\psi_P)) \leq C \min_{K \in \mathcal{M}_P} h_K, \tag{3.1}$$

Fig. 3.1 The left mesh is not a K-mesh if it refines close to P. The right mesh is a K-mesh if it refines close to upper-right corner. The shadow region is the support of ψ_P.

Fig. 3.2 An example of S_P with P is the vertex of (a) one element, (b) two elements, and (c) three elements.

where $\mathcal{M}_P := \{K \in \mathcal{M}, \ K \subset \mathrm{supp}(\psi_P)\}$.

We refer to [4, §1.4] for further properties of K-meshes and Bonito and Nochetto [7, §6] for a refinement algorithm to enforce the assumption (H3) in practical computations.

The a posteriori error analysis depends on a suitable quasi-interpolation operator. In Melenk [40], a Clément type hp-quasi-interpolation is constructed for conforming meshes. The following lemma shows that a similar construction leads to a hp-quasi-interpolation operator on K-meshes.

Lemma 3.1 *Let* $\mathbb{V}_p(\mathcal{M}) = \Pi_{K \in \mathcal{M}} Q_p(K)$. *There exists a quasi-interpolation operator* $\Pi_h : H_0^1(\Omega) \to \mathbb{V}_p(\mathcal{M}) \cap H_0^1(\Omega)$ *such that for any* $v \in H_0^1(\Omega)$,

$$\|D^m(v - \Pi_h v)\|_{L^2(K)} \le C(h_K/p)^{1-m}\|\nabla v\|_{L^2(\omega(K))}, \quad m = 0, 1,$$

$$\|v - \Pi_h v\|_{L^2(\partial K)} \le C(h_K/p)^{1/2}\|\nabla v\|_{L^2(\omega(K))}.$$

Here for any $K \in \mathcal{M}$, $\omega(K)$ *is a union of a discrete set of elements including* K *such that* $\mathrm{diam}(\omega(K)) \le Ch_K$. *The constant* C *is independent of* h_K, p.

Proof The second estimate follows from the first one by the multiplicative trace inequality (2.8). We now describe how to construct the operator which satisfies the first estimate by the method in [40]. For any $P \in \mathcal{N}^0$, denote

$\Omega_P = (\mathrm{supp}(\psi_P))^\circ$, the interior of $\mathrm{supp}(\psi_P)$, and $h_P = \mathrm{diam}(\Omega_P)$. For any $v \in H_0^1(\Omega)$, which is extended to be zero outside Ω, we define

$$I_h v = \sum_{P \in \mathcal{N}^0} (I_P v) \psi_P, \tag{3.2}$$

where $I_P : H_0^1(\Omega) \to \mathbb{V}_{p-1}(\mathcal{M}_P)$, is defined by using local projection and polynomial lifting. More precisely, denote S_P the rectangle centered at P which includes Ω_P and has minimum size. Let $J_P : H^1(S_P) \to Q_{p-1}(S_P)$ be the polynomial approximation operator on rectangles in [40, Theorem 5.1] which satisfies

$$\|D^m(v - J_P v)\|_{L^2(S_P)} \le C(h_P/p)^{1-m} \|\nabla v\|_{L^2(S_P)}, \quad m = 0, 1. \tag{3.3}$$

Notice that $J_P v$ does not vanish on the boundary. Let $P \in \partial\Omega \cap \mathcal{N}^0$ and $\Gamma_P = \partial\Omega \cap \bar{S}_P$. Since $v = 0$ on $\partial\Omega$, we obtain from (3.3) that

$$\|(h/p)^{-1/2} J_P v\|_{L^2(\Gamma_P)} + \|J_P v\|_{H^{1/2}(\Gamma_P)} \le C\|\nabla v\|_{L^2(S_P)}.$$

We observe that if $P \in \partial\Omega$ is the vertex of only one element or two elements, S_P can be chosen to be inside Ω (see Figure 3.2). Thus one can use the polynomial lifting theorem in [40, Proposition 5.3] to obtain a $v_P \in Q_{4(p-1)}(S_P)$ such that

$$\begin{aligned}
&(h_P/p)^{-1} \|v_P\|_{L^2(S_P)} + \|\nabla v_P\|_{L^2(S_P)} \\
&\le C\|(h/p)^{-1/2} J_P v\|_{L^2(\Gamma_P)} + C\|J_P v\|_{H^{1/2}(\Gamma_P)} \\
&\le C\|\nabla v\|_{L^2(S_P)}.
\end{aligned} \tag{3.4}$$

If $P \in \partial\Omega$ is the vertex of three elements, then $S_P \cap \Omega$ is the union of three rectangles S_P^j, $j = 1, 2, 3$, such that each element in $\mathcal{M}_P$ is included in one of these three elements (see Figure 3.2). In this case, one can use the argument in [40, Lemma 5.8] to conclude that there exists a $v_P \in \left[\Pi_{j=1,2,3} Q_{4(p-1)}(S_P^j)\right] \cap H^1(S_P)$ such that (3.4) is valid.

Now we define $I_P v = J_P v$ if $P \in \mathcal{N}^0$ is an interior node and $I_P v = J_P v - v_P$ if $P \in \mathcal{N}^0$ is a node on the boundary. By using the partition of unity (3.1), (3.3) and (3.4), we obtain easily

$$\|D^m(v - I_h v)\|_{L^2(K)} \le C(h_K/p)^{1-m} \|\nabla v\|_{L^2(\omega(K))}, \quad m = 0, 1.$$

Finally, since $I_h v \in \mathbb{V}_{4(p-1)+1}(\mathcal{M}) \cap H_0^1(\Omega)$, we define Π_h by replacing p in (3.2) by $\lfloor (p-1)/4 \rfloor + 1$. This proves the lemma. $\square$

Remark 3.1 We know from the proof of Lemma 3.1 that for any $K \in \mathcal{M}$,

$$\omega(K) = \{K' \in \mathcal{M} : K' \subset S_P, \forall P \in \mathcal{N}^0 \text{ such that } \psi_P|_K \ne 0\}.$$

The following local smoothing operator on K-meshes extends the construction in Burman and Ern [9], Houston, Schötzau and Wihler [29] for conforming meshes and Zhu and Schötzau [55] for 1-irregular meshes.

Lemma 3.2 *There exists an interpolation operator $\pi_h : \mathbb{V}_p(\mathcal{M}) \to \mathbb{V}_p(\mathcal{M}) \cap H^1(\Omega)$ such that for any $v \in \mathbb{V}_p(\mathcal{M})$,*

$$\|v - \pi_h v\|_{L^2(K)} \leq C\|p^{-1}h^{1/2}[\![v]\!]\|_{L^2(\sigma(K))},$$
$$\|\nabla(v - \pi_h v)\|_{L^2(K)} \leq C\|ph^{-1/2}[\![v]\!]\|_{L^2(\sigma(K))},$$

where $\sigma(K) = \{e \in \mathcal{E}^{\text{side}} : e \subset \widetilde{\omega}(K)\}$, $\widetilde{\omega}(K)$ is a set of elements including K such that $\text{diam}(\widetilde{\omega}(K)) \leq Ch_K$. The constant C is independent of h_K, p. Moreover, $\pi_h v \in H_0^1(\Omega)$ if $v = 0$ on $\partial\Omega$.

Proof Let $\hat{K} = I \times I$, $I = (-1, 1)$, be the reference element. Let $\widehat{\mathcal{N}}_p$ be the Gauss-Legendre-Lobatto grid of $\hat{K}$, that is, $\widehat{\mathcal{N}}_p = \{(\xi_i, \xi_j)^T \in \hat{K} : 0 \leq i \leq p\}$, where $\xi_i, 0 \leq i \leq p$, are the zeros of the polynomial $(1 - \xi^2)L_p'(\xi)$. Here $\{L_n\}_{n \geq 0}$ is the set of Legendre polynomials. Let $\{\hat{\phi}_i\}_{i=0}^p$ be the set of Lagrange interpolation functions in $Q_p(\Lambda)$ corresponding to the Gauss-Legendre-Lobatto nodes, that is, $\hat{\phi}_i \in Q_p(\Lambda)$, $\hat{\phi}_i(\xi_j) = \delta_{ij}$, $0 \leq i, j \leq p$. Here δ_{ij} is the Kronecker delta.

It is known by the differential equation satisfied by the Legendre polynomials that

$$\hat{\phi}_i(\xi) = \frac{-1}{p(p+1)}\frac{(1 - \xi^2)L_p'(\xi)}{(\xi - \xi_i)L_p(\xi_i)}, \quad 0 \leq i \leq p.$$

Notice that $\|L_p'\|_{L^2(\Lambda)} = \sqrt{p(p+1)}$, $L_p(\pm 1) = (\pm 1)^p$, we have

$$\|\hat{\phi}_0\|_{L^2(\Lambda)} \leq [p(p+1)]^{-1}\|(1 - \xi)L_p'\|_{L^2(\Lambda)} \leq 2/\sqrt{p(p+1)}. \quad (3.5)$$

Similarly, $\|\hat{\phi}_p\|_{L^2(\Lambda)} \leq 2/\sqrt{p(p+1)}$.

For any $K \in \mathcal{M}$, let $F_K : \hat{K} \to K$ be the affine mapping. Denote $\mathcal{N}_p(K) = F_K(\widehat{\mathcal{N}}_p)$ the set of Gauss-Legendre-Lobatto nodes on K. The degrees of freedom of a function in $Q_p(K)$ are its nodal values at $\mathcal{N}_p(K)$. The set of basis functions of $Q_p(K)$ is $\{\phi_P = \hat{\phi}_{\hat{P}} \circ F_K^{-1} : P = F_K(\hat{P})\}$. Here $\hat{\phi}_{\hat{P}}$ is the nodal basis of $\hat{Q}_p(\hat{K})$ corresponding to $\hat{P} \in \widehat{\mathcal{N}}_p$.

To construct the interpolation operator, we classify the set of nodes and sides of the mesh $\mathcal{M}$. Let $\mathcal{N}^0$ be the set of conforming nodes. For $k \geq 1$, let $\mathcal{N}^k$ be the subset of nodes that are located on some side $e \in \mathcal{E}^{\text{side}}$ whose end points are in $\mathcal{N}^m, 0 \leq m \leq k-1$, and with at least one end point in $\mathcal{N}^{k-1}$. By the assumption (H3), the maximum number of levels L of the classification of the nodes is uniformly bounded.

For $1 \leq k \leq L + 1$, we denote $\mathcal{E}^k \subset \mathcal{E}^{\text{side}}$ the collection of sides whose end points are in $\mathcal{N}^m, 0 \leq m \leq k-1$, and with at least one end point in $\mathcal{N}^{k-1}$. Clearly, $\mathcal{E}^k \cap \mathcal{E}^l = \emptyset$ if $k \neq l$ and $\mathcal{E}^1$ is the set of sides whose end points are conforming nodes. For any $v \in \mathbb{V}_p(\mathcal{M})$, we define $\pi_h^k v \in P_p(\mathcal{E}^k)$, the set of polynomials of order p in each side of $\mathcal{E}^k$, successively as follows.

1. If $e \in \mathcal{E}^1$ whose end points $P_1, P_2 \in \mathcal{N}^0$, $e = \partial K \cap \partial K'$, $K, K' \in \mathcal{M}$, and K' is the element such that the length of its side including e is larger or equal to $|e|$, we define

$$\pi_h^1 v = v|_{K'} + \sum_{i=1}^{2} \left[(\pi_h^0 v)(P_i) - (v|_{K'})(P_i) \right] \phi_{P_i} \quad \text{on } e, \qquad (3.6)$$

where for $P \in \mathcal{N}^0$, $(\pi_h^0 v)(P) = \frac{1}{\#\{K \in \mathcal{M}: P \in \bar{K}\}} \sum_{K \in \mathcal{M}, P \in \bar{K}} (v|_K)(P)$, the local average of v sharing P as the common vertex. Here the boundary value of $v|_K$ is understood as its trace.

2. For $k \geq 2$, $e \in \mathcal{E}^k$ whose end points $P_i \in \mathcal{N}^{m_i} (i = 1, 2)$, $e = \partial K \cap \partial K'$, $K, K' \in \mathcal{M}$, and K' is the element such that the length of its side including e is larger or equal to $|e|$, we define

$$\pi_h^k v = v|_{K'} + \sum_{i=1}^{2} [(\pi_h^{m_i} v)(P_i) - (v|_{K'})(P_i)] \phi_{P_i} \quad \text{on } e. \qquad (3.7)$$

Since for $e \in \mathcal{E}^k$, $0 \leq m_i \leq k - 1$, $i = 1, 2$, (3.7) is well defined. Obviously, $(\pi_h^k v)(P_i) = (\pi_h^{m_i} v)(P_i)$, $i = 1, 2$.

We define $(\pi_h v)|_e = (\pi_h^k v)|_e$ if $e \in \mathcal{E}^k, 1 \leq k \leq L+1$. Then $\pi_h v$ is piecewise polynomial of order p and continuous on $\mathcal{E}^{\text{side}}$. Moreover, $\pi_h v = 0$ on $\partial\Omega$ if $v = 0$ on $\partial\Omega$. Having defined the $\pi_h v$ on $\mathcal{E}^{\text{side}}$ we now define $\pi_h v$ on each element $K \in \mathcal{M}$ as

$$\pi_h v = \sum_{P \in \mathcal{N}_p(K), P \notin \partial K} v(P) \phi_P + \sum_{P \in \mathcal{N}_p(K), P \in \partial K} (\pi_h v)(P) \phi_P.$$

Then $v - \pi_h v \in Q_p(K)$ and vanishes in all interior Gauss-Legrendre-Lobatto nodes, by the inverse trace inequality in Burman and Ern [9, Lemma 3.1], we have

$$\|v - \pi_h v\|_{L^2(K)} \leq C p^{-1} h_K^{1/2} \sum_{e \subset \partial K} \|v|_K - \pi_h v\|_{L^2(e)}. \qquad (3.8)$$

Let $e \subset \partial K$ and $e \in \mathcal{E}^k$ for some $1 \leq k \leq L+1$. There exists a conforming node P such that $e \in \mathcal{E}_P = \{e \in \mathcal{E}^{\text{side}} : e \subset \text{supp}(\psi_P)\}$. By definition, e has the end points $P_i \in \mathcal{N}^{m_i}, m_i \leq k - 1$, $i = 1, 2$, and one of m_1, m_2 is $k - 1$. If $P_i \notin \mathcal{N}^0$, then $\psi_P(P_i) \neq 0$ and it is a hanging node of some $e_i' \in \mathcal{E}^{m_i}$. The crucial observation is that $e_i' \in \mathcal{E}_P$. Thus by (3.7) and using (3.5) we have

$$\|v|_K - \pi_h v\|_{L^2(e)} = \|v|_K - \pi_h^k v\|_{L^2(e)}$$

$$\leq \|[\![v]\!]\|_{L^2(e)} + C p^{-1} h_K^{1/2} \sum_{i=1}^{2} |(v|_{e_i'} - \pi_h^{m_i} v)(P_i)|.$$

By the inverse estimate

$$|(v|_{e_i'} - \pi_h^{m_i} v)(P_i)| \leq \|v - \pi_h^{m_i} v\|_{L^\infty(e_i')} \leq C p h_K^{-1/2} \|v - \pi_h^{m_i} v\|_{L^2(e_i')}.$$

Combining above two inequalities we obtain

$$\|v|_K - \pi_h v\|_{L^2(e)} \leq \|[\![v]\!]\|_{L^2(e)} + C \max_{\substack{e' \in \mathcal{E}_P \\ e' \in \mathcal{E}^m, 1 \leq m \leq k-1}} \|v - \pi_h^m v\|_{L^2(e')}$$
$$+ Cp^{-1} h_K^{1/2} \max_{Q \in \mathcal{N}^0, Q \in \operatorname{supp}(\psi_P)} |(v - \pi_h^0 v)(Q)|.$$

By the mathematical induction, since $k \leq L+1$ and L is uniformly bounded according to (H3), we obtain

$$\|v|_K - \pi_h v\|_{L^2(e)} \leq \|[\![v]\!]\|_{\mathcal{E}_P} + C \max_{e' \in \mathcal{E}_P, e' \in \mathcal{E}^1} \|v - \pi_h^0 v\|_{L^2(e')}$$
$$+ Cp^{-1} h_K^{1/2} \max_{Q \in \mathcal{N}^0, Q \in \operatorname{supp}(\psi_P)} |(v - \pi_h^0 v)(Q)|$$
$$\leq \|[\![v]\!]\|_{\mathcal{E}_P} + Cp^{-1} h_K^{1/2} \max_{Q \in \mathcal{N}^0, Q \in \operatorname{supp}(\psi_P)} |(v - \pi_h^0 v)(Q)|,$$

where we have used (3.6) in the second estimate. Since $(\pi_h^0 v)(Q)$ is the local average of v sharing Q as the common vertex, we have

$$|(v - \pi_h^0 v)(Q)| \leq \sum_{Q \in \bar{e}', e' \in \mathcal{E}^{\text{side}}} \|[\![v]\!]\|_{L^\infty(e')} \leq C \sum_{Q \in \bar{e}', e' \in \mathcal{E}^{\text{side}}} \|ph^{-1/2} [\![v]\!]\|_{L^2(e')}.$$

By using the assumption (H3), we conclude that

$$\|v|_K - \pi_h v\|_{L^2(e)} \leq C \|[\![v]\!]\|_{L^2(\sigma(K))},$$

where $\sigma(K)$ is set of sides included in some $\widetilde{\omega}(K)$ which is a union of elements surrounding K whose diameter is bounded by Ch_K. This shows the first estimate of the lemma by (3.8). The second estimate can be proved by the standard inverse estimate

$$\|\nabla(v - \pi_h v)\|_{L^2(K)} \leq Cp^2 h_K^{-1} \|v - \pi_h v\|_{L^2(K)} \leq C \|ph^{-1/2} [\![v]\!]\|_{L^2(\sigma(K))}.$$

This completes the proof. □

Let Σ be a Lipschitz curve in $\mathbb{R}^2$, we recall the definition of the Aronszaja-Slobodeckij norm $\|v\|_{H^{1/2}(\Sigma)} = (\|v\|_{L^2(\Sigma)}^2 + |v|_{H^{1/2}(\Sigma)}^2)^{1/2}$, where

$$|v|_{H^{1/2}(\Sigma)}^2 = \int_\Sigma \int_\Sigma \frac{|v(x) - v(y)|^2}{|x - y|^2} ds(x) ds(y).$$

The following Gagliardo-Nirenberg type estimate for $H^{1/2}$-seminorm is well known (see e.g., Triebel [49]).

Lemma 3.3 *Let the interval* $(a, b) \subset \mathbb{R}$ *and* $v \in H^1(a, b)$. *Then* $|v|_{H^{1/2}(a,b)} \leq C \|v\|_{L^2(a,b)}^{1/2} \|v'\|_{L^2(a,b)}^{1/2}$ *for some constant* C *independent of* (a, b).

By definition, any function $v \in \mathbb{X}_p(\mathcal{M})$ can be written as $v = v_1 \chi_{\Omega_1} + v_2 \chi_{\Omega_2}$ for some $v_i \in \mathbb{V}_p(\mathcal{M}_i)$. In the following, we still denote by v_i the function in $\mathbb{V}_p(\mathcal{M})$ which is obtained by zero extension of v_i outside Ω_i^h, $i = 1, 2$.

Lemma 3.4 *There exists a linear operator $\pi_h^c : \mathbb{X}_p(\mathcal{M}) \to H^1(\Omega)$ such that*

$$\|a^{1/2}\nabla_h(v - \pi_h^c v)\|_{\mathcal{M}} \le C \left(\sum_{i=1}^{2} \|\hat{a}^{1/2}ph^{-1/2}[\![v_i]\!]\|_{\mathcal{E}_i^{\text{side}}} + \|\hat{a}^{1/2}ph^{-1/2}[\![v]\!]\|_{\mathcal{E}^\Gamma} \right)$$
$$+ C\|\hat{a}^{1/2}p^{-1}h^{1/2}\nabla_\Gamma[\![v]\!]\|_{\mathcal{E}^\Gamma}.$$

Here ∇_Γ is the tangential gradient on Γ. Moreover, $\pi_h^c v = \pi_h v_i$ on $\partial\Omega$ if $\partial\Omega_i \cap \partial\Omega \ne \emptyset$, $i = 1, 2$.

Proof Without loss of generality, we assume $a_1 \le a_2$. By Lemma 3.2, for $v_i \in \mathbb{V}_p(\mathcal{M}_i)$, $i = 1, 2$, there exists $\pi_h v_i \in \mathbb{V}_p(\mathcal{M}_i) \cap H^1(\Omega_i^h)$ such that for any $K \in \mathcal{M}_i$,

$$\|v_i - \pi_h v_i\|_{L^2(K)} \le C\|p^{-1}h^{1/2}[\![v_i]\!]\|_{L^2(\sigma(K))}, \qquad (3.9)$$

$$\|\nabla(v_i - \pi_h v_i)\|_{L^2(K)} \le C\|ph^{-1/2}[\![v_i]\!]\|_{L^2(\sigma(K))}. \qquad (3.10)$$

Let $w_1 \in H^1(\Omega_1)$ satisfy

$$-\Delta w_1 = 0 \ \text{ in } \Omega_1, \quad w_1 = [\![\pi_h v]\!]_\Gamma \ \text{ on } \Gamma, \quad w_1 = 0 \ \text{ on } \partial\Omega_1 \backslash \Gamma.$$

We define $\pi_h^c v := (\pi_h v_1 - w_1)\chi_{\Omega_1} + (\pi_h v_2)\chi_{\Omega_2}$. Obviously, $\pi_h^c v \in H^1(\Omega)$. By (3.10),

$$\|a^{1/2}\nabla_h(v - \pi_h^c v)\|_{\mathcal{M}}$$
$$\le C \left(a_1^{1/2}\|[\![\pi_h v]\!]\|_{H^{1/2}(\Gamma)} + \sum_{i=1}^{2} \|ph^{-1/2}[\![a_i^{1/2}v_i]\!]\|_{\mathcal{E}_i^{\text{side}}} \right). \qquad (3.11)$$

We now estimate $\|[\![\pi_h v]\!]\|_{H^{1/2}(\Gamma)}$. We know from the construction of the finite element space that $\Gamma = \cup_{K \in \mathcal{M}}\Gamma_K$. Since K is large with respect to both Ω_1, Ω_2, the partition $\{\Gamma_K, K \in \mathcal{M}\}$ of Γ is shape regular in the sense that

$$|\Gamma_K|/|\Gamma_{K'}| \le C_0, \quad \forall K, K' \in \mathcal{M}^\Gamma, \ \ K, K' \text{ are adjacent}. \qquad (3.12)$$

Let

$$\omega(\Gamma_K) = \cup\{\Gamma_{K'} : \bar{K}' \cap \bar{K} \ne \emptyset\}$$

be the set of neighboring curve segment of Γ_K. By the localization lemma of the $H^{1/2}$ semi-norm in Faermann [24, Lemma 2.3], we know that

$$|[\![\pi_h v]\!]|^2_{H^{1/2}(\Gamma)} \le \sum_{K \in \mathcal{M}} |[\![\pi_h v]\!]|^2_{H^{1/2}(\omega(\Gamma_K))} + C \sum_{K \in \mathcal{M}} h_K^{-1}\|[\![\pi_h v]\!]\|^2_{L^2(\Gamma_K)},$$

where the constant C depends on the Lipschitz constant of the curve Γ and the shape regularity constant C_0 in (3.12). Now by Lemma 3.3 we obtain easily

$$\sum_{K \in \mathcal{M}} |[\![\pi_h v]\!]|^2_{H^{1/2}(\omega(\Gamma_K))} \le C \sum_{K \in \mathcal{M}} \|[\![\pi_h v]\!]\|_{L^2(\Gamma_K)}\|\nabla_\Gamma[\![\pi_h v]\!]\|_{L^2(\Gamma_K)}.$$

Therefore,

$$
\begin{aligned}
&\|[\![\pi_h v]\!]\|^2_{H^{1/2}(\Gamma)} \\
&\le C \sum_{K \in \mathcal{M}} \left(\|[\![\pi_h v]\!]\|_{L^2(\Gamma_K)} \|\nabla_\Gamma [\![\pi_h v]\!]\|_{L^2(\Gamma_K)} + h_K^{-1} \|[\![\pi_h v]\!]\|^2_{L^2(\Gamma_K)} \right) \quad (3.13)
\end{aligned}
$$

It is easy to see that

$$
\|\nabla_\Gamma [\![\pi_h v]\!]\|_{L^2(\Gamma_K)} \le \sum_{i=1}^{2} \|\nabla(v_i - \pi_h v_i)\|_{L^2(\Gamma_K)} + \|\nabla_\Gamma [\![v]\!]\|_{L^2(\Gamma_K)}.
$$

By Lemma 2.6, the trace inequality (2.8), the inverse estimate, and Lemma 3.2 we have

$$
\begin{aligned}
&\|\nabla(v_i - \pi_h v_i)\|_{L^2(\Gamma_K)} \\
&\le C \left(h_K^{-1/2} \|\nabla(v_i - \pi_h v_i)\|_{L^2(K)} + \|\nabla(v_i - \pi_h v_i)\|^{1/2}_{L^2(K)} \|D^2(v_i - \pi_h v_i)\|^{1/2}_{L^2(K)} \right) \\
&\le C p h_K^{-1/2} \|\nabla(v_i - \pi_h v_i)\|_{L^2(K)} \\
&\le C p^2 h_K^{-1/2} \|h^{-1/2}[\![v_i]\!]\|_{L^2(\sigma(K))}.
\end{aligned}
$$

Thus

$$
\|\nabla_\Gamma [\![\pi_h v]\!]\|_{L^2(\Gamma_K)} \le \|\nabla_\Gamma [\![v]\!]\|_{L^2(\Gamma_K)} + C \sum_{i=1}^{2} p^2 h_K^{-1/2} \|h^{-1/2}[\![v_i]\!]\|_{L^2(\sigma(K))}.
$$

Similarly,

$$
\|[\![\pi_h v]\!]\|_{L^2(\Gamma_K)} \le \|[\![v]\!]\|_{L^2(\Gamma_K)} + C \sum_{i=1}^{2} \|[\![v_i]\!]\|_{L^2(\sigma(K))}.
$$

By substituting above two estimates into (3.13) we have

$$
\begin{aligned}
\|[\![\pi_h v]\!]_{H^{1/2}(\Gamma)} &\le C \|p h^{-1/2}[\![v]\!]\|_{L^2(\Gamma_K)} + C \|p^{-1} h^{1/2} \nabla_\Gamma [\![v]\!]\|_{L^2(\Gamma_K)} \\
&\quad + C \sum_{i=1}^{2} \|p h^{-1/2}[\![v_i]\!]\|_{\mathcal{E}^{\text{side}}}.
\end{aligned}
$$

This completes the proof by (3.11) and the fact that $a_1 \le \hat{a}_e \ \ \forall e \in \mathcal{E}^\Gamma \cup \mathcal{E}_1^{\text{side}}$, and $a_2 \le 2\hat{a}_e \ \ \forall e \in \mathcal{E}_2^{\text{side}}$. □

Let $U \in \mathbb{X}_p(\mathcal{M})$ be the solution of the problem (2.3), we define the element and jump residuals

$$
\begin{aligned}
R(U)|_K &= f + \text{div}_h(a \nabla_h U) \quad \forall K \in \mathcal{M}, \\
J(U)|_e &= [\![a \nabla_h U \cdot n]\!]_e \quad \forall e \in \mathcal{E}^{\text{side}} \cup \mathcal{E}^\Gamma.
\end{aligned}
$$

We also define the functions $\Lambda : \Pi_{K \in \mathcal{M}} L^2(K) \to \mathbb{R}$ and $\hat{\Lambda} : \Pi_{e \in \mathcal{E}} L^2(e) \to \mathbb{R}$ as

$$\Lambda|_K = \|a^{1/2}\|_{L^\infty(K)} \|a^{-1/2}\|_{L^\infty(\omega(K))} \quad \forall K \in \mathcal{M},$$
$$\hat{\Lambda}|_e = \max\{\Lambda_K : e \cap \bar{K} \neq \emptyset\} \quad \forall e \in \mathcal{E}.$$

Here $\omega(K)$ is defined in Remark 3.1. We remark that $\Lambda, \hat{\Lambda}$ are one on the elements or sides away from the interface.

The following theorem is the main result of this section.

Theorem 3.1 *Let $u \in H^1(\Omega)$ be the weak solution of (1.1)-(1.3) with $g \in H^1(\partial\Omega)$ and $U \in \mathbb{X}_p(\mathcal{M})$ be the solution of (2.3). Then there exists a constant C independent of the coefficient a, the mesh $\mathcal{M}$, the interface Γ, and the ratio $\max(a_1, a_2)/\min(a_1, a_2)$ such that*

$$\|u - U\|_{\mathrm{DG}} \le C \left(\sum_{K \in \mathcal{M}} \xi_K^2 \right)^{1/2},$$

where for each $K \in \mathcal{M}$, the local a posteriori error estimator

$$\xi_K^2 = \left(\|a^{-1/2}(h/p)\Lambda R(U)\|_K^2 + \|\hat{a}^{-1/2}(h/p)^{1/2}\hat{\Lambda}J(U)\|_{\mathcal{E}_K \cup \Gamma_K}^2 \right)$$
$$+ \left(\sum_{i=1}^2 \|\alpha^{1/2}\hat{\Lambda}[\![U_i]\!]\|_{\mathcal{E}_K^i}^2 + \|\alpha^{1/2}\hat{\Lambda}[\![U]\!]\|_{\Gamma_K}^2 + \|\alpha^{1/2}(U-g)\|_{\partial K \cap \partial\Omega}^2 \right)$$
$$+ \left(\|\hat{a}^{1/2}p^{-1}h^{1/2}\nabla_\Gamma[\![U]\!]\|_{\Gamma_K}^2 + \|\hat{a}^{1/2}p^{-1}h^{1/2}\nabla_{\partial\Omega}(U-g)\|_{\partial K \cap \partial\Omega}^2 \right),$$

$\nabla_{\partial\Omega}$ is the tangential derivative on the boundary $\partial\Omega$, $\mathcal{E}_K = \{e \in \mathcal{E}^{\mathrm{side}} : e \subset \partial K\}$, and $\mathcal{E}_K^i = \{e \in \mathcal{E}_i^{\mathrm{side}} : e \subset \partial K\}$.

We remark that by (2.4), the sum of the second term in ξ_K^2 over $K \in \mathcal{M}$ is equivalent to $\|\alpha^{1/2}[\![u-U]\!]\|_{\bar{\mathcal{E}}}^2$ up to the factor $\hat{\Lambda}^2$. The sum of the third term in ξ_K^2 over $K \in \mathcal{M}$ is roughly of the same order as the sum of the second term. The local lower bounds of the first term in ξ_K^2 will be studied in the next section.

We also remark that the factors $\Lambda, \hat{\Lambda}$ in the theorem are absent in the a posteriori error estimate in Cai, Ye and Zhang [14] under the assumption that the mesh fits the interface and the coefficient is quasi-monotone with respect to each node of the mesh. The quasi-monotone property of the diffusion coefficient was first introduced in Petzoldt [44] and it also played an important role in Chen and Dai [16] for the study of coefficient robust a posteriori error estimates for conforming finite element methods.

Proof Let $\tilde{U} \in H^1(\Omega)$ satisfy $\tilde{U} = g$ on $\partial\Omega$, and

$$\int_\Omega a\nabla\tilde{U} \cdot \nabla v \, dx = \int_\Omega a\nabla_h U \cdot \nabla v \, dx \quad \forall v \in H_0^1(\Omega). \tag{3.14}$$

By the Lax-Milgram lemma, $\tilde{U} \in H^1(\Omega)$ is well defined. By the triangle inequality, we have

$$
\begin{aligned}
&\|u - U\|_{\mathrm{DG}} \\
&\leq \|u - \tilde{U}\|_{\mathrm{DG}} + \|U - \tilde{U}\|_{\mathrm{DG}} \\
&\leq \|a^{1/2}\nabla(u - \tilde{U})\|_{\mathcal{M}} + \|a^{1/2}\nabla_h(U - \tilde{U})\|_{\mathcal{M}} + \|\alpha^{1/2}[\![U - \tilde{U}]\!]\|_{\bar{\mathcal{E}}}. \quad (3.15)
\end{aligned}
$$

By the definition in (2.4)

$$
\|\alpha^{1/2}[\![U - \tilde{U}]\!]\|_{\bar{\mathcal{E}}}^2 = \sum_{i=1}^{2} \|\alpha^{1/2}[\![U_i]\!]\|_{\mathcal{E}_i^{\mathrm{side}}}^2 + \|\alpha^{1/2}[\![U]\!]\|_{\mathcal{E}^{\Gamma}}^2 + \|\alpha^{1/2}(U - g)\|_{\mathcal{E}^{\mathrm{bdy}}}^2.
$$

Thus we are left to bound the first two terms in (3.15) since $\hat{\Lambda} \geq 1$ on $\mathcal{E}$.

1° We first estimate the conforming component $\|a^{1/2}\nabla(u - \tilde{U})\|_{\mathcal{M}}$ of the error. For any $w \in H_0^1(\Omega)$, we take $w_h = \Pi_h w \in \mathbb{V}_p(\mathcal{M}) \cap H_0^1(\Omega) \subset \mathbb{X}_p(\mathcal{M})$. Since $\mathsf{L}(w_h) = 0$ we obtain from the discrete equation (2.3) that

$$
(a\nabla_h U, \nabla_h w_h)_{\mathcal{M}} - (a\mathsf{L}(U), \nabla_h w_h)_{\mathcal{M}} = (f, w_h)_{\mathcal{M}} - (a\mathsf{L}_1(g), \nabla_h w_h)_{\mathcal{M}}.
$$

This yields by (3.14) that

$$
\begin{aligned}
(a\nabla(u - \tilde{U}), \nabla w)_{\mathcal{M}} &= (f, w)_{\mathcal{M}} - (a\nabla_h U, \nabla w)_{\mathcal{M}} \\
&= (f, w - w_h)_{\mathcal{M}} - (a\nabla_h U, \nabla(w - w_h))_{\mathcal{M}} \\
&\quad - (a\mathsf{L}(U), \nabla_h w_h)_{\mathcal{M}} + (a\mathsf{L}_1(g), \nabla_h w_h)_{\mathcal{M}}.
\end{aligned}
$$

Since $w - w_h \in H_0^1(\Omega)$, by doing integration by parts we have

$$
\begin{aligned}
(a\nabla(u - \tilde{U}), \nabla w)_{\mathcal{M}} &= (R(U), w - w_h)_{\mathcal{M}} - \langle J(U), w - w_h \rangle_{\mathcal{E}} \\
&\quad - (a\mathsf{L}(U), \nabla_h w_h)_{\mathcal{M}} + (a\mathsf{L}_1(g), \nabla_h w_h)_{\mathcal{M}} \\
&:= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4.
\end{aligned}
$$

By Lemma 3.1 we have

$$
\begin{aligned}
|\mathrm{I}_1 + \mathrm{I}_2| &\leq C\|a^{-1/2}(h/p)\Lambda R(U)\|_{\mathcal{M}} \|a^{1/2}\nabla w\|_{\mathcal{M}} \\
&\quad + C\|\hat{a}^{-1/2}(h/p)^{1/2}\hat{\Lambda}J(U)\|_{\mathcal{E}^{\mathrm{side}} \cup \mathcal{E}^{\Gamma}} \|a^{1/2}\nabla w\|_{\mathcal{M}}.
\end{aligned}
$$

Moreover, by Lemma 3.1,

$$
\begin{aligned}
|\mathrm{I}_3 + \mathrm{I}_4| &= |-\langle [\![U]\!], \widehat{a\nabla_h w_h} \cdot n \rangle_{\mathcal{E}} + \langle g, \widehat{a\nabla_h w_h} \cdot n \rangle_{\mathcal{E}^{\mathrm{bdy}}}| \\
&\leq C\left(\|\alpha^{1/2}\hat{\Lambda}[\![U]\!]\|_{\mathcal{E}^{\mathrm{side}} \cup \mathcal{E}^{\Gamma}} + \|\alpha^{1/2}(U - g)\|_{\mathcal{E}^{\mathrm{bdy}}}\right) \|a^{1/2}\Lambda^{-1}\nabla_h w_h\|_{\mathcal{M}} \\
&\leq C\left(\|\alpha^{1/2}\hat{\Lambda}[\![U]\!]\|_{\mathcal{E}^{\mathrm{side}} \cup \mathcal{E}^{\Gamma}} + \|\alpha^{1/2}(U - g)\|_{\mathcal{E}^{\mathrm{bdy}}}\right) \|a^{1/2}\nabla w\|_{\mathcal{M}}.
\end{aligned}
$$

This shows

$$
\begin{aligned}
&\|a^{1/2}\nabla(u - \tilde{U})\|_{L^2(\Omega)} \\
&\leq C\|a^{-1/2}(h/p)\Lambda R(U)\|_{\mathcal{M}} + C\|\hat{a}^{-1/2}(h/p)^{1/2}\hat{\Lambda}J(U)\|_{\mathcal{E}^{\mathrm{side}} \cup \mathcal{E}^{\Gamma}} \\
&\quad + C\|\alpha^{1/2}\hat{\Lambda}[\![U]\!]\|_{\mathcal{E}^{\mathrm{side}} \cup \mathcal{E}^{\Gamma}} + C\|\alpha^{1/2}(U - g)\|_{\mathcal{E}^{\mathrm{bdy}}}. \quad (3.16)
\end{aligned}
$$

$2°$ We next estimate the nonconforming component $\|a^{1/2}\nabla_h(U-\tilde{U})\|_{\mathcal{M}}$ of the error in (3.15). By (3.14) we know that

$$\|a^{1/2}\nabla_h(U-\tilde{U})\|_{\mathcal{M}} \leq \inf_{\substack{w\in H^1(\Omega) \\ w=g \text{ on } \partial\Omega}} \|a^{1/2}\nabla_h(U-w)\|_{\mathcal{M}}$$

$$\leq \|a^{1/2}\nabla_h(U-\pi_h^c U)\|_{\mathcal{M}} + \inf_{\substack{w\in H^1(\Omega) \\ w=g \text{ on } \partial\Omega}} \|a^{1/2}\nabla(\pi_h^c U - w)\|_{\mathcal{M}}.$$

Let $\psi \in H^1(\Omega)$ satisfy $-\Delta\psi = 0$ in Ω, $\psi = \pi_h^c U - g \in H^{1/2}(\partial\Omega)$. Then $\|\psi\|_{H^1(\Omega)} \leq C\|\pi_h^c U - g\|_{H^{1/2}(\partial\Omega)}$. Thus $w = \pi_h^c U - \psi \in H^1(\Omega)$ satisfies $w = g$ on $\partial\Omega$, which yields

$$\inf_{\substack{w\in H^1(\Omega) \\ w=g \text{ on } \partial\Omega}} \|a^{1/2}\nabla(\pi_h^c U - w)\|_{\mathcal{M}} \leq Ca_j^{1/2}\|\pi_h^c U - g\|_{H^{1/2}(\partial\Omega)},$$

where $j = 1,2$ such that $\partial\Omega_j \cap \partial\Omega \neq \emptyset$. Similar to the argument in the proof of Lemma 3.4, we can use the localization lemma of the $H^{1/2}$ semi-norm in Faermann [24, Lemma 2.3] and Lemma 3.3 to obtain

$$\|\pi_h^c U - g\|_{H^{1/2}(\partial\Omega)}$$
$$\leq C(\|ph^{-1/2}(\pi_h^c U - g)\|_{\mathcal{E}^{\text{bdy}}} + \|p^{-1}h^{1/2}\nabla_{\partial\Omega}(\pi_h^c U - g)\|_{\mathcal{E}^{\text{bdy}}}).$$

Since by Lemma 3.4, $\pi_h^c U = \pi_h U_j$ on $\partial\Omega$ for $\partial\Omega_j \cap \partial\Omega \neq \emptyset$, we have by the triangle inequality that

$$\|\pi_h^c U - g\|_{H^{1/2}(\partial\Omega)}$$
$$\leq C(\|ph^{-1/2}(\pi_h U_j - U_j)\|_{\mathcal{E}^{\text{bdy}}} + \|p^{-1}h^{1/2}\nabla_{\partial\Omega}(\pi_h U_j - U_j)\|_{\mathcal{E}^{\text{bdy}}})$$
$$+ C(\|ph^{-1/2}(U - g)\|_{\mathcal{E}^{\text{bdy}}} + \|p^{-1}h^{1/2}\nabla_{\partial\Omega}(U - g)\|_{\mathcal{E}^{\text{bdy}}}).$$

By inverse trace inequality in Lemma 2.7 and Lemma 3.2,

$$\|ph^{-1/2}(\pi_h U_j - U_j)\|_{\mathcal{E}^{\text{bdy}}} + \|p^{-1}h^{1/2}\nabla_{\partial\Omega}(\pi_h U_j - U_j)\|_{\mathcal{E}^{\text{bdy}}}$$
$$\leq C(\|p^2 h^{-1}(\pi_h U_j - U_j)\|_{\mathcal{M}_j} + \|\nabla_h(\pi_h U_j - U_j)\|_{\mathcal{M}_j}$$
$$\leq C\|ph^{-1/2}[\![U_j]\!]\|_{\mathcal{E}_j^{\text{side}}}.$$

Combining above estimates and using Lemma 3.4, we conclude

$$\|a^{1/2}\nabla_h(U-\tilde{U})\|_{\mathcal{M}}$$
$$\leq C\left(\sum_{i=1}^2 \|\alpha^{1/2}[\![U_i]\!]\|_{\mathcal{E}_i^{\text{side}}} + \|\alpha^{1/2}[\![U]\!]\|_{\mathcal{E}^{\Gamma}} + \|\alpha^{1/2}(U-g)\|_{\mathcal{E}^{\text{bdy}}}\right)$$
$$+ C\left(\|\hat{a}^{1/2}p^{-1}h^{1/2}\nabla_{\Gamma}[\![U]\!]\|_{\mathcal{E}^{\Gamma}} + \|\hat{a}^{1/2}p^{-1}h^{1/2}\nabla_{\partial\Omega}(U-g)\|_{\mathcal{E}^{\text{bdy}}}\right).$$

This completes the proof by (3.15) and (3.16). $\square$

To conclude this section we refer to Sacchi and Veeser [45] for a different approach to deal with the non-homogeneous Dirichlet boundary condition in the finite element a posteriori error analysis where the localization of the $H^{1/2}$ semi-norm also plays a crucial role.

4 A posteriori error estimation: efficiency

In this section we derive the lower bound of the a posteriori error estimate proved in Theorem 3.1 by using the domain inverse estimate in Lemma 2.5. We start with the residual $R(U)$.

Lemma 4.1 *For any $K \in \mathcal{M}$, there exists a constant C independent of p and K such that*

$$
(h_K/p)\|a^{-1/2}R(U)\|_{L^2(K)}
$$
$$
\leq C\Theta_K^{1/2}\left(p\|a^{1/2}\nabla_h(u-U)\|_{L^2(K)} + (h_K/p)\|a^{-1/2}(f-f_K)\|_{L^2(K)}\right),
$$

where $f_K = P_K(f|_K)$, $P_K : L^2(K) \to Q_{p-1}(K)$ is the L^2 projection operator and Θ_K is defined in (2.14).

Proof Without loss of generality, we only consider the case when Γ intersects with ∂K at two opposite sides. We also use the notation in Lemma 2.8, see Figure 2.4. Denote $V = f_K + \mathrm{div}_h(a\nabla_h U)$ in K. Since $K_i \subset \Delta A_i CD \cup \Delta A_i B''C''$, by Lemma 2.4,

$$
\|V\|_{L^2(K_i)} \leq \|V\|_{L^2(\Delta A_i CD)} + \|V\|_{L^2(\Delta A_i B''C'')}
$$
$$
\leq \|V\|_{L^2(\Delta A_i CD)} + \mathsf{T}\left(\frac{1+3\eta_K}{1-\eta_K}\right)^{2p+3/2}\|V\|_{L^2(\Delta)}, \quad (4.1)
$$

where $\Delta = \Delta A_i B'C'$ which is shape regular and $h_\Delta \geq Ch_K$. For any $\epsilon > 0$ sufficiently small, denote $\Delta_\epsilon = \{x \in \Delta : \mathrm{dist}(x, \partial\Delta) > \epsilon\}$ and $\chi_\epsilon \in C_0^\infty(\Delta)$ the cut-off function such that $\chi_\epsilon = 1$ in Δ_ϵ, $0 \leq \chi_\epsilon \leq 1$, and $|\nabla\chi_\epsilon| \leq C\epsilon^{-1}$ in Δ.

Let $v = V\chi_\epsilon \in H_0^1(\Delta) \subset H_0^1(K_i)$. Since $\Delta \subset K_i$ in which $a = a_i$, by the domain inverse estimate in Lemma 2.5

$$
\|V\|_{L^2(\Delta)} \leq C(1 + C\sqrt{\epsilon/h_K})^{2p}\|V\|_{L^2(\Delta_\epsilon)}. \quad (4.2)
$$

On the other hand, since the solution u satisfies (1.1)-(1.3),

$$
\|V\|_{L^2(\Delta_\epsilon)}^2 \leq \int_\Delta V^2\chi_\epsilon dx
$$
$$
= \int_\Delta (f_K + \mathrm{div}(a\nabla U))v dx
$$
$$
= \int_\Delta (f_K - f)v dx + \int_\Delta a\nabla(u-U)\cdot\nabla v dx.
$$

Since $\nabla V \in Q_{p-2}(\Delta)$, by the inverse estimate,

$$
\|\nabla v\|_{L^2(\Delta)} \leq \|\nabla V\|_{L^2(\Delta)} + C\epsilon^{-1}\|V\|_{L^2(\Delta)}
$$
$$
\leq C(p^2 h_K^{-1} + \epsilon^{-1})\|V\|_{L^2(\Delta)}.
$$

Thus if we choose $\epsilon = c_0 h_K (p+1)^{-2}$ for some constant $c_0 > 0$ depending possibly on $\delta_0 \in (0, 1/2)$ in Definition 2.1 so that $\epsilon < \rho_\Delta / 2$, where ρ_Δ is the radius of the maximal inscribed circle of Δ, we obtain

$$\|V\|_{L^2(\Delta_\epsilon)}^2 \leq C \|a^{-1/2}(f - f_K)\|_{L^2(\Delta)} \|a_i^{1/2} V\|_{L^2(\Delta)}$$
$$+ C p^2 h_K^{-1} \|a^{1/2} \nabla(u - U)\|_{L^2(\Delta)} \|a_i^{1/2} V\|_{L^2(\Delta)}.$$

Noticing that $(1 + C\sqrt{\epsilon/h_K})^{2p} \leq (1 + Cp^{-1})^{2p} \leq C$, by (4.2) we have

$$(h_K/p)\|V\|_{L^2(\Delta)}$$
$$\leq C\Theta_K^{1/2} \left((h_K/p)\|a^{-1/2}(f - f_K)\|_{L^2(\Delta)} + p\|a^{1/2}\nabla(u - U)\|_{L^2(\Delta)} \right).$$

A similar argument shows the same estimate holds when Δ is replaced by $\Delta A_i C D$. This completes the proof by (4.1). $\square$

To derive a lower bound for the jump residual, we need the following extension lemma.

Lemma 4.2 *Let $\mathcal{D}$ be a bounded Lipschitz domain in $\mathbb{R}^d$ $(d \geq 2)$. For any $g \in H^1(\partial\mathcal{D})$ and any $\epsilon > 0$, there exists a function $v \in H^1(\mathcal{D})$ such that $v = g$ on $\partial\mathcal{D}$, and*

$$\|v\|_{L^2(\mathcal{D})} \leq C\epsilon \|g\|_{L^2(\partial\mathcal{D})}, \quad \|\nabla v\|_{L^2(\mathcal{D})} \leq C\epsilon^{-1}\|g\|_{L^2(\partial\mathcal{D})} + C\epsilon|g|_{H^1(\partial\mathcal{D})},$$

where the constant C depends on the Lipschitz constant of $\partial\mathcal{D}$ and is independent of v and ϵ.

Proof The proof depends on the classical argument of flattening the boundary. Since $\partial\mathcal{D}$ is Lipschitz continuous, there is a set of sub-domains $\{U_j\}_{j=1}^r$ that covers $\partial\mathcal{D}$ and a partition of unity $\{\phi_j\}_{j=1}^r$ subordinated to $\{U_j\}_{j=1}^r$, that is, $\phi_j \in C_0^\infty(U_j)$, $0 \leq \phi_j \leq 1$, $\sum_{j=1}^r \phi_j = 1$ in $\cup_{j=1}^r U_j$. Moreover, there exist bi-jective Lipschitz mappings $\Phi_j : U_j \to V_j$, $V_j \subset \mathbb{R}^d$, such that $\Phi_j(\mathcal{D} \cap U_j) = \mathbb{R}_+^d \cap V_j$ and $\Phi_j(U_j \cap \partial\mathcal{D}) = \partial\mathbb{R}_+^d \cap V_j$, $j = 1, \cdots, r$, see e.g., Evans [23, §C.1]. Here $\mathbb{R}_+^d = \{x \in \mathbb{R}^d : x_d > 0\}$.

For any $y = (y', 0)^T \in \partial\mathbb{R}_+^d \cap V_j$, $j = 1, \cdots, r$, let $\hat{g}_j(y') = g(\Phi_j^{-1}(y'))$. We define the extension of $\hat{g}_j$ by

$$\hat{v}_j(y', y_d) = \hat{g}_j(y')e^{-\frac{y_d}{\epsilon^2}}, \quad \forall y = (y', y_d)^T \in V_j.$$

It is easy to see that

$$\|\hat{v}_j\|_{L^2(V_j \cap \mathbb{R}_+^d)} \leq \epsilon \|\hat{g}_j\|_{L^2(\partial\mathbb{R}_+^d \cap V_j)},$$
$$\|\nabla_y \hat{v}_j\|_{L^2(V_j \cap \mathbb{R}_+^d)} \leq \epsilon |\hat{g}_j|_{H^1(\partial\mathbb{R}_+^d \cap V_j)} + \epsilon^{-1}\|\hat{g}_j\|_{L^2(\partial\mathbb{R}_+^d \cap V_j)}.$$

This completes the proof by letting $v(x) = \sum_{j=1}^r \hat{v}_j(\Phi_j(x))\phi_j(x), \forall x \in \mathcal{D} \cap \left(\cup_{j=1}^r U_j\right)$ and $v(x) = 0, \forall x \in \mathcal{D} \backslash \left(\cup_{j=1}^r \bar{U}_J\right)$. $\square$

For any $K \in \mathcal{M}$, let $L_K = |\Gamma_K|$ and $\Phi_K : (0, L_K) \to \Gamma_K$ be the arc length parametrization of Γ_K. We define the L^2 projection $P_{\Gamma_K} : L^2(\Gamma_K) \to Q_p(\Gamma_K) = Q_p(0, L_K) \circ \Phi_K^{-1}$ as follows: For any $g \in L^2(\Gamma_K)$, $P_{\Gamma_K} g \circ \Phi_K \in Q_p(0, L_K)$ such that

$$\int_0^{L_K} (P_{\Gamma_K} g \circ \Phi_K) v ds = \int_0^{L_K} (g \circ \Phi_K) v ds, \quad \forall v \in Q_p(0, L_K).$$

Lemma 4.3 *For any $K \in \mathcal{M}^\Gamma$, there exists a constant C independent of p and K such that*

$$(h_K/p)^{1/2} \|\hat{a}^{-1/2} J(U)\|_{L^2(\Gamma_K)}$$
$$\leq C (h_K/p)^{1/2} \|\hat{a}^{-1/2}(J(U) - J_{\Gamma_K}(U))\|_{L^2(\Gamma_K)}$$
$$+ C \left(p \|a^{1/2} \nabla_h(u - U)\|_{L^2(K)} + \|a^{-1/2}(h/p) R(U)\|_{L^2(K)} \right),$$

where $J_{\Gamma_K}(U) = P_{\Gamma_K}(J(U)|_{\Gamma_K})$.

Proof Let $\sigma = c_0(p+1)^{-2} h_K$ for some constant $c_0 > 0$ such that σ is less than half of the minimum length of the sides of K, and denote $K_\sigma = \{x \in K : \text{dist}(x, \partial K) > \sigma\}$. Let $(t_1, t_2) \subset (0, L_K)$ such that Φ_K maps (t_1, t_2) to $\Gamma_K \cap K_\sigma$. Obviously, $t_1 \leq C_1 \sigma, L_K - t_2 \leq C_2 \sigma$ for some constants $C_1, C_2 > 0$. Since $J_{\Gamma_K} \circ \Phi_K^{-1} \in Q_p(0, L_K)$, we use the domain inverse Lemma 2.3 to obtain

$$\|J_{\Gamma_K}(U)\|_{L^2(\Gamma_K)} \leq C \|J_{\Gamma_K}(U) \circ \Phi_K^{-1}\|_{L^2(0, L_K)}$$
$$\leq C \mathsf{T}(1 + C\sigma/L_K)^{2p+1} \|J_{\Gamma_K}(U) \circ \Phi_K^{-1}\|_{L^2(t_1, t_2)}$$
$$\leq C \|J_{\Gamma_K}(U)\|_{L^2(\Gamma_K \cap K_\sigma)}, \qquad (4.3)$$

where we have used the fact that $\mathsf{T}(\lambda) = 1 + \sqrt{\lambda - 1}(\sqrt{\lambda - 1} + \sqrt{1 + \lambda})$ and $(1 + C\sqrt{\sigma/h_K})^{2p+1} = (1 + Cp^{-1})^{2p+1} \leq C$ for some constant C independent of p.

Since $J_{\Gamma_K}(U) \circ \Phi_K^{-1} \in Q_p(0, L_K)$, by the inverse estimate we have

$$\|\nabla_\Gamma J_{\Gamma_K}(U)\|_{L^2(\Gamma_K)} \leq C |J_{\Gamma_K}(U) \circ \Phi_K^{-1}|_{H^1(0, L_K)}$$
$$\leq C p^2 L_K^{-1} \|J_{\Gamma_K}(U) \circ \Phi_K^{-1}\|_{L^2(0, L_K)}$$
$$\leq C p^2 h_K^{-1} \|J_{\Gamma_K}(U)\|_{L^2(\Gamma_K)}. \qquad (4.4)$$

Let $\chi_\sigma \in C_0^\infty(K_\sigma)$ be the cut-off function satisfying $\chi_\sigma = 1$ in K_σ, $0 \leq \chi \leq 1$, $|\nabla \chi_\sigma| \leq C\sigma^{-1}$ in K. Let $v_\sigma \in H^1(\Omega)$ be such that $v_\sigma|_{\Omega_i} \in H^1(\Omega_i), i = 1, 2$, is the extension of $J_{\Gamma_K}(U)\chi_\sigma \in H^1(\Gamma)$ defined in Lemma 4.2 with $\epsilon = \sqrt{h_K/p}$, then

$$\|v_\sigma\|_{L^2(\Omega_i)} \leq C(h_K/p)^{1/2} \|J_{\Gamma_K}(U)\chi_\sigma\|_{L^2(\Gamma)} \leq C(h_K/p)^{1/2} \|J_{\Gamma_K}(U)\|_{L^2(\Gamma_K)},$$

and

$$\|\nabla v_\sigma\|_{L^2(\Omega_i)}$$
$$\leq C(h_K/p)^{-1/2} \|J_{\Gamma_K}(U)\chi_\sigma\|_{L^2(\Gamma_K)} + C(h_K/p)^{1/2} \|\nabla_\Gamma(J_{\Gamma_K}(U)\chi_\sigma)\|_{L^2(\Gamma_K)}$$
$$\leq C p (h_K/p)^{-1/2} \|J_{\Gamma_K}(U)\|_{L^2(\Gamma_K)},$$

where we have used (4.4) in the last inequality. Let $w_\sigma = v_\sigma \chi_\sigma$. Then $w_\sigma \in H^1_0(K)$ and satisfies

$$\|w_\sigma\|_{L^2(K)} \leq C(h_K/p)^{1/2}\|J_{\Gamma_K}(U)\|_{L^2(\Gamma_K)}, \qquad (4.5)$$

$$\|\nabla w_\sigma\|_{L^2(K)} \leq Cp(h_K/p)^{-1/2}\|J_{\Gamma_K}(U)\|_{L^2(\Gamma_K)}. \qquad (4.6)$$

Now by (4.3)

$$\|J_{\Gamma_K}(U)\|^2_{L^2(\Gamma_K)} \leq C\int_{\Gamma_K}|J_{\Gamma_K}(U)|^2\chi_\sigma^2 = C\int_{\Gamma_K}J_{\Gamma_K}(U)\cdot w_\sigma.$$

By using the equation (1.1)-(1.3) and integration by parts

$$\|J_{\Gamma_K}(U)\|^2_{L^2(\Gamma_K)} \leq C\|J(U) - J_{\Gamma_K}(U)\|_{L^2(\Gamma_K)}\|w_\sigma\|_{L^2(\Gamma_K)} + \int_{\Gamma_K}J(U)w_\sigma$$

$$\leq C\|(h/p)^{1/2}(J(U) - J_{\Gamma_K}(U))\|_{L^2(\Gamma_K)}\|J_{\Gamma_K}(U)\|_{L^2(\Gamma_K)}$$

$$+ \int_K a\nabla_h(U-u)\cdot\nabla w_\sigma - \int_K R(U)w_\sigma.$$

This completes the proof by using (4.5)-(4.6). □

The following lemma can be proved by the method in Lemma 4.3. We omit the details.

Lemma 4.4 *For any $e \in \mathcal{E}^{\mathrm{side}}$, $e = \partial K \cap \partial K'$, $K, K' \in \mathcal{M}$, we have*

$$\|\hat{a}^{-1/2}(h/p)^{1/2}J(U)\|_{L^2(e)}$$
$$\leq C\|\hat{a}^{-1/2}(h/p)^{1/2}(J(U) - P_eJ(U))\|_{L^2(e)} + Cp\|a^{1/2}\nabla_h(u-U)\|_{L^2(K\cup K')}$$
$$+ C\|a^{-1/2}(h/p)R(U)\|_{L^2(K\cup K')} + C\|\hat{a}^{-1/2}(h/p)^{1/2}J(U)\|_{L^2(\Gamma_K\cup\Gamma_{K'})},$$

where $P_e : L^2(e) \to Q_p(e)$ is the L^2 projection operator.

Let $\mathbb{P} : \Pi_{K\in\mathcal{M}}L^2(K) \to \mathbb{V}_{p-1}(\mathcal{M})$ be defined elementwise as $\mathbb{P}|_K = P_K$ and $\mathbb{Q} : \Pi_{e\in\mathcal{E}}L^2(e) \to Q_p(e)$ be defined as $\mathbb{Q}|_e = P_e$.

The following theorem which is the main result of this section can be proved by combining Lemma 4.1, Lemma 4.3 and Lemma 4.4.

Theorem 4.1 *Let $u \in H^1(\Omega)$ be the weak solution of (1.1)-(1.3) and $U \in \mathbb{X}_p(\mathcal{M})$ be the solution of (2.3). We have*

$$\|a^{-1/2}\Theta^{-1}(h/p)R(U)\|_{\mathcal{M}} + \|\alpha^{-1/2}(h/p)^{1/2}J(U)\|_{\mathcal{E}}$$
$$\leq C\left(p\|a^{1/2}\nabla_h(u-U)\|_{\mathcal{M}} + \mathrm{osc}(f,U,\Gamma)\right).$$

where $\mathrm{osc}(f,U,\Gamma)$ is the data oscillation defined as

$$\mathrm{osc}(f,U,\Gamma) = \|a^{-1/2}(h/p)(f - \mathbb{P}f)\|_{\mathcal{M}} + \|\hat{a}^{-1/2}(h/p)^{1/2}(J(U) - \mathbb{Q}J(U))\|_{\mathcal{E}}.$$

We remark that the factor p in the front of $\|a^{1/2}\nabla_h(u-U)\|_{\mathcal{M}}$ is well-known for residual type hp a posteriori error estimates, see Melenk and Wohlmuth [39], in which hp a posteriori error estimation was first studied for elliptic equations on conforming meshes based on polynomial inverse estimates. Our argument is different by using the domain inverse estimate and is slightly better in the sense that the additional factor p^ϵ in the local lower bound in [39] is removed in our analysis.

5 Numerical examples

In this section, we present several numerical examples to illustrate the performance of the proposed adaptive unfitted finite element method. The computations are carried out using MATLAB on a workstation with Intel(R) i9-9900 CPU 2.70GHz and 64GB memory. The basis functions of $Q_p(K)$ are the Lagrange interpolation polynomials through the local Gauss-Lobatto-Legendre (GLL) integration points in each element K.

For each $K \in \mathcal{M}$, we compute the local a posteriori error estimator ξ_K as in Theorem 3.1 and define the global a posteriori error estimate $\eta = \left(\sum_{K\in\mathcal{M}} \xi_K^2\right)^{1/2}$.

We first describe the adaptive unfitted finite element algorithm.

Algorithm 5.1 Given a tolerance TOL > 0, $N_0 \geq 1$ a fixed number, and an initial conforming Cartesian mesh $\mathcal{T}$.

1. Construct the induced mesh $\mathcal{M}$ by Algorithm 5.2 so that each element K in $\mathcal{M}$ is large with respect to both Ω_1, Ω_2 and satisfies (2.1).
2. Solve the discrete problem (2.3) on $\mathcal{M}$.
3. Compute the local error estimator ξ_K on each $K \in \mathcal{M}$ and the global error estimate η.
4. While $\eta > $ TOL do
   − Mark the elements in $\hat{\mathcal{M}} \subset \mathcal{M}$ such that:

$$\left(\sum_{K\in\hat{\mathcal{M}}} \xi_K^2\right)^{1/2} \geq \frac{1}{2}\eta.$$

   − Refine the elements in $\hat{\mathcal{T}} = \{K \in \mathcal{T} : K \subset K', K' \in \hat{\mathcal{M}}\}$ by quad refinement to obtain a new mesh $\hat{\hat{\mathcal{T}}}$.
   − Refine $\hat{\hat{\mathcal{T}}}$ to obtain a new mesh $\mathcal{T}$ such that each side of $\mathcal{T}$ includes at most N_0 hanging nodes, which makes $\mathcal{T}$ a K-mesh satisfying the Assumption (H3).
   − Construct the induced mesh $\mathcal{M}$ by Algorithm 5.2 so that each element $K \in \mathcal{M}$ is large with respect to both Ω_1, Ω_2, and $\mathcal{M}$ satisfies (2.1).
   − Solve the discrete problem (2.3) on $\mathcal{M}$.
   − Compute the local error estimator ξ_K on each $K \in \mathcal{M}$ and the global error estimate η.

end while

The following algorithm is used to construct the induced mesh $\mathcal{M}$ from a Cartesian mesh $\mathcal{T}$ so that each element K in $\mathcal{M}$ is large with respect to both Ω_1, Ω_2 and $\mathcal{M}$ satisfies (2.1). We use the notation $\mathcal{T}_i^{\text{large}} := \{K \in \mathcal{T} : K \text{ is large with respect to } \Omega_i\}, i = 1, 2$, according to Definition 2.1 with the parameter $\delta_0 \in (0, 1/2)$.

Algorithm 5.2 Given $\delta_0 \in (0, 1/2)$, $N_0 \geq 1$ a fixed number, and a Cartesian mesh $\mathcal{T}$.

1. Mark all small elements in $\mathcal{T}_{\text{small}} \subset \mathcal{T}$, where

$$\mathcal{T}_{\text{small}} = \{K \in \mathcal{T} : K \cap \Gamma \neq \emptyset, K \notin \mathcal{T}_1^{\text{large}} \cap \mathcal{T}_2^{\text{large}}\}.$$

2. If $\mathcal{T}_{\text{small}} \neq \emptyset$, for each $K \in \mathcal{T}_{\text{small}}$, $K \notin \mathcal{T}_i^{\text{large}}$, $i = 1, 2$, do
   – If K has a neighboring element $K' \in \mathcal{T}_i^{\text{large}}$ whose size is the same as that of K and the minimum rectangle containing K, K' is large with respect to Ω_i, then merge K and K'.
   – Else if K has a neighboring element $K' \in \mathcal{T}_i^{\text{large}}$ whose size is larger than that of K, add K' to $\mathcal{T}_{\text{refine}}$.
   – Else if K has a neighboring element $K' \in \mathcal{T}_i^{\text{large}}$ whose size is smaller than that of K, add K to $\mathcal{T}_{\text{refine}}$.
   – Otherwise, add K and all its neighboring elements in $\mathcal{T}_i^{\text{large}}$ to $\mathcal{T}_{\text{refine}}$.
3. While $K \in \mathcal{T} \backslash \mathcal{T}_{\text{small}}$, $\eta_K > \max(1/2, (1 - \delta_0)/(1 + \delta_0))$, do $i = 1, 2$
   – If K does not include singular points of Γ or K is an irregular large element with respect to Ω_i, add K to $\mathcal{T}_{\text{refine}}$.
   – Else if K has two vertices in Ω_i and there exists a neighboring element $K' \subset \Omega_i$ whose size is the same as that of K, then merge K and K'.
   – Else if K has three vertices in Ω_i and there exist three neighboring elements $K', K'', K''' \subset \Omega_i$ whose sizes are the same as that of K, then merge K and K', K'', K'''.
   – Otherwise, add the elements with the largest size among K and its neighboring elements to $\mathcal{T}_{\text{refine}}$.
   end while
4. If $\mathcal{T}_{\text{refine}} \neq \emptyset$, refine the elements in $\mathcal{T}_{\text{refine}}$ and their neighboring elements to obtain a new mesh $\mathcal{T}$ such that each side of $\mathcal{T}$ includes at most N_0 hanging nodes, go to 1.

We remark that if each side of a mesh $\mathcal{T}$ includes at most N_0 hanging nodes, the induced mesh $\mathcal{M}$ from $\mathcal{T}$ by Algorithm 5.2 is also a K-mesh with the constant C in Assumption (H3) depending only on N_0.

Now we present three examples to demonstrate the efficiency of our adaptive algorithm. We consider the case of high contrast coefficient $a(x)$ in Example 2 and the case of non-smooth interface in Example 3.

In all examples we set the computational domain $\Omega = (-2, 2) \times (-2, 2)$. In our theory, the penalty parameter α_0 can be any fixed positive constant and the constant δ_0 in Definition 2.1 can be any constant in $(0, 1/2)$. Clearly,

a larger δ_0 will lead to more small elements to be merged with neighboring elements. Here we take the natural choice $\alpha_0 = 1$ and $\delta_0 = 1/4$. We always set the maximal number of hanging nodes in each side of the mesh $N_0 = 3$.

(a)   (b)   (c)

Fig. 5.1 The interface used in Example 1 (left), Example 2 (middle), and Example 3 (right).

Example 1. We first consider a problem whose exact solution is known to illustrate the effectivity index of the a posteriori error estimate. Let the interface Γ be the circle centered at $(0,0)^T$ with radius $r = 1.1$. We define $\Omega_1 = \{x : |x| < r\}$ and $\Omega_2 = \Omega \backslash \bar{\Omega}_1$, as shown in Figure 5.1 (a). Set $a_1 = 10$ and $a_2 = 1$. The right-hand side f and boundary condition g are computed such that the exact solution is

$$u(x) = \begin{cases} e^{|x|^2 - r^2} + 10r^2 - 1, & \text{if } |x| \leq r, \\ 10|x|^2, & \text{otherwise.} \end{cases}$$

Figure 5.2 depicts the surface plot of the exact solution and one discrete solution. Figure 5.3 shows the quasi-optimal decay $O(N^{-p/2})$ of both the error $\|u - U\|_{DG}$ and the a posterior error estimate η for $p = 1, 2, 3$, respectively. Effectivity indexes eff $= \eta / \|u - U\|_{DG}$ for $p = 1, 2, 3$ are evaluated in Figure 5.4, which keep nearly constant as the number of degrees of freedom (#DoFs) increases.

In Table 5.1, we display #DoFs, η, and eff of uniform refinements and adaptive refinements. Figure 5.5 shows some examples of adaptive meshes and corresponding zoomed meshes. It is clear that much less number of degrees of freedom are needed to reach nearly the same error when using higher order methods. We remark that using higher degree polynomials yields higher accuracy but requires more computational cost. Appropriate balance of these two factors in practical computations is an interesting question that requires further investigation.

Example 2. In this example, we assume the interface Γ to be the union of two closely located circles of radius $r = 0.51$. The distance between two circles is $d = 0.02$. Ω_1 is the union of the interior of the two disks and $\Omega_2 = \Omega \backslash \bar{\Omega}_1$ (see Figure 5.1 (b)). To evaluate the effect of high contrast coefficients, we set $a_1 = 100, a_2 = 1$. We set $f = 1$ and $g = 0$.

Although a_1 is fairly large, the quasi-optimal decay of the global a posteriori error estimate for $p = 1, 2, 3$ is observed (Figure 5.6). Figure 5.7 shows some

Fig. 5.2 Example 1: (a) The exact solution. (b) The discrete solution on the mesh of 4184 elements when $p = 3$.

Fig. 5.3 Example 1: (a) The error $\|u - U\|_{\mathrm{DG}}$ for $p = 1, 2, 3$ by uniform refinements. (b) A priori and a posterior error estimates η for $p = 1, 2, 3$ by adaptive refinements.

Fig. 5.4 Example 1: The effectivity index eff $= \eta/\|u - U\|_{DG}$ against the degrees of freedom for $p = 1, 2, 3$.

examples of the adaptive meshes and the zoomed meshes. The discrete solution on the mesh of 2855 elements is shown in Figure 5.8 (a).

Example 3. We consider a non-smooth interface defined by

$$\Gamma = \left\{ (x, y) : |y| = \frac{4\sqrt{2}}{9} \cos\left(\frac{\sqrt{2}\pi}{3} x\right) + \frac{2\sqrt{2}}{9} \right\}.$$

Table 5.1 Comparison between uniform refinements and adaptive refinements.

$p = 1$				
Refinement Strategy	#DoFs	$\|u - U\|_{\mathrm{DG}}$	η	eff
Uniform	103792	8.43e-1	-	-
Adaptive	103344	8.39e-1	4.63	5.52
$p = 2$				
Refinement strategy	#DoFs	$\|u - U\|_{\mathrm{DG}}$	η	eff
Uniform	363852	6.04e-4	-	-
Adaptive	93357	6.21e-4	4.67e-3	7.52
$p = 3$				
Refinement strategy	#DoFs	$\|u - U\|_{\mathrm{DG}}$	η	eff
Uniform	150848	4.60e-5	-	-
Adaptive	59704	4.32e-5	4.54e-4	10.50

Note that the interface is singular at the points $(\pm\sqrt{2}, 0)$ (see 5.1 (c)). We set $a_1 = 10, a_2 = 1$, the right-hand side $f = 1$ and boundary condition $g = 0$.

The quasi-optimal decay of the a posteriori error estimate are clearly observed in Figure 5.9. Figure 5.10 shows some examples of the adaptive meshes and parts of the zoomed meshes for $p = 1, 2, 3$, respectively. We observe that the meshes are mainly refined around the sharp corners where the solution is singular. The discrete solution on the mesh of 2749 elements is depicted in Figure 5.8 (b).

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(a)

(b)

(c)

(d)

Fig. 5.5 Example 1: Adaptive meshes. (a) Mesh for $p = 1$, #DoFs=29036, $\|u - U\|_{\mathrm{DG}} = 1.6532$, and $\eta = 9.0598$. (b) Mesh for $p = 2$, #DoFs=31923, $\|u - U\|_{\mathrm{DG}} = 2.7700e - 3$, and $\eta = 2.0696e - 2$. (c) Mesh for $p = 3$, #DoFs=31088, $\|u - U\|_{\mathrm{DG}} = 2.4554e - 4$, and $\eta = 2.3696e - 3$. (d) The corresponding local mesh for $p = 3$ within $(0.7, 1.5) \times (-0.4, 0.4)$.

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Fig. 5.6 Example 2: The quasi-optimal decay of the a posteriori error estimate η for $p = 1, 2, 3$.

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(a)   (b)

(c)   (d)

Fig. 5.7 Example 2: Adaptive meshes. (a) Mesh for $p = 1$, #DoFs=35116, and $\eta = 2.4384e - 1$. (b) Mesh for $p = 2$, #DoFs=35235, and $\eta = 3.2069e - 3$. (c) Mesh for $p = 3$, #DoFs=30304, and $\eta = 4.6183e - 4$. (d) The corresponding local mesh for $p = 3$ within $(-0.4, 0.4) \times (-0.4, 0.4)$

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(a)   (b)

Fig. 5.8 (a) Example 2: The discrete solution on the mesh of 2855 elements for $p = 3$. (b) Example 3: The discrete solution on the mesh of 2749 elements for $p = 3$.

Fig. 5.9 Example 3: The quasi-optimal decay of the a posteriori error estimate η for $p = 1, 2, 3$.

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Fig. 5.10 Example 3: Adaptive meshes (left) and corresponding local meshes within $(1.40, 1.42) \times (-0.01, 0.01)$ (right). (a)&(b) The case $p = 1$, #DoFs=37684, and $\eta = 2.3391e - 1$. (c)&(d) The case $p = 2$, #DoFs=31302, and $\eta = 5.2027e - 3$. (e)&(f) The case $p = 3$, #DoFs=32128, and $\eta = 2.7504e - 3$.

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