# An additivity theorem for plain Kolmogorov complexity * 

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#### Abstract

We prove the formula $C(a, b)=K(a \mid C(a, b))+C(b \mid a, C(a, b))+O(1)$ that expresses the plain complexity of a pair in terms of prefix-free and plain conditional complexities of its components.


The well known formula from Shannon information theory states that $H(\xi, \eta)=H(\xi)+H(\eta \mid \xi)$. Here $\xi, \eta$ are random variables and $H$ stands for the Shannon entropy. A similar formula for algorithmic information theory was proven by Kolmogorov and Levin [5] and says that

$$
C(a, b)=C(a)+C(b \mid a)+O(\log n)
$$

where $a$ and $b$ are binary strings of length at most $n$ and $C$ stands for Kolmogorov complexity (as defined initially by Kolmogorov [4]; now this version is usually called plain Kolmogorov complexity). Informally, $C(u)$ is the minimal length of a program that produces $u$, and $C(u \mid v)$ is the minimal length of a program that transforms $v$ to $u$; the complexity $C(u, v)$ of a pair $(u, v)$ is defined as the complexity of some standard encoding of this pair.

This formula implies that $I(a: b)=I(b: a)+O(\log n)$ where $I(u: v)$ is the amount of information in $u$ about $v$ defined as $C(v)-C(v \mid u)$; this property is often called "symmetry of information". The term $O(\log n)$, as was noted in [5], cannot be replaced by $O(1)$. Later Levin found an $O(1)$-exact version of this formula that uses the so-called prefix-free version of complexity:

$$
K(a, b)=K(a)+K(b \mid a, K(a))+O(1)
$$

this version, reported in [2], was also discovered by Chaitin [1]. In the definition of prefix-free complexity we restrict ourselves to self-delimiting programs: reading a program from left to right, the interpreter determines where it ends. See, e.g., [7] for the definitions and proofs of these results.

In this note we provide a somewhat similar formula for plain complexity (also with $O(1)$-precision):

## Theorem 1.

$$
C(a, b)=K(a \mid C(a, b))+C(b \mid a, C(a, b))+O(1) .
$$

Proof. The proof is not difficult after the formula is presented. The $\leq$-inequality is a generalization of the inequality $C(x, y) \leq K(x)+C(y)$ and can be proven in the same way. Assume that $p$ is a self-delimiting program that maps $C(a, b)$ to $a$, and $q$ is a (not necessarily self-delimiting) program that maps $a$ and $C(a, b)$

[^0]to $b$. The natural idea is to concatenate $p$ and $q$; since $p$ is self-delimiting, given $p q$ one may find where $p$ ends and $q$ starts, and then use $p$ to get $a$ and $q$ to get $b$. However, this idea needs some refinement: in both cases we need to know $C(a, b)$ in advance; one may use the length of $p q$ as a replacement for it, but since we have not yet proven the equality, we have no right to do so.

So more caution is needed. Assume that the $\leq$-inequality is not true and $C(a, b)$ exceeds $K(a \mid C(a, b))+$ $C(b \mid a, C(a, b))$ by some $d$. Then we can concatenate prefix-free description $\bar{d}$ of $d$ (that has length $O(\log d)$, then $p$ and then $q$. Now we have enough information: first we find $d$, then $C(a, b)=|p|+|q|+d$, then $a$, and finally $b$. Therefore $C(a, b)$ does not exceed $O(\log d)+|p|+|q|+O(1)$, therefore $d \leq$ $O(\log d)+O(1)$ and $d=O(1)$. The $\leq$-inequality is proven.

Let us prove the reverse inequality. In this proof we use the interpretation of prefix-free complexity as the logarithm of a priori probability (see, e.g., [7] for details). If $n=C(a, b)$ is given, one can start enumerating all pairs $(x, y)$ such that $C(x, y) \leq n$; there are at most $2^{n+1}$ of them and the pair $(a, b)$ is among them. For fixed $x$, for each pair $(x, y)$ in this enumeration we add $2^{-n-1}$ to the probability of $x$; in this way we approximate (from below) the semimeasure $P(x \mid n)=N_{x} 2^{-n-1}$. Therefore, we get an upper bound for $K(a \mid n)$ :

$$
K(a \mid n) \leq-\log P(a \mid n)+O(1)=n-\log _{2} N_{a}+O(1)
$$

where $N_{a}$ is the number of $y$ 's such that $C(a, y) \leq n$. On the other hand, given $a$ and $n$, we can enumerate all these $y$, and $b$ is among them, so $b$ can be described by its ordinal number in this enumeration, therefore

$$
C(b \mid a, n) \leq \log _{2} N_{a}+O(1)
$$

Summing these two inequalities, we get the desired result.
We can now get several known $O(1)$-equalities for complexities as corollaries of Theorem 1

- Recall that $C(a, C(a))=C(a)$, and $K(a, K(a))=K(a)$ (the $O(1)$-additive terms are omitted here and below), since the shortest program for $a$ also describes its own length.
- For empty $b$ we get $C(a)=K(a \mid C(a))$, see also [3, 6].
- For empty $a$ we get $C(b)=C(b \mid C(b))$, see also [3] 6].
- The last two equalities imply that $C(u \mid C(u))=K(u \mid C(u))$.

The direct proof for last three statements is also easy. To show that $C(a) \leq C(a \mid C(a))$, assume that some program $p$ maps $C(a)$ to $a$ and is $d$ bits shorter than $C(a)$. Then we add a prefix $\bar{d}$ of length $O(\log d)$ that describes $d$ in a self-delimiting way, and note that $\bar{d} p$ determines first $C(a)$ and then $a$, so $d \leq O(\log d)+O(1)$ and $d=O(1)$. To show that $K(a \mid C(a)) \leq C(a \mid C(a))$ we note that in the presence of $C(a)$ every program of length $C(a)$ can be considered as a self-delimiting one, since its length is known.
Levin also pointed out that $C(a)$ can be defined in terms of prefix-free complexity as a minimal $i$ such that $K(a \mid i) \leq i$. (Indeed, for $i=C(a)$ both sides differ by $O(1)$, and changing right hand side by $d$, we change left hand side by $O(\log d)$, so the intersection point is unique up to $O(1)$-precision. In other terms, $K(a \mid i)=i+O(1)$ implies $C(a)=i+O(1)$.)

- More generally, we may let $a$ be some fixed computable function of $b$ : if $a=f(b)$, we get $C(b)=$ $K(f(b) \mid C(b))+C(b \mid f(b), C(b))$.

One can also see that Theorem 1 can be formally derived from Levin's results mentioned above. To show that

$$
C(b \mid a, C(a, b))=C(a, b)-K(a \mid C(a, b))
$$

we need to show that the right hand side $i=C(a, b)-K(a \mid C(a, b))$ satisfies the equality $K(b \mid a, C(a, b), i)=$ $i$ with $O(1)$-precision, which implies $C(b \mid a, C(a, b))=i$. (We omit all $O(1)$-terms, as usual.) In the condition of the last inequality we may replace $i$ by $K(a \mid C(a, b))$ since $C(a, b)$ is already in the condition. Therefore, we need to show that

$$
K(b \mid a, C(a, b), K(a \mid C(a, b)))=C(a, b)-K(a \mid C(a, b))
$$

or

$$
K(b \mid a, C(a, b), K(a \mid C(a, b)))+K(a \mid C(a, b))=C(a, b)
$$

But the sum in the left hand side equals $K(a, b \mid C(a, b))$ due to the formula for prefix complexity of a pair $(a, b)$ relativized to the condition $C(a, b)$, and it remains to note that $K(a, b \mid C(a, b))=C(a, b)$. (This alternative proof was suggested by Peter Gacs.)

We can obtain a different version of Theorem 1 .

## Proposition 1.

$$
C(a, b)=K(a \mid C(a, b))+C(b \mid a, K(a \mid C(a, b)))+O(1)
$$

Proof. Indeed, the $\leq$-inequality can be shown in the same way as the $\leq$-inequality in the proof of Theorem 1 hence it remains to show the $\geq$-inequality. Let $p$ be a program of length $C(b \mid a, C(a, b))$ that computes $b$ given $a$ and $C(a, b)$. (The program $p$ is not assumed to be self-delimiting.) Knowing $p$, we can also compute $b$ given $a$ and $K(a \mid C(a, b))$. First, we compute $|p|+K(a \mid C(a, b))$, and this sum equals $C(a, b)$ (Theorem(1). Then, using $a$ again, we compute $b$. Hence $C(b \mid a, C(a, b)) \geq C(b \mid a, K(a \mid C(a, b)))$.

One may complain that Theorem 1 is a bit strange since it uses prefix-free complexity in one term and plain complexity in the second (conditional) part. As we have already noted, one cannot use $C$ in both parts: $C(a, b)$ can exceed even $C(a)+C(b)$ by a logarithmic term. One may then ask whether it is possible to exchange plain and prefix-free complexity in the two terms we have and prove that $C(a, b)$ equals something like

$$
C(a \mid C(a, b))+K(b \mid a, C(a, b))
$$

It turns out that it is not possible: even the inequality $C(a, b) \leq C(a)+K(b \mid a)+O(1)$ is not true. At first it seems that one could concatenate a self-delimiting program $q$ that produces $b$ given $a$ and a (plain) program $p$ that produces $a$, in the hope that the endpoint of $q$ can be reconstructed, and then the rest is $p$. However, this idea does not work: the program $q$ is self-delimiting only when $a$ is known; to know $a$ we need to have $p$, and to know $p$ we need to know where $q$ ends, so there is a vicious circle here.

Let us show that the problem is unavoidable and that for infinitely many pairs $(x, y)$ we have

$$
C(x, y) \geq C(x)+K(y \mid x)+\log n-2 \log \log n-O(1)
$$

where $n=|x|+|y|$ is the total length of both strings. To construct such a pair, let $n=2^{k}$ for some $k$, and choose a string $r$ of length $n$ and a natural number $i<n$ such that $C(r, i) \geq n+\log n$. (For every $n$, there are $n 2^{n}$ pairs $(r, i)$, so one of them has high complexity.)

Let $x=r_{1} \ldots r_{i}$ and $y=r_{i+1} \ldots r_{n}$. Note that $C(x, y)=C(r, i) \geq n+\log n$ and that $C(x) \leq i$. Furthermore, $K(y \mid x) \leq K(y \mid x, n)+K(n)$. Here $K(y \mid x, n) \leq|y|=n-i$, since $x$ and $n$ determine $|y|$ and $K(y||y|) \leq|y|$; on the other hand, $K(n) \leq 2 \log \log n{ }^{1}$

There is still some chance to get a formula for the plain complexity of a pair $(x, y)$ that involves only plain complexities, assuming that we add some condition in the left hand side, i.e., to get some formula of the type $C(a, b \mid ?)=$ ?. Unfortunately, the best result in this direction that we managed to get is the following observation:

Proposition 2. For all $x, y$ there exists a (unique up to $O(1)$-precision) pair $(k, l)$ such that $C(x \mid l)=k$, $C(y \mid x, k)=l$. For such a pair we have $C(x \mid l)=k, C(y \mid x, k)=l$ and this implies $C(x, y \mid k, l)=C(x, y \mid k)=$ $C(x, y \mid l)=l+k($ all with $O(1)$-precision $)$.

Proof. The pair in question is a fixed point of $F:(k, l) \mapsto(C(x \mid l), C(y \mid x, k))$. It exists and is unique since $F$ maps points at distance $d$ into points at distance $O(\log d)$. (Here "distance" means geometric distance between points in $\mathbb{Z}^{2}$.)

[^1]Using the relativized version of the statement $C(z)=C(z \mid C(z))$, we conclude that $C(x \mid k, l)=k$ and $C(y \mid x, k, l)=l$. Let us prove now that $C(x, y \mid k, l)=k+l$. Indeed, the standard proof of Kolmogorov-Levin theorem shows that for any $x, y, k^{\prime}, l^{\prime}$ such that

$$
C\left(x, y \mid k^{\prime}, l^{\prime}\right) \leq k^{\prime}+l^{\prime}
$$

we have either

$$
C\left(x \mid k^{\prime}, l^{\prime}\right) \leq k^{\prime} \quad \text { or } \quad C\left(y \mid x, k^{\prime}, l^{\prime}\right) \leq l^{\prime}
$$

Hence if $C(x \mid k, l)=k$ and $C(y \mid x, k, l)=l$ for some $k$ and $l$, we have $C(x, y \mid k, l) \geq k+l$ (otherwise $k$ and $l$ can be decreased to get a contradiction). By concatenation we obtain also that $C(x, y \mid k, l) \leq k+l$, so $C(x, y \mid k, l)=k+l$ (all equations with $O(1)$-precision).

It remains to show that $C(x, y \mid k, l)=k+l$ implies $C(x, y \mid k)=k+l$ and, similarly, $C(x, y \mid l)=k+l$. Indeed, a program of length $k+l$ that maps $(k, l)$ to $(x, y)$, can be used to map $k$ (or $l$ ) to $(x, y)$ : knowing the length of the program and one of the values of $k$ and $l$, we reconstruct the other value.

Remark 1. One can ask what can be said about pairs $\left(k^{\prime}, l^{\prime}\right)$ such that $C\left(x \mid l^{\prime}\right) \leq k^{\prime}$ and $C\left(y \mid x, k^{\prime}\right) \leq l^{\prime}$. The pair ( $k, l$ ) given by the theorem is not necessarily coordinate-wise minimal: for example, taking a large $k^{\prime}$ that contains full information about $y$ we may let $l^{\prime}=0$. Indeed, $C(x \mid 0) \leq k^{\prime}$ (since $k^{\prime}$ is large) and $C\left(y \mid x, k^{\prime}\right) \leq 0$ (since $k^{\prime}$ determines $y$ ). However, to get some decrease in $k^{\prime}$ (compared to $k$ ) or $l^{\prime}$ (compared to $l$ ) we need to change the other parameter by an exponentially bigger quantity, since the information distance between $i$ and $i^{\prime}$ is $O\left(\log \left|i-i^{\prime}\right|\right)$. The change in the other parameter should be its increase, otherwise we could repeat the arguments exchanging $k$ and $l$ and get a contradiction (each of two changes could not be exponentially big compared to the other one).

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[^1]:    ${ }^{1}$ As a byproduct of this example and the discussion above we conclude that $K(x \mid y)$ cannot be defined as minimal prefix-free complexity of a program that maps $y$ to $x$ : the value $K(y \mid x)$ can be smaller than $\min \{K(p): U(p, x)=y\}$, where $U$ is the universal function. Indeed, in this case we would have the inequality $C(x, y) \leq C(x)+K(y \mid x)$, since the prefix-free description of a program that maps $x$ to $y$ and a shortest description for $x$ can be concatenated into a description of the pair $(x, y)$.

