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# Improved Lower Bound for Online Strip Packing

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**Abstract** We study the online strip packing problem and derive an improved lower bound of $\rho \geq 2.589\ldots$ for the competitive ratio of this problem. The construction is based on modified "Brown-Baker-Katseff sequences" (Brown et al. in Acta Inform. 18:207–225, 1982) using only two types of rectangles. In addition, we present an online algorithm with competitive ratio $(3+\sqrt{5})/2 = 2.618\ldots$ for packing instances of this type.

**Keywords** Strip packing · Rectangle packing · Online algorithms · Lower bounds

## 1 Introduction

In the two-dimensional strip packing problem a number of rectangles have to be packed without rotation or overlap into a strip such that the height of the strip used is minimal. The width of the rectangles is bounded by 1 and the strip has width 1 and infinite height. Baker, Coffman and Rivest [2] show that this problem is NP-hard, while Kenyon and Remila [3] present an approximation scheme for solving this problem.

We study the online version of this packing problem. In the online version the rectangles are given to the online algorithm one by one from a list, and the next rectangle is given as soon as the current rectangle is irrevocably placed into the strip. To evaluate the performance of an online algorithm we employ competitive analysis.

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For a list of rectangles $L$, the height of a strip used by online algorithm ALG and by the optimal solution are denoted by ALG$(L)$ and OPT$(L)$, respectively. The optimal solution is not restricted in any way by the ordering of the rectangles in the list. Competitive analysis measures the absolute worst-case performance of online algorithm ALG by its competitive ratio

$$\rho_{\mathrm{ALG}} = \sup_L \left\{ \frac{\mathrm{ALG}(L)}{\mathrm{OPT}(L)} \right\}.$$

*Known Results* Regarding the upper bound on the competitive ratio for online strip packing, recent advances have been made by Ye, Han and Zhang [4] and Hurink and Paulus [5]. Independently they showed that a modification of the well-known shelf algorithm yields an online algorithm with competitive ratio $7/2 + \sqrt{10} \approx 6.6623$, improving an earlier "shelf type algorithm" by Baker and Schwarz [6]. Another line of research deals with the so-called asymptotic competitive ratio, *cf.* [6–8].

In the early 80's, Brown, Baker and Katseff [1] derived a lower bound $\rho \geq 2$ on the competitive ratio of any online algorithm by constructing certain (adversary) sequences in a fairly straightforward way—see Sect. 2. These sequences, that we call BBK sequences in the sequel, were further studied by Johannes [9] and Hurink and Paulus [10], who derived improved lower bounds of 2.25 and 2.43, respectively. (Both results are computer aided and presented in terms of online parallel machine scheduling, a closely related problem.) The paper of Hurink and Paulus [10] also presents an upper bound of $\rho \leq 2.5$ for packing BBK sequences. Kern and Paulus [11] finally settled the question of how well the BBK sequences can be packed by providing matching upper and lower bounds of $\rho_{\mathrm{BBK}} = 3/2 + \sqrt{33}/6 \approx 2.457$.

*Our Contribution* Using modified BBK sequences we show an improved lower bound of $2.589\ldots$ on the absolute competitive ratio of this problem. The modified sequences that we use consist solely of two types of items, namely, *thin* items that have negligible width (and thus can all be packed in parallel) and *blocking* items that have width 1. The advantage of these sequences is that the structure of the optimal packing is simple, i.e., the optimal packing height is the sum of the heights of the blocking items plus the maximal height of the thin items. Therefore, we call such sequences *primitive*. We like to stress that all instances used so far to derive lower bounds are primitive.

On the positive side, we present an online algorithm for packing primitive sequences with competitive ratio $(3 + \sqrt{5})/2 = 2.618\ldots$. This result shows that our lower bound analysis of modified BBK sequences is fairly tight and, secondly, that in order to derive new lower bounds for strip packing that are larger than $2.618\ldots$ (and thus to significantly reduce the gap to the general upper bound of 6.6623), instances with a more complex structure (not just thin and blocking items) must be analyzed. In this sense, the upper bound result can thus be taken as a hint to future research directions, possibly leading to improved lower bounds.

The present paper is a journal version of an extended abstract that was earlier published (without proof) in the proceedings of WAOA 2011 [12].

*Organization* We start our presentation with a description of the Brown-Baker-Katseff sequences and their modification in Sect. 2. In Sect. 3 we present our lower bound based on these modifications and in Sect. 4 we describe our algorithm for packing primitive sequences. A detailed proof of the main result (lower bound) is presented in Sect. 5.

## 2 Sequence Construction

In this paper we denote the thin items by $p_i$ and the blocking items by $q_i$ (adopting the notation from [11]). As already mentioned in the introduction, we assume that the width of the thin items is negligible and thus all thin items can be packed next to each other. Moreover, the width of the blocking items $q_i$ is always 1, so that no item can be packed next to any blocking item in parallel. Therefore, all items are characterized by their heights and we refer to their heights by $p_i$ and $q_i$ as well. By definition, for any list $L = q_1, q_2, \dots, q_k, p_1, p_1, \dots, p_\ell$ consisting of thin and blocking items we have

$$\mathrm{OPT}(L) = \sum_{i=1}^{k} q_i + \max_{i=1,\dots,\ell} p_i .$$

To prove the desired lower bound we assume the existence of a $\rho$-competitive algorithm ALG for some $\rho < 2.589\dots$ (the exact value of this bound is specified later) and construct an adversary sequence depending on the packing that ALG generates.

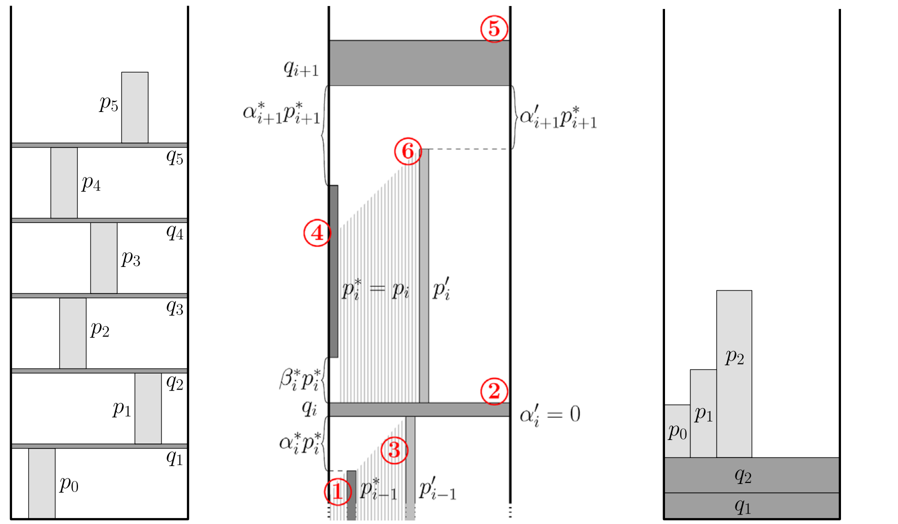
To motivate the construction, let us first consider the GREEDY algorithm for online strip packing, which packs every item as low as possible—see Fig. 1(a). This algorithm is not competitive (i.e., has unbounded competitive ratio): Indeed, consider the list $L_n = p_0, q_1, p_1, q_2, p_2, \dots, q_n, p_n$ of items with

$$\begin{aligned} p_0 &:= 1, \\ q_i &:= \varepsilon && \text{for } 1 \le i \le n, \\ p_i &:= p_{i-1} + \varepsilon && \text{for } 1 \le i \le n \end{aligned}$$

for some $\varepsilon > 0$. GREEDY would pack each item on top of the preceding ones and thus generate a packing of height $\mathrm{GREEDY}(L_n) = \sum_{i=0}^{n} p_i + \sum_{i=1}^{n} q_i = n + 1 + \Omega(n^2\varepsilon)$, whereas the optimum clearly has height $1 + 2n\varepsilon$.

The GREEDY algorithm illustrates that any competitive online algorithm needs to create gaps in the packing. These gaps work as a buffer to accommodate small blocking items—or, viewed another way, force the adversary to release larger blocking items.

*BBK Sequences* The idea of Brown, Baker and Katseff [1] was to try to cheat an arbitrary (non-greedy) online packing algorithm ALG in a similar way by constructing an alternating sequence $p_0, q_1, p_1, \dots$ of thin and blocking items. The heights $p_i$, respectively $q_i$ are determined so as to force the online algorithm ALG to put each item above the previous ones—see Fig. 1(b) for an illustration. To describe the heights of




(a) A greedy packing (b) A packing of a BBK sequence (c) An optimal packing of a BBK sequence

**Fig. 1** Online and optimal packings

the items formally, we consider the gaps that ALG creates between the items. We distinguish two types of gaps, namely gaps below and gaps above a blocking item, and refer to theses gaps as $\alpha$- and $\beta$-gaps, respectively. These gaps also play an important role in our analysis of the modified BBK sequences. We describe the height of the gaps around the blocking item $q_i$ relative to the thin item $p_i$. Thus, we denote the height of the $\alpha$-gap below $q_i$ by $\alpha_i p_i$ and the height of the $\beta$-gap above $q_i$ by $\beta_i p_i$. Using this notation, we are ready to formally describe the BBK sequences $L = p_0, q_1, p_1, q_2, \ldots$ with

$$\begin{aligned} p_0 &:= 1, \\ q_1 &:= \beta_0 p_0 + \varepsilon, \\ p_i &:= \beta_{i-1} p_{i-1} + p_{i-1} + \alpha_i p_i + \varepsilon && \text{for } i \geq 1, \\ q_i &:= \max(\alpha_{i-1} p_{i-1}, \beta_{i-1} p_{i-1}, q_{i-1}) + \varepsilon && \text{for } i \geq 2. \end{aligned}$$

In other words, each $q_i$ is chosen such that it is just too high to fit into one of the preceding gaps. This is equivalent to saying that $q_i$ exceeds the preceding $\alpha$- and $\beta$-gaps as well as $q_{i-1}$ (which in turn exceeds all previous gaps). Similarly, each $p_i$ (except the first $p_0$) is chosen just too large to fit into one of the gaps between two consecutive blocking items. As mentioned in the introduction, Brown, Baker and Katseff [1] used these sequences to derive a lower bound of 2 before Kern and Paulus [11] recently showed that the competitive ratio for packing them is $\rho_{\mathrm{BBK}} = 3/2 + \sqrt{33}/6 \approx 2.457$.

The optimal online algorithm for BBK sequences that Kern and Paulus [11] describe generates packings with striking properties: No $\alpha$- and $\beta$-gaps are created except the first possible gap $\beta_0 = \rho_{\mathrm{BBK}} - 1$ and the second $\alpha$-gap $\alpha_2 = 1/(\rho_{\mathrm{BBK}} - 1)$, which are chosen as large as possible while remaining $\rho_{\mathrm{BKK}}$-competitive. Observing this behavior of the optimal algorithm led us to the modification of the BBK sequences.

*Modified BBK Sequences* By definition, the decisions of the online algorithm, in particular, the gaps it creates, influence the sequence (construction): Creating large $\alpha$- or $\beta$-gaps "forces" the adversary to provide large blocking items in the next step. When packing BBK sequences, a good online algorithm should be eager to "enforce" blocking items of relatively large size (as each blocking item of size $q$ increases the optimal packing by $q$ as well). Thus a good online algorithm should seek to create large gaps.

Modified BBK sequences are designed to counter this strategy: Each time the online algorithm places a blocking item $q_i$, the adversary, rather than immediately releasing a thin item $p_{i+1}$ (of height defined as in standard BBK sequences) that does not fit in between the last two blocking items, generates a whole sequence of slowly growing thin items, which "continuously" grow from $p_i$ to $p_{i+1}$. Packing this subsequence causes additional problems for the online algorithm: If the algorithm fits the whole subsequence into the last interval between $q_{i-1}$ and $q_i$, it would fill out the whole interval and create an $\alpha$-gap of 0 below $q_i$. More generally, if the algorithm fits a large part of the subsequence into the last interval between $q_{i-1}$ and $q_i$, it would create a rather small $\alpha$-gap below $q_i$. On the other extreme, if ALG would pack a thin item of height slightly larger than $p_i$ above $q_i$, then the (relative) $\beta$-gap it can generate is much less compared to what it could have achieved with a thin item of larger height $p_{i+1}$ (assuming that the $p$-items are packed as high as possible, subject to $\rho$-competitiveness). Thus letting thin items grow continuously from $p_i$ to $p_{i+1}$ forces the online algorithm to either create smaller $\alpha$- or smaller $\beta$-gaps. The next blocking item $q_{i+1}$ will be released as soon as the sequence of thin items has grown from $p_i$ to $p_{i+1}$.

This general concept of modified BBK sequences applies after the first blocking item $q_1$ is released. Since subsequences of thin items and single blocking items are released alternatingly from this point on, we refer to this phase as the *alternating* phase. Before that, we have a *starting* phase in which the algorithm is confronted with a "continuously" increasing sequence of thin items. The starting phase ends with the release of the first blocking item $q_1$. The purpose of the starting phase is to prevent the online algorithm from introducing a large gap in the first step (when the first thin item arrives). Indeed, the optimal online algorithm by Kern and Paulus [11] generates an initial gap $\beta_0$ of maximal size to enforce a large first blocking item $q_1$. Since the first item has height 1, it must be "scheduled" at height $\beta_0 = \rho_{\mathrm{BBK}} - 1$ in order to not exceed the optimal ratio $\rho_{\mathrm{BBK}}$ already in the first step. In the starting phase, we seek to prevent the algorithm from creating a large $\beta_0$-gap as described in more detail in the next section.

Summarizing, a *modified BBK sequence* simply consists of a sequence of thin items, continuously growing in height, interleaved with blocking items which (by definition of their height) must be packed above all preceding items, and are released as described above, *i.e.*, when the thin item size has grown up to the largest gap between two blocking items, *cf.* Sects. 3 and 5 for more details.

We will use modified BBK sequences to prove

**Theorem 1** *There exists no algorithm for online strip packing with competitive ratio*

$$\rho < \hat{\rho} = \frac{17}{12} + \frac{1}{48}\sqrt[3]{22\,976 - 768\sqrt{78}} + \frac{1}{12}\sqrt[3]{359 + 12\sqrt{78}} \approx 2.589\ldots.$$

## 3 Lower Bound

For the sake of contradiction, assume there exists an online algorithm ALG that is $\rho$-competitive for online strip packing with $\rho < \hat{\rho}$. Let $\delta = \hat{\rho} - \rho > 0$. *W.l.o.g.* we assume that $\delta$ is sufficiently small. We feed ALG with a sequence $r_1, r_2, \ldots$ of thin items, interleaved with blocking items arriving at certain times as described in the following. The initial subsequence of thin items $r_1, r_2, \ldots, r_i$ that precedes the first blocking item defines the *starting phase*. The basic idea is to prevent ALG from creating a large $\beta$- gap in the first step. So if ALG packs the first thin item $r_1$ "too high", we release a slightly larger thin item $r_2$. The best ALG can do in this case is to bottom-align $r_2$ with $r_1$, yielding a slight decrease in the (relative!) $\beta$-gap. Continuing this way with an increasing sequence $r_3, r_4, \ldots,$ ALG will eventually reduce its $\beta$-gap to almost 0 or decide to "jump", *i.e.*, pack some $r_j$ on top of the current packing, leaving a new gap in between $r_{j-1}$ and $r_j$ of reasonable size. In case this is (still) too large compared to the current height of the packing, we continue with $r_{j+1}$ etc.

Thus in the starting phase we seek to decrease the maximal size (relative to the current packing height) of a gap. More specifically, let

$$\frac{h(\mathrm{maxgap}_{\mathrm{ALG}}(r_i))}{\mathrm{ALG}(r_i)}$$

be the *max-gap-to-height* ratio after packing $r_i$, where $h(\mathrm{maxgap}_{\mathrm{ALG}}(r_i))$ denotes the height of the maximal gap that algorithm ALG created up to item $r_i$ and $\mathrm{ALG}(r_i)$ denotes the height algorithm ALG consumed up to item $r_i$. We say ALG is $(\rho, c)$-competitive in the starting phase if ALG is $\rho$-competitive (i.e., $\mathrm{ALG}(r_i) \le \rho \mathrm{OPT}(r_i)$) and retains a max-gap-to-height ratio of $c$ (i.e., $h(\mathrm{maxgap}_{\mathrm{ALG}}(r_i))/\mathrm{ALG}(r_i) \ge c$ for $i \ge 1$) for all lists $L = r_1, r_2, \ldots$ of thin items.

In the analysis of the starting phase in Sect. 5.1 we show that an increasing sequence of thin items (the starting phase) forces any $\rho$-competitive algorithm to reach a state with max-gap-to-height ratio less than

$$\hat{c} = \frac{\hat{\rho} - 2\sqrt{\hat{\rho} - 1}}{\hat{\rho} - 1}.$$

Thus there must be a first item $r_i$ that ALG packs, causing a max-gap-to-height ratio of less than $\hat{c}$. The starting phase ends with the release of the first blocking item $q_1$ of height $\hat{c} \cdot \mathrm{ALG}(r_i)$ and we enter the second phase which we call the *alternating phase*. (For $\hat{\rho}$ as in Theorem 1 this yields a rather small value of $\hat{c} = 0.04275\ldots.$ This means that ALG might equally well pack the first item $r_1$ of size $r_1 = 1$, say, at height $\hat{c} = 0.04275\ldots$, very close to the bottom of the strip—in which case we would enter the alternating phase immediately.)

In Sect. 5.2 we analyze the alternating phase, more precisely, we investigate, how the competitiveness of ALG in the alternating phase is influenced by its max-gap-to-height ratio in the starting phase. We show that an algorithm with max-gap-to-height ratio of $\hat{c}$ in the starting phase cannot retain $\rho$-competitiveness in the alternating phase for $\rho < \hat{\rho}$ in case

$$\hat{c} = \frac{1 - \sqrt{4\hat{\rho}^2 - 12\hat{\rho} + 5}}{2(\hat{\rho} - 1)}.$$

Thus our two phases fit together for

$$\hat{c} = \frac{\hat{\rho} - 2\sqrt{\hat{\rho} - 1}}{\hat{\rho} - 1} = \frac{1 - \sqrt{4\hat{\rho}^2 - 12\hat{\rho} + 5}}{2(\hat{\rho} - 1)},$$

which is satisfied for

$$\hat{\rho} = \frac{17}{12} + \frac{1}{48}\sqrt[3]{22\,976 - 768\sqrt{78}} + \frac{1}{12}\sqrt[3]{359 + 12\sqrt{78}} \approx 2.589\ldots.$$

We present the proof of Theorem 1 in Sect. 5. We end this section by observing that modified BBK sequences show a completely different behavior as compared to "ordinary" BBK sequences. The optimal online algorithm dealing with ordinary BBK sequences is such that the sequence becomes stationary after a few steps (*cf.* [11]), whereas modified BBK sequences continuously grow to infinity.

## 4 Upper Bound

In this section we present the online algorithm ONL for packing instances that consist solely of thin and blocking items. We prove that the competitive ratio of ONL is $\rho = (3 + \sqrt{5})/2 \approx 2.618$. We distinguish two kinds of packings according to the item on top: If the item on top of the packing is a blocking item, we have a *blocked* packing, otherwise we have an *open* packing. Initially, we have a blocked packing (consider the bottom of the strip as a blocking item of height 0 "on top of" the initial empty packing).

The general idea of the algorithm ONL is pretty straightforward: First note that we might assume that the thin items are increasing in height (a thin item that has smaller size than a previous one can always be packed in parallel to the larger one). If a thin item arrives at a blocked packing and the item does not fit into one of the gaps between two blocking items—we say that a "jump is unavoidable" in this case—then we pack it on top of the current closed packing, leaving a $\beta$-gap of relative height $\rho - 2$ (*i.e.*, $(\rho - 2)$ times the height of the thin item) between the newly placed thin item and the preceding blocking item. Note that placing a thin item on top of a blocked packing results in an open packing. The relative size ($\beta = \rho - 2$) of the gap induced by this new "top" item is determined so as to ensure a competitive ratio of $\rho$ in the long run (*cf.* the proof of Theorem 2 below. Subsequent thin items are placed bottom-aligned with the thin "top" item causing the jump, so as to not deliberately diminish the current $\beta$-gap. Any arriving blocking item is packed as low as possible, *i.e.*, in case it fits into one of the gaps, we pack it there, otherwise it is put on top of the current packing, resulting in a closed packing. The fact that we pack blocking items as low as possible amounts to saying that we work with $\alpha$-gaps equal to 0. Summarizing, we apply the following algorithm.

**Online Algorithm ONL for primitive sequences**

Initially the packing is considered to be blocked.
**WHILE** a rectangle $r_j$ is released
  **IF** $r_j$ is a blocking item, pack $r_j$ at the lowest possible height
  **ELSIF** $r_j$ is a thin item
    **IF** the packing is open, pack $r_j$ bottom-aligned with the top thin item
    **ELSIF** the packing is blocked, try to pack $r_j$ below the top item.
      If this is not possible, pack $r_j$ at distance $(\rho - 2)r_j$ above the packing.
**ENDWHILE**

The above algorithm does not even try to cope with a "starting phase" in any respect. Nonetheless, it turns out to yield a rather good competitive ratio:

**Theorem 2** ONL *is a $\rho$-competitive algorithm for packing primitive sequences for*

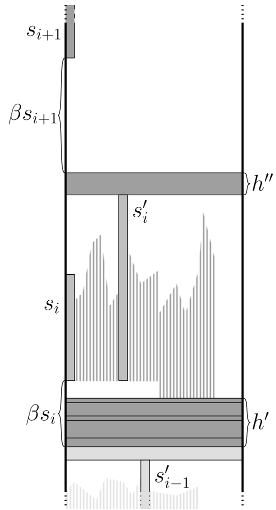
$$\rho = \frac{3+\sqrt{5}}{2} \approx 2.618.$$

*Proof* We show that ONL is $\rho$-competitive for $\rho = (3+\sqrt{5})/2$ by induction on the number of items. As to the inductive step, observe that whenever we pack a blocking item of height, say $q$, then the current height of the ONL-packing increases by at most $q$, whereas the optimum packing height increases by exactly $q$. Further, whenever we pack a thin item into one of the gaps, the ONL-packing height does not increase at all, so the algorithm stays trivially $\rho$-competitive in this case. Summarizing, the only critical case occurs when we pack a thin item $r_j$ at distance $\beta r_j$ with $\beta = \rho - 2$ above the current closed packing, *i.e.*, when a new top item $r_j$ is placed due to an "unavoidable jump".

We denote the thin items that are packed when generating a new gap by $s_i$ for the $i$-th jump. Let $s_i'$ be the highest thin item that is bottom-aligned with $s_i$. Note that the blocking item that blocks the packing after the $i$-th jump is packed directly above $s_i'$. See Fig. 2 for an illustration. We may assume *w.l.o.g.* that the sequence starts with a thin item (otherwise, a blocking item is put on the bottom of the strip in the first step, increasing the height of both ONL as well as OPT without any further consequences). So $s_1$ is the first item and this is packed at distance $(\rho - 2)s_1$ from the bottom line, so the competitive ratio after the first step is $\rho - 1 < \rho$.

For the induction step we assume $\mathrm{ONL}(s_i) \le \rho\,\mathrm{OPT}(s_i)$. Before a jump can become unavoidable, new blocking items of total height greater than $\beta s_i$ (where $\beta = \rho - 2$) need to arrive as otherwise the gap below $s_i$ could accommodate all of them. Let $h'$ be the height of the blocking items that are packed into the $\beta$-gap below $s_i$ and let $h''$ be the total height of blocking items that arrive between $s_i$ and $s_{i+1}$ and are packed above $s_i$. See Fig. 2. We have $h' \le (\rho - 2)s_i$ and $h' + h'' > (\rho - 2)s_i$ as

**Fig. 2** Packing after jump $i+1$. Blocking items released after $s_i$ shown in *darker shade*. By definition, $s_{i+1}$ is the first thin item that does not fit into a gap. Thus, in particular, $s_{i+1} > s_i' + \beta s_i - h'$




otherwise no blocking item would be packed on top. As further blocking items could be packed even below $s_{i-1}'$ we get

$$\begin{aligned}\mathrm{OPT}(s_{i+1}) &\geq \mathrm{OPT}(s_i) + h' + h'' + s_{i+1} - s_i,\\ \mathrm{ONL}(s_{i+1}) &= \mathrm{ONL}(s_i) + s_i' - s_i + h'' + \beta s_{i+1} + s_{i+1}.\end{aligned}$$

And thus we have

$$\begin{aligned}&\mathrm{ONL}(s_{i+1}) \leq \rho \mathrm{OPT}(s_{i+1})\\ &\quad\Leftarrow\quad \mathrm{ONL}(s_i) + s_i' - s_i + h'' + \beta s_{i+1} + s_{i+1}\\ &\qquad\qquad \leq \rho\big(\mathrm{OPT}(s_i) + h' + h'' + s_{i+1} - s_i\big)\\ &\quad\Leftarrow\quad (\rho-1)s_i + s_i' - \rho h' - (\rho-1)h'' \leq (\rho - 1 - \beta)s_{i+1}.\end{aligned}$$

As $\rho - 1 - \beta = 1$ and $s_{i+1} > s_i' + (\rho-2)s_i - h'$ this is satisfied if

$$\begin{aligned}&(\rho-1)s_i + s_i' - \rho h' - (\rho-1)h'' \leq s_i' + (\rho-2)s_i - h'\\ &\quad\Leftrightarrow\quad s_i \leq (\rho-1)\big(h' + h''\big)\\ &\quad\Leftarrow\quad s_i \leq (\rho-1)(\rho-2)s_i = s_i.\end{aligned}$$

The last equality holds since $\rho = (3+\sqrt{5})/2$ and thus $(\rho-1)(\rho-2) = 1$. □

So the true best possible competitive ratio for packing primitive sequences is somewhere in between the two values specified by Theorems 1 and 2. We have reasons to believe that it is strictly in between these two. But perhaps an even more challenging question is whether or not (or to what extent) primitive sequences provide worst case instances for online packing in general. So far, all lower bounds for online strip packing are based on primitive sequences. Theorem 2 states that this approach is limited. In order to achieve significant further improvements (towards the upper bound of 6.6623), suitable *non-primitive sequences* have to be designed.

## 5 Detailed Proof of Theorem 1

### 5.1 The Starting Phase

In this section we describe the lower bound for the starting phase. As we explained before, the key parameter of this phase is the *max-gap-to-height* ratio. We will show that for $\rho < \hat{\rho}$, any $\rho$-competitive algorithm can be forced into a state with max-gap-to-height ratio less than $\hat{c}$. In this section we use the definition

$$\hat{c} = \frac{\hat{\rho} - 2\sqrt{\hat{\rho} - 1}}{\hat{\rho} - 1}.$$

The starting phase ends as soon as a state is reached with a max-gap-to-height ratio less than $\hat{c}$. To derive a contradiction, we assume that the $\rho$-competitive algorithm ALG is $(\rho, \hat{c})$-competitive, i.e., retains a max-gap-to-height ratio of $\hat{c}$.

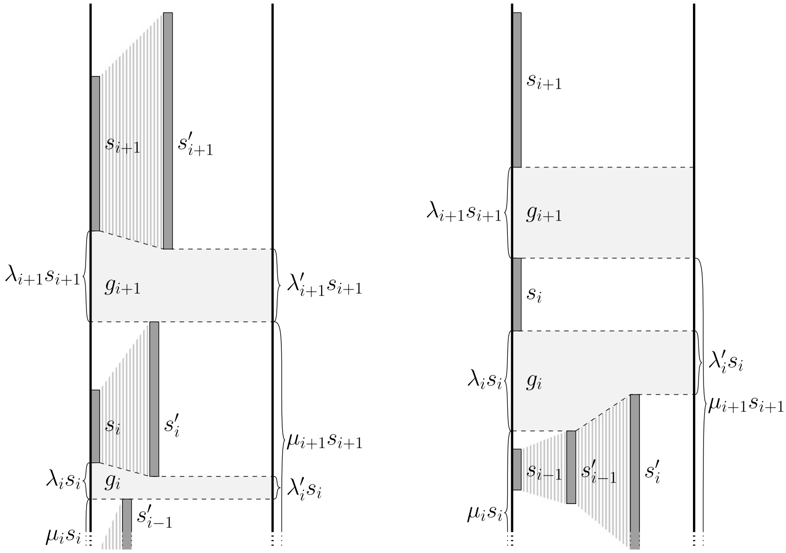
Let $\eta > 0$ be some very small constant and consider the adversary list $L_{\text{start}} = r_1, r_2, \ldots$ consisting of thin items

$$r_1 = 1 \quad \text{and}$$
$$r_i = r_{i-1} + \eta \quad \text{for } i \geq 2.$$

Recall that we denote the thin items by $r_i$ instead of $p_i$ here to be able to designate certain items that correspond to the thin items $p_i$ from the BBK sequence in the analysis of the alternating phase

The sole function of the positive term $\eta$ is to gradually increase the height of items (we substituted $\varepsilon$ from the BBK sequences by $\eta$ because we use $\varepsilon$ later in our analysis). To simplify the calculations, however, we assume that $\eta$ is chosen small enough such that single instances of $\eta$ can be omitted from the analysis. (The careful reader might want to check that the bounds we derive for the competitive ratio are actually continuous functions of $\eta$ and therefore we are well allowed to take the limit $(\eta \rightarrow 0)$.)

In the following analysis we consider the phases between the creation of new gaps. See Fig. 3(a) for an illustration of the following notations. We refer to the first items in each phase as the *jump* items $s_1, s_2, \ldots$ and we denote the last item in each phase by $s'_1, s'_2, \ldots$. As we argued above, we assume $s_{i+1} = s'_i$. Furthermore, we denote the gaps that ALG creates by $g_1, g_2, \ldots$ and refer to the maximal gap after ALG packs an item $r_i$ by $\text{maxgap}_{\text{ALG}}(r_i)$. Note that the height of the gaps might change when further items are packed (in case ALG packs them such that they reach into the gap from above or below). We denote the initial height of gap $g_i$ by $\lambda_i s_i$ and the gap height directly before the next jump, i.e., in the moment $s'_i$ is packed, by $\lambda'_i s_i$. Note that the height of gap $g_i$ is always given relative to the corresponding jump item $s_i$. Finally, we denote the packing height up to gap $g_i$ by $\mu_i s_i$, again relative to $s_i$. We




(a) Notations in the starting phase (b) Situation when ALG packs $s'_i$ *below* $s_i$

**Fig. 3** Starting phase. Lemma 1 shows that the gap sizes are increasing with each jump and Lemma 2 shows that ALG needs to pack $s'_i$ next to $s_i$

have $\mathrm{ALG}(s_i) = \mu_i s_i + \lambda_i s_i + s_i$ and $\mu_i s_i \geq \mathrm{ALG}(s'_{i-1})$ as $s'_{i-1}$ is packed below $g_i$ but other items might even reach higher than $s'_{i-1}$.

Since $\mathrm{OPT}(s_i) = s_i$ and $\mathrm{ALG}(s_i) = (\mu_i + \lambda_i + 1)s_i$ we directly have

$$\begin{aligned} &(\mu_i + \lambda_i + 1)s_i \leq \rho s_i && \text{for all } i \geq 1 \text{ and thus} \\ &\mu_i + \lambda_i \leq \rho - 1 && \text{for all } i \geq 1. \end{aligned} \tag{1}$$

Before we are ready to prove that ALG is forced to reach a state with max-gap-to-height ratio less than $\hat{c}$, we have to show some assumptions that we can make on the algorithm ALG. First, we show that we can assume that ALG generates a packing where the gap preceding $s_i$ is the maximal gap until $s_{i+1}$ is packed for all $i \geq 1$. Or, in other words, ALG generates a packing with increasing gap sizes.

**Lemma 1** *We can assume that* ALG *generates a packing that satisfies*

$$\mathrm{maxgap}_{\mathrm{ALG}}(r_j) = g_i \quad \textit{for } r_j \in \{s_i, \ldots, s'_i\}.$$

*Proof* The intuition of this proof is simple: A new gap $g_i$ that is not maximal (as long as it is the current gap) is unnecessary and can therefore be omitted. We do this by bottom-aligning all items from $s_i$ to $r_j$ with the top of the previous gap.

More formally, let $\mathrm{maxgap}_{\mathrm{ALG}}(r_j) = g_k \neq g_i$ be the first violation of the condition for $r_j \in \{s_i, \ldots, s'_i\}$. The modified algorithm ALG′ simulates ALG with the exception that it bottom-aligns those items from $\{s_i, \ldots, r_j\}$ that were previously packed above $g_k$ with the top of $g_k$.

As items have only been moved downwards, ALG′ remains $\rho$-competitive. Moreover, for the altered algorithm we have

$$\mathrm{maxgap}_{\mathrm{ALG}'}(r_\ell) = \mathrm{maxgap}_{\mathrm{ALG}}(r_j) = g_k \quad \text{for } r_\ell \in \{s_i, \dots, r_j\} \quad \text{and}$$
$$h_{r_\ell}(g_k) \geq h_{r_j}(g_k) \geq \hat{c}\mathrm{ALG}(r_j) \geq \hat{c}\mathrm{ALG}'(r_\ell) \quad \text{for } r_\ell \in \{s_i, \dots, r_j\},$$

where $h_{r_\ell}(g_k)$ denotes the height of gap $g_k$ in the moment $r_\ell$ is packed. The second inequality is due to our assumption that ALGis $(\rho, \hat{c})$-competitive, so the height of the max-gap at any time is at least $\hat{c}$-times the current packing height. The last inequality shows that also the modified algorithm retains a max-gap-to-height ratio of $\hat{c}$. So $(\rho, \hat{c})$-competitiveness is not violated.

In total, the altered algorithm ALG′ potentially even saves packing height in comparison with the original algorithm ALG. We can apply this method to all violations of $\mathrm{maxgap}_{\mathrm{ALG}}(r_j) = g_i$ by induction. □

Now we show that the space *below* a jump item $s_i$ is not large enough to accommodate $s_i'$ before ALG makes the next jump. The implication of this statement is that any $(\rho, \hat{c})$-competitive algorithm needs to place new items next to the current jump item.

**Lemma 2** ALG *cannot generate a gap with an item $s_{i+1}$ when the last item $s_i'$ is packed completely below the previous jump item $s_i$.*

*Proof* For the sake of contradiction assume that ALG generates such a gap with item $s_{i+1}$ while the last item $s_i'$ was packed completely below the previous jump item $s_i$—see Fig. 3(b). As we will see, the proof of this lemma does not require to consider that ALG retains a max-gap-to-height ratio of $\hat{c}$.

By inequality (1) we have $s_i' \leq (\mu_i + \lambda_i)s_i \leq (\rho - 1)s_i$ as $s_i'$ is packed below $s_i$. Thus $s_i \geq s_i'/(\rho - 1)$. With our assumption $s_i' = s_{i+1}$ we have

$$\mathrm{ALG}(s_{i+1}) \geq s_i' + s_i + s_{i+1} \geq \left(2 + \frac{1}{\rho - 1}\right)s_{i+1}.$$

The contradiction follows with $\rho < 2.618\dots$ as

$$\begin{aligned} & 1 > (\rho - 2)(\rho - 1) \\ \Leftrightarrow \quad & \left(2 + \frac{1}{\rho - 1}\right)s_{i+1} > \rho s_{i+1} \\ \Rightarrow \quad & \mathrm{ALG}(s_{i+1}) > \rho \mathrm{OPT}(s_{i+1}). \end{aligned}$$
□

Lemmas 1 and 2 state that each jump is larger than the previous jump and that *w.l.o.g.* ALG bottom-aligns the items next to the current jump item until a subsequent jump is carried out. This gives us sufficient information about the structure of the online packing to derive a contradiction. More specifically, the next two lemmas show that the relative gap height $\lambda_i$ is decreasing by a constant in every step, which contradicts the trivial lower bound of $\lambda_i \geq \hat{c}/(1 - \hat{c}) \cdot (\mu_i + 1) \geq \hat{c}/(1 - \hat{c})$ as $\lambda_i s_i \geq \hat{c}(\mu_i s_i + \lambda_i s_i + s_i)$.

**Lemma 3** $\lambda_1 \leq \rho - 1$ *and for any* $i \geq 1$

$$\lambda_{i+1} \leq \rho - 2 - \frac{\hat{c}(\rho - 1)}{\lambda_i - \hat{c}(\rho - 1)}.$$

*Proof* The first part, $\lambda_1 \leq \rho - 1$, follows directly from the $\rho$-competitiveness.

By Lemma 1 we know that $\mathrm{maxgap}_{\mathrm{ALG}}(s_i') = g_i$. Since ALG preserves a max-gap-to-height ratio of at least $\hat{c}$, we have $\lambda_i' s_i \geq \hat{c}\,\mathrm{ALG}(s_i')$. Moreover, by Lemma 2 we have $\mathrm{ALG}(s_i') \geq \mu_i s_i + \lambda_i' s_i + s_i'$ and thus

$$\begin{aligned} \lambda_i' s_i &\geq \hat{c}\mathrm{ALG}\bigl(s_i'\bigr) \geq \hat{c}\bigl(\bigl(\mu_i + \lambda_i'\bigr)s_i + s_i'\bigr) \\ \Rightarrow \quad s_{i+1} = s_i' &\leq \frac{\lambda_i' s_i - \hat{c}(\mu_i + \lambda_i')s_i}{\hat{c}}. \end{aligned} \tag{2}$$

Now we consider the packing height $\mu_{i+1}s_{i+1}$. We have $\mu_{i+1}s_{i+1} \geq \mathrm{ALG}(s_i') \geq (\mu_i + \lambda_i')s_i + s_i'$ and thus

$$\begin{aligned} \mu_{i+1} &\geq \bigl(\mu_i + \lambda_i'\bigr)\frac{s_i}{s_{i+1}} + \frac{s_i'}{s_{i+1}} \\ &\geq \frac{\hat{c}(\mu_i + \lambda_i')}{\lambda_i' - \hat{c}(\mu_i + \lambda_i')} + 1 \quad \text{by inequality (2)} \\ &\geq \frac{\hat{c}(\rho - 1)}{\lambda_i - \hat{c}(\rho - 1)} + 1. \end{aligned}$$

The last step holds since

$$\begin{aligned} \frac{\partial}{\partial \lambda_i'}\left(\frac{\hat{c}(\mu_i + \lambda_i')}{\lambda_i' - \hat{c}(\mu_i + \lambda_i')}\right) &= \frac{\hat{c}(\lambda_i' - \hat{c}(\mu_i + \lambda_i')) - \hat{c}(\mu_i + \lambda_i')(1 - \hat{c})}{(\lambda_i' - \hat{c}(\mu_i + \lambda_i'))^2} \\ &= \frac{-\hat{c}\,\mu_i}{(\lambda_i' - \hat{c}(\mu_i + \lambda_i'))^2} < 0 \quad \text{as } \mu_i > 0 \end{aligned}$$

and thus $\frac{\hat{c}(\mu_i+\lambda_i')}{\lambda_i'-\hat{c}(\mu_i+\lambda_i')}$ is minimal for $\lambda_i'$ maximal, which is $\lambda_i' = \lambda_i = \rho - 1 - \mu_i$ by inequality (1).

Using this lower bound for $\mu_{i+1}$ we get

$$\begin{aligned} \lambda_{i+1} &\leq \rho - 1 - \mu_{i+1} \quad \text{by inequality (1) for } i + 1 \\ &\leq \rho - 2 - \frac{\hat{c}(\rho - 1)}{\lambda_i - \hat{c}(\rho - 1)}. \end{aligned}$$

□

Using this upper bound for the relative gap height $\lambda_{i+1}$ we will show that no $(\rho, \hat{c})$-competitive algorithm exists. We already gave the lower bound of $\lambda_i \geq \hat{c}/(1 - \hat{c})$. On the other hand, the following lemma shows that the relative gap heights are gradually decreasing over time. This gives a contradiction to the assumption that ALG can retain a max-gap-to-height ratio of $\hat{c}$. Thus ALG is either not $\rho$-competitive or we reach a state with a max-gap-to-height ratio of less than $\hat{c}$, which ends the starting phase.

**Lemma 4** $\lambda_{i+1} \leq \lambda_i - \varepsilon$ *for some fixed* $\varepsilon > 0$.

*Proof* Let $\varepsilon = \varepsilon(\rho) = 2\sqrt{\hat{c}(\rho-1)} - \rho + 2 + \hat{c}(\rho-1)$. By Lemma 3 we have $\lambda_{i+1} \leq \lambda_i - \varepsilon$ since

$$\begin{aligned}
&\rho - 2 - \frac{\hat{c}(\rho-1)}{\lambda_i - \hat{c}(\rho-1)} \leq \lambda_i - 2\sqrt{\hat{c}(\rho-1)} + \rho - 2 - \hat{c}(\rho-1) \\
&\quad\Leftrightarrow \quad -\frac{\hat{c}(\rho-1)}{\lambda_i - \hat{c}(\rho-1)} \leq \lambda_i - 2\sqrt{\hat{c}(\rho-1)} - \hat{c}(\rho-1) \\
&\quad\Leftrightarrow \quad \lambda_i^2 - \bigl(2\sqrt{\hat{c}(\rho-1)} + \hat{c}(\rho-1)\bigr)\lambda_i \\
&\qquad\qquad \geq -\hat{c}(\rho-1) - 2\sqrt{\hat{c}(\rho-1)}\hat{c}(\rho-1) - \hat{c}^2(\rho-1)^2 \\
&\quad\Leftarrow \quad \lambda_i^2 - \bigl(2\sqrt{\hat{c}(\rho-1)} + 2\hat{c}(\rho-1)\bigr)\lambda_i \\
&\qquad\qquad \geq -\hat{c}(\rho-1) - 2\sqrt{\hat{c}(\rho-1)}\hat{c}(\rho-1) - \hat{c}^2(\rho-1)^2 \\
&\quad\Leftrightarrow \quad \bigl(\lambda_i - \bigl(\sqrt{\hat{c}(\rho-1)} + \hat{c}(\rho-1)\bigr)\bigr)^2 \\
&\qquad\qquad \geq \bigl(\sqrt{\hat{c}(\rho-1)} + \hat{c}(\rho-1)\bigr)^2 - \hat{c}(\rho-1) \\
&\qquad\qquad\quad - 2\sqrt{\hat{c}(\rho-1)}\hat{c}(\rho-1) - \hat{c}^2(\rho-1)^2 \\
&\qquad\qquad = 0.
\end{aligned}$$

Thus it only remains to show that $\varepsilon(\rho) > 0$.

With $\hat{c} = \frac{\hat{\rho}-2\sqrt{\hat{\rho}-1}}{\hat{\rho}-1}$ we have $\varepsilon(\hat{\rho}) = 0$ since

$$\begin{aligned}
\varepsilon(\hat{\rho}) &= 2\sqrt{\hat{c}(\hat{\rho}-1)} - \hat{\rho} + 2 + \hat{c}(\hat{\rho}-1) \\
&= 2\sqrt{\hat{\rho} - 2\sqrt{\hat{\rho}-1}} - \hat{\rho} + 2 + \hat{\rho} - 2\sqrt{\hat{\rho}-1}
\end{aligned}$$

and

$$\begin{aligned}
&2\sqrt{\hat{\rho} - 2\sqrt{\hat{\rho}-1}} - \hat{\rho} + 2 + \hat{\rho} - 2\sqrt{\hat{\rho}-1} = 0 \\
&\quad\Leftrightarrow \quad 2\sqrt{\hat{\rho} - 2\sqrt{\hat{\rho}-1}} = 2\sqrt{\hat{\rho}-1} - 2 \\
&\quad\Leftarrow \quad 4\bigl(\hat{\rho} - 2\sqrt{\hat{\rho}-1}\bigr) = 4(\hat{\rho}-1) - 8\sqrt{\hat{\rho}-1} + 4.
\end{aligned}$$

Note that this calculation actually defines the lower bound of $\frac{\hat{\rho}-2\sqrt{\hat{\rho}-1}}{\hat{\rho}-1}$ for $\hat{c}$. Now observe that $\hat{c} = \frac{\hat{\rho}-2\sqrt{\hat{\rho}-1}}{\hat{\rho}-1}$ does not depend on $\rho$ and thus we have

$$\frac{\partial}{\partial\rho}\bigl(2\sqrt{\hat{c}(\rho-1)} - \rho + 2 + \hat{c}(\rho-1)\bigr) = \frac{\hat{c}}{\sqrt{\hat{c}(\rho-1)}} - 1 + \hat{c}.$$

This derivative is negative as

$$\begin{aligned}
&\frac{\hat{c}}{\sqrt{\hat{c}(\rho-1)}} < 1-\hat{c} \\
\Leftarrow \quad & \frac{\hat{c}}{\rho-1} < (1-\hat{c})^2 \\
\Leftrightarrow \quad & \frac{\hat{\rho}-2\sqrt{\hat{\rho}-1}}{(\rho-1)(\hat{\rho}-1)} < \frac{(\hat{\rho}-1-\hat{\rho}+2\sqrt{\hat{\rho}-1})^2}{(\hat{\rho}-1)^2} \qquad \text{by definition of } \hat{c} \\
\Leftrightarrow \quad & \hat{\rho}-2\sqrt{\hat{\rho}-1} < (2\sqrt{\hat{\rho}-1}-1)^2 \cdot \frac{\rho-1}{\hat{\rho}-1} \\
\Leftarrow \quad & \hat{\rho}-2\sqrt{\hat{\rho}-1} < \frac{4(\hat{\rho}-1)-4\sqrt{\hat{\rho}-1}+1}{2} \qquad \text{as } \frac{\rho-1}{\hat{\rho}-1} > \frac{1}{2} \text{ for } \rho \geq 2 \\
\Leftrightarrow \quad & \frac{3}{2} < \hat{\rho}.
\end{aligned}$$

Thus $\varepsilon(\rho)$ is strictly decreasing with respect to $\rho$ and $\varepsilon(\hat{\rho}) = 0$. Hence $\varepsilon(\rho) > 0$ for $\rho < \hat{\rho}$ and the lemma follows. □

Thus the $\lambda_i$ decrease by a fixed amount in each step, contradicting the lower bound $\lambda_i \geq \hat{c}/(1-\hat{c})$ for any $i \geq 1$. Thus our original assumption that /ALG/ cannot be true. In other words, we have proved

**Lemma 5** *Any $\rho$-competitive algorithm* ALG *can be forced to reach a state where the max-gap-to-height ratio is less than*

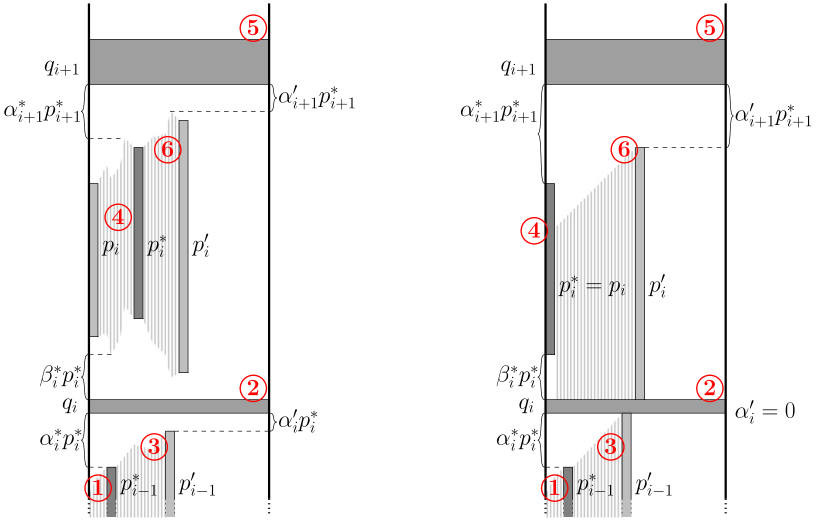
$$\hat{c} = \frac{\hat{\rho}-2\sqrt{\hat{\rho}-1}}{\hat{\rho}-1}.$$

### 5.2 The Alternating Phase

In this section we describe the lower bound in the alternating phase. In this phase we use that by Lemma 5, any $\rho$-competitive algorithm ALG is forced to reach a state where the max-gap-to-height ratio is less than $\hat{c} = \frac{\hat{\rho}-2\sqrt{\hat{\rho}-1}}{\hat{\rho}-1}$. As explained earlier in Sect. 3, we seek to analyze how $\hat{c}$ and $\hat{\rho}$ must be related so that an algorithm that finishes the starting phase with a max-gap-to-height-ratio of $\hat{c}$ cannot stay $\rho$-competitive for $\rho < \hat{\rho}$ in the alternating phase. The outcome will be that the two values must be related by the equation

$$\hat{c} = \frac{1-\sqrt{4\hat{\rho}^2-12\hat{\rho}+5}}{2(\hat{\rho}-1)}.$$

Thus, if $\hat{c}$ satisfies both equations above, then no $\rho$-competitive algorithm can exist for $\rho < \hat{\rho}$. Solving the two equations above yields $\hat{\rho} \approx 2.589\ldots$ and $\hat{c} \approx 0.04275\ldots$.




(a) Notations. $\alpha$-gaps below blocking items with height $\alpha^*_i p^*_i$ in the moment $q_i$ is packed and final height $\alpha'_i p^*_i$. $\beta$-gaps above blocking items with height $\beta^*_i p^*_i$

(b) Structured packing according to the assumptions by Lemmas 6, 7. Here we illustrate the case $\beta^*_i > 0$ and thus $\alpha'_i = 0$ and $p^*_i = p_i$

**Fig. 4** Order of the released items. (1) Thin items up to $p^*_{i-1}$; (2) blocking item $q_i$; (3) and (4) thin items up to $p^*_i$ (including the jump item $p_i$); (5) blocking item $q_{i+1}$ and (6) further thin items up to $p'_i$

Our adversary sequence in the alternating phase starts with the first blocking item $q_1$ and then continues with the list of thin items of gradually increasing height from the starting phase interleaved with further blocking items. Let $\eta > 0$ be some very small constant and let $r_k$ be the last item that was released in the starting phase. Then we continue with the list $L_{\text{alternating}} = q_1, r_{k+1}, r_{k+2}, \ldots$ where

$$q_1 = \hat{c} \cdot \text{ALG}(r_k) \quad \text{and}$$
$$r_i = r_{i-1} + \eta \quad \text{for } i \geq k+1.$$

To understand when the blocking items are inserted, let us first introduce the notations in this phase—see Fig. 4(a).

Similar to the starting phase, we consider the jump items, i.e., the thin items that are the first to be packed above a blocking item $q_i$, and denote them by $p_i$. The thin item directly before the jump item is denoted by $p'_{i-1}$ (we will later see that we can actually assume that $p'_{i-1}$ is the last item that is packed below $q_i$). We denote the interval between the blocking items $q_{i-1}$ and $q_i$ by $I_i$. As in the standard BBK sequences, the thin item whose height exceeds the height of the previous interval plays an important role. We denote the first item that exceeds the height of $I_{i-1}$ by $p^*_i$ and describe all further heights relative to these designated items.

As described in the introduction, we distinguish $\alpha$-gaps (directly below blocking items) and $\beta$-gaps (directly above blocking items). As the gap heights can change during the packing (as further thin items are packed into the same interval) we have to be specific about the moment in which we consider these heights. Let $\alpha^*_i p^*_i$ be the height of the $\alpha$-gap below $q_i$ in the moment $q_i$ is packed and let $\alpha_i' p_i^*$ be the final height of the $\alpha$-gap below $q_i$, i.e., the height in the moment $p_i^*$ is packed (as afterwards no further item can be packed into $I_{i-1}$). The notation is due to our assumption that $p_{i-1}'$ is the last item that is packed into $I_{i-1}$ (which we show later). Regarding the $\beta$-gap we get along with a single definition: Let $\beta_i^* p_i^*$ be the height of the $\beta$-gap above $q_i$ in the moment $p_i^*$ is packed.

The blocking item $q_{i+1}$ is released directly after $p_i^*$. This ensures that the online algorithm jumps before a new blocking item is released (as the height of $p_i^*$ exceeds the height of the previous interval). We set the height of the blocking items to

$$q_1 := \hat{c} \cdot \mathrm{ALG}(r_k) \quad \text{as already mentioned above and}$$

$$q_i := \max\bigl(\alpha_{i-1}' p_{i-1}^*, \beta_{i-1}^* p_{i-1}^*, q_{i-1}\bigr) + \eta \quad \text{for } i \geq 2.$$

Note that we use the *final* height $\alpha_i' p_i^*$ of the $\alpha$-gap in this definition. This definition ensures that the blocking items are always packed above all previous items.

Again the function of the positive term $\eta$ is to gradually increase the height of thin items and to ensure that the blocking items are always packed above all previous items. As before, we make the assumption that $\eta$ is chosen small enough to be omitted from the analysis. Thus we assume that $q_i = \max(\alpha_{i-1}' p_{i-1}^*, \beta_{i-1}^* p_{i-1}^*, q_{i-1})$ and that the height of $p_i^*$ equals the height of the previous interval $I_{i-1}$ throughout this section. (Again, this is justified by taking the limit $(\eta \to 0)$.)

We use $\mathrm{succ}(r_i)$ and $\mathrm{prec}(r_i)$ to denote the thin item that succeeds and that precedes $r_i$, respectively. Using this notations we can rephrase the input list including the blocking items to

$$L_{\text{alternating}} = q_1, r_{k+1}, \ldots, p_1^*, q_1, \mathrm{succ}\bigl(p_1^*\bigr), \ldots, p_2^*, q_2, \mathrm{succ}\bigl(p_2^*\bigr), \ldots.$$

We also refer to Fig. 4 for an illustration of the order in which the items are released.

### *5.2.1 Overview*

We prove by contradiction that no $\rho$-competitive algorithm exists for $\rho < \hat{\rho}$. Thus we assume to the contrary that a $\rho$-competitive algorithm ALG exists. By the analysis of the starting phase we already know that we can force ALG to reach a state with a max-gap-to-height ratio less than $\hat{c}$. In accordance with the notation given above we introduce the parameter $\gamma_i^*$ to measure how much ALG improves upon the $\rho$-competitiveness. Let $\gamma_i^*$ be defined through

$$\mathrm{ALG}\bigl(p_i^*\bigr) + \gamma_i^* p_i^* = \rho\, \mathrm{OPT}\bigl(p_i^*\bigr).$$

Using this value, we introduce the potential function

$$\Phi_i = \gamma_i^* + \beta_i^*.$$

Note that moving $p_i^*$ (together with all other thin items packed on top of $q_i$) up or down will increase *resp.* decrease the value of $\beta_i^*$ and, at the same time, decrease *resp.* increase the value of $\gamma_i^*$ by the same amount. Thus, such a move would not affect the potential $\Phi_i$. (This phenomenon seems to be a characteristic feature of suitable potentials, *cf.* also, *e.g.*, [11] or [13].) Yet, of course, the online algorithm's decision on where to pack the thin items above $q_i$ determines the current $\beta$-gap and influences subsequent items and potential values.

Obviously, any $\rho$-competitive algorithm needs to keep $\Phi_i$ non-negative over time. We aim at deriving a contradiction by showing that $\Phi_i$ decreases by a constant amount in every step. Unfortunately, there is one possible exception to this rule, making the proof substantially more involved: $\Phi_i$ might increase exactly once. We will show that even in this case, $\Phi_i$ is properly bounded from above and cannot increase a second time.

We start our analysis with some preliminary results (Lemmas 6–11) on the structure of a packing generated by ALG. The general theme is that if a $\rho$-competitive algorithm exists, then there also exists a $\rho$-competitive algorithm that generates packings with the assumed structure. In other words: If ALG does not generate such a packing, we can alter the packing (or rather the algorithm) such that the conditions are satisfied and $\rho$-competitiveness is not violated at any point. In addition, a few helpful estimates are derived, essentially lower and upper bounds for $\alpha$-gaps: Lemma 7 states that for the online algorithm it is not wise to generate both a nonzero $\alpha$ *and* a nonzero $\beta$-gap, as one large $\beta$-gap is more promising. Lemma 8 states that the online algorithm should better not work with small nonzero $\alpha$-gaps, as these could better be replaced by $\beta$-gaps. Thus the size of nonzero $\alpha$-gaps can be bounded from below and, of course, also from above (Lemmas 9 and 10), as the online algorithm is assumed to be $\rho$-competitive. The "preliminaries section" then concludes with Lemma 11, saying how exactly $\Phi_{i+1}$ depends on $\Phi_i$ and the parameters (gap- and item sizes) in step $i$.

Then follows the subsection "induction" (Lemmas 12–16) where we prove that the potential decreases by some fixed amount in each step (except possibly once). The proof will be by induction as we need to upper-bound the potential—as well as current $q/p$-values—in each step. Intuitively, a small potential indicates that the online algorithm is in a bad position, as the potential upper bounds both the (relative) distance from the allowed packing height ($\rho$-times OPT ) and the current $\beta$-gap. Lemma 12 states that the initial potential is small, due to the fact that the online algorithm enters the alternating phase with a small max-gap-to-height-ratio. In Lemmas 13 to 16 we investigate the change in the potential depending on how $q^*_{i+1}$ is defined (via the $\alpha$- or $\beta$-gap or $q_i$). In each case we conclude that the potential decreases, assuming that it was low already. The only exception is when $q_{i+1} = q_i$. This has also been the critical case for standard BBK-sequences, where the optimal online algorithm's strategy would create a sequence that becomes stationary after a few steps and items are packed without any gaps. This works only if (in the stationary part) the blocking items $q$ are sufficiently large compared to the thin items $p$. (In the stationary part, OPT increases by $q$ in each step, while the online height increases with $p + q$, so that $q + p \leq \rho q$, *i.e.*, $\frac{q}{p} \geq \frac{1}{\rho-1}$ is required.)

In the case of modified BBK-sequences that we consider here, we shall see that such relatively large blocking items cannot be generated and therefore the optimum online algorithm will never induce a stationary sequence. Indeed, the blocking items stay small in size (relative to the corresponding thin items, *cf.* Lemmas 12–16 below).

Even in the critical case where $q_{i+1} = q_i$, the potential decreases in case the current $q/p$ ratio is small enough (Lemma 15) or—in case the $q/p$ ratio is slightly larger—it will get small enough in the next step (Lemma 16), so that, eventually, the potential is shown to decrease in each except possibly one single step.

### *5.2.2 Preliminaries*

The following lemmas (6–11) provide some simplifications, *i.e.*, "*w.l.o.g.* assumptions" on the structure of the packing that ALG generates in this phase—see Fig. 4(b) for an illustration.

**Lemma 6** *We can assume that* ALG *generates a packing such that*

1. *the items* $p_i, \ldots, p_i^*, \ldots, p_i'$ *lie in interval* $I_i$,
2. *the items* $p_i, \ldots, p_i^*$ *are bottom-aligned,*
3. *the items* $\mathrm{succ}(p_i^*), \ldots, p_i'$ *are bottom-aligned at the top of* $q_i$.

*Proof* By definition the items $p_i^*, \ldots, p_i'$ are taller than the previous interval $I_{i-1}$ and thus all lie in interval $I_i$. Assume that an item from $p_i, \ldots, \mathrm{prec}(p_i^*)$ does not lie in interval $I_i$ and let $r_j$ be the tallest such item. Then we can move down the items $p_i, \ldots, \mathrm{prec}(r_j)$ and bottom-align them with $r_j$. This redefines $p_i$ to $\mathrm{succ}(r_j)$ and hereby satisfies condition 1. Observe that moving down the items $p_i, \ldots, \mathrm{proc}(r_j)$ does not violate $\rho$-competitiveness and as $\alpha_i'$ and $\beta_i^*$ are not changed, the further packing remains unchanged.

If the items $p_i, \ldots, p_i^*$ are not packed bottom-aligned, we move them downwards until they are aligned with the lowest item of this list in order to satisfy condition 2. And to satisfy condition 3 we move the items $\mathrm{succ}(p_i^*), \ldots, p_i'$ down until they are aligned with the top of $q_i$ if these items are not bottom-aligned at the top of $q_i$. In both cases the alteration is possible as the height of the interval $I_i$ and thus the height of $p_{i+1}^*$ remains unchanged. Moreover, the height of $q_i$ does not change (as $\beta_i^*$ is not changed). The values of $\alpha_{i+1}^*$ and $\alpha_{i+1}'$ can actually change, but only become larger. But as the heights of $I_i$ and $p_{i+1}^*$ remain unchanged, the parameter $\alpha_{i+1}'$ only affects $q_{i+1}$ and the value of $q_{i+1}$ contributes to the packing height of OPT and ALG to the same extent. Thus increased values of $\alpha_{i+1}^*$ and $\alpha_{i+1}'$ cannot cause a violation of the $\rho$-competitiveness. □

Recall that $q_{i+1} = \max(\alpha_i' p_i^*, \beta_i^* p_i^*, q_i)$. Depending on the way in which $q_{i+1}$ is actually defined, we can assume that the other value(s) are zero as the following lemma shows. (Intuitively, as a good online algorithm should seek to create large gaps, it does not make sense to create *both* an $\alpha$- and a $\beta$-gap.)

**Lemma 7** *We can assume that* ALG *generates a packing such that*

1. *if* $q_{i+1} = \max(\beta_i^* p_i^*, q_i)$, *then we have* $\alpha_i' = 0$,
2. *if* $q_{i+1} = \max(\alpha_i' p_i^*, q_i)$, *then we have* $\beta_i^* = 0$.

*Proof* First, assume that $q_{i+1} = \max(\beta_i^* p_i^*, q_i)$ and $\alpha_i' > 0$. By construction of the adversary sequence, the height of $p_i^*$ does not depend on $\alpha_i'$ and is predetermined at the moment $q_i$ is packed. Thus a reduction of $\alpha_i'$, which corresponds to packing further thin items into the previous interval, does not change $q_{i+1}$ and $p_i^*$. So we can alter ALG such that all items from $\mathrm{succ}(p_{i-1}'), \ldots, \mathrm{pre}(p_i^*)$, are packed into $I_{i-1}$. This reduces $\alpha_i'$ to 0 and thus satisfies condition 1 without implying any change to the packing after $p_i^*$.

Now assume that $q_{i+1} = \max(\alpha_i' p_i^*, q_i)$ and $\beta_i^* > 0$. In this case a reduction of $\beta_i^*$ does not change $q_{i+1}$ and $p_i^*$. So we can alter ALG to set $\beta_i^*$ to 0, i.e., bottom-align the items $p_i, \ldots, p_i^*$ with the top of $q_i$, without implying any change to the packing after $p_i^*$ and hereby satisfy condition 2. This alteration increases $\alpha_{i+1}^*$ and might increase $\alpha_{i+1}'$ as well—as we saw in Lemma 6, this does not violate $\rho$-competitiveness. $\square$

Observe that by Lemma 6 we have $p_{i+1}^* = \beta_i^* p_i^* + p_i^* + \alpha_{i+1}^* p_{i+1}^*$ and thus

$$p_{i+1}^* = \frac{1 + \beta_i^*}{1 - \alpha_{i+1}^*} p_i^*. \tag{3}$$

Using this equation, we are ready to show the following assumption.

**Lemma 8** *We can assume that* ALG *generates a packing such that if* $\alpha_i' > 0$, *then we have*

$$\alpha_{i+1}^* > \frac{(\rho - 1)\alpha_i'}{1 + (\rho - 1)\alpha_i'}.$$

*Proof* We assume that $\alpha_i' > 0$ and $\alpha_{i+1}^* \le (\rho - 1)\alpha_i'/(1 + (\rho + 1)\alpha_i')$. By Lemma 7 condition 2 we have $\beta_i^* = 0$ and thus $p_{i+1}^* = p_i^*/(1 - \alpha_{i+1}^*)$. We can alter ALG to save a packing height of $\alpha_i' p_i^*$ without violating $\rho$-competitiveness by changing the $\alpha$-gap to a $\beta$-gap. To do that, we move down $q_i$ and all items that are released after $p_{i-1}'$ with the exception of $p_i$ by $\alpha_i' p_i^*$. In other words, we close the $\alpha_i' p_i^*$ gap between $p_{i-1}'$ and $q_i$ by moving down $q_i$ and all items above $q_i$. The only exception is the item $p_i$ that we keep at its position to retain a $\beta$-gap at the moment this item is packed. Hereby, we keep a gap of the original size $\alpha_i' p_i^*$ *above* $q_i$. See Fig. 5 for an illustration of the altered packing.

Note that this alteration changes the adversary sequence: As there does not remain any $\alpha_i'$-gap, the item $q_{i+1}$ is released directly after $p_i$ is packed—also redefining $p_i^*$ to $p_i$. This is the only change in the adversary sequence since the size of $q_{i+1}$ is not changed and also the height of interval $I_i$ stays constant. Since the optimal value changed as $q_{i+1}$ is released earlier than before, we have to check whether the altered packing is actually feasible.

We denote the optimal algorithm for the altered instance by OPT$'$ and the altered algorithm by ALG$'$. With $\alpha_i' p_i^*$ we refer to the height *before* the alteration. The height $\alpha_{i+1}^* p_{i+1}^*$ remains unchanged. We have $\mathrm{OPT}'(q_{i+1}) = \mathrm{OPT}(p_i) + q_{i+1} = \mathrm{OPT}(p_i) + \alpha_i' p_i^*$ and $\mathrm{ALG}'(q_{i+1}) = \mathrm{ALG}(p_i) + \alpha_{i+1}^* p_{i+1}^* + q_{i+1} = \mathrm{ALG}(p_i) +$

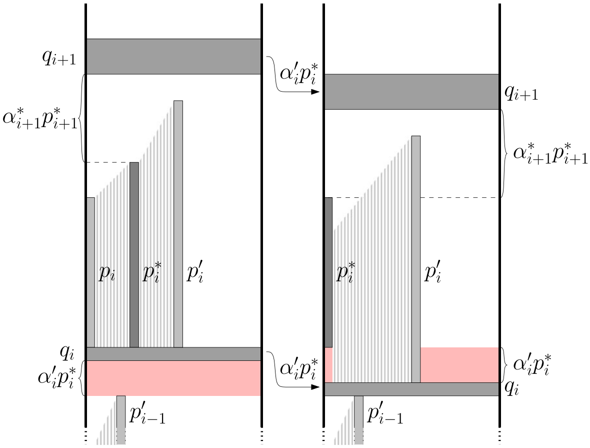



**Fig. 5** If $\alpha^*_{i+1} \le (\rho-1)\alpha'_i/(1+(\rho-1)\alpha'_i)$, then we can move down $q_i$ and all items that are released after $p'_{i-1}$, with the exception of $p_i$, by $\alpha'_i p^*_i$. Hereby the $\alpha$-gap becomes a $\beta$-gap and $p_i$ becomes the new $p^*_i$ as the interval $I_{i-1}$ shrinks

$\alpha^*_{i+1} p^*_{i+1} + \alpha'_i p^*_i$. Thus

$$\begin{aligned}
&\mathrm{ALG}'(q_{i+1}) \le \rho\,\mathrm{OPT}'(q_{i+1})\\
&\quad\Leftrightarrow\quad \alpha^*_{i+1} p^*_{i+1} \le \underbrace{\rho\,\mathrm{OPT}(p_i) - \mathrm{ALG}(p_i)}_{\ge 0} + (\rho-1)\alpha'_i p^*_i\\
&\quad\Leftarrow\quad \alpha^*_{i+1} p^*_{i+1} \le (\rho-1)\alpha'_i p^*_i\\
&\quad\Leftrightarrow\quad \alpha^*_{i+1} p^*_{i+1} \le (\rho-1)\alpha'_i\bigl(1-\alpha^*_{i+1}\bigr)p^*_{i+1} \qquad \text{by (3) with } \beta^*_i = 0\\
&\quad\Leftrightarrow\quad \alpha^*_{i+1} \le \frac{(\rho-1)\alpha'_i}{1+(\rho-1)\alpha'_i},
\end{aligned}$$

which we assumed to be true. Thus $q_{i+1}$ can actually be packed by the altered algorithm. The feasibility for all other items in the altered packing is obvious. □

On the other hand, it is not possible for ALG to create an arbitrarily large gap when packing a blocking item $q_{i+1}$. We capture this fact in the following lemma.

**Lemma 9** *We have*

$$\alpha^*_{i+1} \le \frac{\gamma^*_i + (\rho-1)\frac{q_{i+1}}{p^*_i}}{1+\beta^*_i+\gamma^*_i+(\rho-1)\frac{q_{i+1}}{p^*_i}}.$$

*Proof* The value of $\alpha^*_{i+1}$ can be bounded by observing the moment when $q_{i+1}$ is packed. We have

$$\begin{aligned}
\mathrm{OPT}(q_{i+1}) &= \mathrm{OPT}\bigl(p^*_i\bigr) + q_{i+1},\\
\mathrm{ALG}(q_{i+1}) &= \mathrm{ALG}\bigl(p^*_i\bigr) + \alpha^*_{i+1} p^*_{i+1} + q_{i+1}.
\end{aligned}$$

And since $q_{i+1}$ needs to be packed $\rho$-competitively by ALG we get

$$\begin{aligned} &\mathrm{ALG}(q_{i+1}) \le \rho\,\mathrm{OPT}(q_{i+1}) \\ &\quad\Leftrightarrow\quad \alpha^*_{i+1} p^*_{i+1} \le \gamma^*_i p^*_i + (\rho-1) q_{i+1} \\ &\quad\Leftrightarrow\quad \frac{\alpha^*_{i+1}}{1-\alpha^*_{i+1}} \le \frac{\gamma^*_i + (\rho-1)\frac{q_{i+1}}{p^*_i}}{1+\beta^*_i} \quad \text{by (3)} \\ &\quad\Leftrightarrow\quad \alpha^*_{i+1} \le \frac{\gamma^*_i + (\rho-1)\frac{q_{i+1}}{p^*_i}}{1+\beta^*_i+\gamma^*_i+(\rho-1)\frac{q_{i+1}}{p^*_i}}. \end{aligned}$$

□

The parameter $\alpha'_i$ plays an important role in the analysis as the height of the preceding blocking item depends on it. With the next lemma we get an upper bound for this parameter.

**Lemma 10** *We have* $(\rho-1)\alpha'_i \le \gamma^*_i$.

*Proof* The idea of the bound is that if ALG jumps early, i.e., with an $\alpha'_i > 0$, then it generates a packing where $p^*_i = p'_i + \alpha'_i p^*_i$. This additional height directly contributes to the value of $\gamma^*_i$ with a factor of $\rho - 1$ (as ALG and OPT increase by the same amount).

Formally, we have $\mathrm{ALG}(p^*_i) = \mathrm{ALG}(p_i) + \alpha'_i p^*_i$, $\mathrm{OPT}(p^*_i) = \mathrm{OPT}(p_i) + \alpha'_i p^*_i$ and $\mathrm{ALG}(p^*_i) + \gamma^*_i p^*_i = \rho\,\mathrm{OPT}(p^*_i)$. And since $p_i$ was feasible we have $\mathrm{ALG}(p_i) \le \rho\,\mathrm{OPT}(p_i)$ and get $(\rho-1)\alpha_i p^*_i \le \gamma^*_i p^*_i$. □

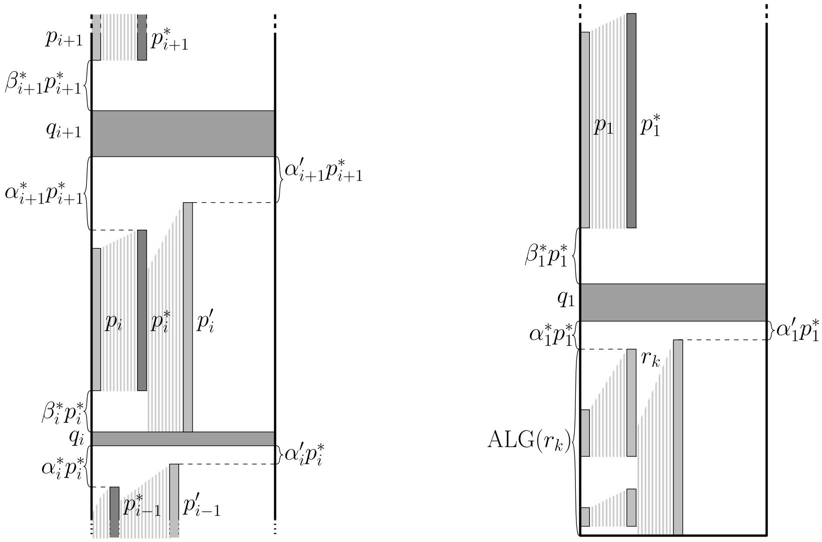
Similar to Kern and Paulus [11] we get the following lemma that bounds the potential function in terms of the parameters of the previous interval.

**Lemma 11** *We have*

$$\begin{aligned} \Phi_{i+1} = \gamma^*_{i+1} + \beta^*_{i+1} = {} & \frac{\gamma^*_i + (\rho-1)\frac{q_{i+1}}{p^*_i} + (\rho-1)\beta^*_i - 1}{1+\beta^*_i}\bigl(1-\alpha^*_{i+1}\bigr) \\ & + (\rho-2)\alpha^*_{i+1}. \end{aligned}$$

*Proof* See Fig. 6(a) for an illustration of the packing. We consider the change between $p^*_i$ and $p^*_{i+1}$ and with $p^*_{i+1} = \beta^*_i p^*_i + p^*_i + \alpha^*_{i+1} p^*_{i+1}$ from (3) we have

$$\begin{aligned} \mathrm{OPT}\bigl(p^*_{i+1}\bigr) &= \mathrm{OPT}\bigl(p^*_i\bigr) + q_{i+1} + p^*_{i+1} - p^*_i \\ &= \mathrm{OPT}\bigl(p^*_i\bigr) + q_{i+1} + \beta^*_i p^*_i + \alpha^*_{i+1} p^*_{i+1}, \\ \mathrm{ALG}\bigl(p^*_{i+1}\bigr) &= \mathrm{ALG}\bigl(p^*_i\bigr) + \alpha^*_{i+1} p^*_{i+1} + q_{i+1} + \beta^*_{i+1} p^*_{i+1} + p^*_{i+1} \\ &= \mathrm{ALG}\bigl(p^*_i\bigr) + \alpha^*_{i+1} p^*_{i+1} + q_{i+1} + \beta^*_{i+1} p^*_{i+1} + \beta^*_i p^*_i + p^*_i + \alpha^*_{i+1} p^*_{i+1}. \end{aligned}$$




(a) In this illustration we disregard Lemma 7 (which would give $\alpha'_i = 0$ or $\beta^*_i = 0$) to show the general case

(b) Here we show that ALG can pack further items aligned with the bottom of the strip before generating the gap $\beta^*_1 p^*_1$

**Fig. 6** Illustrations for Lemmas 11 and 12

Thus with $\gamma^*_i p^*_i = \rho\,\mathrm{OPT}(p^*_i) - \mathrm{ALG}(p^*_i)$ we get

$$\begin{aligned} &\mathrm{ALG}\big(p^*_{i+1}\big) + \gamma^*_{i+1} p^*_{i+1} = \rho \mathrm{OPT}\big(p^*_{i+1}\big) \\ \Leftrightarrow\quad & \gamma^*_{i+1} p^*_{i+1} + \beta^*_{i+1} p^*_{i+1} - (\rho - 2)\alpha^*_{i+1} p^*_{i+1} \\ &\quad = \gamma^*_i p^*_i + (\rho - 1) q_{i+1} + (\rho - 1)\beta^*_i p^*_i - p^*_i. \end{aligned}$$

By (3) we have $(1 - \alpha^*_i) p^*_{i+1} = (\beta^*_i + 1) p^*_i$ and finally get

$$\frac{\gamma^*_{i+1} + \beta^*_{i+1} - (\rho - 2)\alpha^*_{i+1}}{1 - \alpha^*_{i+1}} = \frac{\gamma^*_i + (\rho - 1)\frac{q_{i+1}}{p^*_i} + (\rho - 1)\beta^*_i - 1}{1 + \beta^*_i}. \qquad \square$$

This completes our preparations for the induction that we show next.

### 5.2.3 The Induction

In this section we show the intended contradiction: Any $\rho$-competitive algorithm needs to satisfy $\Phi_i \geq 0$, however the potential $\Phi_i$ decreases indefinitely.

We start the induction with the next lemma, giving a maximal initial value of $\rho - 2 + (\rho - 1)\hat{c}$ for the potential. Afterwards, we distinguish three cases according to the definition of $q_{i+1}$. If $q_{i+1} = \beta^*_i p^*_i$ or $q_{i+1} = \alpha'_i p^*_i$ we show $\Phi_{i+1} \leq \Phi_i - \varepsilon$ for some $\varepsilon > 0$. The case $q_{i+1} = q_i$ is more involved. Either we also get a decreasing potential or the potential might actually rise, but is still lower than the initial value.

Therefore, this rise can only happen once, as we finally show when we bring together all parts.

In the following calculations (which are very technical) we basically derive a series of upper bounds on the potential $\Phi_{i+1}$. In detail, we get

$$\Phi_{i+1} \le \frac{2(\rho-1)\Phi_i - 1}{1+\Phi_i} \qquad \text{in case } q_{i+1} = \beta_i^* p_i^* \tag{4}$$

$$\text{and} \quad \Phi_{i+1} \le \frac{\rho(\rho-1)\Phi_i - 1}{1+\rho\Phi_i} \qquad \text{in case } q_{i+1} = \beta_i^* p_i^*$$

$$\text{and} \quad \Phi_{i+1} \le \frac{2(\rho-1)\Phi_i - 1}{1+2\Phi_i} \qquad \text{in case } q_{i+1} = \alpha_i' p_i^*$$

$$\text{and} \quad \Phi_{i+1} < \frac{\rho(\rho-1)\Phi_i + \rho^2 - 3\rho + 1}{\rho\Phi_i + 2\rho - 1} \qquad \text{in case } q_{i+1} = q_i$$

$$\text{and} \quad \Phi_{i+1} \le \frac{(\Phi_i + (\rho-1)\hat{c} - 1)(\rho-1)}{\rho(\rho - 2 + (\rho-1)\hat{c})} + \rho - 2 \qquad \text{in case } q_{i+1} = q_i.$$

All these conditions eventually imply $\Phi_{i+1} \le \Phi_i - \varepsilon$ for some $\varepsilon > 0$ and $\rho < \hat{\rho}$. Just for condition (4) we additionally require the induction hypothesis $\Phi_i \le \rho - 2 + (\rho-1)\hat{c}$. This is actually exactly the condition that gives us the value of $\hat{c} = \frac{1-\sqrt{4\hat{\rho}^2 - 12\hat{\rho} + 5}}{2(\hat{\rho}-1)}$.

We now start with the induction hypothesis. Not only do we give an upper bound for the initial potential $\Phi_1$, but also for the ratio $q_1/p_1^*$. This is needed later when we bring the different parts together.

**Lemma 12** *We have*

$$\Phi_1 \le \rho - 2 + (\rho-1)\hat{c} \quad \textit{and}$$
$$\frac{q_1}{p_1^*} < \frac{1}{\rho}.$$

*Proof* Consider the packing of ALG and the optimal packing after $p_1^*$ is released—see Fig. 6(b). Recall that $\mathrm{ALG}(r_k)$ is the packing height at the end of the starting phase and that $q_1 = \hat{c}\,\mathrm{ALG}(r_k)$. As $p_1^*$ equals the height of the interval below $q_1$ we have

$$\mathrm{OPT}(p_1^*) = p_1^* + q_1 = p_1^* + \hat{c}\,\mathrm{ALG}(r_k) \quad \text{and}$$
$$\mathrm{ALG}(p_1^*) = p_1^* + q_1 + \beta_1^* p_1^* + p_1^* = 2p_1^* + \hat{c}\mathrm{ALG}(r_k) + \beta_1^* p_1^*.$$

Moreover, we have $p_1^* = \mathrm{ALG}(r_k) + \alpha_1^* p_1^*$ and thus $p_1^* = \frac{\mathrm{ALG}(r_k)}{1-\alpha_1^*}$. We get

$$\gamma_1^* p_1^* = \rho\mathrm{OPT}(p_1^*) - \mathrm{ALG}(p_1^*)$$
$$= (\rho-2)p_1^* + (\rho-1)\hat{c}\mathrm{ALG}(r_k) - \beta_1^* p_1^*$$

$$\Rightarrow \quad \Phi_i = \gamma_1^* + \beta_1^* = \rho - 2 + (\rho - 1)\hat{c}\big(1 - \alpha_1^*\big) \quad \text{since } p_1^* = \frac{\mathrm{ALG}(r_k)}{1 - \alpha_1^*}$$
$$\leq \rho - 2 + (\rho - 1)\hat{c}.$$

Finally, observe that

$$\frac{q_1}{p_1^*} \leq \frac{\hat{c}\,\mathrm{ALG}(r_k)}{\mathrm{ALG}(r_k)} = \hat{c} < \frac{1}{\rho}. \qquad \square$$

With the next two lemmas we show that the potential decreases if $q_{i+1} = \beta_i^* p_i^*$ or $q_{i+1} = \alpha_i' p_i^*$. At the same time we show that $q_{i+1}/p_{i+1}^*$ is bounded, which we need in the last case $q_{i+1} = q_i$.

**Lemma 13** *If $\Phi_i \leq \rho - 2 + (\rho - 1)\hat{c}$ and $q_{i+1} = \beta_i^* p_i^*$, then*

$$\Phi_{i+1} \leq \Phi_i - \varepsilon \quad \textit{for some } \varepsilon > 0 \quad \textit{and}$$
$$\frac{q_{i+1}}{p_{i+1}^*} \leq \frac{\Phi_{i+1}}{\rho - 1} \quad \textit{or} \quad \frac{q_{i+1}}{p_{i+1}^*} < \frac{1}{\rho}.$$

*Proof* By Lemma 7, condition 1 we can assume $\alpha_i' = 0$. Thus Lemma 11 yields

$$\Phi_{i+1} = \frac{\gamma_i^* + 2(\rho - 1)\beta_i^* - 1}{1 + \beta_i^*}\big(1 - \alpha_{i+1}^*\big) + (\rho - 2)\alpha_{i+1}^*.$$

Note that this function is linear in $\alpha_{i+1}^*$. Thus $\Phi_{i+1}$ attains its maximum for maximal or minimal $\alpha_{i+1}^*$. We show for both cases that $\Phi_{i+1} \leq \Phi_i - \varepsilon$ for some $\varepsilon > 0$.

If $\Phi_{i+1}$ is non-increasing in $\alpha_{i+1}^*$ we have

$$\begin{aligned} \Phi_{i+1} &\leq \frac{\gamma_i^* + 2(\rho - 1)\beta_i^* - 1}{1 + \beta_i^*} && \text{as } \alpha_{i+1}^* \geq 0 \\ &= \frac{\Phi_i + (2\rho - 3)\beta_i^* - 1}{1 + \beta_i^*} && \text{as } \gamma_i^* + \beta_i^* = \Phi_i \\ &\leq \frac{2(\rho - 1)\Phi_i - 1}{1 + \Phi_i}. \end{aligned}$$

The last step holds as $\beta_i^* \leq \Phi_i$ and the function is increasing with respect to $\beta_i^*$. With $\varepsilon = \varepsilon(\rho) = \frac{\hat{c}^2(\rho-1)^2 - \hat{c}(\rho-1) - (\rho-1)(\rho-2) + 1}{(\rho-1)(1+\hat{c})}$ we have $\Phi_{i+1} \leq \Phi_i - \varepsilon$ since

$$\begin{aligned} &\frac{2(\rho - 1)\Phi_i - 1}{1 + \Phi_i} \leq \Phi_i - \varepsilon \\ &\quad \Leftrightarrow \quad \Phi_i^2 - (2\rho - 3 + \varepsilon)\Phi_i \geq \varepsilon - 1 \\ &\quad \Leftarrow \quad \big(\rho - 2 + (\rho - 1)\hat{c}\big)^2 - (2\rho - 3 + \varepsilon)\big(\rho - 2 + (\rho - 1)\hat{c}\big) \geq \varepsilon - 1 \\ &\qquad\qquad \text{as } \Phi_i \leq \rho - 2 + (\rho - 1)\hat{c} \text{ and } 2\Phi_i - 2\rho + 3 - \varepsilon \leq 2(\rho - 1)\hat{c} - 1 < 0 \end{aligned}$$

$$\Leftrightarrow \quad \frac{(\rho-2+(\rho-1)\hat{c})^2-(2\rho-3)(\rho-2+(\rho-1)\hat{c})+1}{1+\rho-2+(\rho-1)\hat{c}} \geq \varepsilon$$
$$\Leftarrow \quad \frac{\hat{c}^2(\rho-1)^2-\hat{c}(\rho-1)-(\rho-1)(\rho-2)+1}{(\rho-1)(1+\hat{c})} = \varepsilon.$$

It remains to show $\varepsilon = \varepsilon(\rho) > 0$. We have $\varepsilon(\hat{\rho}) = 0$ since

$$\hat{c}^2(\hat{\rho}-1)^2-\hat{c}(\hat{\rho}-1)-(\hat{\rho}-1)(\hat{\rho}-2)+1=0$$

for $\hat{c} = \frac{1-\sqrt{4\hat{\rho}^2-12\hat{\rho}+5}}{2(\hat{\rho}-1)}$. Now observe that $\varepsilon$ is strictly decreasing with $\rho$ since

$$\frac{\partial}{\partial\rho}\big(\varepsilon(\rho)\big) = \frac{(\hat{c}^2-1)(\rho-1)^2-(\rho^2-2\rho+2)}{(\rho-1)^2(1+\hat{c})} < 0$$

as $\hat{c}^2-1<0$ and $\rho^2-2\rho+2>0$. Thus we have $\varepsilon = \varepsilon(\rho) > \varepsilon(\hat{\rho}) = 0$ in this case.

Now, if $\Phi_{i+1}$ is increasing in $\alpha^*_{i+1}$, we use Lemma 9 to get

$$\begin{aligned}
\Phi_{i+1} &\leq \frac{\gamma_i^*+2(\rho-1)\beta_i^*-1}{1+\beta_i^*}\cdot\left(1-\frac{\gamma_i^*+(\rho-1)\beta_i^*}{1+\gamma_i^*+\rho\beta_i^*}\right)+(\rho-2)\cdot\frac{\gamma_i^*+(\rho-1)\beta_i^*}{1+\gamma_i^*+\rho\beta_i^*}\\
&\leq \frac{\gamma_i^*+2(\rho-1)\beta_i^*-1}{1+\beta_i^*}\cdot\frac{1+\beta_i^*}{1+\gamma_i^*+\rho\beta_i^*}+(\rho-2)\cdot\frac{\gamma_i^*+(\rho-1)\beta_i^*}{1+\gamma_i^*+\rho\beta_i^*}\\
&= \frac{(\rho-1)\gamma_i^*+\rho(\rho-1)\beta_i^*-1}{1+\gamma_i^*+\rho\beta_i^*}\\
&= \frac{(\rho-1)\Phi_i+(\rho-1)^2\beta_i^*-1}{1+\Phi_i+(\rho-1)\beta_i^*} \quad \text{as } \gamma_i^*+\beta_i^*=\Phi_i\\
&\leq \frac{\rho(\rho-1)\Phi_i-1}{1+\rho\Phi_i}.
\end{aligned}$$

Again, the last step holds as $\beta_i^* \leq \Phi_i$ and the function is increasing with respect to $\beta_i^*$. With $\varepsilon = 3-\rho-1/\rho > 0$ (for $\rho < \hat{\rho}$) we have $\Phi_{i+1} \leq \Phi_i - \varepsilon$ since

$$\frac{\rho(\rho-1)\Phi_i-1}{1+\rho\Phi_i} \leq \Phi_i-3+\rho+\frac{1}{\rho}$$
$$\Leftrightarrow \quad \rho(\rho-1)\Phi_i-1 \leq \Phi_i-3+\rho+\frac{1}{\rho}+\rho\Phi_i^2-3\rho\Phi_i+\rho^2\Phi_i+\Phi_i$$
$$\Leftrightarrow \quad \Phi_i^2+\left(\frac{2-2\rho}{\rho}\right)\Phi_i \geq \frac{2-\rho-\frac{1}{\rho}}{\rho}$$
$$\Leftrightarrow \quad \left(\Phi_i^2+\frac{1-\rho}{\rho}\right)^2 \geq \left(\frac{1-\rho}{\rho}\right)^2+\frac{2-\rho-\frac{1}{\rho}}{\rho}=0.$$

Thus in both cases we have $\Phi_{i+1} \leq \Phi_i - \varepsilon$ for some $\varepsilon > 0$.

It remains to show

$$\frac{q_{i+1}}{p^*_{i+1}} \leq \frac{\Phi_{i+1}}{\rho - 1} \quad \text{or} \quad \frac{q_{i+1}}{p^*_{i+1}} < \frac{1}{\rho}.$$

We have $\frac{q_{i+1}}{p^*_{i+1}} = \frac{\beta^*_i}{1+\beta^*_i}(1-\alpha^*_{i+1})$ by (3). If $\beta^*_i < \frac{1}{\rho-1}$ we have

$$\frac{q_{i+1}}{p^*_{i+1}} = \frac{\beta^*_i}{1+\beta^*_i}\big(1-\alpha^*_{i+1}\big) \leq \frac{\beta^*_i}{1+\beta^*_i} < \frac{1}{\rho-1} \cdot \frac{1}{1+\frac{1}{\rho-1}} = \frac{1}{\rho-1} \cdot \frac{1}{\frac{\rho}{\rho-1}} = \frac{1}{\rho}.$$

Otherwise, we have $\beta^*_i \geq \frac{1}{\rho-1}$ and thus

$$\begin{aligned}
&\frac{q_{i+1}}{p^*_{i+1}} = \frac{\beta^*_i}{1+\beta^*_i}\big(1-\alpha^*_{i+1}\big) \leq \frac{\Phi_{i+1}}{\rho-1} \\
&\Leftrightarrow \quad (\rho-1)\beta^*_i \leq \gamma^*_i + 2(\rho-1)\beta^*_i - 1 + \underbrace{(\rho-2)(1+\beta^*_i)\frac{\alpha^*_{i+1}}{1-\alpha^*_{i+1}}}_{\geq 0} \\
&\Leftarrow \quad 1 \leq \gamma^*_i + (\rho-1)\beta^*_i \\
&\Leftarrow \quad \beta^*_i \geq \frac{1}{\rho-1}.
\end{aligned}$$

□

**Lemma 14** *If* $\Phi_i \leq \rho - 2 + (\rho-1)\hat{c}$ *and* $q_{i+1} = \alpha'_i p^*_i$ *then*

$$\Phi_{i+1} \leq \Phi_i - \varepsilon \quad \textit{for some } \varepsilon > 0, \textit{ and}$$
$$\frac{q_{i+1}}{p^*_{i+1}} < \frac{1}{\rho}.$$

*Proof* In this case we can assume $\beta^*_i = 0$ (by Lemma 7, condition 2) and hereby have $\Phi_i = \gamma^*_i$. Thus by Lemma 11 and with $(\rho-1)\alpha'_i \leq \gamma^*_i = \Phi_i$ by Lemma 10 we have

$$\begin{aligned}
\Phi_{i+1} &= \big(\Phi_i + (\rho-1)\alpha'_i - 1\big)\big(1-\alpha^*_{i+1}\big) + (\rho-2)\alpha^*_{i+1} \\
&\leq (2\Phi_i - 1)\big(1-\alpha^*_{i+1}\big) + (\rho-2)\alpha^*_{i+1}.
\end{aligned}$$

We consider the derivative with respect to $\alpha^*_{i+1}$ and with $\Phi_i \leq \rho - 2 + (\rho-1)\hat{c}$ we get

$$\begin{aligned}
\frac{\partial}{\partial \alpha^*_{i+1}}\big((2\Phi_i - 1)\big(1-\alpha^*_{i+1}\big) + (\rho-2)\alpha^*_{i+1}\big) &= \rho - 1 - 2\Phi_i \\
&\geq 3 - \rho - 2(\rho-1)\hat{c} > 0.
\end{aligned}$$

Thus $\Phi_{i+1}$ increases with $\alpha^*_{i+1}$ and since by Lemma 9

$$\alpha^*_{i+1} \leq \frac{\gamma^*_i + (\rho-1)\alpha'_i}{1+\gamma^*_i + (\rho-1)\alpha'_i} \leq \frac{2\gamma^*_i}{1+2\gamma^*_i} = \frac{2\Phi_i}{1+2\Phi_i}$$

we get

$$\begin{aligned}\Phi_{i+1} &\le (2\Phi_i - 1)\left(1 - \frac{2\Phi_i}{1+2\Phi_i}\right) + (\rho - 2)\frac{2\Phi_i}{1+2\Phi_i}\\ &= \frac{2(\rho-1)\Phi_i - 1}{1+2\Phi_i}.\end{aligned}$$

With $\varepsilon = \sqrt{2\rho} - \rho + 1/2 > 0$ (for $\rho < \hat{\rho}$) we have $\Phi_{i+1} \le \Phi_i - \varepsilon$ since

$$\begin{aligned}&\frac{2(\rho-1)\Phi_i - 1}{1+2\Phi_i} \le \Phi_i - \sqrt{2\rho} + \rho - \frac{1}{2}\\ \Leftrightarrow \quad & 2\Phi_i^2 + \bigl(2 - 2\sqrt{2\rho}\bigr)\Phi_i \ge \sqrt{2\rho} - \rho - \frac{1}{2}\\ \Leftrightarrow \quad & \left(\Phi_i + \frac{1-\sqrt{2\rho}}{2}\right)^2 \ge \left(\frac{1-\sqrt{2\rho}}{2}\right)^2 + \frac{\sqrt{2\rho}-\rho-\frac{1}{2}}{2} = 0.\end{aligned}$$

Thus we proved the first part of the lemma.

It remains to show

$$\frac{q_{i+1}}{p^*_{i+1}} < \frac{1}{\rho}.$$

With $\beta^*_i = 0$ we have $p^*_{i+1} = p^*_i/(1 - \alpha^*_{i+1})$ by (3). Using $\alpha^*_{i+1} \ge \frac{(\rho-1)\alpha'_i}{1+(\rho-1)\alpha'_i}$ by Lemma 8 and $\alpha'_i \le \alpha^*_i < 1$ (by definition of $\alpha^*_i$ as a fraction of $p^*_i$) we get

$$\frac{q_{i+1}}{p^*_{i+1}} = \frac{\alpha'_i p^*_i}{p^*_{i+1}} = \alpha'_i\bigl(1 - \alpha^*_{i+1}\bigr) \le \frac{\alpha'_i}{1+(\rho-1)\alpha'_i} < \frac{1}{\rho}.$$

This finishes the proof of this lemma. □

Finally, we consider the case $q_{i+1} = q_i$. First, we show that the potential definitely decreases if $q_i/p^*_i < 1/\rho$.

**Lemma 15** *If $\Phi_i \le \rho - 2 + (\rho-1)\hat{c}$ and $\frac{q_i}{p^*_i} < \frac{1}{\rho}$ and $q_{i+1} = q_i$ then*

$$\begin{aligned}&\Phi_{i+1} \le \Phi_i - \varepsilon \quad \textit{for some } \varepsilon > 0, \textit{ and}\\ &\frac{q_{i+1}}{p^*_{i+1}} < \frac{1}{\rho}.\end{aligned}$$

*Proof* The second part is trivial since $q_{i+1} = q_i$, $p^*_{i+1} \ge p^*_i$ and $q_i/p^*_i < 1/\rho$.

To show the first part we assume $\alpha'_i = 0$ and $\beta^*_i = 0$ according to conditions 1 and 2 of Lemma 7. Thus we have $\Phi_i = \gamma^*_i$ and with Lemma 11 we get

$$\Phi_{i+1} = \left(\Phi_i + (\rho-1)\frac{q_i}{p^*_i} - 1\right)\bigl(1 - \alpha^*_{i+1}\bigr) + (\rho-2)\alpha^*_{i+1}.$$

We consider the derivative with respect to $\alpha^*_{i+1}$ and get

$$\frac{\partial}{\partial \alpha^*_{i+1}}(\Phi_{i+1}) = \rho - 1 - \Phi_i - (\rho-1)\frac{q_i}{p^*_i} > 1 - (\rho-1)\hat{c} - \frac{\rho-1}{\rho} > 0.$$

Thus $\Phi_{i+1}$ increases with $\alpha^*_{i+1}$ and since by Lemma 9

$$\alpha^*_{i+1} \le \frac{\Phi_i + (\rho-1)\frac{q_i}{p^*_i}}{1 + \Phi_i + (\rho-1)\frac{q_i}{p^*_i}}$$

we get

$$\begin{aligned}
\Phi_{i+1} &\le \frac{(\rho-1)\Phi_i + (\rho-1)^2\frac{q_i}{p^*_i} - 1}{1 + \Phi_i + (\rho-1)\frac{q_i}{p^*_i}} \\
&< \frac{(\rho-1)\Phi_i + \frac{(\rho-1)^2}{\rho} - 1}{\Phi_i + \frac{2\rho-1}{\rho}} \quad \text{as } \frac{q_i}{p^*_i} < \frac{1}{\rho} \\
&= \frac{\rho(\rho-1)\Phi_i + \rho^2 - 3\rho + 1}{\rho\Phi_i + 2\rho - 1}.
\end{aligned}$$

With

$$\varepsilon = \frac{3\rho - \rho^2 - 1}{\rho\Phi_i + 2\rho - 1} \ge \frac{3\rho - \rho^2 - 1}{\rho(\rho + (\rho-1)\hat{c}) - 1} > 0,$$

as $\Phi_i \le \rho - 2 + (\rho-1)\hat{c}$ and $3\rho - \rho^2 - 1 > 0$ for $\rho < \hat{\rho}$, we have $\Phi_{i+1} \le \Phi_i - \varepsilon$ since

$$\begin{aligned}
&\frac{\rho(\rho-1)\Phi_i + \rho^2 - 3\rho + 1}{\rho\Phi_i + 2\rho - 1} \le \Phi_i - \frac{3\rho - \rho^2 - 1}{\rho\Phi_i + 2\rho - 1} \\
\Leftrightarrow \quad & \rho(\rho-1)\Phi_i + \rho^2 - 3\rho + 1 \le \rho\Phi_i^2 + (2\rho-1)\Phi_i - 3\rho + \rho^2 + 1 \\
\Leftrightarrow \quad & 0 \le \rho\Phi_i^2 + \left(3\rho - \rho^2 - 1\right)\Phi_i
\end{aligned}$$

which is satisfied as $3\rho - \rho^2 - 1 > 0$ for $\rho < \hat{\rho}$. □

If we do not have $q_i/p^*_i < 1/\rho$ we can still assume that $q_i/p^*_i \le (\rho - 2 + (\rho-1)\hat{c})/(\rho-1)$ by Lemmas 13 and 14 (as $\Phi_i \le \rho - 2 + (\rho-1)\hat{c}$ and this ratio does not increase in case $q_{i+1} = q_i$).

We use this bound to show that either the potential still decreases or we can bound the potential by $\rho - 2$ and the $q_{i+1}/p^*_{i+1}$ ratio is less than $1/\rho$. So from this point on we remain in the case of the previous lemma and the potential decreases by a constant in every step.

**Lemma 16** *If $\Phi_i \le \rho - 2 + (\rho-1)\hat{c}$ and $\frac{q_i}{p_i^*} \le \frac{\rho-2+(\rho-1)\hat{c}}{\rho-1}$ and $q_{i+1} = q_i$ then either*

$$\Phi_{i+1} < \rho - 2 \quad \textit{and}$$
$$\frac{q_{i+1}}{p_{i+1}^*} < \frac{1}{\rho},$$

*or*

$$\Phi_{i+1} \le \Phi_i - \varepsilon \quad \textit{for some } \varepsilon > 0, \ \textit{and}$$
$$\frac{q_{i+1}}{p_{i+1}^*} \le \frac{\rho - 2 + (\rho-1)\hat{c}}{\rho - 1}.$$

*Proof* As in Lemma 15 we have $\alpha_i' = 0$, $\beta_i^* = 0$, $\Phi_i = \gamma_i^*$ and

$$\Phi_{i+1} = \left(\Phi_i + (\rho-1)\frac{q_i}{p_i^*} - 1\right)\bigl(1 - \alpha_{i+1}^*\bigr) + (\rho-2)\alpha_{i+1}^*.$$

Again, $\Phi_{i+1}$ increases with $\alpha_{i+1}^*$. We distinguish two cases according to the value of $\alpha_{i+1}^*$.

If $\alpha_{i+1}^* > 1 - \frac{\rho-1}{\rho(\rho-2+(\rho-1)\hat{c})}$ then

$$\frac{q_{i+1}}{p_{i+1}^*} = \frac{q_i}{p_i^*}\bigl(1 - \alpha_{i+1}^*\bigr) < \frac{\rho - 2 + (\rho-1)\hat{c}}{\rho-1} \cdot \frac{\rho-1}{\rho(\rho - 2 + (\rho-1)\hat{c})} = \frac{1}{\rho}.$$

As $\Phi_{i+1}$ is increasing with $\alpha_{i+1}^*$ we use Lemma 9 to get

$$\Phi_{i+1} \le \frac{(\rho-1)\Phi_i + (\rho-1)^2\frac{q_i}{p_i^*} - 1}{1 + \Phi_i + (\rho-1)\frac{q_i}{p_i^*}}$$
$$\le \frac{2(\rho-1)(\rho-2+(\rho-1)\hat{c}) - 1}{1 + 2(\rho - 2 + (\rho-1)\hat{c})}$$

as $\Phi_i \le \rho - 2 + (\rho-1)\hat{c}$ and $\frac{q_i}{p_i^*} \le \frac{\rho-2+(\rho-1)\hat{c}}{\rho-1}$. We have

$$\frac{2(\rho-1)(\rho-2+(\rho-1)\hat{c}) - 1}{1 + 2(\rho-2+(\rho-1)\hat{c})} < \rho - 2$$
$$\Leftrightarrow \quad 2(\rho-1)\bigl(\rho - 2 + (\rho-1)\hat{c}\bigr) - 1 < \rho - 2 + 2(\rho-2)\bigl(\rho - 2 + (\rho-1)\hat{c}\bigr)$$
$$\Leftrightarrow \quad 2\bigl(\rho - 2 + (\rho-1)\hat{c}\bigr) < \rho - 1$$
$$\Leftrightarrow \quad 2(\rho-1)\hat{c} < 3 - \rho,$$

which holds for $\rho < \hat{\rho}$. We showed that if $\alpha_{i+1}^* > 1 - \frac{\rho-1}{\rho(\rho-2+(\rho-1)\hat{c})}$, then we have $\Phi_{i+1} < \rho - 2$ and $q_{i+1}/p_{i+1}^* < 1/\rho$.

Otherwise, we have $\alpha^*_{i+1} \le 1 - \frac{\rho-1}{\rho(\rho-2+(\rho-1)\hat{c})}$ and get

$$\begin{aligned}\Phi_{i+1} &\le \frac{(\Phi_i + (\rho-1)\frac{q_i}{p_i^*} - 1)(\rho-1)}{\rho(\rho-2+(\rho-1)\hat{c})} + \rho - 2 - \frac{(\rho-2)(\rho-1)}{\rho(\rho-2+(\rho-1)\hat{c})}\\ &\le \frac{(\Phi_i + (\rho-1)\hat{c} - 1)(\rho-1)}{\rho(\rho-2+(\rho-1)\hat{c})} + \rho - 2 \quad \text{as } \frac{q_i}{p_i^*} \le \frac{\rho-2+(\rho-1)\hat{c}}{\rho-1}.\end{aligned}$$

With $\varepsilon = \Phi_i + \frac{(1-(\rho-1)\hat{c}-\Phi_i)(\rho-1)}{\rho(\rho-2+(\rho-1)\hat{c})} - \rho + 2$ we have $\Phi_{i+1} \le \Phi_i - \varepsilon$ since

$$\begin{aligned}&\frac{(\Phi_i + (\rho-1)\hat{c} - 1)(\rho-1)}{\rho(\rho-2+(\rho-1)\hat{c})} + \rho - 2\\ &\quad \le \Phi_i - \Phi_i - \frac{(1-(\rho-1)\hat{c}-\Phi_i)(\rho-1)}{\rho(\rho-2+(\rho-1)\hat{c})} + \rho - 2.\end{aligned}$$

It remains to show that $\varepsilon > 0$ . To see this, observe that $\varepsilon$ is increasing with respect to $\Phi_i$ as

$$\frac{\partial}{\partial \Phi_i}\left(\Phi_i + \frac{(1-(\rho-1)\hat{c}-\Phi_i)(\rho-1)}{\rho(\rho-2+(\rho-1)\hat{c})} - \rho + 2\right) = 1 - \frac{\rho-1}{\rho(\rho-2+(\rho-1)\hat{c})} > 0$$

for $2.55 \le \rho < \hat{\rho}$ (here we assume $\delta = \hat{\rho} - \rho$ is sufficiently small). As $\Phi_i \ge 0$ we have

$$\varepsilon \ge \frac{(1-(\rho-1)\hat{c})(\rho-1)}{\rho(\rho-2+(\rho-1)\hat{c})} - \rho + 2 > 0.$$

Of course,

$$\frac{q_{i+1}}{p_{i+1}^*} \le \frac{q_i}{p_i^*} \le \frac{\rho-2+(\rho-1)\hat{c}}{\rho-1}$$

holds trivially. We showed that if $\alpha^*_{i+1} \le 1 - \frac{\rho-1}{\rho(\rho-2+(\rho-1)\hat{c})}$, then we have $\Phi_{i+1} \le \Phi_i - \varepsilon$ for $\varepsilon > 0$ and $q_{i+1}/p^*_{i+1} \le (\rho-2+(\rho-1)\hat{c})/(\rho-1)$. This finishes the proof of this lemma. □

This ends this extensive induction. Let us summarize the complete induction and show that it actually gives the desired contradiction.

Recall that our induction hypothesis in Lemma 12 states that $\Phi_1 \le \rho-2+(\rho-1)\hat{c}$ and $q_1/p_1^* < 1/\rho$.

First assume that whenever we need to apply Lemma 16, then the second condition holds, i.e., $\Phi_{i+1} \le \Phi_i - \varepsilon$ and $q_{i+1}/p^*_{i+1} \le (\rho-2+(\rho-1)\hat{c})/(\rho-1)$. Then Lemma 13 (for $q_{i+1} = \beta_i^* p_i^*$), Lemma 14 (for $q_{i+1} = \alpha_i' p_i^*$) and Lemmas 15 and 16 (for $q_{i+1} = q_i$) show that the potential decreases by a constant in every step.

Now if Lemma 16 is applied and the second condition does not hold, then $\Phi_{i+1} < \rho - 2$ and $q_{i+1}/p^*_{i+1} < 1/\rho$. Thus Lemma 13 (for $q_{i+1} = \beta_i^* p_i^*$), Lemma 14 (for $q_{i+1} = \alpha_i' p_i^*$) and Lemmas 15 (for $q_{i+1} = q_i$) show that the $q_i/p_i^*$ ratio remains less than $1/\rho$. So the precondition for Lemma 15 is always satisfied if $q_{i+1} = q_i$ and we do not need to apply Lemma 16 anymore. Thus from this point on, the potential decreases by a constant in every further step.

Summarizing, we derived a contradiction to $\Phi_i \geq 0$ for all $i \geq 1$, thereby proving Theorem 1.

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