# An Improved Deterministic Parameterized Algorithm for Cactus Vertex Deletion 

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#### Abstract

A cactus is a connected graph that does not contain $K_{4}-e$ as a minor. Given a graph $G=(V, E)$ and an integer $k \geq 0$, Cactus Vertex Deletion (also known as Diamond Hitting SET) is the problem of deciding whether $G$ has a vertex set of size at most $k$ whose removal leaves a forest of cacti. The previously best deterministic parameterized algorithm for this problem was due to Bonnet et al. [WG 2016], which runs in time $26^{k} n^{O(1)}$, where $n$ is the number of vertices of $G$. In this paper, we design a deterministic algorithm for Cactus Vertex Deletion, which runs in time $17.64^{k} n^{O(1)}$. As an almost straightforward application of our algorithm, we also give a deterministic $17.64^{k} n^{O(1)}$-time algorithm for Even Cycle Transversal, which improves the previous running time $50^{k} n^{O(1)}$ of the known deterministic parameterized algorithm due to Misra et al. [WG 2012].


## 1 Introduction

A connected graph is a cactus if every edge belongs to at most one cycle. A cactus forest is a graph such that every connected component is a cactus. In this paper, we consider the following problem.

Definition 1 (Cactus Vertex Deletion). Given a graph $G=(V, E)$ and an integer $k \geq 0$, the problem asks whether $G$ has a vertex set $X \subseteq V$ with $|X| \leq k$ whose removal leaves a cactus forest.

The problem is one of vertex deletion problems for hereditary properties, which have been both intensively and extensively studied in the field of parameterized algorithms and complexity. The best known problem in this context is Vertex Cover. The problem asks whether there is a vertex set of size at most $k$ whose removal leaves an edge-less graph. A naive algorithm solves Vertex Cover in $O^{*}\left(2^{k}\right)$ time ${ }^{1}$, and after a series of improvements, the fastest known algorithm is due to Chen et al. [4], which runs in time $O^{*}\left(1.2738^{k}\right)$.

Another example of this kind of problems is Feedback Vertex Set. The problem asks whether an input graph $G=(V, E)$ has a vertex set of size at most $k$ that hits all the cycles in the graph. In other words, the goal of this problem is to compute $X \subseteq V$ with $|X| \leq k$ such that the graph obtained from $G$ by deleting $X$ is a forest. This problem is also intensively studied, and several deterministic and randomized algorithms have been proposed so far [1, 5, 6, 12, 13, 15]. The current best running

[^0]time is due to Iwata and Kobayashi [13] for deterministic algorithms and Li and Nederlof [15] for randomized algorithms, which run in time $O^{*}\left(3.460^{k}\right)$ and $O^{*}\left(2.7^{k}\right)$, respectively.

The gap between the running time of deterministic and randomized algorithms sometimes emerges for vertex deletion problems to "sparse" hereditary classes of graphs, such as Feedback Vertex Set. For instance, Pseudo Forest Vertex Deletion can be solved deterministically in time $O^{*}\left(3^{k}\right)$ [2] and randomizedly in time $O^{*}\left(2.85^{k}\right)$ [11 and Bounded Degree-2 Vertex Deletion can be solved deterministically in time $O^{*}\left(3.0645^{k}\right)$ [18] and randomizedly in time $O^{*}\left(3^{k}\right)$ [7. Among others, the known gap on Cactus Vertex Deletion is remarkable: Bonnet et al. [3 presented a deterministic $O^{*}\left(26^{k}\right)$-time algorithm, while Kolay et al. [14] presented a randomized $O^{*}\left(12^{k}\right)$-time algorithm.

In this paper, we narrow the gap between the running time of deterministic and randomized algorithms by giving an improved deterministic algorithm for Cactus Vertex Deletion.

Theorem 1. Cactus Vertex Deletion can be solved deterministically in time $O^{*}\left(17.64^{k}\right)$.
As a variant of Cactus Vertex Deletion, we consider Even Cycle Transversal defined as follows. A cactus is called an odd cactus if every cycle in it has an odd number of vertices.

Definition 2 (Even Cycle Transversal). Given a graph $G=(V, E)$ and an integer $k \geq 0$, the problem asks whether $G$ has a vertex set $X \subseteq V$ with $|X| \leq k$ whose removal leaves a forest of odd cacti.

Note that a graph has no cycles of even length if and only if it is a forest of odd cacti [14. Kolay et al. [14] gave an $O^{*}\left(12^{k}\right)$-time randomized algorithm and Misra et al. 16] gave an $O^{*}\left(50^{k}\right)$-time deterministic algorithm for Even Cycle Transversal. In this paper, we improve the running time of the deterministic algorithm for Even Cycle Transversal as well as Cactus Vertex Deletion.

Theorem 2. Even Cycle Transversal can be solved deterministically in time $O^{*}\left(17.64^{k}\right)$.
The idea of our algorithms follows that used in 3. We solve the disjoint version of Cactus Vertex Deletion with a branching algorithm. In this version, given a vertex subset $S \subseteq V$ such that $|S| \leq k+1$ and the subgraph induced by $V \backslash S$, denoted $G[V \backslash S]$, is a cactus forest, the problem asks whether there is a vertex subset $X \subseteq V \backslash S$ such that $|X| \leq k$ and $G[V \backslash X]$ is a cactus forest. To solve this problem, Bonnet et al. [3] gave a branching algorithm with the measure and conquer analysis [9. They used measure $k+\mathrm{cc}(G[S])$, where $\mathrm{cc}(G[S])$ is the number of connected components in $G[S]$, and proved that each branch of their algorithm strictly decreases this measure. The main difficulty with using this measure is that when we consider a vertex $v \in V \backslash S$ such that $v$ has at least two neighbors only in a single connected component in $G[S]$, then one of the branch, for which $v$ is determined to be not deleted, does not decrease the measure. We also use the measure and conquer analysis with a slightly elaborate measure $\alpha k+\beta \cdot \mathrm{cc}(G[S])+\gamma \cdot \mathrm{b}(G[S])$, where $\alpha, \beta, \gamma$ are some constants and $\mathrm{b}(G[S])$ is the number of bridges in $G[S]$, which allows us to decrease the measure efficiently: When $v$ is determined to be not deleted, the number of bridges in $G[S]$ is decreased in the above situation since otherwise $v$ belongs to a $K_{4}-e$ minor. We believe that although our measure is slightly involved compared to that in [3], the algorithm itself and its analysis would be simpler than theirs.

## 2 Preliminaries

Graphs. Throughout the paper, graphs have no self-loops but may have multiedges. Let $G=(V, E)$ be a graph. We write $V(G)$ and $E(G)$ to denote the sets of vertices and edges of $G$, respectively. For two distinct vertices $u, v$ in $G$, we denote by $m(u, v)$ the number of edges between $u$ and $v$. Let $v \in V$. The degree of $v$ is the number of edges incident to it. We denote by $N_{G}(v)$ the set of neighbors of $v$ in $G$. Note that as $G$ may have multiedges, $\left|N_{G}(v)\right|$ may not be equal to its degree. For $X \subseteq V$, the subgraph of $G$ induced by $X$ is denoted as $G[X]$. We denote by $\operatorname{cc}(G)$ the number of connected
components in $G$. A vertex $v \in V$ is called a cut vertex of $G$ if $\operatorname{cc}(G[V \backslash\{v\}])>\operatorname{cc}(G)$ and an edge $e \in E$ is called a bridge of $G$ if $\operatorname{cc}(G-e)>\operatorname{cc}(G)$, where $G-e$ is the graph obtained from $G$ by deleting $e$. Note that it holds that $\operatorname{cc}(G-e)=\operatorname{cc}(G)+1$ when $e$ is a bridge of $G$. The number of bridges in $G$ is denoted by $\mathrm{b}(H)$.

Lemma 1. Let $H$ be a multigraph with $h$ vertices. Then, it holds that $\mathrm{cc}(H)+\mathrm{b}(H) \leq h$.
Proof. Let $H^{\prime}$ be the graph obtained from $H$ by removing all bridges of $H$. Then, $\mathrm{cc}(H)+\mathrm{b}(H)=$ $\operatorname{cc}\left(H^{\prime}\right) \leq h$.

A block of a graph $G$ is a maximal vertex set $B$ of $G$ such that $G[B]$ is connected and has no cut vertices. Note that a graph consisting of two vertices with at least one edge is a block. It is easy to see that every block in a cactus forest is either a cycle, an edge, or an isolated vertex. In particular, we call $B$ a leaf block if it has at most one cut vertex. We say that vertices $v_{1}, \ldots, v_{t} \in V(B)$ are consecutive in $B$ if for each $1 \leq i<t, v_{i}$ is adjacent to $v_{i+1}$ in $B$.

Iterative compression. Our algorithm employs the well-known iterative compression technique invented by Reed, Smith, and Vetta [17. They gave an algorithm for Odd Cycle Transversal based on this technique. The essential idea can be generalized as follows. Let $\mathcal{C}$ be a hereditary class of graphs, that is, for $G \in \mathcal{C}$, every induced subgraph of $G$ also belongs to $\mathcal{C}$. The technique is widely used for designing algorithms of vertex deletion problems to hereditary classes of graphs. The crux of the technique can be described as the following lemma.

Lemma 2 ([17]). Let $\mathcal{C}$ be a hereditary class of graphs. Given a graph $G=(V, E)$ and an integer $k$, the problem of computing $X \subseteq V$ with $|X| \leq k$ such that $G[V \backslash X] \in \mathcal{C}$ can be solved in time $O^{*}\left((c+1)^{k}\right)$ if one can solve the following problem in time $O^{*}\left(c^{k}\right)$ : Given a subset $S \subseteq V$ of cardinality at most $k+1$ with $G[V \backslash S] \in \mathcal{C}$, the problem asks to find $X \subseteq V \backslash S$ with $|X| \leq k$ such that $G[V \backslash X] \in \mathcal{C}$.

For Cactus Vertex Deletion, the latter problem is defined as follows.
Definition 3 (Disjoint Cactus Vertex Deletion). Given a graph $G=(V, E)$, an integer $k \geq 0$, and $S \subseteq V$ such that $G[V \backslash S]$ is a cactus forest, the problem asks to find a vertex set $X \subseteq V \backslash S$ with $|X| \leq k$ whose removal leaves a cactus forest.

Let us note that we can assume that $G[S]$ is also a cactus forest as otherwise the problem is trivially infeasible.

Measure and conquer analysis. Our algorithm for Disjoint Cactus Vertex Deletion is based on a standard branching algorithm with the measure and conquer analysis [9]. Given an instance $I$ of the problem, we define a measure $\mu(I)$ that is non-negative real and design a branching algorithm that generates subinstances $I_{1}, \ldots, I_{t}$ with $\mu(I)>\mu\left(I_{i}\right)$ for $1 \leq i \leq t$. To measure the running time of the algorithm, we use a branching factor $\left(b_{1}, \ldots, b_{t}\right)$, where $\mu(I)-\mu\left(I_{i}\right) \geq b_{i}$ for each $i$. It is known that the total running time of this branching algorithm is upper bounded by $O^{*}\left(c^{\mu(I)}\right)$, where $c$ is the unique positive real root of equation

$$
x^{-b_{1}}+x^{-b_{2}}+\cdots+x^{-b_{t}}=1
$$

assuming that from any instance $I$ with $\mu(I)>0$, its subinstances can be generated in polynomial time and for any instance $I$ with $\mu(I)=0$, the problem can be solved in polynomial time. We refer the reader to the book [10] for a detailed exposition for the measure and conquer analysis.

## 3 An improved algorithm for Disjoint Cactus Vertex Deletion

This section is devoted to developing an algorithm for Disjoint Cactus Vertex Deletion that runs in time $O^{*}\left(16.64^{k}\right)$, proving Theorem 1 by Lemma 2 .

Lemma 3. Suppose that $|S| \leq k+1$. Then, Disjoint Cactus Vertex Deletion can be solved in time $O^{*}\left(16.64^{k}\right)$.

Let $I=(G, S, k)$ be an instance of Disjoint Cactus Vertex Deletion, where $G=(V, E)$ is a multigraph, $S \subseteq V$. Recall that we assume $G[V \backslash S]$ and $G[S]$ are both cactus forests as otherwise the problem is trivially infeasible.

Let $\mu(I)=\alpha \cdot k+\beta \cdot \mathrm{cc}(G[S])+\gamma \cdot \mathrm{b}(G[S])$, where $\alpha, \beta, \gamma$ are chosen later. In the following, we assume that $\beta \geq \gamma$. For the sake of simplicity, we write, for $X \subseteq V, \mathrm{cc}(X)$ and $\mathrm{b}(X)$ to denote $\operatorname{cc}(G[X])$ and $\mathrm{b}(G[X])$, respectively.

As $G$ may have multiedges, every cactus forest can be characterized as the following form.
Proposition 1 ( 8 ). Let $D$ be the graph of two vertices and three parallel edges between them. A graph is a cactus forest if and only if it does not contain a subgraph isomorphic to any subdivision of D.

We call a subdivision of $D$ an obstruction. In particular, $D$ itself is also an obstruction.
The algorithm consists of several branching rules and reduction rules. We say that a reduction rule is safe if the original instance has a yes-instance if and only if the instance obtained by applying the rule is a yes-instance. We also say that a branching rule is safe if the original instance is a yesinstance if and only if at least one of the instances obtained by applying the rule is a yes-instance. Our algorithm described below determines whether $G$ has a solution $X$ for Disjoint Cactus Vertex Deletion. However, the algorithm easily turns into one that finds an actual solution if the answer is affirmative. We apply these rules in the order of their appearance. The algorithm terminates if $V(G)=S$ or $k=0$, and it answers "YES" if and only if $k \geq 0$ and $G$ is a cactus forest.

The following reduction and branching rules are trivially safe.
Reduction rule 1. If $G[V \backslash S]$ contains a component $C$ that has no neighbors in $S$, then delete all the vertices in $C$.

Reduction rule 2. If $G[V \backslash S]$ contains a vertex of degree one in $G$, then delete it.
Reduction rule 3. If $G[V \backslash S]$ contains a vertex $v$ such that $G[S \cup\{v\}]$ is not a cactus forest, then delete $v$ and decrease $k$ by one.

Branching rule 1. If $G[V \backslash S]$ contains vertices $u, v \in V \backslash S$ with $m(u, v) \geq 3$, branch into two cases: (1) delete $u$ and decrease $k$ by one; (2) delete $v$ and decrease $k$ by one.

The branching factor of Branching rule 1 is $(\alpha, \alpha)$. By applying these rules, we make the following assumption on each vertex in $V \backslash S$.

Assumption 1. Every vertex $v \in V \backslash S$ has degree at least two in $G$ and there are at most two edges between two vertices.

As $G$ is a multigraph, some vertex may have only one neighbor even if its degree is greater than one. If $G[V \backslash S]$ contains a vertex $v$ with $\left|N_{G}(v)\right|=1$, this vertex also can be removed since it is not a part of an obstruction, assuming that $m(u, v) \leq 2$ with $u \in N_{G}(v)$. This implies the following reduction rule.

Reduction rule 4. If $G[V \backslash S]$ contains a vertex $v$ with $\left|N_{G}(v)\right|=1$, then delete it.
Thus, we further make the following assumption on each vertex in $V \backslash S$.

Assumption 2. Every vertex $v \in V \backslash S$ has at least two neighbors in $G$.
Suppose that there is a vertex $v \in V \backslash S$ that has at least two neighbors in $S$. By Reduction rule 3 . there is no component in $G[S]$ that contains at least three vertices of $N_{G}(v) \cap S$. Let $W=N_{G}(v) \cap S$. We denote by $t_{1}$ (resp. by $t_{2}$ ) the number of components in $G[S]$ that contain exactly one vertex (resp. two vertices) of $W$. Let $C$ be a component in $G[S]$ that has at least one vertex of $W$. If $|W \cap C|=2$, say $w, w^{\prime} \in W \cap C$, every edge on the path between $w$ and $w^{\prime}$ in $G[C]$ is a bridge as otherwise $G[C \cup\{v\}]$ contains an obstruction, which implies that $v$ is removed by Reduction rule 3 . Then, there is at least one bridge on the path between $w$ and $w^{\prime}$ in $G[C]$. Thus, $\mathrm{b}(C \cup\{v\}) \leq \mathrm{b}(C)-1$. If $|W \cap C|=1, G[C \cup\{v\}]$ has $\mathrm{b}(C)+1$ bridges. Hence, we have

$$
\begin{aligned}
\beta \cdot \mathrm{cc}(S \cup\{v\}) & \leq \beta \cdot \mathrm{cc}(S)-\beta\left(t_{1}+t_{2}-1\right) \\
\gamma \cdot \mathrm{b}(S \cup\{v\}) & \leq \gamma \cdot \mathrm{b}(S)+\gamma\left(t_{1}-t_{2}\right)
\end{aligned}
$$

Consider the value $t_{1}(\beta-\gamma)+t_{2}(\beta+\gamma)-\beta$, that is, a lower bound of $\mu((G, S, k))-\mu((G, S \cup\{v\}, k))$. If $t_{1}+t_{2} \geq 2$, the value is at least $\beta-2 \gamma$. This follows from the fact that the value is minimized when $t_{1}=2$ and $t_{2}=0$ under $\beta \geq \gamma \geq 0$. If $t_{1}+t_{2}=1, t_{1}$ must be zero since $|W| \geq 2$. In this case, the value is at least $\gamma$. This implies the following branching rule, which is clearly safe, has branching factor $(\alpha, \min (\beta-2 \gamma, \gamma))$.

Branching rule 2. Suppose $G[V \backslash S]$ contains a vertex $v$ that has at least two neighbors in $S$. Then, branch into two cases: (1) delete $v$ and decrease $k$ by one; (2) put $v$ into $S$.

Thus, we make the following assumption on each vertex in $V \backslash S$.
Assumption 3. Every vertex $v \in V \backslash S$ has at least two neighbors in $G$ and at most one of them belongs to $S$.

We can remove a vertex having exactly two neighbors by adding an edge between its neighbors. The following lemma justifies this reduction.

Lemma 4. Let $v \in V \backslash S$ be a vertex with exactly two neighbors $u, w$ in $G$. Suppose that $p=$ $\max (m(u, v), m(v, w)) \leq 2$. Let $G^{\prime}$ be the graph obtained from $G$ by deleting $v$ and adding $p$ parallel edges between $u$ and $w$. Then, $G$ has a cactus deletion set of size at most $k$ if and only if $G^{\prime}$ has a cactus deletion set of size at most $k$.

Proof. Since every obstruction in $G$ containing $v$ also has both $u$ and $w$, there is a smallest cactus deletion set $X$ that does not contain $v$. Such a set is also a cactus deletion set of $G^{\prime}$ and vise versa.

By Lemma 4, the following reduction rule is safe.
Reduction rule 5. Suppose that $G[V \backslash S]$ contains a vertex $v$ with $N_{G}(v)=\{u, w\}$. Then delete $v$ and add $\max (m(u, v), m(v, w))$ parallel edges between $u$ and $w$.

This implies that the following assumption is made.
Assumption 4. Every vertex $v \in V \backslash S$ has at least three neighbors. Moreover, at most one of them belongs to $S$ and hence at least two of them belong to $V \backslash S$.

Since $G[V \backslash S]$ is a cactus forest, there is a leaf block $B$. By Assumption 4, $B$ contains at least three vertices. Suppose that $B$ has exactly three vertices $u, v, w$. As $B$ is a leaf block, we can assume that both $u$ and $v$ are not cut vertices of $G[V \backslash S]$. By Assumption 3, both $u$ and $v$ have exactly one neighbor in $S$, which can be an identical vertex. If there is a component $C$ in $G[S]$ that contains both a neighbor of $u$ and a neighbor of $v$, then $G[C \cup\{u, v, w\}]$ has an obstruction, which yields the following branching rule with branching factor $(\alpha, \alpha, \alpha)$.


Figure 1: An Illustration of four branching cases under Assumption 5
Branching rule 3. Suppose that there is a leaf block $B$ with $V(B)=\{u, v, w\}$ in $G[V \backslash S]$. Suppose moreover that each of $u$ and $v$ has exactly one neighbor in $S$ and that these neighbors belong to a single component in $G[S]$. Then, branch into three cases: (1) delete $u$; (2) delete $v$; (3) delete $w$. For each case, decrease $k$ by one.

Otherwise, the neighbors of $u$ and $v$ belong to distinct components in $G[S]$. Let $C_{u}$ and $C_{v}$ be the components of $G[S]$ that have neighbors of $u$ and $v$, respectively. If $w$ has a neighbor in $C_{u}$ or $C_{v}$, then $G[S \cup\{u, v, w\}]$ contains an obstruction. In this case, we apply Branching rule 3 as well. Thus, either $w$ has no neighbor in $S$ or $w$ has exactly one neighbor in a component $C_{w}$ in $G[S]$ with $C_{w} \neq C_{u}$ and $C_{w} \neq C_{v}$. In both cases, we apply the following branching rule.

Branching rule 4. Suppose that there is a leaf block $B$ with $V(B)=\{u, v, w\}$ in $G[V \backslash S]$. Suppose moreover that each of $u$ and $v$ has exactly one neighbor in $S$ and that these neighbors belong to distinct components in $G[S]$. Then, branch into four cases: (1) delete $u$; (2) delete $v$; (3) delete $w$; (4) put $u, v$, and $w$ into $S$. For (1), (2), and (3), decrease $k$ by one.

To see the branching factor of this rule, suppose first that $w$ has no neighbor in $S$. Then, $G[S \cup$ $\{u, v, w\}]$ contains $\mathrm{cc}(S)-1$ components and $\mathrm{b}(S)+2$ bridges. Thus, Branching rule 4 has branching factor ( $\alpha, \alpha, \alpha, \beta-2 \gamma$ ). Suppose otherwise that $w$ has an exactly one neighbor in a component $C_{w}$ in $G[S]$ with $C_{w} \neq C_{u}$ and $C_{w} \neq C_{v}$. Then, $G[S \cup\{u, v, w\}]$ contains $\mathrm{cc}(S)-2$ components and $\mathrm{b}(S)+3$ bridges. Thus, Branching rule 4 has branching factor ( $\alpha, \alpha, \alpha, 2 \beta-3 \gamma$ ).

By Branching rules 3 and 4 the following assumption is made.
Assumption 5. Every leaf block $B$ in $G[V \backslash S]$ contains three consecutive vertices, each of which is not a cut vertex in $G[V \backslash S]$ and has exactly one neighbor in $S$.

Let $u, v, w$ be three consecutive vertices in $B$, each of which is not a cut vertex in $G[V \backslash S]$ and has exactly one neighbor in $S$. Let $u^{\prime}, v^{\prime}, w^{\prime}$ be the neighbors of $u, v, w$ in $S$, respectively. There are four cases (Figure 11).

Suppose that there is a component $C$ in $G[S]$ that contains these neighbors ((a) in Figure 1$]$ ). Then, $G[C \cup\{u, v, w\}]$ has an obstruction, yielding the following Branching rule 5 that has branching factor $(\alpha, \alpha, \alpha)$.
Branching rule 5. Suppose that there is a leaf block $B$ with $|V(B)| \geq 4$ in $G[V \backslash S]$. Let $u, v, w$ be three consecutive vertices in $B$, each of which is not a cut vertex in $G[V \backslash S]$ and has exactly one neighbor in $S$. Suppose that these neighbors belong to a single component in $G[S]$. Then, branch into three cases: (1) delete $u$; (2) delete $v$; (3) delete $w$. For each case, decrease $k$ by one.

Suppose next that exactly two of $u^{\prime}, v^{\prime}, w^{\prime}$ are contained in a single component $C$ in $G[S]$. There are essentially two cases: (1) $u^{\prime}$ and $v^{\prime}$ are contained in $C$ ((b) in Figure 1) or (2) $u^{\prime}$ and $w^{\prime}$ are contained in $C$ ((c) in Figure 1). In case (1), $G[S \cup\{u, v, w\}]$ contains $\mathrm{cc}(S)-1$ components and $\mathrm{b}(S)+2$ bridges. In case (2), $G[S \cup\{u, v, w\}]$ contains $\mathrm{cc}(S)-1$ components and $\mathrm{b}(S)+1$ bridges. For these cases, we apply the following Branching rule 6, which has branching factors ( $\alpha, \alpha, \alpha, \beta-2 \gamma$ ) and ( $\alpha, \alpha, \alpha, \beta-\gamma$ ) for these cases.

Branching rule 6. Suppose that there is a leaf block $B$ with $|V(B)| \geq 4$ in $G[V \backslash S]$. Let $u, v, w$ be three consecutive vertices in $B$, each of which is not a cut vertex in $G[V \backslash S]$ and has exactly one neighbor in $S$. Suppose that these neighbors are not contained in a single component in $G[S]$. Then, branch into four cases: (1) delete $u$; (2) delete $v$; (3) delete $w$; (4) put $u$, $v$, and $w$ into $S$. For (1), (2), and (3), decrease $k$ by one.

Finally, suppose any two of $u^{\prime}, v^{\prime}, w^{\prime}$ are not contained in a single component in $G[S]$ ((d) in Figure 11. Again, we apply Branching rule 6 to this case. Since $G[S \cup\{u, v, w\}]$ contains $\operatorname{cc}(S)-2$ components and $\mathrm{b}(S)+5$ bridges, Branching rule 6 has branching factor ( $\alpha, \alpha, \alpha, 2 \beta-5 \gamma$ ). The entire algorithm for Disjoint Cactus Vertex Deletion is given in Algorithm 1.

The reduction and branching rules cover all cases for the instance $I$ and all the rules are safe. Thus, the algorithm correctly computes a cactus deletion set $X \subseteq V \backslash S$ with $|X| \leq k$ if it exists. By choosing $\alpha=1, \beta=0.4052, \gamma=0.0726$, the running time is dominated by the branching factor $(\alpha, \alpha, \alpha, \beta-2 \gamma)=(1,1,1,0.26)$. By Lemma 1 1 , we have $\beta \cdot \mathrm{cc}(S)+\gamma \cdot \mathrm{b}(S) \leq \beta(k+1)$. Therefore, the running time of the algorithm is

$$
\begin{aligned}
O^{*}\left(c^{\mu(I)}\right) & \subseteq O^{*}\left(c^{\alpha \cdot k+\beta \cdot c c(S)+\gamma \cdot b(S)}\right) \\
& \subseteq O^{*}\left(c^{1.4052 k}\right)
\end{aligned}
$$

where $c<7.3961$ is the unique positive real root of equation $3 x^{-1}+x^{-0.26}=1$. This yields the running time bound $O^{*}\left(16.64^{k}\right)$ for Disjoint Cactus Vertex Deletion.

## 4 An improved algorithm for Even Cycle Transversal

Recall that Even Cycle Transversal asks whether, given a graph $G=(V, E)$ and an integer $k$, $G$ has a vertex set $X$ of size at most $k$ such that $G[V \backslash X]$ is a forest of odd cacti. As in the previous section, we solve the disjoint version of Even Cycle Transversal and give an $O\left(16.64^{k}\right)$-time algorithm for it, assuming that $|S| \leq k+1$.

Definition 4 (Disjoint Even Cycle Transversal). Given a graph $G=(V, E)$, an integer $k \geq 0$, and $S \subseteq V$ such that $G[V \backslash S]$ is a forest of odd cacti, the problem asks to find a vertex set $X \subseteq V \backslash S$ with $|X| \leq k$ whose removal leaves a forest of odd cacti.

A key difference from Disjoint Cactus Vertex Deletion is that we need to take the length of cycles into account. However, in Reduction rule 5, we replace (a chain of) cycles with two multiple edges between two extreme vertices, which does not preserve the length of cycles in the original graph. Given this, we consider a slightly generalized problem. In addition to the input of Disjoint Even Cycle Transversal, we are given a binary weight function $\omega: E \rightarrow\{0,1\}$ on edges, and the length of a cycle is defined to be the total weight of edges in it. Indeed, when $\omega(e)=1$ for all $e \in E$, the problem corresponds to Disjoint Even Cycle Transversal.

Let $S \subseteq V$ such that $G[V \backslash S]$ is a forest of odd cacti. We first apply Reduction rules 1, 2, 3, and 4 and Branching rules 1 and 2 which are trivially safe for Disjoint Even Cycle Transversal. Moreover, we add the following reduction rule, which is also trivially safe.

Reduction rule 6. If there is a vertex $v \in V \backslash S$ such that $G[S \cup\{v\}]$ has a cycle of even length, then delete it and decrease $k$ by one.

We can check in linear time whether an edge-weighted graph is a forest of odd cacti and hence Reduction rule 6 can be applied in linear time as well.

Up to this point, Assumption 3 is made. By Reduction rule 3 and Branching rule 1 , we also assume that $m(u, v) \leq 2$ for every pair of vertices in $G$. Suppose that $m(u, v)=2$. If the parities of two edges between $u$ and $v$ are the same, the length of the cycle consisting of these edges is even. Thus, we apply the following branching rule in this case.

Branching rule 7. Suppose that $u, v \in V \backslash S$ and $m(u, v)=2$ for some $u$. Let $f, f^{\prime}$ be the edges between $u$ and $v$. If $\omega(f)=\omega\left(f^{\prime}\right)$, branch into two cases: (1) delete $u$ and decrease $k$ by one; (2) delete $v$ and decrease $k$ by one.

By Reduction rule 6 and Branching rule 7, the following assumption is made.
Assumption 6. For every pair of vertices $u, v$ with $m(u, v)=2$, the parities of the weights of edges between them are opposite.

Now, let us consider a vertex $v \in V \backslash S$ that has exactly two neighbors in $G$. Let $u$ and $w$ be the neighbors of $v$. By Assumption 3, at least one of $u$ and $w$ belongs to $V \backslash S$. Similarly to Lemma 4, we define a graph $G^{\prime}$ by deleting $v$ from $G$ and adding $p$ parallel edges between $u$ and $v$, where $p=\max (m(u, v), m(v, w))$. We define the weight function $w^{\prime}$ for $G^{\prime}$ as follows. If $p=1$, we set the weight of the introduced edge $e=\{u, w\}$ as $\omega^{\prime}(e)=\omega(f)+\omega\left(f^{\prime}\right)$, where $f$ (resp. $f^{\prime}$ ) is the edge between $u$ and $v$ (resp. between $v$ and $w$ ) and the sum is taken under addition modulo two. If $p=2$, at least one of the pairs $\{u, v\}$ or $\{v, w\}$ has multiple edges. By Assumption 6, these two edges have different parities. A crucial observation is that if there is a cycle passing through exactly one of these edges, there is another cycle passing through the other edges, which has the different parity. By setting $\omega^{\prime}(e)=0$ and $\omega^{\prime}\left(e^{\prime}\right)=1$, such cycles are preserved in $G^{\prime}$.

Lemma 5. The instance ( $G, \omega, S, k$ ) is a yes-instance if and only if ( $G^{\prime}, \omega^{\prime}, S, k$ ) is a yes-instance.
Proof. Consider a cycle $C$ of even length that passes through $v$ in $G$. By Assumption 6, $C$ must pass through both $u$ and $w$. Thus, there is a feasible solution $X \subseteq V \backslash S$ for $(G, \omega, S, k)$ with $v \notin X$ and $\{u, w\} \cap X \neq \emptyset$. By the construction of $G^{\prime}$, the cycle obtained from $C$ by omitting $v$ is an even cycle of $G^{\prime}$. Hence, $X$ is a feasible solution for $\left(G^{\prime}, \omega^{\prime}, S, k\right)$. It is not hard to see that this correspondence is reversible and hence the lemma follows.

This lemma ensures that the weighted version of Reduction rule 5 is safe for Even Cycle Transversal, and then Assumption 4 is made as well. The rest of branching rules are the same with Disjoint Cactus Vertex Deletion, which yields an $O^{*}\left(16.64^{k}\right)$-time algorithm that solves Disjoint Even Cycle Transversal as well.

Lemma 6. Suppose $|S| \leq k+1$. Then, Disjoint Even Cycle Transversal can be solved in time $O^{*}\left(16.64^{k}\right)$.

By Lemma 2, Even Cycle Transversal can be solved in time $O^{*}\left(17.64^{k}\right)$, completing the proof of Theorem 2.

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Algorithm 1 A pseudocode of the algorithm for Disjoint Cactus Vertex Deletion
    procedure \(\operatorname{DCVD}(G=(V, E), S, k)\)
        if \(k \geq 0\) and \(V=S\) then
            return true
        if \(k<0\) then
            return false
        if \(G[V \backslash S]\) has a component \(C\) that has no neighbors in \(S\) then \(\triangleright\) Reduction rule 1
            return \(\operatorname{DCVD}(G[V \backslash C], S, k)\)
        if \(G[V \backslash S]\) has a vertex \(v\) of degree one in \(G\) then \(\triangleright\) Reduction rule 2
            return \(\operatorname{DCVD}(G[V \backslash\{v\}], S, k)\)
        if \(G[V \backslash S]\) has \(v\) such that \(G[V \cup\{v\}]\) is not a cactus forest then \(\triangleright\) Reduction rule 3
            return \(\operatorname{DCVD}(G[V \backslash\{v\}], S, k-1)\)
        if \(G[V \backslash S]\) has vertices \(u\) and \(v\) with \(m(u, v) \geq 3\) then \(\triangleright\) Branching rule 1
            return \(\operatorname{DCVD}(G[V \backslash\{u\}], S, k-1) \vee \operatorname{DCVD}(G[V \backslash\{v\}], S, k-1)\)
        if \(G[V \backslash S]\) has a vertex \(v\) with \(\left|N_{G}(v)\right|=1\) then \(\quad \triangleright\) Reduction rule 4
            return \(\operatorname{DCVD}(G[V \backslash\{v\}], S, k)\)
        if \(G[V \backslash S]\) has a vertex \(v\) having at least two neighbors in \(S\) then \(\triangleright\) Branching rule 2
            return \(\operatorname{DCVD}(G[V \backslash\{v\}], S, k-1) \vee \operatorname{DCVD}(G[V], S \cup\{v\}, k)\)
        if \(G[V \backslash S]\) has a vertex \(v\) with \(N_{G}(v)=\{u, w\}\) then \(\triangleright\) Reduction rule 5
            Let \(G^{\prime}=G[V \backslash\{v\}]\).
            Add \(\max \{m(u, v), m(v, w)\}\) parallel edges between \(u\) and \(w\) to \(G^{\prime}\).
            return \(\operatorname{DCVD}\left(G^{\prime}, S, k\right)\)
        if \(G[V \backslash S]\) has a leaf block \(B\) with \(V(B)=\{u, v, w\}\) then \(\quad \triangleright\) Branching rules 3 and 4
            for \(x \in V(B)\) do
                if \(\operatorname{DCVD}(G[V \backslash\{x\}], S, k-1)\) then
                return true
            if \(G[S \cup V(B)]\) is a cactus forest then
                return \(\operatorname{DCVD}(G[V], S \cup V(B), k)\)
            return false
        if \(G[V \backslash S]\) has a leaf block \(B\) with \(|V(B)| \geq 4\) then \(\quad \triangleright\) Branching rules 5 and 6
            Let \(B^{\prime}=\{u, v, w\}\) be consecutive vertices in \(B\) that are not cut vertices in \(G[V \backslash S]\).
            for \(x \in V\left(B^{\prime}\right)\) do
                if \(\operatorname{DCVD}(G[V \backslash\{x\}], S, k-1)\) then
                return true
            if \(G\left[S \cup V\left(B^{\prime}\right)\right]\) is a cactus forest then
                return \(\operatorname{DCVD}\left(G[V], S \cup V\left(B^{\prime}\right), k\right)\)
            return false
```


[^0]:    ${ }^{1}$ The notation $O^{*}$ suppresses a polynomial factor of the input size.

