# A note on the factorization conjecture * 

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#### Abstract

We give partial results on the factorization conjecture on codes proposed by Schützenberger. We consider a family of finite maximal codes $C$ over the alphabet $A=\{a, b\}$ and we prove that the factorization conjecture holds for these codes. This family contains $(p, 4)$-codes, where a ( $p, 4$ )-code $C$ is a finite maximal code over $A$ such that each word in $C$ has at most four occurrences of $b$ and $a^{p} \in C$, for a prime number $p$. We also discuss the structure of these codes. The obtained results once again show relations between factorizations of finite maximal codes and factorizations of finite cyclic groups.


## 1 Introduction

The theory of variable-length codes is a topic with elegant mathematical results and strong connections with automata theory. The theory originated at the end of the 60's with Schützenberger, who proposed in [28] the semigroup theory as a mathematical setting for the study of the uniquely decipherable sets of words in the context of information theory (see [2] for a complete treatment of this topic and also [1] for a viewpoint focused on applications of codes). In this paper we follow this algebraic approach and codes are defined as the bases of the free submonoids of a free monoid.

A well known class of codes is that of prefix codes, i.e., codes such that none of their words is a left factor of another. A classical representation of a finite prefix code $C$ over an alphabet $A$ is as a set of leaves on a tree. In this case, the set of the internal nodes represents the set of the proper left factors $P$ of $C$ and $C$ is maximal (i.e., $C$ is not properly contained in any other code over $A$ ) if and only if each internal node has a number of children equal to the cardinality of $A$. Thus, it is clearly evident that $C=P A \backslash P$ or, in terms of noncommutative polynomials, $\underline{C}-1=\underline{P}(\underline{A}-1)$ (here 1 is the empty word and $\underline{\mathrm{X}}$ denotes the characteristic polynomial of a finite language $X$, i.e., the formal sum of its elements).

One of the conjectures proposed by Schützenberger, known as the factorization conjecture, asks whether a more general equation can be stated for a finite (not necessarily prefix) maximal code $C$, namely whether finite subsets $P, S$ of $A^{*}$ exist such that $\underline{C}-1=\underline{P}(\underline{A}-1) \underline{S}$ [22, 29]. This longstanding open question, one of the most important in the theory of codes, is inspired by a problem of information theory [23].

[^0]Only partial results are known (see [2]). The major contribution to this conjecture is due to Reutenauer [26, 27]. In particular, he proved that for any finite maximal code $C$ over $A$, there exist polynomials $P, S \in \mathbb{Z}\langle A\rangle$ such that $\underline{C}-1=P(\underline{A}-1) S$. We call $(P, S)$ a factorization for $C$. Moreover we say that a factorization $(P, S)$ for $C$ is positive if $P, S$ or $-P,-S$ have coefficients $0,1.1$

Positive factorizations always exist for a finite maximal code $C$ over a one-letter alphabet and these positive factorizations have all been constructed in [18]. However, factorizations which are not positive also exist, even for these simple codes. In the case of a one-letter alphabet, it has also been conjectured that if $(P, S)$ is a factorization for $C$ and $S$ has coefficients 0,1 , then the same holds for $P$ [19].

There are not many examples of factorizations which are not positive. On the contrary, every factorization for $C$ is positive if $C$ is a finite maximal code over a two-letter alphabet $\{a, b\}$ with $m \leq 3$ occurrences of the letter $b$ in its words [9, 16, 24]. In this paper, we investigate this further and we prove the results which are briefly explained below.

Let $A=\{a, b\}$. For a polynomial $S \in \mathbb{Z}\langle A\rangle$, we denote by $\operatorname{supp}(S)$ the set of words in $A^{*}$ having a non-zero coefficient in $S$. Let $C$ be a finite maximal code over $A$ such that $a^{p} \in C$, for a prime number $p$. Let $(P, S)$ be a factorization for $C$ such that, for any word $w \in A^{*}$, if $w b a^{j}$ is in $\operatorname{supp}(S)$ then $a^{j}$ is also in $\operatorname{supp}(S)$.

First, we show that if $S \in \mathbb{N}\langle A\rangle$, then $(P, S)$ is positive (Theorem4.1). This result is related to the above-mentioned conjecture in [19]. Second, we prove that if $\operatorname{supp}(S) \subseteq a^{*} \cup a^{*} b a^{*}$ then $(P, S)$ is positive (Theorem 4.2). Moreover, in this case we may inductively construct all these factorizations ( $P, S$ ) (Section 7).

A ( $p, 4$ )-code $C$ is a finite maximal code over $A$ containing $a^{p}$ and such that each word in $C$ has at most four occurrences of $b$. A corollary of the previous results is that if $C$ is a $(p, 4)$-code, for a prime number $p$, then each factorization for $C$ is positive (Theorem 4.3).

Finally, for a polynomial $P \in \mathbb{Z}\langle A\rangle$, let $P_{g}$ be polynomials such that a word $w \in A^{*}$ has a non-zero coefficient $\alpha$ in $P_{g}$ if and only if $w$ has $g$ occurrences of the letter $b$ and $w$ has the same non-zero coefficient $\alpha$ in $P$. Let $(P, S)$ be a factorization for a finite maximal code $C$ over $\{a, b\}$ such that if $a^{i} b a^{j}$ is in $\operatorname{supp}\left(S_{1}\right)$ then $a^{j}$ is in $\operatorname{supp}\left(S_{0}\right)$. We prove that if $P_{0}, S_{0}, S_{1}$ have coefficients 0,1 , then $P_{1}$ has nonnegative coefficients (Theorem 4.4).

Another objective is the description of the structure of the (positively) factorizing codes, i.e., codes satisfying the factorization conjecture. There are several papers devoted to this problem [8, 9, 10, 11, 12, 13, 14, 15, 24]. In particular, the structure of $m$-codes, $m \leq 3$, has been characterized, as well as that of codes $C$ such that $\underline{C}=\underline{P}(\underline{A}-1) \underline{S}+1$, with $P \subseteq A^{*}, S \subseteq a^{*}$. In all these cases, there are relations between (positive) factorizations of finite maximal codes and factorizations of cyclic groups. We tackle this problem for $(p, 4)$-codes and the results proved in this paper once again show these relations.

The paper is organized as follows. In Section 2, we set up the basic definitions and known results we need. In Section 3, we give an outline of the results on the factorization conjecture and we prove these results in Section 4. In Section 5, we recall some known results and give an outline of new results on positively factorizing codes. The new results will be stated in Sections 6. 77 and 8. Finally, in Section 9, we discuss some open problems that follow on from these results.

[^1]
## 2 Basics

### 2.1 Codes and words

Let $A^{*}$ be the free monoid generated by a finite alphabet $A$ and let $A^{+}=A^{*} \backslash 1$ where 1 is the empty word. For a word $w \in A^{*}$ and a letter $a \in A$, we denote by $|w|$ the length of $w$ and by $|w|_{a}$ the number of the occurrences of $a$ in $w$. The reversal of a word $w=a_{1} \ldots a_{n}, a_{i} \in A$, is the word $w^{\sim}=a_{n} \ldots a_{1}$ and we set $X^{\sim}=\left\{w^{\sim} \mid w \in X\right\}$.

A code $C$ is a subset of $A^{*}$ such that, for all $h, k \geq 0$ and $c_{1}, \ldots, c_{h}, c_{1}^{\prime}, \ldots, c_{k}^{\prime} \in C$, we have

$$
c_{1} \cdots c_{h}=c_{1}^{\prime} \cdots c_{k}^{\prime} \quad \Rightarrow \quad h=k \quad \text { and } \quad c_{i}=c_{i}^{\prime} \quad \text { for } \quad i=1, \ldots, h .
$$

A set $C \subseteq A^{+}$, such that $C \cap C A^{+}=\emptyset$, is a prefix code. $C$ is a suffix code if $C^{\sim}$ is a prefix code and $C$ is a biprefix code when $C$ is both a suffix and a prefix code. A code $C$ is a maximal code over $A$ if for each code $C^{\prime}$ over $A$ such that $C \subseteq C^{\prime}$ we have $C=C^{\prime}$.

### 2.2 Polynomials

Let $\mathbb{Z}\langle A\rangle$ (resp. $\mathbb{N}\langle A\rangle)$ denote the semiring of the polynomials with noncommutative variables in $A$ and integer (resp. nonnegative integer) coefficients. For a finite subset $X$ of $A^{*}, \underline{X}$ denotes its characteristic polynomial, defined by $\underline{X}=\sum_{x \in X} x$. Therefore, "characteristic polynomial" will be synonymous with "polynomial with coefficients 0,1 ". For a polynomial $P$ and a word $w \in A^{*},(P, w)$ denotes the coefficient of $w$ in $P$ and we set $\operatorname{supp}(P)=\left\{w \in A^{*} \mid(P, w) \neq 0\right\}$. If $\operatorname{supp}(P)=\emptyset$, then $P=0$ is the null polynomial. When we write $P \geq Q$, with $P, Q \in \mathbb{Z}\langle A\rangle$, we mean that $(P, w) \geq(Q, w)$, for any $w \in A^{*}$. In particular, $P \geq 0$ means that $P \in \mathbb{N}\langle A\rangle$. Furthermore, $P^{\sim}$ is defined by $\left(P^{\sim}, w^{\sim}\right)=(P, w)$, for each $w \in A^{*}$. For $P \in \mathbb{Z}\langle A\rangle, A=\{a, b\}$ and $g \in \mathbb{N}$, we denote by $P_{g}$ polynomials such that

$$
\forall w \in A^{*} \quad\left(P_{g}, w\right)= \begin{cases}(P, w) & \text { if }|w|_{b}=g \\ 0 & \text { otherwise }\end{cases}
$$

Then for any $P \in \mathbb{Z}\langle A\rangle$, there exists $h \in \mathbb{N}$ such that $P=P_{0}+\ldots+P_{h}$. We write, as usual, $\mathbb{Z}[a]$ and $\mathbb{N}[a]$ instead of $\mathbb{Z}\langle a\rangle$ and $\mathbb{N}\langle a\rangle$. The map which associates the polynomial $\sum_{n \in \mathbb{N}}(H, n) a^{n} \in \mathbb{N}[a]$ to a finite multiset $H$ of nonnegative integers, is a bijection between the set of the finite multisets $H$ of nonnegative integers and $\mathbb{N}[a]$. We represent this bijection by the notation $a^{H}=\sum_{n \in \mathbb{N}}(H, n) a^{n}$. For example, $a^{\{0,0,1,1,1,3,4\}}=2+3 a+a^{3}+a^{4}$. Consequently, the following computation rules are defined: $a^{M+L}=a^{M} a^{L}, a^{M \cup L}=a^{M}+a^{L}, a^{\emptyset}=0, a^{0}=1$.

### 2.3 Factorization conjecture

Conjecture 2.1, given in a weaker form in [23], is among the most difficult, unsolved problems in the theory of codes. This conjecture was formulated by Schützenberger but, as far as we know, it does not appear explicitly in any of his papers. It was quoted as the factorization conjecture in [22] for the first time and then also reported in [2, 3, 7].

Conjecture 2.1 [29] Given a finite maximal code $C$, there are finite subsets $P, S$ of $A^{*}$ such that:

$$
\underline{C}-1=\underline{P}(\underline{A}-1) \underline{S} .
$$

Each code $C$ verifying the previous conjecture is finite, maximal and is called a (positively) factorizing code.

Finite maximal prefix codes are the simplest examples of positively factorizing codes. Indeed, $C$ is a finite maximal prefix code if and only if $\underline{C}=\underline{P}(\underline{A}-1)+1$ for a finite subset $P$ of $A^{*}[2]$. In the previous relation, $P$ is the set of the proper prefixes of the words in $C$. More interesting constructions of factorizing codes can be found in [4, 5, 6], whereas the result which is closest to a solution of the conjecture is reported in Theorem 2.1 and was obtained by Reutenauer [2, 3, 26, 27].

Theorem 2.1 [27] Let $C \in \mathbb{N}\langle A\rangle$, with $(C, 1)=0$, and let $P, S \in \mathbb{Z}\langle A\rangle$ be such that $C=$ $P(\underline{A}-1) S+1$. Then, $C$ is the characteristic polynomial of a finite maximal code. Furthermore, if $P, S \in \mathbb{N}\langle A\rangle$, then $P, S$ are polynomials with coefficients 0,1 . Conversely, for any finite maximal code $C$ there exist $P, S \in \mathbb{Z}\langle A\rangle$ such that $\underline{C}=P(\underline{A}-1) S+1$.

Given a finite maximal code $C$, a factorization $(P, S)$ for $C$ is a pair of polynomials $P, S \in \mathbb{Z}\langle A\rangle$ such that $\underline{C}=P(\underline{A}-1) S+1$. Of course, $(P, S)$ is a factorization for $C$ if and only if the same holds for $(-P,-S)$ and, moreover, $\left(S^{\sim}, P^{\sim}\right)$ is a factorization for $C^{\sim}$. We say that a factorization $(P, S)$ for $C$ is positive if $P, S$ or $-P,-S$ have coefficients 0,1 . From now on, $A=\{a, b\}$ will be a two-letter alphabet.

## 3 Outline of the results on the factorization conjecture

Let $C$ be a finite maximal code over $A$, let $(P, S)$ be a factorization for $C$ (Theorem 2.1). Then $P, S \in \mathbb{Z}\langle A\rangle$ are such that $\underline{C}=P(\underline{A}-1) S+1$. Thus, the characteristic polynomial $\underline{C}_{r}$ of the set $C_{r}=\left\{\left.w \in C| | w\right|_{b}=r\right\}$ of the words in $C$ with $r$ occurrences of $b$, is the sum of the terms of degree $r$ with respect to the variable $b$ in the polynomial $P(\underline{A}-1) S+1$, i.e.,

$$
\begin{align*}
\underline{C}_{0} & =P_{0}(a-1) S_{0}+1,  \tag{3.1}\\
\forall r \geq 0 \quad \underline{C}_{r+1} & =\sum_{i+j=r} P_{i} b S_{j}+\sum_{i+j=r+1} P_{i}(a-1) S_{j} . \tag{3.2}
\end{align*}
$$

Example 3.1 Consider the finite maximal code defined by the relation $\underline{C}=P(\underline{A}-1) S+1$, with

$$
\begin{aligned}
P & =1+a^{2} b a^{\{0,1,2,3,4,5,6\}}+a^{2} b a^{3} b a^{\{0,1,2,3,4,5,6\}} \\
S & =a^{\{0,1,2,3,4\}}+a^{\{0,1\}} b a^{\{0,1,2,3,4\}}
\end{aligned}
$$

Then we have

$$
\begin{aligned}
& P_{0}=1, \quad P_{1}=a^{2} b a^{\{0,1,2,3,4,5,6\}}, \quad P_{2}=a^{2} b a^{3} b a^{\{0,1,2,3,4,5,6\}}, \\
& S_{0}=a^{\{0,1,2,3,4\}}, \quad S_{1}=a^{\{0,1\}} b a^{\{0,1,2,3,4\}} .
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\underline{C}_{0}= & P_{0}(a-1) S_{0}+1=(a-1) a^{\{0,1,2,3,4\}}+1=a^{5} \\
\underline{C}_{1}= & P_{0} b S_{0}+P_{1}(a-1) S_{0}+P_{0}(a-1) S_{1} \\
= & b a^{\{0,1,2,3,4\}}+a^{2} b a^{\{0,1,2,3,4,5,6\}}(a-1) a^{\{0,1,2,3,4\}}+(a-1) a^{\{0,1\}} b a^{\{0,1,2,3,4\}} \\
= & a^{2} b a^{\{7,8,9,10,11\}} \\
\underline{C}_{2}= & P_{1} b S_{0}+P_{0} b S_{1}+P_{1}(a-1) S_{1}+P_{2}(a-1) S_{0} \\
= & a^{2} b a^{\{0,1,2,3,4,5,6\}} b a^{\{0,1,2,3,4\}}+b a^{\{0,1\}} b a^{\{0,1,2,3,4\}}+ \\
& a^{2} b a^{\{0,1,2,3,4,5,6\}}(a-1) a^{\{0,1\}} b a^{\{0,1,2,3,4\}}+a^{2} b a^{3} b a^{\{0,1,2,3,4,5,6\}}(a-1) a^{\{0,1,2,3,4\}} \\
= & b a^{\{0,1\}} b a^{\{0,1,2,3,4\}}+a^{2} b a^{\{2,4,5,6,7,8\}} b a^{\{0,1,2,3,4\}}+a^{2} b a^{3} b a^{\{7,8,9,10,11\}} \\
= & P_{1} b S_{1}+P_{2} b S_{0}+P_{2}(a-1) S_{1} \\
= & a^{2} b a^{\{0,1,2,3,4,5,6\}} b a^{\{0,1\}} b a^{\{0,1,2,3,4\}}+a^{2} b a^{3} b a^{\{0,1,2,3,4,5,6\}} b a^{\{0,1,2,3,4\}}+ \\
\underline{C}_{3}= & a^{2} b a^{3} b a^{\{0,1,2,3,4,5,6\}}(a-1) a^{\{0,1\}} b a^{\{0,1,2,3,4\}} \\
= & a^{2} b a^{\{0,1,2,3,4,5,6\}} b a^{\{0,1\}} b a^{\{0,1,2,3,4\}}+a^{2} b a^{3} b a^{\{2,3,4,5,6,7,8\}} b a^{\{0,1,2,3,4\}} \\
= & P_{2} b S_{1}=a^{2} b a^{3} b a^{\{0,1,2,3,4,5,6\}} b a^{\{0,1\}} b a^{\{0,1,2,3,4\}} .
\end{aligned}
$$

The factorization $(P, S)$ for $C$ in Example 3.1 is positive. We notice that $a^{p} \in C$, with $p$ a prime number, $S=S_{0}+S_{1}$, and words $a^{i} b a^{j}$ in $\operatorname{supp}(S)$ are such that $a^{j}$ is also in $\operatorname{supp}(S)$. We will prove that each factorization $(P, S)$, for a code $C$ satisfying these hypotheses, is necessarily positive.

In the proof of this result we may assume that $P=P_{0}+\ldots+P_{k}$ with $k \geq 2$ and $S=S_{0}+S_{1}$ with $S_{1} \neq 0$. Indeed, we recall below that $(P, S)$ is always positive if $k \leq 1$ (and $S=S_{0}+S_{1}$ ) or if $S \in \mathbb{Z}[a]$. Given $m \in \mathbb{N}$, an $m$-code $C$ is a finite maximal code over $\{a, b\}$ such that each word in $C$ has at most $m$ occurrences of $b$, and at least one word of $C$ contains exactly $m$ occurrences of $b$. The following result has been proved in [24] for $m=1$, in [16] for $m=2$ and in [9] for $m=3$.

Theorem 3.1 Let $m \in \mathbb{N}, m \leq 3$. Any $m$-code $C$ is positively factorizing. Moreover, each factorization for $C$ is a positive factorization for $C$.

The following results have been proved in [9].
Theorem 3.2 Let $C$ be a finite maximal code, let $(P, S)$ be a factorization for $C$. If $P \in \mathbb{Z}\langle a\rangle$ or $S \in \mathbb{Z}\langle a\rangle$, then $(P, S)$ is a positive factorization for $C$.

Theorem 3.3 Let $C$ be a finite maximal code, let $(U, V)$ be a factorization for $C$. Then either $(P, S)=(U, V)$ or $(P, S)=(-U,-V)$ satisfies the following conditions, where $P=\sum_{i=0}^{k} P_{i}$, $S=\sum_{i=0}^{h} S_{i}$.
(i) $P_{k}$ and $S_{h}$ have coefficients 0,1 .
(ii) $P_{k-1} \in \mathbb{N}\langle A\rangle \backslash\{0\}$ and there are finite sets $L_{p}$ of nonnegative integers, for $p \in \operatorname{supp}\left(P_{k-1}\right)$, such that $P_{k}=\sum_{p \in \operatorname{supp}\left(P_{k-1}\right)} p b a^{L_{p}}$ or $S_{h-1} \in \mathbb{N}\langle A\rangle \backslash\{0\}$ and there are finite sets $M_{s}$ of nonnegative integers, for $s \in \operatorname{supp}\left(S_{h-1}\right)$, such that $S_{h}=\sum_{s \in \operatorname{supp}\left(S_{h-1}\right)} a^{M_{s}} b s$.

Theorem 3.3 states that $P_{k}, S_{1}$ or $-P_{k},-S_{1}$ always have coefficients 0,1 . If $a^{p} \in C$, we have $\underline{C}_{0}=a^{p}=P_{0}(a-1) S_{0}+1$, i.e., $P_{0} S_{0}=1+a+\ldots+a^{p-1}$. Since for a prime number $p$, the polynomial $1+a+\ldots+a^{p-1}$ is irreducible in $\mathbb{Z}[a]$ (see Example, p. 129 in [21]), one of the
pairs $\left(P_{0}, S_{0}\right),\left(S_{0}, P_{0}\right),\left(-P_{0},-S_{0}\right),\left(-S_{0},-P_{0}\right)$ is equal to the pair $\left(1,1+a+\ldots+a^{p-1}\right)$. In Section 4.1] we will prove that if $S_{1}$ has coefficients 0,1 , then the same holds for polynomials $P_{0}, S_{0}$ (Lemma 4.1).

In conclusion, $P_{0}$ and $S=S_{0}+S_{1}$ or $-P_{0}$ and $-S$ have coefficients 0,1 . Next, we will consider factorizations $(P, S)$ for $C$, with $a^{p} \in C$ and where $S=S_{0}+\ldots+S_{t}, t \geq 1$, is such that, for any word $w$, if $w b a^{j} \in \operatorname{supp}(S)$ then $a^{j} \in \operatorname{supp}\left(S_{0}\right)$. In Section 4.1, we will prove that if $S \in \mathbb{N}\langle A\rangle$, then $P$ is also in $\mathbb{N}\langle A\rangle$ (Theorem 4.1). Hence, $P, S$ have coefficients 0,1 (Theorem 2.11). As a consequence, we state our main result: if $S=S_{0}+S_{1}$, then $S \in \mathbb{N}\langle A\rangle$ and $(P, S)$ is positive (Theorem 4.2).

Regarding the factorization conjecture, we will prove another result. In Section 4.2, we consider ( $p, 4$ )-codes, i.e., 4 -codes $C$ such that $C \cap a^{*}=\left\{a^{p}\right\}$. If $(P, S)$ is a factorization for $C$ such that neither $P$ nor $S$ is in $\mathbb{Z}[a]$, then either $P=P_{0}+P_{1}+P_{2}, S=S_{0}+S_{1}$ with $P_{2} \neq 0$ and $S_{1} \neq 0$ or $P=P_{0}+P_{1}, S=S_{0}+S_{1}+S_{2}$ with $P_{1} \neq 0$ and $S_{2} \neq 0$ (see Example 3.1). Assume $S=S_{0}+S_{1}$. One of the two cases in item (ii) of Theorem 3.3 applies to $(P, S)$. In the second of these cases, the above-mentioned results show that $(P, S)$ is positive. We will easily prove that $(P, S)$ is positive in the first case also. The same arguments apply if $P=P_{0}+P_{1}$. Therefore, we show that all $(p, 4)$-codes have only positive factorizations. Notice that in [30] it has been proved that an $m$-code $C$ is positively factorizing if $b^{m} \in C$ and $m$ is a prime number or $m=4$.

Finally, in Section 4.3 we will prove that $P_{1}$ has nonnegative coefficients under weaker hypotheses on $P, S, C$. More precisely, we remove the hypothesis on the power of $a$ in $C$. We assume that $(P, S)$ is a factorization for $C$ such that if $a^{i} b a^{j}$ is in $\operatorname{supp}\left(S_{1}\right)$ then $a^{j}$ is in $\operatorname{supp}\left(S_{0}\right)$. We prove that if $P_{0}, S_{0}, S_{1}$ have coefficients 0,1 , then $P_{1}$ has nonnegative coefficients (Theorem 4.4). In the proof of this result, we point out properties of $P_{1}$ and $S_{1}$ that will be used in Sections 6 and 8 for the construction of factorizing codes.

## 4 Main results

### 4.1 Factorizations $(P, S)$ with one $b$ in $S$

In this section, we prove our main result. We consider a factorization $(P, S)$ for a finite maximal code $C$ over $A$, with $C \cap a^{*}=\left\{a^{p}\right\}$, for a prime number $p$, and $S=S_{0}+S_{1}$. We assume that $a^{i} b a^{j} \in \operatorname{supp}\left(S_{1}\right)$ implies $a^{j} \in \operatorname{supp}\left(S_{0}\right)$. We prove that $(P, S)$ is positive (Theorem 4.2). This result is a direct consequence of Lemma 4.1 and Theorem 4.1.

Lemma 4.1 Let $C$ be a finite maximal code over $A$ with $C \cap a^{*}=\left\{a^{p}\right\}$, for a prime number p. Let $(P, S)$ be a factorization of $C$ such that $S=S_{0}+S_{1}$. If $S_{1}$ is a nonnull polynomial with coefficients 0,1 , then $P_{0}, S_{0}$ are also polynomials with coefficients 0,1 .

Proof :
Let $S_{1}=\sum_{h \in H} a^{M_{h}} b a^{h}$, where $H$ and $M_{h}$ are finite, nonempty sets of nonnegative integers, for $h \in H$. Set $P_{1}=\sum_{k \in K} a^{k} b a^{L_{k}}-\sum_{t \in T} a^{t} b a^{L_{t}^{\prime}}$, where $K, T$ are finite sets of nonnegative integers, $L_{k}, L_{t}^{\prime}$ are finite multisets of nonnegative integers, for $k \in K, t \in T$, and $\operatorname{supp}\left(\sum_{k \in K} a^{k} b a^{L_{k}}\right) \cap$ $\operatorname{supp}\left(\sum_{t \in T} a^{t} b a^{L_{t}^{\prime}}\right)=\emptyset$. Suppose that $P_{0}, S_{0}$ are not polynomials with coefficients 0,1 . By Eq. (3.1), we have $P_{0}=-1, S_{0}=-\left(1+a+\ldots+a^{p-1}\right)$ or $P_{0}=-\left(1+a+\ldots+a^{p-1}\right), S_{0}=-1$.

Assume that the first case holds. Of course, $\underline{C}_{1} \geq 0$ and, by Eqs. (3.2), we have

$$
\begin{equation*}
\underline{C}_{1}=b a^{\{0,1, \ldots, p-1\}}-\sum_{h \in H} a^{M_{h}}(a-1) b a^{h}+\sum_{k \in K} a^{k} b a^{L_{k}}\left(1-a^{p}\right)+\sum_{t \in T} a^{t} b a^{L_{t}^{\prime}}\left(a^{p}-1\right) . \tag{4.1}
\end{equation*}
$$

Let $m=\max \left\{m^{\prime} \mid m^{\prime} \in M_{h^{\prime}}, h^{\prime} \in H\right\}$. Thus, for $h \in H$ such that $m=\max M_{h}$, we have $\left(b a^{\{0,1, \ldots, p-1\}}-\sum_{h \in H} a^{M_{h}}(a-1) b a^{h}, a^{m+1} b a^{h}\right)<0$. By Eq. (4.1) this implies

$$
\left(\sum_{k \in K} a^{k} b a^{L_{k}}\left(1-a^{p}\right)+\sum_{t \in T} a^{t} b a^{L_{t}^{\prime}}\left(a^{p}-1\right), a^{m+1} b a^{h}\right)>0 .
$$

Assume $\left(\sum_{k \in K} a^{k} b a^{L_{k}}\left(1-a^{p}\right), a^{m+1} b a^{h}\right)>0$. Hence, $m+1 \in K$ and $L_{m+1} \neq \emptyset$. Let $\ell=$ $\max L_{m+1}$. We now prove that $\left(\underline{C}_{1}, a^{m+1} b a^{\ell+p}\right)<0$, in contradiction with $\underline{C}_{1} \geq 0$. By the definition of $m$, we have $\left(b a^{\{0,1, \ldots, p-1\}}-\sum_{h \in H} a^{M_{h}}(a-1) b a^{h}, a^{m+1} b a^{\ell+p}\right) \leq 0$ and by the definition of $\ell$, we have $\left(\sum_{k \in K} a^{k} b a^{L_{k}}\left(1-a^{p}\right), a^{m+1} b a^{\ell+p}\right)<0$. If we had $\left(\sum_{t \in T} a^{t} b a^{L_{t}^{\prime}}\left(a^{p}-1\right), a^{m+1} b a^{\ell+p}\right)>0$, then we would have $\left(\sum_{t \in T} a^{t} b a^{L_{t}^{\prime}} a^{p}, a^{m+1} b a^{\ell+p}\right) \geq\left(\sum_{t \in T} a^{t} b a^{L_{t}^{\prime}}\left(a^{p}-1\right), a^{m+1} b a^{\ell+p}\right)>0$, hence, $m+1 \in T, \ell \in L_{m+1}^{\prime}$ and $\operatorname{supp}\left(\sum_{k \in K} a^{k} b a^{L_{k}}\right) \cap \operatorname{supp}\left(\sum_{t \in T} a^{t} b a^{L_{t}^{\prime}}\right) \neq \emptyset$, a contradiction.

Therefore, $\left(\sum_{t \in T} a^{t} b a^{L_{t}^{\prime}}\left(a^{p}-1\right), a^{m+1} b a^{h}\right)>0$. Hence, $m+1 \in T$ and $L_{m+1}^{\prime} \neq \emptyset$. Let $\ell=\min L_{m+1}^{\prime}$. Then $\left(b a^{\{0,1, \ldots, p-1\}}-\sum_{h \in H} a^{M_{h}}(a-1) b a^{h}, a^{m+1} b a^{\ell}\right) \leq 0$ by the definition of $m$, and $\left(\sum_{t \in T} a^{t} b a^{L_{t}^{\prime}}\left(a^{p}-1\right), a^{m+1} b a^{\ell}\right)<0$ by the definition of $\ell$. By Eq. (4.1), this implies $\left(\sum_{k \in K} a^{k} b a^{L_{k}}\left(1-a^{p}\right), a^{m+1} b a^{\ell}\right)>0$ which yields $m+1 \in K, \ell \in L_{m+1}$ and $\operatorname{supp}\left(\sum_{k \in K} a^{k} b a^{L_{k}}\right) \cap$ $\operatorname{supp}\left(\sum_{t \in T} a^{t} b a^{L_{t}^{\prime}}\right) \neq \emptyset$, again a contradiction.

Finally, assume $P_{0}=-\left(1+a+\ldots+a^{p-1}\right), S_{0}=-1$. Now $\underline{C}_{1}$ is defined by

$$
\begin{equation*}
\underline{C}_{1}=a^{\{0,1, \ldots, p-1\}} b-\sum_{h \in H} a^{M_{h}}\left(a^{p}-1\right) b a^{h}+\sum_{k \in K} a^{k} b a^{L_{k}}(1-a)+\sum_{t \in T} a^{t} b a^{L_{t}^{\prime}}(a-1) . \tag{4.2}
\end{equation*}
$$

Arguing as before, we consider $a^{m+p} b a^{h}$ with $m=\max \left\{m^{\prime} \mid m^{\prime} \in M_{h^{\prime}}, h^{\prime} \in H\right\}=\max M_{h}$. Since $\left(a^{\{0,1, \ldots, p-1\}} b-\sum_{h \in H} a^{M_{h}}\left(a^{p}-1\right) b a^{h}, a^{m+p} b a^{h}\right)<0$ and $\underline{C}_{1} \geq 0$, by Eq. (4.2) we have $\left(\sum_{k \in K} a^{k} b a^{L_{k}}(1-a)+\sum_{t \in T} a^{t} b a^{L_{t}^{\prime}}(a-1), a^{m+p} b a^{h}\right)>0$.

If $\left(\sum_{k \in K} a^{k} b a^{L_{k}}(1-a), a^{m+p} b a^{h}\right)>0$, then $m+p \in K$ and $L_{m+p} \neq \emptyset$. Hence, for $\ell=\max L_{m+p}$ we have $\left(\sum_{k \in K} a^{k} b a^{L_{k}}(1-a), a^{m+p} b a^{\ell+1}\right)<0$. Moreover, $\left(a^{\{0,1, \ldots, p-1\}} b-\right.$ $\left.\sum_{h \in H} a^{M_{h}}\left(a^{p}-1\right) b a^{h}, a^{m+p} b a^{\ell+1}\right) \leq 0$ by the definition of $m$, and $\left(\sum_{t \in T} a^{t} b a^{L_{t}^{\prime}}(a-1), a^{m+p} b a^{\ell+1}\right) \leq$ 0 , since otherwise $a^{m+p} b a^{\ell} \in \operatorname{supp}\left(\sum_{k \in K} a^{k} b a^{L_{k}}\right) \cap \operatorname{supp}\left(\sum_{t \in T} a^{t} b a^{L_{t}^{\prime}}\right)$. Thus $\left(\underline{C}_{1}, a^{m+p} b a^{\ell+1}\right)<$ 0 , a contradiction. If $\left(\sum_{t \in T} a^{t} b a^{L_{t}^{\prime}}(a-1), a^{m+p} b a^{h}\right)>0$, then a similar argument applies, and for $\ell=\min L_{m+p}^{\prime}$, we get $\left(\underline{C}_{1}, a^{m+p} b a^{\ell}\right)<0$, again a contradiction.

The following lemma is needed for the proof of Theorem 4.1.
Lemma 4.2 Let $S$ be a polynomial in $\mathbb{N}\langle A\rangle$ such that if the word wba ${ }^{j}$ is in $\operatorname{supp}(S)$, then $a^{j}$ is also in $\operatorname{supp}(S)$. Let $S_{0}=a^{J}$ with $J=\{0\}$ or $J=\{0,1, \ldots, p-1\}$, where $p$ is a nonnegative number. Let $r \geq 0$ and assume that $P_{0}, P_{1}, \ldots, P_{r+1} \in \mathbb{Z}\langle A\rangle$ are polynomials such that

$$
\sum_{i+h=r} P_{i} b S_{h}+\sum_{i+h=r+1} P_{i}(a-1) S_{h} \geq 0 .
$$

If $P_{0}, P_{1}, \ldots, P_{r} \in \mathbb{N}\langle A\rangle$, then $P_{r+1} \in \mathbb{N}\langle A\rangle$.

## Proof :

Assume that $J=\{0,1, \ldots, p-1\}$ (if $J=\{0\}$ we may apply the following argument with $p=1$ ). By contradiction, let $P_{r+1}=P_{r+1}^{\prime}-P_{r+1}^{\prime \prime}$ with $P_{r+1}^{\prime}, P_{r+1}^{\prime \prime} \in \mathbb{N}\langle A\rangle, \operatorname{supp}\left(P_{r+1}^{\prime}\right) \cap \operatorname{supp}\left(P_{r+1}^{\prime \prime}\right)=\emptyset$ and $P_{r+1}^{\prime \prime} \neq 0$. Let $\ell=\max \left\{\ell^{\prime} \mid \exists x \in A^{*} x b a^{\ell^{\prime}} \in \operatorname{supp}\left(P_{r+1}^{\prime \prime}\right)\right\}$ and let $x$ be a word such that $y=x b a^{\ell} \in \operatorname{supp}\left(P_{r+1}^{\prime \prime}\right)$.

By hypothesis, for any word $w b a^{j} \in \operatorname{supp}(S)$, the nonnegative integer $j$ is less than $p$. Hence, $\left(\sum_{i+h=r} P_{i} b S_{h}+\sum_{i+h=r+1, h \neq 0} P_{i}(a-1) S_{h}, y a^{p}\right)=0$. By the definition of $y$, we also have $\left(P_{r+1}^{\prime \prime}\left(a^{p}-1\right), y a^{p}\right)=\left(P_{r+1}^{\prime \prime} a^{p}, y a^{p}\right)>0$. Thus

$$
\begin{aligned}
0 & \leq\left(\sum_{i+h=r} P_{i} b S_{h}+\sum_{i+h=r+1} P_{i}(a-1) S_{h}, y a^{p}\right) \\
& =\left(P_{r+1}(a-1) a^{J}, y a^{p}\right)=\left(P_{r+1}\left(a^{p}-1\right), y a^{p}\right) \\
& =\left(P_{r+1}^{\prime}\left(a^{p}-1\right), y a^{p}\right)-\left(P_{r+1}^{\prime \prime}\left(a^{p}-1\right), y a^{p}\right) \\
& \leq\left(P_{r+1}^{\prime} a^{p}, y a^{p}\right)-\left(P_{r+1}^{\prime \prime}\left(a^{p}-1\right), y a^{p}\right) \\
& <\left(P_{r+1}^{\prime} a^{p}, y a^{p}\right)=\left(P_{r+1}^{\prime}, y\right) .
\end{aligned}
$$

In conclusion, $\left(P_{r+1}^{\prime}, y\right)>0$ and $y \in \operatorname{supp}\left(P_{r+1}^{\prime}\right) \cap \operatorname{supp}\left(P_{r+1}^{\prime \prime}\right)$, a contradiction.

Theorem 4.1 Let $C$ be a finite maximal code such that $a^{p} \in C$, for a prime number $p$. Let $P \in \mathbb{Z}\langle A\rangle$ and let $S$ be a polynomial in $\mathbb{N}\langle A\rangle$ such that if the word wba ${ }^{j}$ is in $\operatorname{supp}(S)$ then $a^{j}$ is also in $\operatorname{supp}(S)$. If $(P, S)$ is a factorization for $C$, then $P, S$ have coefficients 0,1 .

Proof:
Let $P=P_{0}+P_{1}+\ldots+P_{k}$ and let $S$ be a polynomial in $\mathbb{N}\langle A\rangle$ such that if the word $w b a^{j}$ is in $\operatorname{supp}(S)$ then $a^{j}$ is also in $\operatorname{supp}(S)$. Assume that $(P, S)$ is a factorization for a finite maximal code $C$ and $a^{p} \in C$, where $p$ is a prime number. Thus $P_{0}$ has coefficients 0,1 and $\underline{C}_{r+1}$ is defined by Eqs. (3.2), for any $r \geq 0$. By using induction and Lemma 4.2, we can prove that $P_{0}, P_{1}, \ldots, P_{k} \in \mathbb{N}\langle A\rangle$. Hence $P \in \mathbb{N}\langle A\rangle$ and, by Theorem [2.1, $P, S$ have coefficients 0,1 .

Remark 4.1 Lemma 4.2 is no longer true if we drop the hypothesis that $w b a^{j} \in \operatorname{supp}(S)$ only if $a^{j} \in \operatorname{supp}(S)$, even if $S$ is a polynomial with coefficients 0,1 . Indeed, let $s, t, n \in$ $\mathbb{N}$, with $s \geq 1, t \geq 0, n \geq 1$, let $P_{0}=1, P_{1}=b a^{\{0,1, \ldots, t\} n}-a^{s} b a^{t n}, S=a^{\{0,1, \ldots, n-1\}}+$ $a^{\{0,1, \ldots, s-1\}} b a^{(t+1) n}$. Then $P_{0} b S_{0}+P_{1}(a-1) S_{0}+P_{0}(a-1) S_{1}=b a^{\{1, \ldots, n-1\}}+a^{s} b a^{t n}$ is a polynomial with coefficients 0,1 . However, we do not know whether Theorem 4.1 is still true without the aforementioned hypothesis. As already stated in Section 1, in [19] the authors formulated the following conjecture: if $(P, S)$ is a factorization for a finite maximal code over a one-letter alphabet and $S$ has coefficients 0,1 , then $(P, S)$ is positive. Notice that Theorem4.1 is connected with a generalization of this conjecture to alphabets with size greater than one.

Theorem 4.2 Let $C$ be a finite maximal code such that $a^{p} \in C$, for a prime number $p$. Let $S=S_{0}+S_{1}$ be a polynomial such that if the word $a^{i} b a^{j}$ is in $\operatorname{supp}(S)$ then the word $a^{j}$ is also in $\operatorname{supp}(S)$. If $(P, S)$ is a factorization for $C$, then $(P, S)$ is positive.

## Proof :

Assume that $C$ and $S$ are as in the statement. Let $(P, S)$ be a factorization for $C$. If $S_{1}=0$ then by Theorem $3.2(P, S)$ is positive. Otherwise, by Theorem [3.3, either $S_{1}$ or $-S_{1}$ has coefficients 0,1 . In the first case, by Lemma 4.1, $P_{0}, S_{0}$ have coefficients 0,1 . Thus, Theorem 4.1 applies to $(P, S)$ and $P, S$ have coefficients 0,1 . In the second case, arguing as before on the factorization $(-P,-S)$ for $C$, we can prove that $-P,-S$ have coefficients 0,1 . Hence, in both cases $(P, S)$ is positive.

## $4.2(p, 4)$-codes are positively factorizing

In this section, we consider ( $p, 4$ )-codes, i.e., 4 -codes $C$ such that $C \cap a^{*}=\left\{a^{p}\right\}$, for a prime number $p$. We show that they are positively factorizing and have only positive factorizations. Looking at Eqs. (3.2), factorizations for a 4 -code may be divided into two sets, as described in Lemma 4.3 .

Lemma 4.3 Let $C$ be a 4-code, let $(U, V)$ be a factorization for $C$. Then for $(P, S)=(U, V)$ or for $(P, S)=\left(V^{\sim}, U^{\sim}\right)$ one of the following two conditions is satisfied.
(1) $P=P_{0}+P_{1}+P_{2}+P_{3}, S=S_{0}$, with $P_{3} \neq 0, S_{0} \neq 0$.
(2) $P=P_{0}+P_{1}+P_{2}, S=S_{0}+S_{1}$, with $P_{2} \neq 0, S_{1} \neq 0$.

Theorem 4.3 Let p be a prime number. Any ( $p, 4$ )-code $C$ is positively factorizing. Moreover, each factorization for $C$ is a positive factorization for $C$.

Proof :
Let $p$ be a prime number, let $C$ be a $(p, 4)$-code and let $(U, V)$ be a factorization for $C$. By Lemma4.3, $(P, S)=(U, V)$ or $(P, S)=\left(V^{\sim}, U^{\sim}\right)$, satisfies item (1) or item (2) in this lemma. If $(P, S)=(U, V)$ satisfies item (1) in Lemma 4.3, then $(P, S)$ is positive by Theorem 3.2. Assume that $(P, S)=(U, V)$ satisfies item (2) in Lemma 4.3. By Theorem [3.3, $(P, S)$ also satisfies one of the following four conditions:
(i) $P_{2}, S_{1}$ have coefficients 0,1 and $P_{1} \in \mathbb{N}\langle A\rangle$.
(ii) $P_{2}, S_{1}$ have coefficients 0,1 and if the word $a^{i} b a^{j}$ is in $\operatorname{supp}(S)$, then the word $a^{j}$ is also in $\operatorname{supp}(S)$.
(iii) $-P_{2},-S_{1}$ have coefficients 0,1 and $-P_{1} \in \mathbb{N}\langle A\rangle$.
(ii) $-P_{2},-S_{1}$ have coefficients 0,1 and if the word $a^{i} b a^{j}$ is in $\operatorname{supp}(S)$, then the word $a^{j}$ is also in $\operatorname{supp}(S)$.

In view of Lemma 4.1 and Theorem 2.1, it is obvious that if $(P, S)$ satisfies item (i), then $P, S$ have coefficients 0,1 . By Theorem 4.2, it is also clear that if $(P, S)$ satisfies item (ii), then $P, S$ have coefficients 0,1 . Finally, if $(P, S)$ satisfies item (iii) or (iv), then $-P,-S$ have coefficients 0,1 and $(P, S)$ is positive.

Assume now that $(P, S)=\left(V^{\sim}, U^{\sim}\right)$ satisfies item (1) or item (2) in Lemma 4.3, The pair $\left(V^{\sim}, U^{\sim}\right)$ is a factorization for the $(p, 4)$-code $C^{\sim}$ and the above arguments prove that ( $V^{\sim}, U^{\sim}$ ) is positive. Hence $(U, V)$ is also positive.

### 4.3 Partially positive factorizations

We end this section with another result: if $(P, S)$ is a factorization for $C$ such that $P_{0}=a^{I}$, $S_{0}=a^{J}$ and $S_{1}=\sum_{j \in J} a^{M_{j}} b a^{j}$, where $I, J, M_{j}$, for $j \in J$, are finite subsets of $\mathbb{N}$, then $P_{1}$ has nonnegative coefficients (Theorem (4.4). Let $(P, S)$ be a factorization for $C$ satisfying the above conditions. Set $P_{1}=\sum_{i \in I^{\prime}} a^{i} b\left(a^{L_{i}}-a^{L_{i}^{\prime}}\right)$, where $I^{\prime}$ is a finite subset of $\mathbb{N}$ and $a^{L_{i}}, a^{L_{i}^{\prime}}$ are polynomials in $\mathbb{N}[a]$ such that $L_{i} \cap L_{i}^{\prime}=\emptyset$, for each $i \in I^{\prime}$. By Eqs. (3.2) for $r=0$, we have

$$
\begin{equation*}
a^{I} b a^{J}+\sum_{i \in I^{\prime}} a^{i} b\left(a^{L_{i}}-a^{L_{i}^{\prime}}\right)(a-1) a^{J}+\sum_{j \in J} a^{I}(a-1) a^{M_{j}} b a^{j} \geq 0 . \tag{4.3}
\end{equation*}
$$

Notice that $P_{0}(a-1) S_{0}=a^{n}-1$, i.e., $a^{I} a^{J}=1+a+\ldots+a^{n-1}$, where $a^{n} \in C$. These pairs $(I, J)$ of subsets of $\mathbb{N}$ can be constructed by a method given in [18] and here they will be called Krasner factorizations of the finite cyclic group $\mathbb{Z}_{n}$ of order $n$. Theorem 4.4 will be proved through Lemmas 4.4-4.7. These lemmas will also be used in Sections 6 and 8 .

Lemma 4.4 Let $(I, J)$ be a Krasner factorization of $\mathbb{Z}_{n}$. Let $I^{\prime}, M_{j}$, with $j \in J$, be finite subsets of $\mathbb{N}$, let $a^{L_{i}}, a^{L_{i}^{\prime}} \in \mathbb{N}[a]$, with $i \in I^{\prime}$. Assume that $E q$. (4.3) holds. Then, for each $i \in I^{\prime}$, there exists $k_{i} \in \mathbb{N}$ such that

$$
a^{L_{i}}(a-1) a^{J}-a^{L_{i}^{\prime}}(a-1) a^{J}+k_{i} a^{J} \geq 0 .
$$

Proof :
Assume that $I^{\prime}, M_{j}$, with $j \in J, a^{L_{i}}, a^{L_{i}^{\prime}}$, with $i \in I^{\prime}$, are as in the statement. By using Eq. (4.3), we have:

$$
\begin{aligned}
& \forall i \in I^{\prime} \quad 0 \leq\left(a^{I}, a^{i}\right) a^{J}+\left(a^{L_{i}}-a^{L_{i}^{\prime}}\right)(a-1) a^{J}+\sum_{j \in J}\left(a^{I}(a-1) a^{M_{j}}, a^{i}\right) a^{j} \\
& \leq\left(a^{I}, a^{i}\right) a^{J}+\left(a^{L_{i}}-a^{L_{i}^{\prime}}\right)(a-1) a^{J}+\sum_{j \in J,\left(a^{I}(a-1) a^{M_{j}}, a^{i}\right) \geq 0}\left(a^{I}(a-1) a^{M_{j}}, a^{i}\right) a^{j} \\
& \leq a^{L_{i}}(a-1) a^{J}-a^{L_{i}^{\prime}}(a-1) a^{J}+k_{i} a^{J},
\end{aligned}
$$

where $k_{i}=k_{i}^{\prime}+\left(a^{I}, a^{i}\right), k_{i}^{\prime}=\max \Gamma_{i}, \Gamma_{i}=\{0\} \cup\left\{\gamma_{j} \mid \gamma_{j}=\left(a^{I}(a-1) a^{M_{j}}, a^{i}\right) \geq 0, j \in J\right\}$.
Lemma 4.5 Let $k \in \mathbb{N}$. Let $a^{X}$, $a^{X^{\prime}} \in \mathbb{N}[a]$, with $X \cap X^{\prime}=\emptyset$. If we have

$$
\begin{equation*}
a^{X}(a-1)-a^{X^{\prime}}(a-1)+k \geq 0 \tag{4.4}
\end{equation*}
$$

then $a^{X^{\prime}}=0$. Furthermore, if $a^{X}$ is a nonnull polynomial then $k>0$.
Proof:
By contradiction, assume that $a^{X^{\prime}} \neq 0$ and let $x=\max \left\{x^{\prime} \mid x^{\prime} \in X^{\prime}\right\}$. Thus we have $x+1>0$ and $\left(-a^{X^{\prime}}(a-1), a^{x+1}\right)<0$. Hence, in view of Eq. (4.4), we have

$$
\begin{aligned}
0 & \leq\left(a^{X}(a-1)-a^{X^{\prime}}(a-1)+k, a^{x+1}\right) \\
& =\left(a^{X}(a-1)-a^{X^{\prime}}(a-1), a^{x+1}\right) \\
& <\left(a^{X}(a-1), a^{x+1}\right) \\
& \leq\left(a^{X} a, a^{x+1}\right),
\end{aligned}
$$

which yields $x \in X \cap X^{\prime}$, a contradiction. Thus $a^{X^{\prime}}=0$. If $a^{X} \neq 0$, let $x=\min \left\{x^{\prime} \mid x^{\prime} \in X\right\}$. Since $\left(a^{X}(a-1), a^{x}\right)<0$, we have $k>0$.

Let $a^{H} \in \mathbb{N}[a], n \in \mathbb{N}$ and $t \in\{0, \ldots, n-1\}$. We set $[H]_{t}=\{h \in H \mid h=t(\bmod n)\}$. Notice that $[H]_{t}$ could be a multiset: any element $h \in H$ such that $h=t(\bmod n)$ is in $[H]_{t}$ with the same multiplicity as in $H$.

Lemma 4.6 Let $(I, J)$ be a Krasner factorization of $\mathbb{Z}_{n}$, let $k \in \mathbb{N}$. Let $a^{L}$, $a^{L^{\prime}} \in \mathbb{N}[a]$, with $L \cap L^{\prime}=\emptyset$. If we have

$$
\begin{equation*}
a^{L}(a-1) a^{J} a^{I}-a^{L^{\prime}}(a-1) a^{J} a^{I}+k a^{J} a^{I} \geq 0 \tag{4.5}
\end{equation*}
$$

then $a^{L^{\prime}}=0$. Furthermore, if $a^{L}$ is a nonnull polynomial then $k>0$.

Proof :
By Eq. (4.5), since $(I, J)$ is a Krasner factorization of $\mathbb{Z}_{n}$, we have:

$$
\begin{aligned}
0 & \leq a^{L}(a-1) a^{J} a^{I}-a^{L^{\prime}}(a-1) a^{J} a^{I}+k a^{J} a^{I} \\
& =a^{L}\left(a^{n}-1\right)-a^{L^{\prime}}\left(a^{n}-1\right)+k \frac{a^{n}-1}{a-1},
\end{aligned}
$$

which yields:

$$
\begin{equation*}
\forall t \in\{0, \ldots, n-1\}, \quad a^{[L]_{t}}\left(a^{n}-1\right)-a^{\left[L^{\prime}\right]_{t}}\left(a^{n}-1\right)+k a^{t} \geq 0 . \tag{4.6}
\end{equation*}
$$

By erasing $a^{t}$ and by changing $a^{n}$ with $a$ in each term of this inequality, we get an inequality as in Eq. (4.4). Precisely, for all $t \in\{0, \ldots, n-1\}$, let $a^{X_{t}}, a^{X_{t}^{\prime}} \in \mathbb{N}[a]$ be defined as follows:

$$
a^{X_{t}}=\sum_{x \in \mathbb{N}}\left(a^{[L] t}, a^{t+x n}\right) a^{x}, \quad a^{X_{t}^{\prime}}=\sum_{x^{\prime} \in \mathbb{N}}\left(a^{\left[L^{\prime}\right] t}, a^{t+x^{\prime} n}\right) a^{x^{\prime}} .
$$

As a direct consequence we have:

$$
X_{t} \neq \emptyset \Leftrightarrow[L]_{t} \neq \emptyset, \quad X_{t}^{\prime} \neq \emptyset \Leftrightarrow\left[L^{\prime}\right]_{t} \neq \emptyset .
$$

Furthermore, for any $x \in \mathbb{N}$, we also have

$$
\left(a^{[L]_{t}}\left(a^{n}-1\right)-a^{\left[L^{\prime}\right]_{t}}\left(a^{n}-1\right)+k a^{t}, a^{x n+t}\right)=\left(a^{X_{t}}(a-1)-a^{X_{t}^{\prime}}(a-1)+k, a^{x}\right) .
$$

Since for any $y, t \in \mathbb{N}$ we have $\left(a^{[L]_{t}}\left(a^{n}-1\right)-a^{\left[L^{\prime}\right]_{t}}\left(a^{n}-1\right)+k a^{t}, a^{y}\right) \neq 0$ if and only if $y=x n+t$, the above relation and Eq. (4.6) show that $a^{X_{t}}(a-1)-a^{X_{t}^{\prime}}(a-1)+k \geq 0$. Thus, in view of Lemma 4.5, $a^{X_{t}^{\prime}}=0$, for all $t$. The latter relation yields $a^{\left[L^{\prime}\right] t}=0$ for all $t$, i.e., $a^{L^{\prime}}=0$. Finally, if $a^{L} \neq 0$ then there is $t$ such that $[L]_{t} \neq \emptyset$. Consequently, $a^{X_{t}} \neq 0$ and $k>0$, once again by Lemma 4.5 .

Lemma 4.7 Let $(I, J)$ be a Krasner factorization of $\mathbb{Z}_{n}$, let $k \in \mathbb{N}$. Let $a^{L}$, $a^{L^{\prime}} \in \mathbb{N}[a]$, with $L \cap L^{\prime}=\emptyset$. If we have

$$
\begin{equation*}
a^{L}(a-1) a^{J}-a^{L^{\prime}}(a-1) a^{J}+k a^{J} \geq 0, \tag{4.7}
\end{equation*}
$$

then $a^{L^{\prime}}=0$. Furthermore, if $a^{L}$ is a nonnull polynomial then $k>0$.
Proof:
By Eq. (4.7) and since $a^{I} \geq 0$, we have:

$$
a^{L}(a-1) a^{J} a^{I}-a^{L^{\prime}}(a-1) a^{J} a^{I}+k a^{J} a^{I} \geq 0
$$

Thus the conclusion follows by Lemma 4.6.
We have proved the following result.
Theorem 4.4 Let $(P, S)$ be a factorization for $C$ such that $P_{0}=a^{I}, S_{0}=a^{J}$ and $S_{1}=$ $\sum_{j \in J} a^{M_{j}} b a^{j}$, where $(I, J)$ is a Krasner factorization of $\mathbb{Z}_{n}$ and $M_{j}$ is a finite subset of $\mathbb{N}$, for $j \in J$. Then $P_{1}$ has nonnegative coefficients.

Proof :
Let $(P, S)$ be a factorization for $C$ with $P_{0}, S_{0}, S_{1}$ as in the statement. Let $I^{\prime}$ be a finite subset of $\mathbb{N}$ and let $a^{L_{i}}, a^{L_{i}^{\prime}}$ be polynomials in $\mathbb{N}[a]$ such that $P_{1}=\sum_{i \in I^{\prime}} a^{i} b\left(a^{L_{i}}-a^{L_{i}^{\prime}}\right)$, where $L_{i} \cap L_{i}^{\prime}=\emptyset$, for $i \in I^{\prime}$. Thus $(P, S)$ satisfies Eq. (4.3). By Lemma 4.4, for each $i \in I^{\prime}$ there exists $k_{i} \in \mathbb{N}$ such that $a^{L_{i}}(a-1) a^{J}-a^{L_{i}^{\prime}}(a-1) a^{J}+k_{i} a^{J} \geq 0$. Thus, $a^{L_{i}^{\prime}}=0$ for each $i \in I^{\prime}$ (Lemma 4.7) and $P_{1} \in \mathbb{N}\langle A\rangle$.

## 5 Positively factorizing codes

As observed in [2], the aim of the theory of codes is to give a structural description of codes in a way that allows their construction. This has not yet been accomplished, except for some special families of codes. In particular, a still open problem is a structural description of positively factorizing codes. Once again, this description has been achieved for particular classes of codes, through the construction of their factorizations. In this paper, we extend this construction to a larger class, as outlined in Section 5.2. The obtained results once again show relations between factorizations of finite maximal codes and factorizations of finite cyclic groups, whose definition is recalled in Section 5.1.

### 5.1 Factorizations of cyclic groups

A pair $(T, R)$ of subsets of $\mathbb{N}$ is a factorization of $\mathbb{Z}_{n}$ if, for each $i$ in $\{0, \ldots, n-1\}$, there exists a unique pair $(t, r) \in T \times R$ such that $i=t+r(\bmod n)$. We are interested in a special class of factorizations, defined in [17] and called Hajós factorizations here. There are relations between the structure of positively factorizing codes and Hajós factorizations of $\mathbb{Z}_{n}$. These relations have been highlighted by a characterization of Hajós factorizations given in [10]: $(T, R)$ is a Hajós factorization of $\mathbb{Z}_{n}$ if and only if there is a Krasner factorization $(I, J)$ of $\mathbb{Z}_{n}$ and a pair ( $M, L$ ) of finite subsets of $\mathbb{N}$ such that $a^{T}=a^{M}(a-1) a^{I}+a^{I} \geq 0, a^{R}=a^{L}(a-1) a^{J}+a^{J} \geq 0$. The structure of the above subsets $M, L$, and therefore of the pairs $(T, R)$, has been described in [8]. As a consequence, the following result has been stated (see Remark 6.7 in [8]).

Proposition 5.1 Let $(I, J)$ be a Krasner factorization of $\mathbb{Z}_{n}$, let $M$ be a finite subset of $\mathbb{N}$ and let $a^{T}=a^{M}(a-1) a^{I}+a^{I}$. If $a^{T}$ has nonnegative coefficients, then $a^{T}$ has coefficients 0,1 .

As stated in Proposition 5.2, subsets $M, L$ satisfy another equation too. We also need the following known result (Lemma 3.2 (ii), (iii), (iv) in [9]).

Lemma 5.1 Let $k, n \in \mathbb{N}$.
(i) If $H$ is a finite subset of $\mathbb{N}$ and $\left(a^{H}(a-1)+k\right)\left(a^{n}-1\right) /(a-1) \geq 0$, then the polynomial $\left(a^{H}(a-1)+1\right)\left(a^{n}-1\right) /(a-1)$ has coefficients 0,1 .
(ii) If $a^{H} \in \mathbb{N}[a]$ and $\left(a^{H}(a-1)+1\right)\left(a^{n}-1\right) /(a-1) \geq 0$, then $a^{H}$ has coefficients 0,1 .

Proposition 5.2 Let $(I, J)$ be a Krasner factorization of $\mathbb{Z}_{n}$, let $M, L$ be finite subsets of $\mathbb{N}$ such that

$$
a^{T}=a^{M}(a-1) a^{I}+a^{I} \geq 0, \quad a^{R}=a^{L}(a-1) a^{J}+a^{J} \geq 0
$$

Then $a^{M}(a-1) a^{L}+a^{M}+a^{L}$ is a polynomial with coefficients 0,1 .
Proof :
Let $a^{H}, a^{H^{\prime}} \in \mathbb{N}[a]$, with $H \cap H^{\prime}=\emptyset$, and assume $a^{M}(a-1) a^{L}+a^{M}+a^{L}=a^{H}-a^{H^{\prime}}$. An easy computation shows that

$$
a^{T} a^{R}=a^{H}(a-1) a^{J} a^{I}-a^{H^{\prime}}(a-1) a^{J} a^{I}+a^{J} a^{I} \geq 0 .
$$

Thus $a^{H^{\prime}}=0$, by Lemma 4.6,
Then $\left(a^{H}(a-1)+1\right)\left(a^{n}-1\right) /(a-1) \geq 0$ and $a^{H}=a^{M}(a-1) a^{L}+a^{M}+a^{L}$ has coefficients 0,1 by Lemma 5.1 (ii).

We will see that our construction of positive factorizations is strongly related to Hajós factorizations $(T, R)$ satisfying an additional hypothesis: for the corresponding pair ( $M, L$ ) we have $a^{M}(a-1) a^{L}+a^{L} \geq 0$ or $a^{M}(a-1) a^{L}+a^{M} \geq 0$. In this case we say that $(T, R)$ is a strong Hajós factorization.

In [20], the author gave a construction of an infinite family of Hajós factorizations $(T, R)$ of $\mathbb{Z}_{n}$ which are not strong, i.e., which are such that, for the corresponding pair ( $M, L$ ), neither $a^{M}(a-1) a^{L}+a^{L}$ nor $a^{M}(a-1) a^{L}+a^{M}$ is a polynomial with nonnegative coefficients. The non-strong Hajós factorization $(T, R)=(\{0,4,8,12,16,20\},\{0,3,6,21\})$ of $\mathbb{Z}_{24}$ is an element of this family and the corresponding pair $(M, L)$ is $(\{2,3\},\{1,9,11,13\})$.

### 5.2 Outline of the results on positively factorizing codes

So far, factorizations $(P, S)$ for finite maximal codes $C$ have been constructed for 3-codes or when $\operatorname{supp}(P)$ or $\operatorname{supp}(S)$ is a subset of $a^{*}$. Factorizations for 1 - and 2 -codes belong to the latter family. As said, all these factorizations are positive, i.e., $P, S \in \mathbb{N}\langle A\rangle$ or $-P,-S \in \mathbb{N}\langle A\rangle$. In the description of their structure, as well as for other positive factorizations, we will assume $P, S \in \mathbb{N}\langle A\rangle$.

Let $P, S \in \mathbb{N}\langle A\rangle$. Then $(P, S)$ is a positive factorization for a 1-code if and only if $P=a^{I}$ and $S=a^{J}$, for a Krasner factorization $(I, J)$ of $\mathbb{Z}_{n}[24]$. Starting with these pairs $\left(a^{I}, a^{J}\right)$, one may inductively construct all positive factorizations $(P, S)$ with $P$ or $S$ in $\mathbb{N}[a]$ as follows [8]. Assume $P \in \mathbb{N}[a]$. We have $P=a^{I}, S=S_{0}+S_{1}+\ldots+S_{t}$, with $S_{0}=a^{J}$ and there are finite subsets $M_{w}$ of $\mathbb{N}$ such that $S_{i}=\sum_{w \in \operatorname{supp}\left(S_{i-1}\right)} a^{M_{w}} b w, 1 \leq i \leq t$, with $a^{T_{w}}=a^{M_{w}}(a-1) a^{I}+a^{I} \geq 0$. Therefore $\left(T_{w}, J\right)$ is a strong Hajós factorization of $\mathbb{Z}_{n}$ since for the corresponding pair ( $M_{w}, \emptyset$ ), we have $a^{M_{w}} \geq 0$. Of course, if $S \in \mathbb{N}[a]$, then $\left(S^{\sim}, P^{\sim}\right)$ is as above.

In this paper, starting with the "simplest" pairs $\left(a^{I}, a^{J}+\sum_{j \in J} a^{M_{j}} b a^{j}\right)$ in the above family, we give a recursive construction of all positive factorizations $(P, S)$ for a maximal code $C$ with $S=a^{J}+\sum_{j \in J} a^{M_{j}} b a^{j}$. Notice that when $n$ is a prime number, the factorizations $(P, S)$ are those mentioned in Theorem4.2. We begin with a characterization of the words $C_{1}$ with one occurrence of $b$ in Section 6. Then, in Section 7, we prove that each $m$-code $C$ having a factorization $(P, S)$, with $S$ as above, may be obtained from an $(m-1)$-code $C^{\prime}$ having a factorization $\left(P^{\prime}, S\right)$.

Positive factorizations $(P, S)$ with $\operatorname{supp}(P) \subseteq a^{*} \cup a^{*} b a^{*}$ and $S=a^{J}+\sum_{j \in J} a^{M_{j}} b a^{j}$ have already been characterized in [9]. As a matter of fact, if $(U, V)$ is a positive factorization for a 3-code such that neither $U$ nor $V$ is in $\mathbb{N}[a]$, then for $(P, S)=(U, V)$ or $(P, S)=\left(V^{\sim}, U^{\sim}\right)$, we have $P=a^{I}+\sum_{i \in I^{\prime}} a^{i} b a^{L_{i}}, S=a^{J}+\sum_{j \in J} a^{M_{j}} b a^{j}$, where $I^{\prime}, L_{i}$, for $i \in I^{\prime}$, and $M_{j}$, for $j \in J$, are finite subsets of $\mathbb{N}$ such that

$$
\begin{gather*}
\forall j \in J \quad a^{T_{j}}=a^{M_{j}}(a-1) a^{I}+a^{I} \geq 0,  \tag{5.1}\\
\left\{i \in I^{\prime} \mid L_{i} \neq \emptyset\right\} \subseteq \cup_{j \in J} T_{j},  \tag{5.2}\\
\forall i \in I^{\prime} \quad a^{R_{i}}=a^{L_{i}}(a-1) a^{J}+a^{J} \geq a^{L_{i}}(a-1) a^{J}+a^{J_{i}} \geq 0, \tag{5.3}
\end{gather*}
$$

where $J_{i}=\left\{j \in J \mid i \in T_{j}\right\}$,

$$
\begin{equation*}
\forall j \in J, i \in I^{\prime} \backslash I \quad a^{L_{i}}(a-1) a^{M_{j}}+a^{L_{i}} \geq 0 \tag{5.4}
\end{equation*}
$$

Moreover, there are $i \in I^{\prime}$ and $j \in J$ such that $L_{i} \neq \emptyset$ and $M_{j} \neq \emptyset$.
Remark 5.1 Of course, if $M_{j^{\prime}}=\emptyset$, then $T_{j^{\prime}}=I$ and Eqs. (5.1) and (5.4) are satisfied for $j^{\prime}$. Analogously, if $L_{i^{\prime}}=\emptyset$, then Eqs. (5.3) and (5.4) are satisfied for $i^{\prime}$.

Remark 5.2 Note that in the characterization of the positive factorizations $(P, S)$ given in 9], it is also required that $a^{M_{j}}(a-1) a^{L_{i}}+a^{M_{j}}+a^{L_{i}} \geq 0$, for $i \in I^{\prime} \cap I, j \in J$. In view of Eqs. (5.1), (5.3), Proposition 5.2 applies to $L_{i}, M_{j}$ and shows that the above polynomial has coefficients 0,1 . Therefore, we may omit this condition.

Section 8 deals with some positive factorizations for 4 -codes which are described below. Lemma 5.2 is the counterpart of Lemma 4.3 for positive factorizations.

Lemma 5.2 Let $C$ be a 4-code, let $(U, V)$ be a factorization for $C$, with $U, V \in \mathbb{N}\langle A\rangle$. Then, for $(P, S)=(U, V)$ or $(P, S)=\left(V^{\sim}, U^{\sim}\right)$, one of the following three conditions is satisfied.
(1) $P=P_{0}+P_{1}+P_{2}+P_{3} \in \mathbb{N}\langle A\rangle, S=S_{0} \in \mathbb{N}[a]$, with $P_{3} \neq 0, S_{0} \neq 0$.
(2) $P=a^{I}+\sum_{i \in I^{\prime}} a^{i} b a^{L_{i}}+\sum_{w \in X_{1}} w b a^{L_{w}}, S=a^{J}+\sum_{j \in J} a^{M_{j}} b a^{j}$, where $(I, J)$ is a Krasner factorization of $\mathbb{Z}_{n}, I^{\prime}, L_{i}, M_{j}, L_{w}$ are finite subsets of $\mathbb{N}$, for any $i, j, w$, and $X_{1}$ is a finite subset of $a^{*} b a^{*}$. Moreover, $\sum_{w \in X_{1}} w b a^{L_{w}} \neq 0, \sum_{j \in J} a^{M_{j}} b a^{j} \neq 0$.
(3) $P=a^{I}+\sum_{i \in I^{\prime}} a^{i} b a^{L_{i}}+\sum_{i \in I^{\prime}, \ell \in L_{i}} a^{i} b a^{\ell} b a^{L_{i, \ell}}, S=a^{J}+\sum_{j \in J^{\prime}} a^{M_{j}} b a^{j}$, where $(I, J)$ is a Krasner factorization of $\mathbb{Z}_{n}, I^{\prime}, J^{\prime}, L_{i}, M_{j}, L_{i, \ell}$ are finite subsets of $\mathbb{N}$, for any $i, j, \ell$. Moreover, $\sum_{i \in I^{\prime}, \ell \in L_{i}} a^{i} b a^{\ell} b a^{L_{i, \ell}} \neq 0, \sum_{j \in J^{\prime}} a^{M_{j}} b a^{j} \neq 0$.

Proof :
The statement is a direct consequence of Lemma 4.3 and Theorem 3.3 ,
We have already described the structure of the positive factorizations for 4-codes satisfying item (1) in Lemma 5.2, Furthermore, 4-codes having positive factorizations that satisfy item (2) in Lemma 5.2 belong to the class considered in Section 7 . Positive factorizations $(P, S)$, satisfying item (3) in Lemma 5.2, will be handled in Section 8 when $I^{\prime}=I$, i.e., when all words $a^{i} b w$ in $\operatorname{supp}(P)$ are such that $i \in I$.

Finally, let $\Omega(n)$ be the number of factors in the prime factorization of $n \in \mathbb{N}$. We recall that the structure of the words in $C_{1}=C \cap a^{*} b a^{*}$ has been investigated in [12]. A characterization of the words in $C_{1}$ has been obtained when $a^{n} \in C$ with $\Omega(n) \leq 2$ [15].

## 6 Words in $C$ with one b: a special case

Let $(P, S)$ be a factorization for a finite maximal code $C$ with $P, S \in \mathbb{N}\langle A\rangle$. Set $P_{0}=a^{I}$, $S_{0}=a^{J}$, where $(I, J)$ is a Krasner factorization of $\mathbb{Z}_{n}$. Lemma 6.1 characterizes polynomials $P_{1}, S_{1}$ under the hypothesis $S_{1}=\sum_{j \in J} a^{M_{j}} b a^{j}$ or $P_{1}=\sum_{i \in I} a^{i} b a^{L_{i}}$. Loosely speaking, this result states that the set $C_{1}=C \cap a^{*} b a^{*}$ of the words with one occurrence of $b$ is the same as in a 3 -code. The proof of Lemma 6.1 is the same as in [9] and it is reported here for the sake of completeness. This result will also be used in Sections 7 and 8 ,

Proposition 6.1 Let $(I, J)$ be a Krasner factorization of $\mathbb{Z}_{n}$, let $L$ be a finite subset of $\mathbb{N}$ and let $k$ be a positive integer such that $a^{L}(a-1) a^{J}+k a^{J} \geq 0$. Then, for any $j \in J$, we have $\left(a^{L}(a-1) a^{J}+a^{J}, a^{j}\right) \leq 1$.
Proof:
Assume that there exists $k>0$ such that $a^{L}(a-1) a^{J}+k a^{J} \geq 0$. Thus,

$$
\left(a^{L}(a-1) a^{J}+k a^{J}\right) a^{I}=\left(a^{L}(a-1)+k\right)\left(a^{n}-1\right) /(a-1) \geq 0 .
$$

Therefore, by Lemma $5.1(\mathrm{i}),\left(a^{L}(a-1)+1\right)\left(a^{n}-1\right) /(a-1)$ is a polynomial with coefficients 0,1 . Assume that $\left(a^{L}(a-1) a^{J}+a^{J}, a^{j}\right) \geq 2$ with $j \in J$. Notice that $\left(a^{L}(a-1)+1\right)\left(a^{n}-1\right) /(a-1)=a^{L}(a-1) a^{J} a^{I}+a^{J} a^{I}=a^{L}(a-1) a^{J}+a^{J}+\left(a^{L}(a-1)+1\right) a^{J} a^{I \backslash 0}$. Thus, we obtain $\left(\left(a^{L}(a-1)+1\right) a^{J} a^{I \backslash 0}, a^{j}\right)<0$, i.e.,

$$
\exists q \in \mathbb{N}, i \in I \backslash 0: q+i=j,\left(a^{L}(a-1) a^{J}+a^{J}, a^{q}\right)<0 .
$$

On the other hand, since $a^{L}(a-1) a^{J}+k a^{J} \geq 0$, we have $k>1$ and $q \in J$. This is impossible since $(I, J)$ is a Krasner factorization of $\mathbb{Z}_{n}$ and $i+q=0+j$ with $i \in I \backslash 0,0 \in I, q, j \in J$.

Lemma 6.1 Let $(I, J)$ be a Krasner factorization of $\mathbb{Z}_{n}$, let $I^{\prime}, L_{i}, M_{j}$ be finite subsets of $\mathbb{N}$. We have

$$
\underline{C}_{1}=a^{I} b a^{J}+\sum_{i \in I^{\prime}} a^{i} b a^{L_{i}}(a-1) a^{J}+\sum_{j \in J} a^{I}(a-1) a^{M_{j}} b a^{j} \geq 0
$$

if and only if $I^{\prime}, L_{i}, M_{j}$ satisfy Eqs. (5.1)-(5.3), with $J_{i}=\left\{j \in J \mid i \in T_{j}\right\}$, for $i \in I^{\prime}$.
Proof :
Suppose that

$$
\underline{C}_{1}=a^{I} b a^{J}+\sum_{i \in I^{\prime}} a^{i} b a^{L_{i}}(a-1) a^{J}+\sum_{j \in J} a^{I}(a-1) a^{M_{j}} b a^{j} \geq 0 .
$$

In view of Lemma 4.4 (applied with $L_{i}^{\prime}=\emptyset$ ), for any $i \in I^{\prime}$ there exists $k_{i}>0$ such that $a^{L_{i}}(a-1) a^{J}+k_{i} a^{J} \geq 0$.

Assume that there exist $j \in J$ and $h \in \mathbb{N}$ such that $\left(a^{M_{j}}(a-1) a^{I}+a^{I}, a^{h}\right)<0$. Thus, we have $\left(a^{I} b a^{J}+\sum_{j \in J} a^{I}(a-1) a^{M_{j}} b a^{j}, a^{h} b a^{j}\right)<0$ and so $\left(\sum_{i \in I^{\prime}} a^{i} b a^{L_{i}}(a-1) a^{J}, a^{h} b a^{j}\right)>0$, since $\underline{C}_{1} \geq 0$. Hence, $h \in I^{\prime}$ and $\left(a^{L_{h}}(a-1) a^{J}, a^{j}\right)>0$. Consequently, $\left(a^{L_{h}}(a-1) a^{J}+a^{J}, a^{j}\right) \geq 2$, in contradiction with Proposition 6.1. This proves Eq. (5.1).

Set, as in Eq. (5.1), $a^{T_{j}}=a^{M_{j}}(a-1) a^{I}+a^{I}$. By Proposition 5.1, $T_{j}$ is a subset of $\mathbb{N}$ and we have

$$
\underline{C}_{1}=\sum_{j \in J} a^{T_{j}} b a^{j}+\sum_{i \in I^{\prime}} a^{i} b a^{L_{i}}(a-1) a^{J} \geq 0 .
$$

For any $i \in I^{\prime}$ such that $L_{i}$ is nonempty, let $\ell_{i}=\min L_{i}$. Thus, $\left(\sum_{i \in I^{\prime}} a^{i} b a^{L_{i}}(a-1) a^{J}, a^{i} b a^{\ell_{i}}\right)<0$ and

$$
0 \leq\left(\underline{C}_{1}, a^{i} b a^{\ell_{i}}\right)=\left(\sum_{j \in J} a^{T_{j}} b a^{j}+a^{i} b a^{L_{i}}(a-1) a^{J}, a^{i} b a^{\ell_{i}}\right)<\left(\sum_{j \in J} a^{T_{j}} b a^{j}, a^{i} b a^{\ell_{i}}\right) .
$$

Hence, $\left\{i \in I^{\prime} \mid L_{i} \neq \emptyset\right\} \subseteq \cup_{j \in J} T_{j}$, i.e., Eq. (5.2) holds. If we set $L_{i}=\emptyset$ for $i \in \cup_{j \in J} T_{j} \backslash I^{\prime}$ and $J_{i}=\left\{j \in J \mid i \in T_{j}\right\}$, we have

$$
\underline{C}_{1}=\sum_{j \in J} a^{T_{j}} b a^{j}+\sum_{i \in I^{\prime}} a^{i} b a^{L_{i}}(a-1) a^{J}=\sum_{i \in \cup_{j \in J} T_{j}} a^{i} b\left(a^{L_{i}}(a-1) a^{J}+a^{J_{i}}\right) \geq 0 .
$$

The above relation proves Eq. (5.3).
Conversely, let $(I, J)$ be a Krasner factorization of $\mathbb{Z}_{n}$ and assume that $I^{\prime}, L_{i}, M_{j}$ satisfy Eqs. (5.1)-(5.3). If we set $L_{i}=\emptyset$ for $i \in \cup_{j \in J} T_{j} \backslash I^{\prime}$ and $J_{i}=\left\{j \in J \mid i \in T_{j}\right\}$, we have

$$
\begin{aligned}
\underline{C}_{1} & =a^{I} b a^{J}+\sum_{i \in I^{\prime}} a^{i} b a^{L_{i}}(a-1) a^{J}+\sum_{j \in J} a^{I}(a-1) a^{M_{j}} b a^{j} \\
& =\sum_{j \in J} a^{T_{j}} b a^{j}+\sum_{i \in I^{\prime}} a^{i} b a^{L_{i}}(a-1) a^{J}=\sum_{i \in \mathrm{U}_{j \in J} T_{j}} a^{i} b\left(a^{L_{i}}(a-1) a^{J}+a^{J_{i}}\right) \geq 0
\end{aligned}
$$

and the proof is complete.

## 7 Construction of factorizing codes with $S=a^{J}+\sum_{j \in J} a^{M_{j}} b a^{j}$

In this section, we give a recursive characterization of positive factorizations $(P, S)$ such that $S=a^{J}+\sum_{j \in J} a^{M_{j}} b a^{j}$. Consider again the finite maximal code $\underline{C}$ defined in Example 3.1 by the relation $\underline{C}=P(\underline{A}-1) S+1$, with

$$
\begin{aligned}
P & =1+a^{2} b a^{\{0,1,2,3,4,5,6\}}+a^{2} b a^{3} b a^{\{0,1,2,3,4,5,6\}} \\
S & =a^{\{0,1,2,3,4\}}+a^{\{0,1\}} b a^{\{0,1,2,3,4\}}
\end{aligned}
$$

Set $P^{\prime}=P_{0}+P_{1}=1+a^{2} b a^{\{0,1,2,3,4,5,6\}}$. Equalities defining $\underline{C}_{0}, \underline{C}_{1}, \underline{C}_{2}$ in Example 3.1, show that $\underline{C}^{\prime}=P^{\prime}(\underline{A}-1) S+1$ is a 3 -code. Moreover, $\underline{C}=\underline{C}^{\prime}+z b a^{L_{z}}(\underline{A}-1) S$, with $z b a^{L_{z}}=$ $a^{2} b a^{3} b a^{\{0,1,2,3,4,5,6\}}$ and $z b a^{t}=a^{2} b a^{3} b a^{t} \in C^{\prime}$, for $t \in\{0,1,2,3,4\}$. This construction can be easily generalized and allows us to construct all factorizing codes with $S=a^{J}+\sum_{j \in J} a^{M_{j}} b a^{j}$ (Theorem[7.1). Corollary 7.1]shows that these codes may be recursively constructed. Proposition 6.1 and Lemma 7.1 are the main tools we need in the proof. Lemma 7.2 is a preliminary step.

Lemma 7.1 Let $(P, S)$ be a factorization for a finite maximal code $C$, with $S=a^{J}+\sum_{j \in J} a^{M_{j}} b a^{j} \in$ $\mathbb{N}\langle A\rangle, P=P_{0}+\ldots+P_{k} \in \mathbb{N}\langle A\rangle, k \geq 0$. Then $C_{k+1} \subseteq\left\{w b a^{j} \mid w \in A^{*}, j \in J\right\}$.

Proof:
The conclusion is a direct consequence of the following equation, where it is understood that $P_{k-1}=0$ for $k=0$.

$$
\underline{C}_{k+1}=P_{k} b a^{J}+\sum_{j \in J} P_{k-1} b a^{M_{j}} b a^{j}+\sum_{j \in J} P_{k}(a-1) a^{M_{j}} b a^{j} .
$$

Lemma 7.2 Let $P, P^{\prime}, S$ be polynomials in $\mathbb{Z}\langle A\rangle$, with $P=P_{0}+\ldots+P_{k}+P_{k+1}, P^{\prime}=P_{0}+$ $\ldots+P_{k}, k \geq 0$ and $\operatorname{supp}(S) \subseteq a^{*} \cup a^{*} b a^{*}$. Set $X=P(\underline{A}-1) S+1$ and $Y=P^{\prime}(\underline{A}-1) S+1$. Then $X_{i}=Y_{i}$ for $i \in\{0, \ldots, k\}$, and moreover,

$$
\begin{aligned}
X_{k+1} & =Y_{k+1}+P_{k+1}(a-1) S_{0} \\
X_{k+2} & =Y_{k+2}+P_{k+1}(a-1) S_{1}+P_{k+1} b S_{0} \\
X_{k+3} & =P_{k+1} b S_{1}
\end{aligned}
$$

Proof :
The conclusion is a direct consequence of Eqs. (3.1), (3.2).

Theorem 7.1 Let $(I, J)$ be a Krasner factorization of $\mathbb{Z}_{n}$, let $P, S$ be polynomials in $\mathbb{N}\langle A\rangle$, with $S=a^{J}+\sum_{j \in J} a^{M_{j}} b a^{j}$. Set $P^{\prime}=P_{0}+\ldots+P_{k}$ and $P=P^{\prime}+P_{k+1}$, where $k \geq 0$ and $P_{k+1}$ is a nonnull polynomial. Then $(P, S)$ is a positive factorization for $a(k+3)$-code if and only if the following conditions are satisfied:
(1) $\left(P^{\prime}, S\right)$ is a positive factorization for a $(k+t)$-code $C^{\prime}$, where $t=2$ if $P_{k} \neq 0, t=1$ otherwise.
(2) $P_{k+1}=\sum_{z \in Q_{k}} z b a^{L_{z, k+1}}$, where $Q_{k}=\left\{z \mid \exists j \in J z b a^{j} \in C_{k+1}^{\prime}\right\}$ and $L_{z, k+1}$ is a finite subset of $\mathbb{N}$, for any $z \in Q_{k}$.
(3) For any $z \in Q_{k}$, set $J_{z}=\left\{j \in J \mid z b a^{j} \in C_{k+1}^{\prime}\right\}$. We have

$$
\begin{align*}
& \forall z \in Q_{k} \quad a^{L_{z, k+1}}(a-1) a^{J}+a^{J} \geq a^{L_{z, k+1}}(a-1) a^{J}+a^{J_{z}} \geq 0,  \tag{7.1}\\
& \forall j \in J, z \in Q_{k} \backslash \operatorname{supp}\left(P_{k}\right) \quad a^{L_{z, k+1}}(a-1) a^{M_{j}}+a^{L_{z, k+1}} \geq 0 . \tag{7.2}
\end{align*}
$$

Proof :
Let $(I, J)$ be a Krasner factorization of $\mathbb{Z}_{n}$. Let $P, P^{\prime}, S$ be polynomials in $\mathbb{N}\langle A\rangle$, where $P=P_{0}+$ $\ldots+P_{k}+P_{k+1}, P^{\prime}=P_{0}+\ldots+P_{k}, k \geq 0, P_{k+1}$ is a nonnull polynomial and $S=a^{J}+\sum_{j \in J} a^{M_{j}} b a^{j}$.

Suppose that $\left(P^{\prime}, S\right)$ is a positive factorization for a $(k+t)$-code $C^{\prime}$, with $t=2$ if $P_{k} \neq 0$, $t=1$ otherwise, $P_{k+1}$ is as in item (2) and Eqs. (7.1), (7.2) are satisfied. By Lemma 7.1, $C_{k+1}^{\prime} \subseteq$ $\left\{w b a^{j} \mid w \in A^{*}, j \in J\right\}$. Thus, $\underline{C}_{k+1}^{\prime}=\sum_{z \in Q_{k}} z b a^{J_{z}}$. Let us prove that $(P, S)$ is a positive factorization for a $(k+3)$-code $C$. By Lemma [7.2, since $P_{k+1} b S_{1}=\sum_{j \in J} P_{k+1} b a^{M_{j}} b a^{j} \geq 0$, it suffices to prove that the following relations hold

$$
\begin{align*}
& \underline{C}_{k+1}=\underline{C}_{k+1}^{\prime}+P_{k+1}(a-1) a^{J} \geq 0  \tag{7.3}\\
& \underline{C}_{k+2}=\underline{C}_{k+2}^{\prime}+\sum_{j \in J} P_{k+1}(a-1) a^{M_{j}} b a^{j}+P_{k+1} b a^{J} \geq 0 . \tag{7.4}
\end{align*}
$$

By Eq. (7.1) we have
$\underline{C}_{k+1}=\underline{C}_{k+1}^{\prime}+P_{k+1}(a-1) a^{J}=\underline{C}_{k+1}^{\prime}+\sum_{z \in Q_{k}} z b a^{L_{z, k+1}}(a-1) a^{J}=\sum_{z \in Q_{k}} z b\left(a^{J_{z}}+a^{L_{z, k+1}}(a-1) a^{J}\right) \geq 0$,
hence Eq. (7.3) is satisfied. Next, we have

$$
\begin{aligned}
\underline{C}_{k+2}= & \underline{C}_{k+2}^{\prime}+\sum_{j \in J} P_{k+1}(a-1) a^{M_{j}} b a^{j}+P_{k+1} b a^{J} \\
= & \sum_{j \in J} P_{k} b a^{M_{j}} b a^{j}+\sum_{j \in J, z \in Q_{k}} z b\left(a^{L_{z, k+1}}(a-1) a^{M_{j}}+a^{L_{z, k+1}}\right) b a^{j} \geq \\
& \sum_{j \in J, z \in Q_{k} \cap \operatorname{supp}\left(P_{k}\right)} z b\left(a^{M_{j}}+a^{L_{z, k+1}}(a-1) a^{M_{j}}+a^{L_{z, k+1}}\right) b a^{j} \\
+ & \sum_{j \in J, z \in Q_{k} \backslash \operatorname{supp}\left(P_{k}\right)} z b\left(a^{\left.L_{z, k+1}(a-1) a^{M_{j}}+a^{L_{z, k+1}}\right) b a^{j}}\right.
\end{aligned}
$$

In turn, by Eq. (7.2) we have

$$
\sum_{j \in J, z \in Q_{k} \backslash \operatorname{supp}\left(P_{k}\right)} z b\left(a^{L_{z, k+1}}(a-1) a^{M_{j}}+a^{L_{z, k+1}}\right) b a^{j} \geq 0
$$

then, by Lemma 6.1, $a^{M_{j}}(a-1) a^{I}+a^{I} \geq 0$ for any $j \in J$, and by Proposition 5.2 we have

$$
\sum_{j \in J, z \in Q_{k} \cap \operatorname{supp}\left(P_{k}\right)} z b\left(a^{M_{j}}+a^{L_{z, k+1}}(a-1) a^{M_{j}}+a^{L_{z, k+1}}\right) b a^{j} \geq 0 .
$$

Therefore, Eq. (7.4) is also satisfied and $(P, S)$ is a positive factorization for a $(k+3)$-code $C$.

Conversely, assume that $(P, S)$ is a positive factorization for a $(k+3)$-code $C$. Set $P_{k+1}=$ $\sum_{z \in Q_{k}} z b a^{L_{z, k+1}}$, where $Q_{k}$ is a finite set of words and $L_{z, k+1}$ is a finite nonempty subset of $\mathbb{N}$, for any $z \in Q_{k}$.

We first prove that the polynomial $Y_{k+1}=P_{k} b a^{J}+\sum_{j \in J} P_{k-1} b a^{M_{j}} b a^{j}+\sum_{j \in J} P_{k}(a-1) a^{M_{j}} b a^{j}$ is in $\mathbb{N}\langle A\rangle \backslash\{0\}$. Notice that if $w b a^{j} \in \operatorname{supp}\left(Y_{k+1}\right)$, then $j \in J$. By hypothesis, $\underline{C}_{k+1}=$ $Y_{k+1}+\sum_{z \in Q_{k}} z b a^{L_{z, k+1}}(a-1) a^{J} \geq 0$. Moreover, we have $\left(a^{L_{z, k+1}}(a-1) a^{J}, a^{q}\right)<0$, for $q=$ $\min L_{z, k+1}$. For any $q \in \mathbb{N}$ such that $\left(a^{L_{z, k+1}}(a-1) a^{J}, a^{q}\right)<0$ we get $\left(P_{k+1}(a-1) a^{J}, z b a^{q}\right)<0$. Thus, since $\underline{C}_{k+1} \geq 0$, we also have $\left(Y_{k+1}, z b a^{q}\right)>0$, that is $q \in J$. As a consequence, $a^{L_{z, k+1}}(a-1) a^{J}+k a^{J} \geq 0$, where

$$
k=\max \left\{h \mid\left(a^{L_{z, k+1}}(a-1) a^{J}, a^{q}\right)=-h, q \in J\right\}>0 .
$$

If there existed $w \in A^{*}$ and $j \in \mathbb{N}$ such that $\left(Y_{k+1}, w b a^{j}\right)<0$, since $\underline{C}_{k+1} \geq 0$, we should have $\left(\sum_{z \in Q_{k}} z b a^{L_{z, k+1}}(a-1) a^{J}, w b a^{j}\right)>0$. Thus, there should be $z \in Q_{k}$ such that ( $a^{L_{z, k+1}}(a-$ 1) $\left.a^{J}, a^{j}\right)>0$. Moreover, the word $w b a^{j}$ is in $\operatorname{supp}\left(Y_{k+1}\right)$, hence, as noticed before, $j \in J$ and so $\left(a^{L_{z, k+1}}(a-1) a^{J}+a^{J}, a^{j}\right) \geq 2$. This relation is in contradiction with Proposition 6.1. In conclusion, $Y_{k+1} \in \mathbb{N}\langle A\rangle \backslash\{0\}$. Of course, $Y_{k+2}=\sum_{j \in J} P_{k} b a^{M_{j}} b a^{j} \geq 0$. Finally, $\underline{C}_{i}=Y_{i} \geq 0$, for $i \in\{0, \ldots, k\}$ (Lemma 7.2). Hence, $\left(P^{\prime}, S\right)$ is a positive factorization for a $(k+t)$-code $C^{\prime}$, with $t=2$ if $P_{k} \neq 0, t=1$ otherwise. In turn, this implies $Y_{k+1}=\underline{C}_{k+1}^{\prime}$ and, by Lemma 7.1, we may set $\underline{C}_{k+1}^{\prime}=\sum_{z \in Q_{k}^{\prime}} z b a^{J_{z}}$, where $J_{z}$ is a nonempty subset of $J$.

We have already observed that for any $z \in Q_{k}$, we have $\left(\underline{C}_{k+1}^{\prime}, z b a^{\ell}\right)>0$ for $\ell=\min L_{z, k+1}$. Therefore, $Q_{k} \subseteq Q_{k}^{\prime}$ and we may assume $Q_{k}=Q_{k}^{\prime}$ if we define $L_{z, k+1}=\emptyset$ for any word $z \in Q_{k}^{\prime} \backslash Q_{k}$. Therefore, condition (2) holds. Then by

$$
\underline{C}_{k+1}=\sum_{z \in Q_{k}} z b a^{J_{z}}+\sum_{z \in Q_{k}} z b a^{L_{z, k+1}}(a-1) a^{J} \geq 0
$$

Eq. (7.1) easily follows. Finally, by

$$
\underline{C}_{k+2}=\sum_{j \in J} P_{k} b a^{M_{j}} b a^{j}+\sum_{z \in Q_{k}, j \in J} z b a^{L_{z, k+1}}(a-1) a^{M_{j}} b a^{j}+\sum_{z \in Q_{k}} z b a^{L_{z, k+1}} b a^{J} \geq 0
$$

Eq. (7.2) easily follows.

Corollary 7.1 Let $(I, J)$ be a Krasner factorization of $\mathbb{Z}_{n}$, let $P, S$ be polynomials in $\mathbb{N}\langle A\rangle$, with $S=a^{J}+\sum_{j \in J} a^{M_{j}} b a^{j}$. Set $P=P_{0}+\ldots+P_{k+1}$, with $k \geq 0$. If $(P, S)$ is a positive factorization for a $(k+3)$-code $C$, then $\left(P_{0}+\ldots+P_{r}, S\right)$ is a positive factorization for a $(r+2)$-code for any $r \in\{0, \ldots, k\}$ such that $P_{r} \neq 0$.

Proof :
The conclusion may be easily obtained by using induction and Theorem 7.1,

## 8 Construction of 4-codes

In this section we focus on positive factorizations $(P, S)$ for 4-codes $C$ satisfying item (3) in Lemma 5.2 and such that $I^{\prime}=I$, i.e.,

$$
\begin{equation*}
P=a^{I}+\sum_{i \in I} a^{i} b a^{L_{i}}+\sum_{i \in I, \ell \in L_{i}} a^{i} b a^{\ell} b a^{L_{i, \ell}}, \quad S=a^{J}+\sum_{j \in J^{\prime}} a^{M_{j}} b a^{j}, \tag{8.1}
\end{equation*}
$$

where $(I, J)$ is a Krasner factorization of $\mathbb{Z}_{n}$ and $J^{\prime}, L_{i}, M_{j}, L_{i, \ell}$ are finite subsets of $\mathbb{N}$, for any $i, j, \ell$. We give a characterization of these pairs. Their first property is stated by Lemma 8.1, This lemma, which is true also when $I^{\prime} \neq I$, shows that the following relations hold:

$$
\begin{gather*}
\forall i \in I, \ell \in L_{i}, j \in J^{\prime} \cap J \quad a^{L_{i, \ell}}(a-1) a^{M_{j}}+a^{L_{i, \ell}}+a^{M_{j}} \geq 0,  \tag{8.2}\\
\forall i \in I, \ell \in L_{i}, j \in J^{\prime} \backslash J \quad a^{L_{i, \ell}}(a-1) a^{M_{j}}+a^{M_{j}} \geq 0 . \tag{8.3}
\end{gather*}
$$

Lemma 8.1 Let $I, J^{\prime}, J, L_{i}, M_{j}, L_{i, \ell}$ be finite subsets of $\mathbb{N}$, for any $i, j, \ell$. We have

$$
\begin{aligned}
X_{3}= & \sum_{i \in I, \ell \in L_{i}} a^{i} b a^{\ell} b a^{L_{i, \ell}} b a^{J}+\sum_{i \in I, j \in J^{\prime}} a^{i} b a^{L_{i}} b a^{M_{j}} b a^{j} \\
& +\sum_{i \in I, \ell \in L_{i}, j \in J^{\prime}} a^{i} b a^{\ell} b a^{L_{i, \ell}}(a-1) a^{M_{j}} b a^{j} \geq 0
\end{aligned}
$$

if and only if Eqs. (8.2), (8.3) hold.
Proof :
Let us write the polynomial $X_{3}$ in a different way:

$$
\begin{gathered}
X_{3}=\sum_{i \in I, \ell \in L_{i}, j \in J \backslash J^{\prime}} a^{i} b a^{\ell} b a^{L_{i, \ell}} b a^{j}+\sum_{i \in I, \ell \in L_{i}, j \in J^{\prime} \cap J} a^{i} b a^{\ell} b\left(a^{L_{i, \ell}}+a^{M_{j}}+a^{L_{i, \ell}}(a-1) a^{M_{j}}\right) b a^{j} \\
+\sum_{i \in I, \ell \in L_{i}, j \in J^{\prime} \backslash J} a^{i} b a^{\ell} b\left(a^{L_{i, \ell}}(a-1) a^{M_{j}}+a^{M_{j}}\right) b a^{j} .
\end{gathered}
$$

Then $X_{3}$ is a polynomial in $\mathbb{N}\langle A\rangle$ if and only if the second and the third sum on the right side of the above equation are also polynomials in $\mathbb{N}\langle A\rangle$. Hence Eqs. (8.2), (8.3) easily follow.

Let $(P, S)$ be a factorization for $C$ satisfying Eq. (8.1). Then Lemma 6.1 applies to the factorization ( $S^{\sim}, P^{\sim}$ ) of $C^{\sim}$ and Eqs. (5.1)-(5.3) become:

$$
\begin{gather*}
\forall i \in I \quad a^{R_{i}}=a^{L_{i}}(a-1) a^{J}+a^{J} \geq 0,  \tag{8.4}\\
\left\{j \in J^{\prime} \mid M_{j} \neq \emptyset\right\} \subseteq \cup_{i \in I} R_{i},  \tag{8.5}\\
\forall j \in J^{\prime} \quad a^{T_{j}}=a^{M_{j}}(a-1) a^{I}+a^{I} \geq a^{M_{j}}(a-1) a^{I}+a^{I_{j}} \geq 0, \tag{8.6}
\end{gather*}
$$

where $I_{j}=\left\{i \in I \mid j \in R_{i}\right\}$, for $j \in J^{\prime}$. Proposition 8.1 shows that two further relations are required in order to characterize this family of positive factorizations for a finite maximal code.

Proposition 8.1 Let $P=a^{I}+\sum_{i \in I} a^{i} b a^{L_{i}}+\sum_{i \in I, \ell \in L_{i}} a^{i} b a^{\ell} b a^{L_{i, \ell}}, S=a^{J}+\sum_{j \in J^{\prime}} a^{M_{j}} b a^{j}$, where $(I, J)$ is a Krasner factorization of $\mathbb{Z}_{n}, J^{\prime}, L_{i}, M_{j}, L_{i, \ell}$ are finite subsets of $\mathbb{N}$, for any $i, j, \ell$. Then $(P, S)$ is a positive factorization for a finite maximal code $C$ if and only if Eqs. (8.2)-(8.6) hold and, moreover,

$$
\begin{align*}
& \forall i \in I, \ell \in L_{i} \quad a^{R_{i, \ell}}=a^{L_{i, \ell}}(a-1) a^{J}+a^{J} \geq 0,  \tag{8.7}\\
& \forall i \in I, j \in J^{\prime}, \ell \in L_{i} \quad\left(a^{L_{i}}(a-1) a^{M_{j}}+a^{M_{j}}, a^{\ell}\right)<0 \quad \Rightarrow \quad j \in R_{i, \ell} . \tag{8.8}
\end{align*}
$$

## Proof :

Assume that $P, S$ are as in the statement and Eqs. (8.2)-(8.8) hold. Since $L_{i}, M_{j}$ satisfy Eqs. (8.4)-(8.6), for any $i \in I, j \in J^{\prime}$, the polynomial $a^{L_{i}}(a-1) a^{M_{j}}+a^{L_{i}}+a^{M_{j}}$ has coefficients 0,1 (Proposition 5.2). Therefore, if $\left(a^{L_{i}}(a-1) a^{M_{j}}+a^{M_{j}}, a^{\ell}\right)<0$ then $\ell \in L_{i}$ and $\left(a^{L_{i}}(a-\right.$ 1) $\left.a^{M_{j}}+a^{M_{j}}+a^{\ell}, a^{\ell}\right) \geq 0$.

Let us show that $(P, S)$ is a positive factorization for a code $C$, i.e., $P(A-1) S+1=\underline{C} \geq 0$. We have to prove that $\underline{C}_{h} \geq 0$ for $h \in\{0,1,2,3,4\}$. Of course, $\underline{C}_{0}=a^{I}(a-1) a^{J}+1=a^{n} \geq 0$ and $\underline{C}_{4}=\sum_{i \in I, \ell \in L_{i}, j \in J^{\prime}} a^{i} b a^{\ell} b a^{L_{i, \ell}} b a^{M_{j}} b a^{j} \geq 0$. Furthermore, since Eqs. (8.4)-(8.6) hold, we have $\underline{C}_{1} \geq 0$ (Lemma 6.1 applied to $\underline{C}_{1}^{\sim}$ ) and since Eqs. (8.2), (8.3) hold, we have $\underline{C}_{3} \geq 0$ (Lemma 8.1). Finally, in view of Eqs. (8.7), (8.8) we have

$$
\begin{aligned}
\underline{C}_{2} & =\sum_{i \in I} a^{i} b a^{L_{i}} b a^{J}+\sum_{j \in J^{\prime}} a^{I} b a^{M_{j}} b a^{j}+\sum_{i \in I, j \in J^{\prime}} a^{i} b a^{L_{i}}(a-1) a^{M_{j}} b a^{j} \\
& +\sum_{i \in I, \ell \in L_{i}} a^{i} b a^{\ell} b a^{L_{i, \ell}}(a-1) a^{J} \\
& =\sum_{i \in I, \ell \in L_{i}} a^{i} b a^{\ell} b a^{R_{i, \ell}}+\sum_{i \in I, j \in J^{\prime}} a^{i} b\left(a^{L_{i}}(a-1) a^{M_{j}}+a^{M_{j}}\right) b a^{j} \geq 0 .
\end{aligned}
$$

Conversely, let $P=a^{I}+\sum_{i \in I} a^{i} b a^{L_{i}}+\sum_{i \in I, \ell \in L_{i}} a^{i} b a^{\ell} b a^{L_{i, \ell}}, S=a^{J}+\sum_{j \in J^{\prime}} a^{M_{j}} b a^{j}$, where $(I, J)$ is a Krasner factorization of $\mathbb{Z}_{n}, J^{\prime}, L_{i}, M_{j}, L_{i, \ell}$ are finite subsets of $\mathbb{N}$, for any $i, j, \ell$. Assume $P(A-1) S+1=\underline{C} \geq 0$, thus $\underline{C}_{h} \geq 0$ for $h \in\{0,1,2,3,4\}$. Since $\underline{C}_{1} \geq 0$, by Lemma 6.1 applied to $\underline{C}_{1}^{\sim}$, Eqs. (8.4)-(8.6) hold. Hence, by Proposition 5.2 for any $i \in I, j \in J^{\prime}$, $a^{L_{i}}(a-1) a^{M_{j}}+a^{L_{i}}+a^{M_{j}}$ is a polynomial with coefficients 0,1 . In addition, since $\underline{C}_{3} \geq 0$, Eqs. (8.2), (8.3) hold (Lemma 8.1). Furthermore, we have

$$
\begin{gathered}
\underline{C}_{2}=\sum_{i \in I} a^{i} b a^{L_{i}} b a^{J}+\sum_{j \in J^{\prime}} a^{I} b a^{M_{j}} b a^{j}+\sum_{i \in I, j \in J^{\prime}} a^{i} b a^{L_{i}}(a-1) a^{M_{j}} b a^{j} \\
+\sum_{i \in I, \ell \in L_{i}} a^{i} b a^{\ell} b a^{L_{i, \ell}}(a-1) a^{J} \geq 0 .
\end{gathered}
$$

Assume that there are $i \in I, \ell \in L_{i}, t \in \mathbb{N}$ such that $\left(a^{L_{i, \ell}}(a-1) a^{J}+a^{J}, a^{t}\right)<0$. Thus,

$$
\left(\sum_{i \in I} a^{i} b a^{L_{i}} b a^{J}+\sum_{i \in I, \ell \in L_{i}} a^{i} b a^{\ell} b a^{L_{i, \ell}}(a-1) a^{J}, a^{i} b a^{\ell} b a^{t}\right)<0 .
$$

Since $\underline{C}_{2} \geq 0$, we have

$$
\left(\sum_{j \in J^{\prime}} a^{I} b a^{M_{j}} b a^{j}+\sum_{i \in I, j \in J^{\prime}} a^{i} b a^{L_{i}}(a-1) a^{M_{j}} b a^{j}, a^{i} b a^{\ell} b a^{t}\right)>0
$$

which yields $\left(a^{M_{t}}+a^{L_{i}}(a-1) a^{M_{t}}, a^{\ell}\right)>0$, i.e., $\left(a^{M_{t}}+a^{L_{i}}(a-1) a^{M_{t}}+a^{L_{i}}, a^{\ell}\right)>1$. The last relation is impossible since $a^{L_{i}}(a-1) a^{M_{t}}+a^{L_{i}}+a^{M_{t}}$ is a polynomial with coefficients 0,1 . Therefore, Eq. (8.7) holds. Finally, we have

$$
\underline{C}_{2}=\sum_{i \in I, \ell \in L_{i}} a^{i} b a^{\ell} b a^{R_{i, \ell}}+\sum_{i \in I, j \in J^{\prime}} a^{i} b\left(a^{L_{i}}(a-1) a^{M_{j}}+a^{M_{j}}\right) b a^{j} \geq 0 .
$$

By the above equation, for all $i \in I, j \in J^{\prime}, \ell \in L_{i}$ such that $\left(a^{L_{i}}(a-1) a^{M_{j}}+a^{M_{j}}, a^{\ell}\right)<0$, we have $j \in R_{i, \ell}$, i.e., Eq. (8.8) holds.

Looking at Proposition 8.1, we see that two cases may occur: either for any $i \in I$ and $j \in J^{\prime}$ we have $a^{L_{i}}(a-1) a^{M_{j}}+a^{M_{j}} \geq 0$ or not. Examples 8.1 and 8.2 from [13] illustrate the first and the second case respectively and point out relations between 4 -codes, 3 -codes and Hajós factorizations.

Example 8.1 Let us consider the polynomials:

$$
\begin{aligned}
P= & a^{\{0,2,4,12,14,16\}}+a^{\{0,2,4,12,14,16\}} b a^{\{1,3,5,7,9,11,13,15,17,19\}}+ \\
& a^{\{0,2,4,12,14,16\}} b a^{\{1,3,5,7,9,11,13,15,17,19\}} b a^{\{1,3,5,7,9,11,13,15,17,19\}}, \\
S= & a^{\{0,1,6,7\}}+a^{\{2,3\}} b a^{21} .
\end{aligned}
$$

An easy computation shows that $(P, S)$ is a positive factorization for a 4-code $\underline{C}=P(A-$ 1) $S+1$. The pair $(I, J)=(\{0,2,4,12,14,16\},\{0,1,6,7\})$ is a Krasner factorization of $\mathbb{Z}_{24}$. Moreover, $L_{i}=L_{i, \ell}=L=\{1,3,5,7,9,11,13,15,17,19\}, M_{j}=M=\{2,3\}, J^{\prime}=\{21\}$. There is a strong Hajós factorization $(T, R)$ of $\mathbb{Z}_{24}$ associated with $(P, S)$, namely $(T, R)=$ $(\{0,4,8,12,16,20\},\{0,27,6,21\})$, with the corresponding pair $\left(M, L^{\prime}\right)=(\{2,3\},\{1,3,5,7,9,11$, $13,15,17,19\})$. It is easy to see that the pair $\left(P^{\prime}, S\right)$, with $P^{\prime}=P_{0}+P_{1}$, defines a 3-code.

Example 8.2 Let us consider the polynomials:

$$
\begin{aligned}
P= & a^{\{0,2,4,12,14,16\}}+a^{\{0,2,4,12,14,16\}} b a^{\{1,9,11,13\}}+ \\
& a^{\{0,2,4,12,14,16\}} b a^{\{1,9,11,13\}} b a^{\{1,3,5,7,9,11,13,15,17,19\}}, \\
S= & a^{\{0,1,6,7\}}+a^{\{2,3\}} b a^{21} .
\end{aligned}
$$

An easy computation shows that $(P, S)$ is a positive factorization for a 4-code $\underline{C}=P(A-$ 1) $S+1$. We have $L_{i}=L=\{1,9,11,13\}, M_{j}=M=\{2,3\}, J^{\prime}=\{21\}, L_{i, \ell}=L^{\prime}=$ $\{1,3,5,7,9,11,13,15,17,19\}$. There are two Hajós factorizations of $\mathbb{Z}_{24}$ associated with $(P, S)$ : the strong Hajós factorization $\left(T, R^{\prime}\right)=(\{0,4,8,12,16,20\},\{0,27,6,21\})$ with the corresponding pair $\left(M, L^{\prime}\right)=(\{2,3\},\{1,3,5,7,9,11,13,15,17,19\})$ and the non-strong Hajós factorization $(T, R)=(\{0,4,8,12,16,20\},\{0,3,6,21\})$ with the corresponding pair $(M, L)=(\{2,3\},\{1,9,11,13\})$. Notice that $R \cap R^{\prime}=\{21\}$. The pair $\left(P^{\prime}, S\right)$, with $P^{\prime}=P_{0}+P_{1}$, does not define a 3 -code.

## 9 Conclusions

In this paper we proved that if $(P, S)$ is a factorization for a finite maximal code $C$, with $C \cap a^{*}=\left\{a^{p}\right\}$ for a prime number $p, S=S_{0}+S_{1}$ and if $a^{j} \in \operatorname{supp}(S)$ for any $a^{i} b a^{j} \in \operatorname{supp}(S)$, then $(P, S)$ is positive. We also proved that $(p, 4)$-codes satisfy the factorization conjecture and each factorization $(P, S)$ for a $(p, 4)$-code is positive.

A natural question is to characterize those positively factorizing codes having only positive factorizations. One may conjecture that this is the case for finite maximal codes containing a power of $a$ with a prime exponent. This is a first research direction.

A related problem is to find conditions under which a factorization is positive. In this framework, Hansel and Krob asked the following question, reported in [19]: let $P, Q \in \mathbb{Z}[a]$ be such that $P Q=1+a+\ldots+a^{n}$, with $n \in \mathbb{N}$. If $P$ has coefficients 0,1 then does $Q$ also have coefficients 0,1 ?

As a second direction of research for both the above mentioned problems, one can investigate finite maximal codes containing a power of $a$ with a prime exponent and having a factorization $(P, S)$, where $S=S_{0}+S_{1}+\ldots+S_{k}$ is a polynomial such that if $a^{i} b w \in \operatorname{supp}\left(S_{j}\right)$, then
$w \in \operatorname{supp}\left(S_{j-1}\right)$, for $j \in\{1, \ldots, k\}$. One may ask whether it is still true that $P, S$ are necessarily in $\mathbb{N}\langle A\rangle$. We have already proved that this statement holds if $S$ is a polynomial in $\mathbb{N}\langle A\rangle$ such that $a^{j} \in \operatorname{supp}(S)$, for any $w b a^{j}$ in $\operatorname{supp}(S)$ (Theorem 4.1).

Concerning the structure of positively factorizing codes, our construction of $(p, 4)$-codes is not complete. More generally, a method for constructing all positively factorizing codes is still lacking (see [13, 14, 15] for conjectures and related problems). In this regard, it could be interesting to look for a generalization of the constructions given in Sections 7, 8,

Another related question is to find conditions under which a set of words $C_{1}$ satisfying $\underline{C}_{1}=a^{I}(a-1) S_{1}+P_{1}(a-1) a^{J}+a^{I} b a^{J}$, where $(I, J)$ is a Krasner pair and $P_{1}, S_{1}$ are polynomials with coefficients 0,1 , could be embedded in a factorizing code. Some sufficient conditions have been stated in [10, 12, 14, 15].

Finally, one can investigate whether all the results concerning $m$-codes can be generalized to alphabets having cardinality greater than two.

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[^1]:    ${ }^{1}$ Note that in this paper we use the term "positive factorization" with a slightly different meaning with respect to the definition of the same term in $[2]$.

