# SHARP ESTIMATES FOR THE NUMBER OF DEGREES OF FREEDOM FOR THE DAMPED-DRIVEN 2D NAVIER–STOKES EQUATIONS

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**Abstract.** We derive upper bounds for the number of asymptotic degrees (determining modes and nodes) of freedom for the two-dimensional Navier–Stokes system and Navier–Stokes system with damping. In the first case we obtain the previously known estimates in an explicit form, which are larger than the fractal dimension of the global attractor. However, for the Navier–Stokes system with damping our estimates for the number of the determining modes and nodes are comparable to the sharp estimates for the fractal dimension of the global attractor. Our investigation of the damped-driven 2D Navier–Stokes system is inspired by the Stommel–Charney barotropic model of ocean circulation where the damping represents the Rayleigh friction. We remark that our results equally apply to the Stommel–Charney model.

Key words: Determining modes and nodes, fractal dimension, Stommel-Charney model, Navier–Stokes equations, barotropic ocean circulation model.

AMS subject classification: 35Q30, 76D05, 76U05, 76E20, 86A05, 86A10.

# 1. INTRODUCTION

In this paper we derive estimates for the number of determining modes, nodes and other determining projections for the two-dimensional Navier–Stokes system (2.1) and for the two-dimensional damped-driven Navier–Stokes system (3.1). The latter system is inspired by the viscous Stommel–Charney barotropic ocean circulation model [4],[33],[35]:

$$\partial_t u + \sum_{i=1}^2 u^i \partial_i u + \mathbf{k} l \times u = -\mu u + \nu \Delta u - \nabla p + f,$$
  
div  $u = 0,$  (1.1)

where the damping  $\mu u$  represents the so-called Rayleigh friction term in the ocean circulation model. In recent years there has been some analytical study of the Stommel–Charney model from the dynamical systems point of view (see, for instance, [3], [5], [18], [19], [22], [26], [34], [39]). For the sake of clarity in our mathematical presentation and in order to make a straight forward comparison to the physical literature about the 2D turbulence we focus here on the Navier–Stokes system (2.1) and the damped-driven Navier–Stokes system (3.1). It worth stressing, however, that our results concerning the system (3.1), especially about determining modes, equally applies to the barotropic model (1.1), which we report in a forthcoming paper.

Our paper is organized as follows. In section 2 we obtain previously known estimates reported in [8], [9] and [28] for the number of determining modes, nodes and other determining projections for the two-dimensional space-periodic Navier–Stokes system which are linear with respect to the Grashof number. We use, however, the scalar vorticity formulation, which makes it possible to give all the estimates and constants in an explicit

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form. Furthermore, the dependence on the aspect ratio of the periodic domain is explicitly singled out. For the Dirichlet boundary conditions we obtain an explicit estimate for the number of determining modes which is quadratic with respect to the Grashof number (see, for example, [20], [29]). It is worth mentioning that these best known estimates for the number of explicit determining modes and nodes of the 2D Navier–Stokes equations without damping are still much larger than the dimension of the global attractor.

In section 3 we consider the Navier-Stokes system with damping, subject to periodic boundary conditions and the stress-free boundary conditions, and obtain estimates for the number of the determining modes and nodes that are of the same order as of the sharp estimates for the fractal dimension of the global attractor [26]. These remarkable estimates are extensive, that is, depend linearly on the area of the spatial domain, which is consistent with the physical intuition. We remark, again, that such an observation is not known to exist in the case of the 2D Navier–Stokes equations without damping.

Finally, in the Appendix in section 4 we prove some auxiliary inequalities, namely, we derive sharp constant in the Agmon inequality on the two-dimensional torus and prove a variant of the embedding theorem reported in [28].

Since we are interested in obtaining explicit bounds about the constants involved in our asymptotic estimates for the numbers of degrees of freedom, we focus in this paper on the notions of determining modes and nodes. However, it is worth stressing that our asymptotic estimates, in terms of the physical parameters, are valid for other determining functionals and projections, as it has been demonstrated in [8], [9], (see also [7]), with constants that may vary depending on the underlying chosen determining functionals.

# 2. Determining modes and nodes for two-dimensional Navier–Stokes Equations

**Dirichlet boundary conditions.** We consider in this section the two-dimensional Navier–Stokes system

$$\partial_t u + \sum_{i=1}^2 u^i \partial_i u = \nu \Delta u - \nabla p + f,$$
  
div  $u = 0, \quad u(0) = u_0,$  (2.1)

where u is the velocity vector field satisfying Dirichlet boundary conditions  $u|_{\partial\Omega} = 0$ , p is the pressure, and  $\nu > 0$  is the kinematic viscosity. The right-hand side f = f(x, t) is given and the domain  $\Omega$  is an arbitrary open connected set in  $\mathbb{R}^2$  with finite measure  $|\Omega| < \infty$ .

We use the standard notation and facts from the theory of Navier–Stokes equations (see, for instance, [10], [30], [37], [38]) and denote by P the Helmholtz–Leray orthogonal projection in  $L_2(\Omega)^2$  onto the Hilbert space H which is the closure in  $L_2(\Omega)^2$  of the set of smooth solenoidal vector functions with compact supports in  $\Omega$ . Applying P to the first equation in (2.1), we obtain

$$\partial_t u + B(u, u) + \nu A u = f, \qquad u(0) = u_0,$$
(2.2)

where  $A = -P\Delta$  is the Stokes operator,  $B(u, v) = P\left(\sum_{i=1}^{2} u^{i}\partial_{i}v\right)$  is the nonlinear term, and  $f = Pf \in H$ .

Next, we denote by  $\{\lambda_j\}_{j=1}^{\infty}$ ,  $0 < \lambda_1 \leq \lambda_2 \leq \ldots$  and  $\{w_j\}_{j=1}^{\infty}$  the eigenvalues and the corresponding eigenfunctions of the Stokes operator A:  $A w_j = \lambda_j w_j$ . The asymptotic behavior  $\lambda_k \sim \frac{4\pi k}{|\Omega|}$  as  $k \to \infty$  was established in [32], while in this work we use the

following explicit non-asymptotic lower bounds for the eigenvalues  $\{\lambda_j\}_{j=1}^{\infty}$  (see [23]):

$$\sum_{j=1}^{m} \lambda_j \ge \frac{\pi m^2}{|\Omega|} \quad \text{and, consequently,} \quad \lambda_m \ge \frac{\pi m}{|\Omega|}, \ m \ge 1; \quad \lambda_1 \ge \frac{2\pi}{|\Omega|}.$$
(2.3)

The Hilbert space  $V = D(A^{1/2})$  is the space  $H_0^1(\Omega)^2 \cap H$  with norm

$$||u||_{D(A^{1/2})} = ||\nabla u|| = ||\operatorname{rot} u||,$$

where  $\|\cdot\| = \|\cdot\|_{L_2(\Omega)}$ . The nonlinear operator B(v, v) satisfies the well-known inequalities (see, for instance, [10], [30], [37], [38])

$$|(B(v,v),u)| = |(B(v,u),v)| \le c_1 ||v|| ||\nabla v|| ||\nabla u||,$$
  

$$|(B(u,v),w)| \le c_2 ||u||^{1/2} ||\nabla u||^{1/2} ||\nabla v|| ||w||^{1/2} ||\nabla w||^{1/2},$$
(2.4)

where it was shown in [6] that

$$c_1 =: c_b \le \left(\frac{8}{27\pi}\right)^{1/2} \qquad c_2 \le \sqrt{2}c_b \le \left(\frac{16}{27\pi}\right)^{1/2}.$$
 (2.5)

Let u and v be the solutions of the Navier–Stokes equations

$$\partial_t u + B(u, u) + \nu A u = f, \qquad u(0) = u_0, \partial_t v + B(v, v) + \nu A v = g, \qquad v(0) = v_0,$$
(2.6)

where  $f, g \in L_{\infty}(0, \infty; H)$ .

We denote by  $P_m$  the  $L_2$ -orthogonal projection onto the space  $\text{Span}\{w_1, \ldots, w_m\}$ , and we set  $Q_m = I - P_m$ .

**Definition 2.1.** We call a set of modes  $\{w_j\}_{j=1}^m$  determining (see [12], [15]) if

$$\lim_{t \to \infty} \|u(t) - v(t)\| = 0, \tag{2.7}$$

as long as

$$\lim_{t \to \infty} \|f(t) - g(t)\| = 0 \text{ and } \lim_{t \to \infty} \|P_m(u(t) - v(t))\| = 0$$

Accordingly, a set of points  $\{x^i\}_{i=1}^N \subset \Omega$  is called a set of determining nodes (see [12], [16]) if (2.7) holds as long as

$$\lim_{t \to \infty} \|f(t) - g(t)\| = 0 \text{ and } \lim_{t \to \infty} \eta(u(t) - v(t)) = 0$$

where  $\eta(w) = \max_{j=1,...,N} |w(x^{j})|.$ 

We further suppose that

$$\limsup_{t \to \infty} \|f(t)\| =: \mathbf{f} < \infty \tag{2.8}$$

Subtracting in (2.6) the second equation from the first and setting w(t) = u(t) - v(t)and h(t) = f(t) - g(t) we obtain

$$\partial_t w + \nu A w + B(u, w) + B(w, u) - B(w, w) = h(t)$$

We write w = p + q, where  $p = P_m w$ ,  $q = Q_m w$  and take the scalar product with q:

$$\frac{1}{2}\partial_t \|q\|^2 + \nu \|\nabla q\|^2 + b(q, u, q) = (h, q) - b(u, p, q) - b(p, u, q) + b(p, p, q) + b(q, p, q),$$

where b(u, v, w) = (B(u, v), w).

A variant of the Gronwall lemma [12], [13], [28] is essential in the estimates below.

**Lemma 2.1.** Suppose that  $\alpha(t)$  and  $\beta(t)$  are locally integrable functions on  $(0, \infty)$  satisfying for some T > 0 the following conditions:

$$\liminf_{t \to \infty} \frac{1}{T} \int_t^{t+T} \alpha(\tau) d\tau = \gamma, \ \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \alpha^-(\tau) d\tau = \Gamma, \ \lim_{t \to \infty} \frac{1}{T} \int_t^{t+T} \beta^+(\tau) d\tau = 0,$$

where  $\gamma > 0$ ,  $\Gamma < \infty$  and  $\alpha^- = \max\{-\alpha, 0\}$ ,  $\beta^+ = \max\{\beta, 0\}$ . If  $\xi(t) \ge 0$ , an absolutely continuous function, satisfies

$$\xi' + \alpha \xi \le \beta \quad on \quad (0,\infty),$$

then  $\xi(t) \to 0$  as  $t \to \infty$ .

All the terms on the right-hand side containing h or p can be absorbed in the function  $\beta(t)$  in Lemma 2.1. For example, using the second inequality in (2.4) we obtain

$$\begin{aligned} |b(u, p, q)| &\leq c_2 ||u||^{1/2} ||\nabla u||^{1/2} ||\nabla p|| ||q||^{1/2} ||\nabla q||^{1/2} \\ &\leq c_2 (\lambda_m / \lambda_1)^{1/2} ||p|| ||\nabla u|| ||\nabla q|| \leq (c_2 / 2) (\lambda_m / \lambda_1)^{1/2} ||p|| (||\nabla u||^2 + ||\nabla q||^2). \end{aligned}$$

Our claim follows since  $||p(t)|| \to 0$  and in view of (2.9)

$$\frac{1}{T} \int_{t}^{t+T} \|p(\tau)\| (\|\nabla u(\tau)\|^{2} + \|\nabla q(\tau)\|^{2}) d\tau$$
  
$$\leq \max_{\tau \in [t,t+T]} \|p(\tau)\| \frac{1}{T} \int_{t}^{t+T} (\|\nabla u(\tau)\|^{2} + \|\nabla q(\tau)\|^{2}) d\tau \to 0 \quad \text{as} \quad t \to \infty$$

The remaining terms can be treated in exactly the same way. Therefore we obtain

$$\partial_t ||q||^2 + 2\nu ||\nabla q||^2 + 2b(q, u, q) \le \beta(t),$$

where  $\beta(t) \to 0$  as  $t \to \infty$ .

Using the first inequality in (2.4) we have

$$2|b(q, u, q)| \le \nu \|\nabla q\|^2 + c_b^2 \nu^{-1} \|q\|^2 \|\nabla u\|^2$$

and then using the Poincaré inequality  $\lambda_{m+1} ||q||^2 \leq ||\nabla q||^2$  we obtain

$$\partial_t \|q\|^2 + \alpha(t) \|q\|^2 \le \beta(t), \quad \text{where} \quad \alpha(t) = \nu \lambda_{m+1} - \nu^{-1} c_b^2 \|\nabla u(t)\|^2$$

Since by the well-known estimates for the Navier–Stokes system (see, for instance, [10],[38])

$$\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \|\nabla u(\tau)\|^2 d\tau \le \frac{\mathbf{f}^2}{T\nu^3 \lambda_1^2} + \frac{\mathbf{f}^2}{\nu^2 \lambda_1},$$
(2.9)

it follows that  $\alpha$  satisfies the conditions of Lemma 2.1 provided that T is sufficiently large and

$$\lambda_{m+1} > \frac{c_b^2 \mathbf{f}^2}{\nu^4 \lambda_1}. \tag{2.10}$$

In view of (2.3) and (2.4), (2.5) we have proved the following theorem.

**Theorem 2.1.** The first m eigenfunctions of the Stokes operator are determining for the two-dimensional Navier–Stokes system with Dirichlet boundary conditions if

$$m+1 > \frac{4}{27\pi^3} G^2$$
, where  $G = \frac{\mathbf{f}|\Omega|}{\nu^2}$ . (2.11)

*Remark* 2.1. The above theorem without explicit value of the constant has been mentioned as a remark in [20] and was also proved in [12] and [29].

**Periodic boundary conditions. Determining modes.** We now consider the Navier– Stokes system (2.1) with space periodic boundary conditions  $x \in \Omega = [0, L_1] \times [0, L_2]$ . We set  $L_2 = L$  and  $L_1 = L/\gamma$ . Without loss of generality we assume that  $\gamma \leq 1$ . As before,  $|\Omega|$  denotes the measure of the periodic domain  $\Omega$ :  $|\Omega| = L_1 L_2 = L^2/\gamma$ .

We further assume that u, p and f have mean value zero over the torus. Applying the rot (curl) operator to the first equation in (2.1) we obtain the well-known vorticity equation

$$\partial_t \varphi + J(\Delta^{-1} \varphi, \varphi) - \nu \Delta \varphi = \operatorname{rot} f, \qquad (2.12)$$

where rot  $u = \varphi$ ,  $u = \nabla^{\perp} \Delta^{-1} \varphi$ ,  $J(a, b) = \partial_1 a \partial_2 b - \partial_2 a \partial_1 b = \nabla^{\perp} a \cdot \nabla b$ ,  $\nabla^{\perp} \psi = \mathbf{k} \times \nabla \psi = (-\partial_2 \psi, \partial_1 \psi)$ , and  $\mathbf{k}$  is the vertical unit vector.

We now recall that the spectrum  $\{\lambda_j\}_{j=1}^{\infty}$  of the Stokes operator with periodic boundary conditions coincides with that of the negative scalar Laplacian

$$-\Delta \,\varphi_j = \lambda_j \varphi_j$$

and the corresponding eigenfunctions are as follows:

$$A w_j = \lambda_j w_j, \quad w_j = \lambda_j^{-1/2} \nabla^{\perp} \varphi_j = \lambda_j^{-1/2} (-\partial_2 \varphi_j, \partial_1 \varphi_j).$$

Therefore the modes  $\{w_1, \ldots, w_m\}$  are determining for the Navier–Stokes system (2.1) with periodic boundary conditions if the modes  $\{\varphi_1, \ldots, \varphi_m\}$  are determining for the equation (2.12).

Similarly to (2.6) we write

$$\partial_t \varphi + J(\Delta^{-1} \varphi, \varphi) - \nu \Delta \varphi = \operatorname{rot} f(t), 
\partial_t \psi + J(\Delta^{-1} \psi, \psi) - \nu \Delta \psi = \operatorname{rot} g(t).$$
(2.13)

Setting  $\omega = \varphi - \psi$  and  $H(t) = \operatorname{rot} f(t) - \operatorname{rot} g(t)$  we obtain for  $\omega$  the equation

$$\partial_t \omega - \nu \Delta \omega + J(\Delta^{-1} \varphi, \omega) + J(\Delta^{-1} \omega, \varphi) - J(\Delta^{-1} \omega, \omega) = H.$$
(2.14)

As before, we write  $\omega = p + q$ , where  $p = P_m \omega$  and  $q_m = Q_m \omega$  and where  $P_m$  is the orthogonal projection  $P_m : L_2(\Omega) \to \text{Span}(\varphi_1, \ldots, \varphi_m)$ . Taking the scalar product with q we obtain

$$\begin{aligned} &\frac{1}{2}\partial_t \|q\|^2 + \nu \|\nabla q\|^2 + (J(\Delta^{-1}q,\varphi),q) \\ &= (H,q) - (J(\Delta^{-1}\varphi,p),q) - (J(\Delta^{-1}p,\varphi),q) + (J(\Delta^{-1}p,p),q) + (J(\Delta^{-1}q,p),q) =:\beta(t). \end{aligned}$$

As before, all the terms on the right-hand side containing p can be absorbed in  $\beta(t)$  in Lemma 2.1. Next we have

$$\begin{aligned} |(J(\Delta^{-1}q,\varphi),q)| &\leq \int |\nabla\Delta^{-1}q| |\nabla\varphi| \, |q| \, dx \leq \|\nabla\Delta^{-1}q\|_{L_4} \|\nabla\varphi\| \|q\|_{L_4} \\ &\leq c_{\rm L}(\gamma)^2 \|\nabla\Delta^{-1}q\|^{1/2} \|q\| \|\nabla q\|^{1/2} \|\nabla\varphi\| \leq \lambda_m^{-1/2} c_{\rm L}(\gamma)^2 \|q\| \|\nabla\varphi\| \|\nabla q\|, \end{aligned}$$

where we used the Ladyzhenskaya inequality (see, for instance, [10], [30], [37])

$$\|\varphi\|_{L_4} \le c_{\mathcal{L}}(\gamma) \|\varphi\|^{1/2} \|\nabla\varphi\|^{1/2}, \quad \|\nabla\varphi\|_{L_4} \le c_{\mathcal{L}}(\gamma) \|\nabla\varphi\|^{1/2} \|\Delta\varphi\|^{1/2}$$

We arrive at the same estimate if we use the integral identity

$$(J(f,g),h) = (J(h,f),g)$$

and the Agmon inequality (see Theorem 4.1)

$$\|\varphi\|_{\infty} \le c_{\mathrm{AT}}(\gamma) \|\varphi\|^{1/2} \|\Delta\varphi\|^{1/2}$$

In fact,

$$\begin{split} |(J(\Delta^{-1}q,\varphi),q)| &= |(J(\varphi,q),\Delta^{-1}q)| \le \|\Delta^{-1}q\|_{\infty} \|\nabla \varphi\| \|\nabla q\| \\ &\le c_{\rm AT}(\gamma) \|\Delta^{-1}q\|^{1/2} \|q\|^{1/2} \|\nabla q\| \|\nabla \varphi\| \le \lambda_m^{-1/2} c_{\rm AT}(\gamma) \|q\| \|\nabla \varphi\| \|\nabla q\|, \end{split}$$

which gives

$$|(J(\Delta^{-1}q,\varphi),q)| \le \lambda_m^{-1/2} c_{\mathrm{J}} ||q|| ||\nabla \varphi|| ||\nabla q||, \quad \text{where} \quad c_{\mathrm{J}} = \min(c_{\mathrm{L}}(\gamma)^2, c_{\mathrm{AT}}(\gamma)).$$

As before we obtain the differential inequality

$$\partial_t \|q\|^2 + \alpha(t) \|q\|^2 \le 2\beta(t), \quad \text{where} \quad \alpha(t) = \nu \lambda_{m+1} - \nu^{-1} \lambda_{m+1}^{-1} c_{\mathrm{J}}^2 \|\nabla \varphi(t)\|^2.$$

It follows from the well-known a priory estimate on the time average of the  $H^2$ -norm of a solution u (see, for instance, [2], [10], [28], [38])

$$\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \|\nabla \varphi(\tau)\|^2 d\tau = \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \|A u(\tau)\|^2 d\tau \le \frac{\mathbf{f}^2}{T\nu^3\lambda_1} + \frac{\mathbf{f}^2}{\nu^2}$$
(2.15)

that  $\alpha$  satisfies conditions of Lemma 2.1 provided that T is sufficiently large and

$$\lambda_{m+1}^2 > \frac{c_{\mathbf{J}}^2 \mathbf{f}^2}{\nu^4}.$$
(2.16)

It was shown in [26], [25] that  $c_{\rm L}(\gamma) \leq (6/(\gamma \pi))^{1/4}$ . In the Appendix in section 4 we will show that  $c_{\rm AT}(\gamma) \leq 1/\sqrt{\gamma \pi}$ . Hence we can take  $c_{\rm J} = 1/\sqrt{\gamma \pi}$ . Furthermore, for  $\gamma = 1$  $\lambda_m \geq (\lambda_1/4)m$ , where  $\lambda_1 = 4\pi^2 L^{-2}$ . We obtain the following theorem.

**Theorem 2.2.** The first *m* eigenfunctions of the Stokes operator are determining for the two-dimensional Navier–Stokes system with periodic boundary conditions if

$$\lambda_{m+1} > \left(\frac{1}{\gamma\pi}\right)^{1/2} \frac{\mathbf{f}}{\nu^2}.$$
(2.17)

For a square torus  $(\gamma = 1)$  this condition is satisfied if

$$m+1 > \frac{1}{\pi^{3/2}}G, \quad where \quad G = \frac{\mathbf{f}L^2}{\nu^2}.$$
 (2.18)

Remark 2.2. The first eigenvalues  $\lambda_1, \lambda_2, \ldots$  of the Laplacian on the periodic domain  $\Omega = [0, L/\gamma] \times [0, L]$  are of order  $\gamma^2$  when  $\gamma \ll 1$ . It was shown in [25] (see Proposition 4.1) that if  $m \ge 2/\gamma$ , then

$$\lambda_m \ge \frac{m\gamma}{8} \cdot \frac{4\pi^2}{L^2} = \frac{\pi^2}{2} \cdot \frac{m}{|\Omega|} \,. \tag{2.19}$$

Therefore condition (2.17) is satisfied if

$$m+1 > \frac{2}{\gamma} + \frac{2}{\pi^2} \left(\frac{1}{\gamma\pi}\right)^{1/2} \frac{\mathbf{f}|\Omega|}{\nu^2}.$$
(2.20)

**Determining nodes.** Suppose that the periodic domain  $\Omega$  is divided into N equal squares with side of length l and for each square there is a point  $x^j$ ,  $j = 1, \ldots, N$  chosen arbitrarily in it. For the two solutions u and v of the Navier-Stokes equations we assume that

$$\eta(w(t)) = \max_{j=1,\dots,N} |w(t,x^j)| \to 0 \quad \text{as} \quad t \to \infty, \quad \text{where} \quad w = u - v.$$

We take the scalar product of (2.14) and  $\omega$ :

$$\partial_t \|\omega\|^2 + 2\nu \|\nabla \omega\|^2 = \beta(t) - 2(J(\Delta^{-1}\omega, \varphi), \omega), \quad \text{where} \quad \beta(t) = 2(H, \omega).$$

# For the nonlinear term using inequalities (4.14), (4.17) and Young's inequality we have

$$2|(J(\Delta^{-1}\omega,\varphi),\omega)| \leq 2\|\nabla\Delta^{-1}\omega\|_{\infty}\|\nabla\varphi\|\|\omega\| \leq 2c_{\mathrm{AT}}(\gamma)\|\nabla\Delta^{-1}\omega\|^{1/2}\|\nabla\omega\|^{1/2}\|\omega\|\|\nabla\varphi\| \leq 2c_{\mathrm{AT}}(\gamma)(4|\Omega|)^{1/4}\eta^{1/2}(w)\|\nabla\omega\|^{1/2}\|\omega\|\|\nabla\varphi\| + 2c_{\mathrm{AT}}(\gamma)68^{1/4}l\|\nabla\omega\|\|\omega\|\|\nabla\varphi\| \leq \beta_{1}(t) + c_{\mathrm{AT}}(\gamma)^{2}68^{1/2}l^{2}\nu^{-1}\|\omega\|^{2}\|\nabla\varphi\|^{2} + \nu\|\nabla\omega\|^{2}.$$

Hence

$$\partial_t \|\omega\|^2 + \nu \|\nabla \omega\|^2 \le \beta_1(t) + c_{\rm AT}(\gamma)^2 68^{1/2} l^2 \nu^{-1} \|\omega\|^2 \|\nabla \varphi\|^2.$$

Using inequality (4.16) in the form  $\|\nabla \omega\|^2 \ge 68^{-1/2}l^{-2}\|\omega\|^2 - \beta_2(t)$  to bound from below the second term on the left-hand side we obtain

$$\partial_t \|\omega\|^2 + \alpha(t) \|\omega\|^2 \le \beta(t),$$

where

$$\alpha(t) = 68^{-1/2}\nu l^{-2} - c_{\rm AT}(\gamma)^2 68^{1/2}\nu^{-1} l^2 \|\nabla\varphi\|^2.$$

Taking into account (2.15) we see that if the number of squares  $N = |\Omega|/l^2$  is sufficiently large (or, equivalently, the typical distance l between the nodes is sufficiently small), then  $\alpha$  satisfies conditions of Lemma 2.1 and the corresponding N nodes are determining. We obtained the following result.

#### Theorem 2.3. If

$$N > \left(\frac{68}{\gamma\pi}\right)^{1/2} \frac{\mathbf{f}|\Omega|}{\nu^2} = \left(\frac{68}{\gamma^3\pi}\right)^{1/2} \frac{\mathbf{f}L^2}{\nu^2}$$
(2.21)

equal squares tile  $\Omega = [0, L/\gamma] \times [0, L]$ , then N nodes (chosen arbitrarily one in each square) are determining for the space periodic Navier–Stokes system in  $\Omega$ .

*Remark* 2.3. The estimates for determining modes and nodes of Theorems 2.2 and 2.3 were obtained for  $\gamma = 1$  in [28].

#### 3. Determining modes and nodes for damped Navier-Stokes equations

In this section we consider the damped-driven Navier–Stokes system having important applications in geophysical hydrodynamics [11], [33]

$$\partial_t u + \sum_{i=1}^2 u^i \partial_i u = \nu \Delta u - \mu u - \nabla p + f,$$
  
div  $u = 0, \quad u(0) = u_0.$  (3.1)

**Periodic boundary conditions.** We first consider this system on the torus  $x \in \Omega = [0, L/\gamma] \times [0, L]$ , with space periodic boundary conditions. The right-hand side f = f(t) satisfies the condition

$$\limsup_{t \to \infty} \|\operatorname{rot} f(t)\|_{\infty} =: \mathbf{F}_{\infty} < \infty.$$
(3.2)

**Lemma 3.1.** The following bound holds for u(t):

$$\limsup_{t \to \infty} \|\operatorname{rot} u(t)\|_{\infty} \le \frac{\mathbf{F}_{\infty}}{\mu}.$$
(3.3)

*Proof.* We use the vorticity formulation of the system (3.1)

$$\partial_t \varphi + J(\Delta^{-1} \varphi, \varphi) - \nu \Delta \varphi + \mu \varphi = \operatorname{rot} f$$
(3.4)

and take the scalar product with  $\varphi^{2k-1}$ , where  $k \geq 1$  is integer, and use the identity  $(J(\psi,\varphi),\varphi^{2k-1}) = (2k)^{-1} \int J(\psi,\varphi^{2k}) dx = (2k)^{-1} \int \operatorname{div}(\varphi^{2k} \nabla^{\perp} \psi) dx = 0$ . We obtain

$$\|\varphi\|_{L_{2k}}^{2k-1}\partial_t\|\varphi\|_{L_{2k}} + (2k-1)\nu\int |\nabla\varphi|^2\varphi^{2k-2}dx + \mu\|\varphi\|_{L_{2k}}^{2k} = = (\operatorname{rot} f(t), \varphi^{2k-1}) \le \|\operatorname{rot} f(t)\|_{L_{2k}}\|\varphi\|_{L_{2k}}^{2k-1}.$$

Hence, by Gronwall's inequality

$$\|\varphi(t+\tau)\|_{L_{2k}} \le \|\varphi(\tau)\|_{L_{2k}} e^{-\mu t} + \mu^{-1} \sup_{s \in [\tau,\infty)} \|\operatorname{rot} f(s)\|_{L_{2k}} (1 - e^{-\mu t}),$$

and passing to the limit as  $k \to \infty$  we find

$$\|\varphi(t+\tau)\|_{\infty} \le \|\varphi(\tau)\|_{\infty} e^{-\mu t} + \mu^{-1} \sup_{s \in [\tau,\infty)} \|\operatorname{rot} f(s)\|_{\infty} (1 - e^{-\mu t}).$$

Now, we let  $t \to \infty$  to obtain

$$\limsup_{t \to \infty} \|\varphi(t)\|_{\infty} \leq \frac{\mathbf{F}_{\infty}}{\mu}$$

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We consider the systems (3.1) with right-hand sides f and g such that

$$\lim_{t \to \infty} \|\operatorname{rot}(f(t) - g(t))\|_{\infty} = 0.$$

Similarly to (2.13) for  $\omega = \varphi - \psi$  we obtain the equation

$$\partial_t \omega - \nu \Delta \omega + \mu \omega + J(\Delta^{-1} \varphi, \omega) + J(\Delta^{-1} \omega, \varphi) - J(\Delta^{-1} \omega, \omega) = H, \qquad (3.5)$$

where  $H(t) = \operatorname{rot} f(t) - \operatorname{rot} g(t)$ .

**Determining modes.** We take the scalar product of (3.5) with  $q = Q_m \omega$ :

$$\partial_t \|q\|^2 + 2\nu \|\nabla q\|^2 + 2\mu \|q\|^2 \le 2\beta(t) + 2|(J(\Delta^{-1}q,\varphi),q)| \le 2\beta(t) + 2\|\nabla q\| \|\nabla \Delta^{-1}q\| \|\varphi\|_{\infty} \le 2\beta(t) + \nu \|\nabla q\|^2 + \nu^{-1}\lambda_{m+1}^{-1} \|q\|^2 \|\varphi\|_{\infty}^2,$$

where we used Young's and Poincaré inequalities. Dropping the  $\mu$ -term on the left-hand side and again using the Poincaré inequality we obtain

$$\partial_t \|q\|^2 + \|q\|^2 (\nu \lambda_{m+1} - \nu^{-1} \lambda_{m+1}^{-1} \|\varphi(t)\|_{\infty}^2) \le 2\beta(t).$$

By estimate (3.3) and Lemma 2.1 the first m modes are determining provided

$$\lambda_{m+1} \ge \frac{\mathbf{F}_{\infty}}{\mu\nu} \,. \tag{3.6}$$

Using (2.19) we see that this condition is satisfied if

$$m+1 > \max\left\{\frac{2}{\gamma}, \frac{2}{\pi^2} \frac{\mathbf{F}_{\infty}|\Omega|}{\mu\nu}\right\}.$$
 (3.7)

For a square torus  $\lambda_m \geq \lambda_1/4$ ,  $\lambda_1 = 4\pi^2/L^2$ , hence condition (3.6) is satisfied if

$$m+1 \ge \frac{1}{\pi^2} \frac{\mathbf{F}_{\infty} L^2}{\mu \nu} \,. \tag{3.8}$$

**Determining nodes.** Suppose that our periodic domain  $\Omega$  is divided into N equal squares  $Q_j$  with side of length l, j = 1, ..., N and we chose arbitrarily a point  $x^j \in Q_j$  for each j = 1, ..., N. For  $u \in H^2$  ( $\Omega$ ) we get

For  $u \in H^2_{\text{per}}(\Omega)$  we set

$$\eta(u) = \max_{j=1,\dots,N} |u(x^j)|.$$
(3.9)

Suppose that  $\eta(w(t)) \to 0$  as  $t \to \infty$  for w(t) = u(t) - v(t), where u and w are two solutions of (3.1).

We take the scalar product of (3.5) with  $\omega$ :

$$\frac{1}{2}\partial_t \|\omega\|^2 + \nu \|\nabla \omega\|^2 + \mu \|\omega\|^2 = (H,\omega) - (J(\Delta^{-1}\omega,\varphi),\omega).$$

We estimate the nonlinear term by means of inequality (4.17)

$$\begin{split} |(J(\Delta^{-1}\omega,\varphi),\omega)| &= |(J(\omega,\Delta^{-1}\omega),\varphi)| \le \|\nabla\,\omega\| \|\nabla\Delta^{-1}\omega\| \|\varphi\|_{\infty} \le \\ &2|\Omega|^{1/2}\eta(w)\|\nabla\,\omega\| \|\varphi\|_{\infty} + \sqrt{68}|\Omega|N^{-1}\|\nabla\,\omega\|^2\|\varphi\|_{\infty} \,. \end{split}$$

As a result we obtain

$$\frac{1}{2}\partial_t \|\omega\|^2 + \alpha(t) \|\nabla \omega\|^2 \le \beta(t),$$

where

$$\alpha(t) = \nu - \sqrt{68} |\Omega| N^{-1} ||\varphi(t)||_{\infty}, \quad \beta(t) = (H(t), \omega) + 2 |\Omega|^{1/2} \eta(w(t)) ||\nabla \omega|| ||\varphi(t)||_{\infty}.$$

As before  $\beta(t) \to 0$  as  $t \to \infty$ , while in view of (3.3)  $\alpha(t) \ge \text{const} > 0$  for all t large enough provided that  $N > \sqrt{68} \mathbf{F}_{\infty} |\Omega| / (\mu \nu)$ .

We combine the results so obtained above in the following theorem.

**Theorem 3.1.** The first *m* modes of the Stokes operator are determining for the Navier– Stokes system with damping (3.1) if

$$m+1 > \frac{2}{\gamma} + \frac{2}{\pi^2} \frac{\mathbf{F}_{\infty}|\Omega|}{\mu\nu}.$$
(3.10)

If  $\Omega$  is tiled by N equal squares, then any collection of nodes (one in each square) is determining if

$$N > \sqrt{68} \frac{\mathbf{F}_{\infty}|\Omega|}{\mu\nu} \,. \tag{3.11}$$

Remark 3.1. We observe that estimates for the number of the determining modes and nodes (3.7) and (3.11) depend linearly on the measure of the periodic domain  $|\Omega|$  and depend on the aspect ratio  $\gamma$  of the torus only via  $|\Omega| = L^2/\gamma$ . Furthermore, the characteristic microscopic length l of the lattice of the determining nodes satisfies the following  $\Omega$ -independent estimate from above

$$l < 68^{-1/4} \left(\frac{\mu\nu}{\mathbf{F}_{\infty}}\right)^{1/2}.$$
(3.12)

*Remark* 3.2. It was shown in [26] that the fractal dimension of the global attractor of the autonomous system (3.1) on the torus  $[0, L]^2$  satisfies the estimate

$$\dim_F \mathcal{A} \le \left(\frac{6}{\pi^3}\right)^{1/2} \frac{\|\operatorname{rot} f\|L}{\mu\nu}.$$
(3.13)

It was also shown that for the Kolmogorov forcing of the form

$$f = f_s = \begin{cases} f_1 = A(\mu, \nu) \sin s \frac{2\pi x_2}{L}, \\ f_2 = 0, \end{cases}$$
(3.14)

one has a lower bound

$$\dim_F \mathcal{A} \ge \operatorname{const} \frac{\|\operatorname{rot} f\|L}{\mu\nu}$$
(3.15)

in the limit  $\nu \to 0^+$  (accordingly,  $s = (\mu L^2 / \nu)^{1/2} \to \infty$ ). Since  $\| \operatorname{rot} f \| \leq \| \operatorname{rot} f \|_{\infty} L$ , it follows that

since 
$$\|\operatorname{rot} f\| \leq \|\operatorname{rot} f\|_{\infty} L$$
, it follows that

$$\dim_F \mathcal{A} \le \left(\frac{6}{\pi^3}\right)^{1/2} \frac{\|\operatorname{rot} f\|_{\infty} L^2}{\mu\nu}, \qquad (3.16)$$

and since  $\|\operatorname{rot} f_s\| = (L/\sqrt{2}) \|\operatorname{rot} f_s\|_{\infty}$ , it follows from (3.15) that the estimate (3.16) is also sharp with respect to the dimensionless number  $\frac{\|\operatorname{rot} f\|_{\infty}L^2}{\mu\nu}$ .

Hence the bounds (3.8) and (3.11) for the number of determining modes and nodes for the damped Navier–Stokes system are of the same order as the fractal dimension of the global attractor.

We point out that in general and for the Navier–Stokes equations there is a gap between the number of the determining modes and nodes and the dimension of the global attractor. However, the works [14], [21] and [17] indicate that one can perturb the points or the projections to obtain the number of nodes and the rank of the projections comparable with the dimension of the global attractor. Here, however, we show that there is no need for perturbation and that the usual projections  $P_m$  and any choice of points will do. It will therefore be interesting to understand the role of damping term here in terms of the generic results of [14], [21], [17] which rely heavily on the Mãné embedding theorem.

Stress-free boundary conditions. Let  $\Omega \subset \mathbb{R}^2$  be a bounded simply connected domain with  $C^2$  boundary. Let *n* be the outward unit normal vector. We consider the system (3.1) supplemented with the so-called stress-free boundary conditions

$$u \cdot n|_{\partial\Omega} = 0, \quad \operatorname{rot} u|_{\partial\Omega} = 0.$$
 (3.17)

Then any smooth vector field u, div u = 0, satisfying (3.17) has a unique single valued stream function  $\psi$ ,  $u = \nabla^{\perp} \psi$  with  $\psi|_{\partial\Omega} = 0$  and  $\Delta \psi|_{\partial\Omega} = \operatorname{rot} u|_{\partial\Omega} = 0$ . Therefore, the vorticity formulation for the system (3.1), (3.17) is the equation (3.4) with zero boundary condition both for  $\varphi$  and  $\psi$ :

$$\partial_t \varphi + J(\psi, \varphi) - \nu \Delta \varphi + \mu \varphi = \operatorname{rot} f,$$
  

$$\Delta \psi = \varphi,$$
  

$$\varphi|_{\partial\Omega} = \psi|_{\partial\Omega} = 0.$$
(3.18)

For the Stokes eigenvalue problem with boundary conditions (3.17)

$$-\Delta w_k + \nabla p_k = \lambda_k w_k, \quad \text{div} w_k = 0, w_k \cdot n|_{\partial\Omega} = 0, \quad \text{rot} w_k|_{\partial\Omega} = 0,$$
(3.19)

we have (as in the case of periodic boundary conditions) that  $\{\lambda_k\}_{k=1}^{\infty}$  are the eigenvalues of the scalar Dirichlet problem  $-\Delta \varphi_k = \lambda_k \varphi_k$ ,  $\varphi_k|_{\partial\Omega} = 0$  and  $w_k = \lambda_k^{-1/2} \nabla^{\perp} \varphi_k$ . Hence we can use the Li–Yau lower bound [31] for the eigenvalues  $\lambda_k$ 

$$\lambda_k \ge \frac{2\pi k}{|\Omega|} \,. \tag{3.20}$$

Lemma 3.1 and the subsequent argument for estimates of the determining modes still hold and we obtain that condition (3.6) is sufficient for the first m modes to be determining. In view of (3.20) we obtain the following result.

**Theorem 3.2.** The first m modes of the Stokes operator are determining for the Navier-Stokes system with damping (3.1) with stress-free boundary conditions (3.17) if

$$m+1 > \frac{1}{2\pi} \frac{\mathbf{F}_{\infty}|\Omega|}{\mu\nu}$$

*Remark* 3.3. As in the space-periodic case this estimate agrees with the estimate for the fractal dimension of the attractor [26], but, unlike the latter, does not involve constants depending on the smoothness and shape of the boundary.

Remark 3.4. A similar result holds for the determining nodes and other determining functionals and projections (see [8], [9]) if we use extension operators mapping Sobolev spaces defined on  $\Omega$  to the spaces defined on corresponding periodic rectangular domain containing  $\Omega$ . In this case, however, the estimate involves a constant depending on the smoothness and shape of the boundary.

# 4. Appendix. Proof of auxiliary inequalities

The embedding of the Sobolev space  $H^{l}(M)$  with norm  $||u||_{H^{l}}^{2} = ||u||^{2} + ||(-\Delta)^{l/2}u||^{2}$ into the space of bounded continuous functions C(M), where dim M = n and l > n/2, can be written as a multiplicative inequality

$$||u||_{\infty} \le c_M(l) ||u||^{\theta} ||(-\Delta)^{l/2} u||^{1-\theta}, \text{ where } \theta = (2l-n)/2l.$$
 (4.1)

Inequalities of this type are sometimes called the Agmon inequalities (see [1]). The best constant  $c_M(l)$  for  $M = \mathbb{R}$  in this inequality was found in [36] (the results of [36] can easily be generalized to the case when  $M = \mathbb{R}^n$ ). Sharp constants in inequalities for periodic functions and functions defined on the sphere were found in [24]. Following [24] we consider below the case of a two-dimensional torus.

The constant in inequality (4.1) on a two-dimensional torus clearly depends only on the aspect ratio  $\gamma$  of the torus. We first consider the case of a square torus  $\gamma = 1$ , and then without loss of generality we assume that  $\Omega = T^2 = [0, 2\pi]^2$ .

We consider the negative Laplacian  $-\Delta$  in  $H = L_2(\mathbb{T}^2) \cap \{\varphi, \int \varphi \, dx = 0\}$  and order its eigenvalues according to magnitude and multiplicity:

$$1 = \lambda_1 \le \lambda_2 \le \dots, \qquad \{\lambda_j, \ j = 1, \dots\} = \{k^2 = k_1^2 + k_2^2, \ k = (k_1, k_2) \in \mathbb{Z}_0^2\}, \qquad (4.2)$$

where  $\mathbb{Z}_0^2 = \mathbb{Z}^2 \setminus \{0\}$ . The corresponding basis of orthonormal eigenfunctions  $w_j(x)$ ,  $-\Delta w_j = \lambda_j w_j$ , is the basis of trigonometric functions

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$$\bigcup_{j \in \mathbb{N}} w_j(x) = \bigcup_{k \in \mathbb{Z}^2_+} \left\{ (\sqrt{2}\pi)^{-1} \sin kx, \quad (\sqrt{2}\pi)^{-1} \cos kx \right\},$$

$$\mathbb{Z}^2_+ = \{ k \in \mathbb{Z}^2_0, \quad k_1 \ge 0, \ k_2 \ge 0 \} \cup \{ k \in \mathbb{Z}^2_0, \quad k_1 \ge 1, \ k_2 \le 0 \}.$$
(4.3)

Similarly to (4.2), we write

$$1 = \Lambda_1 \le \Lambda_2 \le \dots, \qquad \{\Lambda_j, \ j = 1, \dots\} = \{k^2, \ k \in \mathbb{Z}^2_+\}$$
(4.4)

and observe that

$$\bigcup_{j=1}^{\infty} \{\lambda_j\} = \bigcup_{l=1}^{\infty} \{\Lambda_l, \Lambda_l\}.$$
(4.5)

Hence, for  $j \ge 1$ , we have  $\Lambda_j = \lambda_{2j} = \lambda_{2j-1}$  and, corresponding to each  $\Lambda = \Lambda_j$ , there are two eigenfunctions  $u_j(x) = (\sqrt{2\pi})^{-1} \sin kx$  and  $v_j(x) = (\sqrt{2\pi})^{-1} \cos kx$  for some uniquely defined  $k_j = (k_1(j), k_2(j))$  with  $k_j^2 = \Lambda_j$ . We obviously have

$$u_j(x)^2 + v_j(x)^2 = \frac{1}{2\pi^2}.$$
(4.6)

**Theorem 4.1.** The sharp constant  $c_{AT}$  in the inequality

$$\|\varphi\|_{\infty} \leq c_{\mathrm{AT}} \|\varphi\|^{1/2} \|\Delta\varphi\|^{1/2}, \quad \varphi \in H \cap H^2_{\mathrm{per}}(\mathrm{T}^2), \tag{4.7}$$

is given by

$$c_{\rm AT}^2 = \frac{1}{\pi^2} \sup_{\mu>0} \mu \sum_{n=1}^{\infty} \frac{1}{\mu^2 + \Lambda_n^2}$$
(4.8)

and, in particular,

$$c_{\rm AT}^2 < \frac{1}{\pi}$$
 (4.9)

*Proof.* Writing  $\varphi$  in terms of the Fourier series  $\varphi(x) = \sum_{n=1}^{\infty} c_n w_n(x)$ , for an arbitrary point  $x_0$  and a positive parameter  $\nu$  we have

$$\varphi(x_0)^2 = \left(\sum_{n=1}^{\infty} c_n w_n(x_0)\right)^2 \le \sum_{n=1}^{\infty} \frac{w_n(x_0)^2}{1 + \nu \lambda_n^2} \sum_{n=1}^{\infty} c_n^2 (1 + \nu \lambda_n^2) = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{1 + \nu \Lambda_n^2} \cdot \left(\|\varphi\|^2 + \nu \|\Delta \varphi\|^2\right),$$
(4.10)

where we used (4.5), (4.6). Since the right-hand side of (4.10) is independent of  $x_0$ , it follows that

$$\|\varphi\|_{\infty}^{2} \leq \frac{1}{2\pi^{2}} \sum_{n=1}^{\infty} \frac{1}{1+\nu\Lambda_{n}^{2}} \cdot \left(\|\varphi\|^{2}+\nu\|\Delta\varphi\|^{2}\right).$$
(4.11)

Let  $x_0$  be fixed. Then there is equality in (4.10), (4.11) if and only if

$$c_n = (1 + \nu \lambda_n^2) w_n(x_0),$$

that is, if

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{w_n(x)w_n(x_0)}{1+\nu\lambda_n^2} = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{\cos(k_n(x-x_0))}{1+\nu\Lambda_n^2}.$$
(4.12)

We now set  $\nu = \nu_* = \|\varphi\|^2 / \|\Delta \varphi\|^2$ . Then  $\|\varphi\|^2 + \nu_* \|\Delta \varphi\|^2 = 2\nu_*^{1/2} \|\varphi\| \|\Delta \varphi\|$  and therefore

$$\|\varphi\|_{\infty}^{2} \leq \frac{1}{\pi^{2}} \nu_{*}^{1/2} \sum_{n=1}^{\infty} \frac{1}{1 + \nu_{*} \Lambda_{n}^{2}} \cdot \|\varphi\| \|\Delta \varphi\| \leq \frac{1}{\pi^{2}} \sup_{\nu > 0} \nu^{1/2} \sum_{n=1}^{\infty} \frac{1}{1 + \nu \Lambda_{n}^{2}} \cdot \|\varphi\| \|\Delta \varphi\|,$$

which shows (with  $\nu = \mu^{-2}$ ) that  $c_{\text{AT}}^2$  is less than or equal to the right-hand side of (4.8). Suppose now that the supremum of the function  $H(\nu) = \nu^{1/2} \sum_{n=1}^{\infty} (1 + \nu \Lambda_n^2)^{-1}$  is attained at a finite point  $\nu_*$ ,  $0 < \nu_* < \infty$ . Then

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{1+\nu_*\Lambda_n^2} = \nu_*\sum_{n=1}^{\infty}\frac{\Lambda_n^2}{(1+\nu_*\Lambda_n^2)^2}, \qquad \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{1+\nu_*\Lambda_n^2} = \sum_{n=1}^{\infty}\frac{1}{(1+\nu_*\Lambda_n^2)^2}.$$
 (4.13)

In fact, the first equality follows from  $H'(\nu_*) = 0$ . Summing the first and the second equalities we obtain a valid identity, hence the second equality also holds.

Next we set  $\nu = \nu_*$  in (4.12) and  $x_0 = 0$ . Then for the corresponding  $\varphi = \varphi_*$  we have

$$\|\varphi_*\|^2 = \sum_{n=1}^{\infty} \frac{1}{(1+\nu_*\Lambda_n^2)^2}, \qquad \|\Delta \varphi_*\|^2 = \sum_{n=1}^{\infty} \frac{\Lambda_n^2}{(1+\nu_*\Lambda_n^2)^2}.$$

Then it follows from (4.13) that  $\|\varphi_*\|^2 / \|\Delta \varphi_*\|^2 = \nu_*$ . Hence

$$\|\varphi_*\|_{\infty}^2 = \frac{1}{\pi^2} H(\nu_*) \|\varphi_*\| \|\Delta \varphi_*\|_{2}$$

which proves the theorem in the case when  $0 < \nu_* < \infty$ .

Suppose now that supremum is attained as  $\nu \to 0$  (observe that  $H(\nu) \to 0$  as  $\nu \to \infty$ ). We consider inequality (4.7) on the finite dimensional space  $T_N = \text{Span}\{\sin kx, \cos kx\}, k^2 \leq N$ . The corresponding sharp constant  $c_{\text{AT}}(N)$  is given by the formula

$$c_{\rm AT}(N)^2 = \frac{1}{\pi^2} \max_{\nu>0} H_N(\nu), \quad H_N(\nu) = \nu^{1/2} \sum_{\Lambda_n \le N} \frac{1}{1 + \nu \lambda_n^2}$$

The maximum is attained since  $H_N(0) = 0$  and  $H_N(\nu) \to 0$  as  $\nu \to \infty$ . Hence there exists an extremal function  $\varphi_*^N \in T_N$ . Since the spaces  $T_N$  are dense in the Sobolev space  $H^2_{\text{per}}(\mathbf{T}^2)$ , it follows that

$$c_{\text{AT}} = \lim_{N \to \infty} c_{\text{AT}}(N), \text{ and } c_{\text{AT}}^2 = \frac{1}{\pi^2} \sup_{\nu > 0} \nu^{1/2} \sum_{n=1}^{\infty} \frac{1}{1 + \nu \Lambda_n^2}.$$

It remains to prove (4.9). Using the lower bound  $\Lambda_n \ge n/2$  (see [26]) we have

$$\nu^{1/2} \sum_{n=1}^{\infty} \frac{1}{1+\nu\Lambda_n^2} = \frac{1}{\mu} \sum_{n=1}^{\infty} \frac{1}{1+(\Lambda_n/\mu)^2} < \frac{1}{\mu} \sum_{n=1}^{\infty} \frac{1}{1+(n/(2\mu))^2} < \int_0^{\infty} f(x) \, dx = \pi,$$

where  $f(x) = 1/(1 + (x/2)^2)$  is monotone decreasing and the third term in the above formula is the Riemann sum with step  $1/\mu$  for the corresponding integral.

Remark 4.1. In the vector case  $u \in H^2_{per}(\mathbb{T}^2)^2$  we have the same constant in the corresponding inequality

$$\|u\|_{\infty} \le c_{\rm AT} \|u\|^{1/2} \|\Delta u\|^{1/2}.$$
(4.14)

In fact, for  $u = \{u^1, u^2\}$  we have

$$\begin{aligned} \|u\|_{\infty}^{2} \leq \|u^{1}\|_{\infty}^{2} + \|u^{2}\|_{\infty}^{2} \leq c_{\mathrm{AT}}^{2}(\|u^{1}\|\|\Delta u^{1}\| + \|u^{2}\|\|\Delta u^{2}\|) \leq \\ \frac{c_{\mathrm{AT}}^{2}}{2}(\varepsilon(\|u^{1}\|^{2} + \|u^{2}\|^{2}) + \varepsilon^{-1}(\|\Delta u^{1}\|^{2} + \|\Delta u^{2}\|^{2})) = \frac{c_{\mathrm{AT}}^{2}}{2}(\varepsilon\|u\|^{2} + \varepsilon^{-1}\|\Delta u\|^{2}) \end{aligned}$$

and minimizing with respect to  $\varepsilon$  we obtain inequality (4.14).

**Corollary 4.1.** The constant  $c_{\text{AT}}(\gamma)$  on the torus  $\Omega = [0, L/\gamma] \times [0, L]$ ,  $\gamma \leq 1$ , satisfies the estimate  $c_{\text{AT}}(\gamma) \leq c_{\text{AT}}/\sqrt{\gamma} \leq 1/\sqrt{\gamma\pi}$ .

Proof. We assume for simplicity that  $1/\gamma$  is integer. Given a function  $\varphi \in H^2_{\text{per}}(\Omega)$  we extent it by periodicity in  $x_2$ -direction  $1/\gamma$  times and denote the function so obtained by  $\tilde{\varphi}$ . Then  $\tilde{\varphi} \in H^2_{\text{per}}(\tilde{\Omega})$ , where  $\tilde{\Omega} = [0, L/\gamma]^2$  is a square-shaped periodic domain so that  $\|\tilde{\varphi}\|_{L_{\infty}(\tilde{\Omega})} \leq c_{\text{AT}} \|\tilde{\varphi}\|_{L_{2}(\tilde{\Omega})}^{1/2} \|\Delta \tilde{\varphi}\|_{L_{2}(\tilde{\Omega})}^{1/2}$ . Since  $\|\varphi\|_{L_{\infty}(\Omega)} = \|\tilde{\varphi}\|_{L_{\infty}(\tilde{\Omega})}, \|\varphi\|_{L_{2}(\Omega)}^2 = \gamma \|\tilde{\varphi}\|_{L_{2}(\tilde{\Omega})}^2$ , and  $\|\Delta \varphi\|_{L_{2}(\Omega)}^2 = \gamma \|\Delta \tilde{\varphi}\|_{L_{2}(\tilde{\Omega})}^2$ , the corollary is proved.

We now prove the remaining two inequalities used for estimates of the number of the determining nodes.

**Lemma 4.1.** (see [12], [28]). Let  $\Omega = [0, L_1] \times [0, L_2]$  be divided into N equal squares  $Q_j$  with side l and let  $x^j \in Q_j$  for  $j = 1, \ldots, N$ . Then for  $u \in H^2_{per}(\Omega)$  the following inequalities hold:

$$||u||^{2} \leq 4l^{2} N \eta^{2}(u) + 68l^{4} ||\Delta u||^{2}, \qquad (4.15)$$

$$\|\nabla u\|^2 \le 2 \cdot 68^{-1/2} N \eta^2(u) + 68^{1/2} l^2 \|\Delta u\|^2, \tag{4.16}$$

where  $\eta(u) = \max_{j=1,...,N} |u(x^{j})|.$ 

*Proof.* We consider the scalar case and prove the first inequality. Let  $u \in H^2(Q)$ , where  $Q = [0, l]^2$ . For any two points  $\mathbf{x} = (x, y)$  and  $\mathbf{x}^0 = (x_0, y_0)$  in Q we have

$$u(\mathbf{x}) - u(\mathbf{x}^0) = \int_{x_0}^x u_x(\xi, y) d\xi + \int_{y_0}^y u_y(x_0, \eta) d\eta.$$

Hence

$$(u(\mathbf{x}) - u(\mathbf{x}^0))^2 \le 2l \int_0^l u_x(\xi, y)^2 d\xi + 2l \int_0^l u_y(x_0, \eta)^2 d\eta,$$

which gives after integration over Q with respect to x, y that

$$||u - u(\mathbf{x}^{0})||_{L_{2}(Q)}^{2} \leq 2l^{2} ||u_{x}||_{L_{2}(Q)}^{2} + 2l^{3} \int_{0}^{l} u_{y}(x_{0}, \eta)^{2} d\eta$$

For the second term on the right we have

$$u_y(x_0,\eta)^2 \le u_y(x,\eta)^2 + 2\int_0^l |u_y(\xi,\eta)| |u_{yx}(\xi,\eta)| d\xi,$$

hence, integrating with respect to x and  $\eta$  over Q we find

$$l\int_{0}^{l} u_{y}(x_{0},\eta)^{2} d\eta \leq \|u_{y}\|_{L_{2}(Q)}^{2} + 2l\int_{0}^{l} \int_{0}^{l} |u_{y}(\xi,\eta)| u_{yx}(\xi,\eta)| d\xi d\eta \leq 2\|u_{y}\|_{L_{2}(Q)}^{2} + l^{2}\|u_{xy}\|_{L_{2}(Q)}^{2}.$$
  
Therefore

Therefore

$$\|u - u(\mathbf{x}^0)\|_{L_2(Q)}^2 \le 4l^2 \|\nabla u\|_{L_2(Q)}^2 + 2l^4 \|u_{xy}\|_{L_2(Q)}^2$$

Temporarily denoting the right-side by K and using Young's inequality we have

$$\begin{aligned} \|u\|_{L_{2}(Q)}^{2} \leq K + 2u(\mathbf{x}^{0}) \int_{Q} u(x,y) dx dy - l^{2} u(\mathbf{x}^{0})^{2} \leq \\ K + 2u(\mathbf{x}^{0}) \, l \|u\|_{L_{2}(Q)} - l^{2} u(\mathbf{x}^{0})^{2} \leq K + l^{2} u(\mathbf{x}^{0})^{2} + \frac{1}{2} \|u\|_{L_{2}(Q)}^{2}. \end{aligned}$$

Hence

$$\|u\|_{L_2(Q)}^2 - 2l^2 u(\mathbf{x}^0)^2 \le 8l^2 \|\nabla u\|_{L_2(Q)}^2 + 4l^4 \|u_{xy}\|_{L_2(Q)}^2.$$

We now divide  $\Omega = [0, L_1] \times [0, L_2]$  into N equal squares of side  $l = (|\Omega|/N)^{1/2}$  and choose a point  $\mathbf{x}^{j}$  in each square  $Q_{j}, j = 1, ..., N$ . Summing over j we obtain

$$||u||^{2} - 2l^{2} \sum_{j=1}^{N} u(\mathbf{x}^{j})^{2} \le 8l^{2} ||\nabla u||^{2} + 2l^{4} ||\Delta u||^{2},$$

where we used the periodic boundary conditions so that  $||u_{xy}||^2 = \int u_{xx} u_{yy} dx dy \leq \frac{1}{2} ||\Delta u||^2$ . Next we use the interpolation inequality

$$\|\nabla u\|^2 \le \|u\| \|\Delta u\| \le \frac{1}{16l^2} \|u\|^2 + 4l^2 \|\Delta u\|^2$$

and finally obtain

$$||u||^{2} \leq 4l^{2} \sum_{j=1}^{N} u(\mathbf{x}^{j})^{2} + 68l^{4} ||\Delta u||^{2} \leq 4l^{2} N \eta^{2}(u) + 68l^{4} ||\Delta u||^{2},$$

which proves inequality (4.15) for the scalar case. For the vector case we apply the above inequality for each component and add up the results.

Finally, if  $u \in H^2_{\text{per}}(\Omega)$ , div u = 0, and rot  $u = \omega$ , then taking into account that  $||u|| = ||\nabla \Delta^{-1} \omega||$  and  $||\nabla \omega|| = ||\Delta u||$  we can write the previous inequality in the form

$$\|\nabla\Delta^{-1}\omega\|^{2} \leq 4l^{2}N\eta^{2}(u) + 68l^{4}\|\nabla\omega\|^{2} = 4|\Omega|\eta^{2}(u) + 68|\Omega|^{2}N^{-2}\|\nabla\omega\|^{2}.$$
 (4.17)

For the proof of (4.16) we have

 $\|\nabla u\|^{2} \leq \|u\| \|\Delta u\| \leq \varepsilon \|u\|^{2} + (4\varepsilon)^{-1} \|\Delta u\|^{2} \leq 4Nl^{2} \varepsilon \eta^{2}(u) + (68l^{4}\varepsilon + (4\varepsilon)^{-1}) \|\Delta u\|^{2},$ which gives (4.16) by setting  $\varepsilon^{-1} = 2 \cdot 68^{1/2}l^{2}.$ 

## Acknowledgments

This work was supported in part by the US Civilian Research and Development Foundation, grant no. RUM1-2654-MO-05 (A.A.I. and E.S.T.), by the Russian Foundation for Fundamental Research, grant no. 03-01-00189 02, and by the RAS Programme 'Modern problems of theoretical mathematics', contract no. 090703-1028 (A.A.I.). The work of E.S.T. was supported in part by the National Science Foundation, grant no. DMS-0204794, the MAOF Fellowship of the Israeli Council of Higher Education, and by the USA Department of Energy under contract W-7405-ENG-36 and the ASCR Program in Applied Mathematical Sciences.

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