# STABILITY OF SOLITARY WAVES FOR A GENERALIZED DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. We consider a derivative nonlinear Schrödinger equation with a general nonlinearity. This equation has a two parameter family of solitary wave solutions. We prove orbital stability/instability results that depend on the strength of the nonlinearity and, in some instances, their velocity. We illustrate these results with numerical simulations.

### 1. INTRODUCTION

The derivative nonlinear Schrödinger (DNLS) equation

(1.1) 
$$i\partial_t u + \partial_x^2 u + i(|u|^2 u)_x = 0.$$

is a nonlinear dispersive wave equation that appears in the long wavelength approximation of Alfvén waves propagating in a plasma [25, 26, 30]. Applying the gauge transformation

(1.2) 
$$\psi = u(x) \exp i \left\{ \frac{1}{2} \int_{-\infty}^{x} |u(\eta)|^2 d\eta \right\},$$

equation (1.1) has the form

(1.3) 
$$i\partial_t \psi + \partial_x^2 \psi + i|\psi|^2 \psi_x = 0, \quad x \in \mathbb{R}.$$

This equation has a Hamiltonian structure and can be written as

$$\frac{\partial \psi}{\partial t} = -iE'(\psi)$$

where the Hamiltonian is

$$E \equiv \frac{1}{2} \int_{-\infty}^{\infty} |\psi_x|^2 dx + \frac{1}{4} \Im \int_{-\infty}^{\infty} |\psi|^2 \bar{\psi} \psi_x dx.$$

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Equation (1.3) and some of its generalizations, also appear in the modeling of ultrashort optical pulses [1, 27]. Furthermore, the DNLS equation has the remarkable property of being integrable by inverse scattering [18]. It admits a two-parameter family of solitary wave solutions of the form: (1.4)

$$u_{\omega,c}(x,t) = \varphi_{\omega,c}(x-ct) \exp i \left\{ \omega t + \frac{c}{2}(x-ct) - \frac{3}{4} \int_{-\infty}^{x-ct} \varphi_{\omega,c}^2(\eta) d\eta \right\},$$

where  $\omega > c^2/4$  and

(1.5) 
$$\varphi_{\omega,c}(y) = \sqrt{\frac{(4\omega - c^2)}{\sqrt{\omega}(\cosh(\sigma\sqrt{4\omega - c^2}y) - \frac{c}{2\sqrt{\omega}})}}$$

is the positive solution to

(1.6) 
$$-\partial_y^2 \varphi_{\omega,c} + (\omega - \frac{c^2}{4})\varphi_{\omega,c} + \frac{c}{2}|\varphi_{\omega,c}|^2 \varphi_{\omega,c} - \frac{3}{16}|\varphi_{\omega,c}|^4 \varphi_{\omega,c} = 0$$

Guo and Wu [11] showed that these solitary waves are *orbitally stable* if c < 0 and  $c^2 < 4\omega$ . Colin and Ohta [2] subsequently extended the result, proving orbital stability for all  $c, c^2 < 4\omega$ .

**Definition 1.1.** Let  $u_{\omega,c}$  be the solitary wave solution of (1.1). The solitary wave  $u_{\omega,c}$  is orbitally stable if, for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $||u_0 - u_{\omega,c}||_{H^1} < \delta$ , then the solution u(t) of (1.1) with initial data  $u(0) = u_0$ , exists globally in time and satisfies

$$\sup_{t\geq 0} \inf_{(\theta,y)\in\mathbb{T}\times\mathbb{R}} \|u(t) - e^{i\theta} u_{\omega,c}(t,\cdot-y)\|_{H^1} < \epsilon.$$

Otherwise,  $u_{\omega,c}$  is said to be *orbitally unstable*.

In an effort to understand the structural properties of DNLS, we study an extension of (1.3) with general power nonlinearity ( $\sigma > 0$ ).

(1.7) 
$$i\partial_t \psi + \partial_x^2 \psi + i|\psi|^{2\sigma} \psi_x = 0$$

Equation (1.7) also admits a two-parameter family of solitary wave solutions,

(1.8)

$$\psi_{\omega,c}(x,t) = \varphi_{\omega,c}(x-ct) \exp i \left\{ \omega t + \frac{c}{2}(x-ct) - \frac{1}{2\sigma+2} \int_{-\infty}^{x-ct} \varphi_{\omega,c}^{2\sigma}(\eta) d\eta \right\},$$

where  $\omega > c^2/4$  and

(1.9) 
$$\varphi_{\omega,c}(y)^{2\sigma} = \frac{(\sigma+1)(4\omega-c^2)}{2\sqrt{\omega}(\cosh(\sigma\sqrt{4\omega-c^2}y) - \frac{c}{2\sqrt{\omega}})}$$

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is the positive solution of (1.10)

$$-\partial_y^2 \varphi_{\omega,c} + (\omega - \frac{c^2}{4})\varphi_{\omega,c} + \frac{c}{2}|\varphi_{\omega,c}|^{2\sigma}\varphi_{\omega,c} - \frac{2\sigma + 1}{(2\sigma + 2)^2}|\varphi_{\omega,c}|^{4\sigma}\varphi_{\omega,c} = 0.$$

It is convenient to define

(1.11) 
$$\phi_{\omega,c}(y) = \varphi_{\omega,c}(y)e^{i\theta_{\omega,c}(y)},$$

with the traveling phase

(1.12) 
$$\theta_{\omega,c}(y) \equiv \frac{c}{2}y - \frac{1}{2\sigma+2}\int_{-\infty}^{y}\varphi_{\omega,c}^{2\sigma}(\eta)d\eta.$$

Clearly,

(1.13) 
$$\psi_{\omega,c}(x,t) = e^{i\omega t}\phi_{\omega,c}(x-ct),$$

and the complex function  $\phi_{\omega,c}(y)$  satisfies

(1.14) 
$$-\partial_y^2 \phi_{\omega,c} + \omega \phi_{\omega,c} + ic \partial_x \phi_{\omega,c} - i |\phi_{\omega,c}|^{2\sigma} \partial_y \phi_{\omega,c} = 0, y \in \mathbb{R}.$$

Provided there is no ambiguity, we write  $\phi, \varphi$  for  $\phi_{\omega,c}, \varphi_{\omega,c}$  respectively. Furthermore, we only consider *admissible* values of  $(\omega, c)$  satisfying the conditions  $\omega > \frac{c^2}{4}$ ,  $c \in \mathbb{R}$ .

1.1. **Main Results.** We investigate the stability of solitary wave solutions  $\psi_{\omega,c}$  to the gDNLS equation. This is determined by both the value of  $\sigma$  and the choice of the soliton parameters, c and  $\omega$ . These results are *conditional* in the sense that for  $\sigma \neq 1$ , we lack a suitable local well-posedness theory. Throughout our study, we assume that given  $\sigma > 0$ , and  $\psi_0 \in H^1(\mathbb{R})$ , there exists a weak solution  $\psi \in C([0,T); H^1(\mathbb{R}))$  of (1.7), for T > 0, which satisfies

(1.15) 
$$\frac{d}{dt} \langle \psi(\cdot, t), f \rangle = \langle E'(\psi(\cdot, t)), -Jf \rangle$$

for appropriate test functions f. E is the energy functional, and J is the symplectic operator. These are defined in Section 2.

Subject to this assumption, we have the following results:

**Theorem 1.2.** For any admissible  $(\omega, c)$  and  $\sigma \geq 2$ , the solitary wave solution  $\psi_{\omega,c}(x,t)$  of (1.7) is orbitally unstable.

For  $\sigma$  between 1 and 2, slow solitons, those with sufficiently low c, will be stable while fast right-moving solitons will be unstable:

**Theorem 1.3** (Numerical). For  $\sigma \in (1, 2)$ , there exists  $z_0 = z_0(\sigma) \in (-1, 1)$  such that:

(i) the solitary wave solution  $\psi_{\omega,c}(x,t)$  of (1.7) is orbitally stable for admissible  $(\omega, c)$  satisfying  $c < 2z_0\sqrt{\omega}$ .

(ii) the solitary wave solution  $\psi_{\omega,c}(x,t)$  of (1.7) is orbitally unstable for admissible  $(\omega,c)$  satisfying  $c > 2z_0\sqrt{\omega}$ .

Our last result concerns  $\sigma < 1$ , where all solitons are stable:

**Theorem 1.4** (Numerical). For admissible  $(\omega, c)$  and  $0 < \sigma < 1$ , the solitary wave solution  $\psi_{\omega,c}(x,t)$  of (1.7) is orbitally stable.

The endpoint,  $\sigma = 1$ , corresponds to the cubic case, which has already been studied in [2, 11].

Theorems 1.3 and 1.4 are fully rigorous up to the determination of the sign of a function of one variable, which is parametrized by  $\sigma$ . The number  $z_0$  in Theorem 1.3 corresponds to a zero crossing. This function, defined below by (4.3), includes improper integrals of transcendental functions.

Theorem 1.3 is particularly noteworthy for distinguishing gNLS from the focusing nonlinear Schrödinger equation (NLS),

(1.16) 
$$i\psi_t + \Delta \psi + |\psi|^{2\sigma} \psi = 0, \quad \psi : \mathbb{R}^{d+1} \to \mathbb{C}.$$

Equation (1.7) is invariant under the scaling transformation  $\psi_{\lambda}(x,t) = \lambda^{\frac{1}{2\sigma}}\psi(\lambda x,\lambda^2 t)$ . This implies that its critical Sobolev exponent is  $s_c = \frac{1}{2} - \frac{1}{2\sigma}$ . Hence, it is  $L^2$ -critical for  $\sigma = 1$ , and it is  $L^2$ -supercritical, energy subcritical for  $\sigma > 1$ . While NLS only admits stable solitons in the  $L^2$ -subcritical regime, gDNLS admits stable solitons not only in the critical regime, but also in the supercritical one.

1.2. **Outline.** Our results are proven using the abstract functional analysis framework of Grillakis, Shatah and Strauss, [7,8]; see [34,35] for related results and [31] for a survey. The test for stability involves two parts: (i) Counting the number of negative eigenvalues of the linearized evolution operator  $H_{\phi}$  near the solitary solution of (1.7), denoted  $n(H_{\phi})$ ; (ii) Counting the number of positive eigenvalues of the Hessian of the scalar function  $d(\omega, c)$  built out of the action functional evaluated at the soliton, denoted p(d''). We give explicit characterizations of  $H_{\phi}$  and  $d(\omega, c)$  in Section 2. We then apply:

**Theorem 1.5** (Grillakis et al. [7,8]).

$$(1.17) p(d'') \le n(H_{\phi})$$

Furthermore, under the condition that d is non-degenerate at  $(\omega, c)$ :

- (i) If  $p(d'') = n(H_{\phi})$ , the solitary wave is orbitally stable;
- (ii) If  $n(H_{\phi}) p(d'')$  is odd, the solitary wave is orbitally unstable.

In Section 3, we show that  $n(H_{\phi}) = 1$ , and in Section 4, we compute the number of positive eigenvalues of the Hessian, d''. We complete the

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proofs of Theorems 1.2 to 1.4 in Section 5. In Section 6, we discuss the results and illustrate them with numerical simulations of both stable and unstable solitons. Algebraic manipulations useful for the evaluation of  $\det[d''(\omega, c)]$  and  $\operatorname{tr}[d''(\omega, c)]$  are presented in the Appendix.

1.3. Remarks on Well Posedness Assumption. The DNLS equation (cubic nonlinearity) has been studied in  $H^1$  and higher regularity spaces, with results for both local and global well posedness results, [13–16, 29, 32, 33]. Much of the analysis relies on a transformation related to (1.2) that turns the equation into two coupled semilinear Schrödinger equations with no derivative. In addition, Hayashi [13, 14, 16] identified a smallness condition on the data

$$(1.18) ||u_0||_{L^2} < \sqrt{2\pi},$$

for which global solutions exist in  $H^s$ ,  $s \in \mathbb{N}$ . The constant  $\sqrt{2\pi}$  is the  $L^2$ -norm of the ground state of the quintic NLS soliton. More recently, the global in time result for data satisfying (1.18) were extended to  $H^s$  spaces with s > 1/2 in [3]. DNLS with low regularity has also been studied on the torus, [9, 10].

There has also been progress beyond the cubic equation in the aforementioned results. Some studies, such as [6, 29, 32], include additive terms to the cubic nonlinearity with derivative. More generally, Kenig, Ponce and Vega [19, 20] used viscosity methods to show that, for general quasilinear Schrödinger with polynomial nonlinearities, local wellposedness holds in Sobolev spaces of high enough index (See Linares-Ponce [22] for a review). In [12], Hao proved that (1.3) is locally well-posed in  $H^{1/2}$  intersected with an appropriate Strichartz space for  $\sigma \geq 5/2$ . Working in the Schwartz space, Lee [21] used the framework of inverse scattering to show that DNLS is globally well-posed for a dense subset of initial conditions, excluding certain non generic ones.

### 2. Problem Setup

In this section, we define the linearized operator, the invariant quantities and the action functional of the soliton. Throughout, we adopt the notation of [8].

We study the problem in the space  $X = H^1(\mathbb{R})$ , with real inner product

(2.1) 
$$(u,v) \equiv \Re \int_{\mathbb{R}} (u_x \bar{v}_x + u\bar{v}) dx.$$

The dual of X is  $X^* = H^{-1}(\mathbb{R})$ . Let  $I : X \to X^*$  be the natural isomorphism defined by

(2.2) 
$$\langle Iu, v \rangle = (u, v),$$

where  $\langle \cdot, \cdot \rangle$  is the pairing between X and  $X^*$ ,

(2.3) 
$$\langle f, u \rangle = \Re \int_{\mathbb{R}} f \bar{u} dx$$

gDNLS can be formulated as the Hamiltonian system

(2.4) 
$$\frac{d\psi}{dt} = JE'(\psi),$$

where the map  $J: X^* \to X$  is J = -i, and E is the Hamiltonian:

(2.5) 
$$E \equiv \frac{1}{2} \int_{-\infty}^{\infty} |\psi_x|^2 dx + \frac{1}{2(\sigma+1)} \Im \int_{-\infty}^{\infty} |\psi|^{2\sigma} \bar{\psi} \psi_x dx. \quad \text{(Energy)}$$

Two other conserved quantities are:

(2.6) 
$$Q \equiv \frac{1}{2} \int_{-\infty}^{\infty} |\psi|^2 dx, \quad (\text{Mass})$$

(2.7) 
$$P \equiv -\frac{1}{2}\Im \int_{-\infty}^{\infty} \bar{\psi}\psi_x dx. \quad (\text{Momentum})$$

Let  $T_1$  and  $T_2$  be the one-parameter groups of unitary operator on X defined by

(2.8a) 
$$T_1(t)\Phi(x) \equiv e^{-it}\Phi(x), \quad \Phi(x) \in X, \quad t \in \mathbb{R}$$

(2.8b) 
$$T_2(t)\Phi(x) \equiv \Phi(x+t).$$

Then

$$\psi_{\omega,c} = T_1(-\omega t)T_2(-ct)\phi_{\omega,c}(x).$$

and

$$\partial_t T_1(\omega t)|_{t=0} = -i\omega, \ \partial_t T_2(ct)|_{t=0} = c\partial_x$$

We define the linear operators

(2.9a) 
$$B_1 \equiv J^{-1} \partial_t T_1(\omega t)|_{t=0} = \omega,$$

(2.9b) 
$$B_2 \equiv J^{-1}\partial_t T_2(ct)|_{t=0} = ic\partial_x.$$

The mass and momentum invariants (2.6) and (2.7) are thus related to the symmetry groups via:

(2.10a) 
$$Q_1 \equiv \frac{1}{2} \langle B_1 \phi, \phi \rangle = \frac{1}{2} \Re \int \omega \phi \bar{\phi} dx = \frac{\omega}{2} \int |\phi|^2 dx = \omega Q,$$
  
(2.10b)  $Q_2 \equiv \frac{1}{2} \langle B_2 \phi, \phi \rangle = \frac{1}{2} \Re \int i c \partial_x \phi \bar{\phi} dx = -\frac{1}{2} c \Im \int \partial_x \phi \bar{\phi} dx = cP.$ 

Computing the first variations of (2.5), (2.6) and (2.7), we have

(2.11a) 
$$E'(\phi) = -\partial_x^2 \phi - i|\phi|^{2\sigma} \partial_x \phi,$$

(2.11b)  $Q'(\phi) = \phi, \quad P'(\phi) = i\partial_x \phi.$ 

The second variations are:

(2.12a) 
$$E''(\phi)v = (-\partial_x^2 - i\sigma|\phi|^{2\sigma-2}\bar{\phi}\partial_x\phi - i|\phi|^{2\sigma}\partial_x)v - i\sigma|\phi|^{2\sigma-2}\phi\partial_x\phi\bar{v},$$

(2.12b) 
$$Q''(\phi)v = v, \quad P''(\phi)v = i\partial_x v.$$

2.1. Linearized Hamiltonian. The linearized Hamiltonian about the soliton  $\phi$  is

(2.13) 
$$H_{\phi}u \equiv [E''(\phi) + Q_1''(\phi) + Q_2''(\phi)] u \\ = [E''(\phi) + \omega Q''(\phi) + cP''(\phi)] u$$

for  $u \in H^2(\mathbb{R})$ . For later use, we give two equivalent expressions of  $H_{\phi}$ . First, we decompose it into complex conjugates:

**Lemma 2.1.** For any function u in the domain of  $H_{\phi}$ ,

$$H_{\phi}u = L_1u + L_2\bar{u},$$

where

(2.14a) 
$$L_1 \equiv -\partial_x^2 + \omega + ic\partial_x - i\sigma|\phi|^{2\sigma-2}\bar{\phi}\partial_x\phi - i|\phi|^{2\sigma}\partial_x,$$

(2.14b) 
$$L_2 \equiv -i\sigma |\phi|^{2\sigma-2} \phi \partial_x \phi$$

Second, we give the expression of  $H_{\phi}$  and the quadratic form it induces, after extraction of the soliton's phase:

**Lemma 2.2.** Let  $u \in H^1$  be decomposed as

(2.15) 
$$u = e^{i\theta}(u_1 + iu_2),$$

where  $\theta$  is given by (1.12), and  $u_1$  and  $u_2$  are the real and imaginary parts of  $ue^{-i\theta}$ . Then

(2.16) 
$$H_{\phi}u = e^{i\theta} \left[ (L_{11}u_1 + L_{21}u_2) + i \left( L_{12}u_1 + L_{22}u_2 \right) \right]$$

and

(2.17) 
$$\langle H_{\phi}u, u \rangle = \langle L_{11}u_1, u_1 \rangle + \langle L_{21}u_2, u_1 \rangle + \langle L_{12}u_1, u_2 \rangle + \langle L_{22}u_2, u_2 \rangle,$$

where

(2.18a) 
$$L_{11} \equiv -\partial_{yy} + \omega - \frac{c^2}{4} + \frac{c(2\sigma+1)}{2}\varphi^{2\sigma} - \frac{4\sigma^2 + 6\sigma + 1}{4(\sigma+1)^2}\varphi^{4\sigma},$$

(2.18b) 
$$L_{21} \equiv -\frac{\sigma}{\sigma+1}\varphi^{2\sigma-1}\varphi_y + \frac{\sigma}{\sigma+1}\varphi^{2\sigma}\partial_y$$

(2.18c) 
$$L_{12} \equiv -\frac{(2\sigma+1)\sigma}{\sigma+1}\varphi^{2\sigma-1}\varphi_y - \frac{\sigma}{\sigma+1}\varphi^{2\sigma}\partial_y,$$

(2.18d) 
$$L_{22} \equiv -\partial_{yy} + \omega - \frac{c^2}{4} + \frac{c}{2}\varphi^{2\sigma} - \frac{2\sigma + 1}{4(\sigma + 1)^2}\varphi^{4\sigma}.$$

Proof. Using (1.12),

(2.19) 
$$\theta_y = \frac{c}{2} - \frac{1}{2\sigma + 2}\varphi^{2\sigma}, \quad \theta_{yy} = -\frac{\sigma}{\sigma + 1}\varphi^{2\sigma - 1}\varphi_y.$$

We can then rewrite the operators  $L_1$  and  $L_2$  from Lemma 2.1 in terms of  $\varphi$  as

$$L_{1} = -\partial_{y}^{2} + \omega + ic\partial_{y} - i\varphi^{2\sigma}\partial_{y} + \frac{\sigma c}{2}\varphi^{2\sigma} - \frac{\sigma}{2\sigma + 2}\varphi^{4\sigma} - i\sigma\varphi^{2\sigma - 1}\varphi_{y}, L_{2} = \left[\frac{c\sigma}{2}\varphi^{2\sigma} - \frac{\sigma}{2\sigma + 2}\varphi^{4\sigma} - i\sigma\varphi^{2\sigma - 1}\varphi_{y}\right]e^{2i\theta}$$

Letting  $\chi = u e^{-i\theta}$ , we have

$$(2.20) H_{\phi}u = e^{i\theta} \left[ -\partial_{yy} + \omega - \frac{c^2}{4} + \frac{c(\sigma+1)}{2}\varphi^{2\sigma} - \frac{i\sigma^2}{\sigma+1}\varphi^{2\sigma-1}\varphi_y - \frac{i\sigma}{\sigma+1}\varphi^{2\sigma}\partial_y - \frac{2\sigma^2 + 4\sigma + 1}{4(\sigma+1)^2}\varphi^{4\sigma} \right] \chi + e^{i\theta} \left[ \frac{c\sigma}{2}\varphi^{2\sigma} - \frac{\sigma}{2\sigma+2}\varphi^{4\sigma} - i\sigma\varphi^{2\sigma-1}\varphi_y \right] \bar{\chi}.$$

Since  $\chi = u_1 + iu_2$ , we get (2.16) and (2.17) by grouping terms appropriately.

2.2. Scalar Soliton Function. Using (1.14), we observe that when evaluated at the soliton  $\phi$ ,

(2.21) 
$$E' + \omega Q' + cP' = 0.$$

For any  $\omega > c^2/4$ , we define the scalar function

(2.22) 
$$d(\omega, c) \equiv E(\phi_{\omega,c}) + Q_1(\phi_{\omega,c}) + Q_2(\phi_{\omega,c}),$$

which is the action functional evaluated at the soliton. It has the following properties:

Lemma 2.3.

(2.23) 
$$d(\omega, c) = E(\phi) + \omega Q(\phi) + cP(\phi),$$

(2.24) 
$$\partial_{\omega} d(\omega, c) = Q(\phi) > 0,$$

(2.25) 
$$\partial_c d(\omega, c) = P(\phi).$$

The Hessian is

(2.26) 
$$d''(\omega,c) = \begin{pmatrix} \partial_{\omega}Q(\phi) & \partial_{c}Q(\phi) \\ \partial_{\omega}P(\phi) & \partial_{c}P(\phi) \end{pmatrix}.$$

*Proof.* Using (2.10a), (2.10b) and (2.22), we have (2.23). Differentiating (2.23) with respect to  $\omega$  and c respectively and using (2.21), we obtain (2.24) and (2.25). The expression for the Hessian follows.

#### 3. Spectral Decomposition of the Linearized Operator

This section provides a full description of the spectrum of the linearized operator  $H_{\phi}$ . In particular, we prove:

**Theorem 3.1.** For all values of  $\sigma > 0$  and admissible  $(\omega, c)$ , the space  $X = H^1$  can be decomposed as the direct sum

$$(3.1) X = N + Z + P,$$

where the three subspaces intersect trivially and:

(i) N is a one dimensional subspace such that for  $u \in N$ ,  $u \neq 0$ ,

(3.2) 
$$\langle H_{\phi}u, u \rangle < 0.$$

(ii) Z is the two dimensional kernel of  $H_{\phi}$ .

(iii) P is a subspace such that for  $p \in P$ ,

(3.3) 
$$\langle H_{\phi}p,p\rangle \ge \delta \|p\|_X^2$$

where the constant  $\delta > 0$  is independent of p.

**Corollary 3.2.** For all values of  $\sigma > 0$  and admissible  $(\omega, c)$ ,

$$n(H_{\phi}) = 1.$$

An important ingredient of the proof involves rewriting the quadratic form (2.17) induced by  $H_{\phi}$  in a more favorable form. This rearrangement, inspired by [11], expresses it as a sum of a quadratic form involving an operator with exactly one negative eigenvalue and a nonnegative term.

## Lemma 3.3. Let

$$u = e^{i\theta}(u_1 + iu_2),$$

where  $\theta, u_1, u_2$  are defined the same as Corollary 2.2, then (3.4)

$$\langle H_{\phi}u,u\rangle = \langle \widetilde{L}_{11}u_1,u_1\rangle + \int_{-\infty}^{\infty} \left[\varphi\left(\varphi^{-1}u_2\right)_y + \frac{\sigma}{(\sigma+1)}\varphi^{2\sigma}u_1\right]^2 dy,$$

where

(3.5) 
$$\widetilde{L}_{11} \equiv -\partial_{yy} + \omega - \frac{c^2}{4} + \frac{c(2\sigma+1)}{2}\varphi^{2\sigma} - \frac{8\sigma^2 + 6\sigma + 1}{4(\sigma+1)^2}\varphi^{4\sigma}.$$

*Proof.* Recall the terms in the quadratic form (2.18). We first examine  $L_{11}$ . The relationship between  $L_{11}$  and  $\tilde{L}_{11}$  is

(3.6) 
$$L_{11} = \widetilde{L}_{11} + \frac{\sigma^2}{(\sigma+1)^2} \varphi^{4\sigma}$$

Next, consider  $L_{22}$ . From (1.10),

$$(3.7) L_{22}\varphi = 0.$$

Letting  $\tilde{u}_2 = \varphi^{-1} u_2$ , we can then write (3.8)

$$\begin{split} \langle L_{22}u_2, u_2 \rangle &= \langle -\partial_{yy}u_2, u_2 \rangle + \left\langle \left(\omega - \frac{c^2}{4} + \frac{c}{2}\varphi^{2\sigma} - \frac{2\sigma + 1}{4(\sigma + 1)^2}\varphi^{4\sigma}\right)u_2, u_2 \right\rangle \\ &= \langle -\varphi_{yy}\tilde{u}_2 - 2\varphi_y\tilde{u}_{2y} - \varphi\tilde{u}_{2yy}, \varphi\tilde{u}_2 \rangle \\ &+ \langle \left(\omega - \frac{c^2}{4} + \frac{c}{2}\varphi^{2\sigma} - \frac{2\sigma + 1}{4(\sigma + 1)^2}\varphi^{4\sigma}\right)\varphi\tilde{u}_2, \varphi\tilde{u}_2 \rangle \\ &= \langle \tilde{u}_2 L_{22}\varphi, \varphi\tilde{u}_2 \rangle + \langle -2\varphi_y\tilde{u}_{2y} - \varphi\tilde{u}_{2yy}, \varphi\tilde{u}_2 \rangle \\ &= \langle -(\varphi^2\tilde{u}_{2y})_y, \tilde{u}_2 \rangle = \langle \varphi\tilde{u}_{2y}, \varphi\tilde{u}_{2y} \rangle, \end{split}$$

where  $u_{2y}$  and  $u_{2yy}$  denote  $\partial_y u_2$  and  $\partial_{yy} u_2$ , respectively. Lastly, we simplify the off diagonal entries,  $L_{21}$  and  $L_{12}$ . Integrating by parts, we have

$$\begin{split} \langle L_{12}u_1, u_2 \rangle &= \left\langle \left( -\frac{(2\sigma+1)\sigma}{\sigma+1} \varphi^{2\sigma-1} \varphi_y - \frac{\sigma}{\sigma+1} \varphi^{2\sigma} \partial_y \right) u_1, u_2 \right\rangle \\ &= -\frac{2\sigma+1}{2(\sigma+1)} \left\langle (\varphi^{2\sigma})_y, u_1 u_2 \right\rangle - \frac{\sigma}{\sigma+1} \left\langle \varphi^{2\sigma} u_{1y}, u_2 \right\rangle \\ &= \frac{2\sigma+1}{2(\sigma+1)} \left\langle \varphi^{2\sigma} u_{2y}, u_1 \right\rangle + \frac{2\sigma+1}{2(\sigma+1)} \left\langle \varphi^{2\sigma} u_{1y}, u_2 \right\rangle - \frac{\sigma}{\sigma+1} \left\langle \varphi^{2\sigma} u_{1y}, u_2 \right\rangle \\ &= \frac{2\sigma+1}{2(\sigma+1)} \left\langle \varphi^{2\sigma} u_{2y}, u_1 \right\rangle + \frac{1}{2(\sigma+1)} \left\langle \varphi^{2\sigma} u_{1y}, u_2 \right\rangle. \end{split}$$

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Similarly,

$$\begin{split} \langle L_{21}u_2, u_1 \rangle &= \left\langle \left( -\frac{\sigma}{\sigma+1} \varphi^{2\sigma-1} \varphi_y + \frac{\sigma}{\sigma+1} \varphi^{2\sigma} \partial_y \right) u_2, u_1 \right\rangle \\ &= -\frac{1}{2(\sigma+1)} \left\langle (\varphi^{2\sigma})_y, u_1 u_2 \right\rangle + \frac{\sigma}{\sigma+1} \left\langle \varphi^{2\sigma} u_{2y}, u_1 \right\rangle \\ &= \frac{1}{2(\sigma+1)} \left\langle \varphi^{2\sigma} u_{2y}, u_1 \right\rangle + \frac{1}{2(\sigma+1)} \left\langle \varphi^{2\sigma} u_{1y}, u_2 \right\rangle + \frac{\sigma}{\sigma+1} \left\langle \varphi^{2\sigma} u_{2y}, u_1 \right\rangle \\ &= \frac{2\sigma+1}{2(\sigma+1)} \left\langle \varphi^{2\sigma} u_{2y}, u_1 \right\rangle + \frac{1}{2(\sigma+1)} \left\langle \varphi^{2\sigma} u_{1y}, u_2 \right\rangle. \end{split}$$

The off diagonal terms then sum to

$$\left\langle L_{12}u_1, u_2 \right\rangle + \left\langle L_{21}u_2, u_1 \right\rangle = \frac{2\sigma + 1}{\sigma + 1} \left\langle \varphi^{2\sigma}u_{2y}, u_1 \right\rangle + \frac{1}{\sigma + 1} \left\langle \varphi^{2\sigma}u_{1y}, u_2 \right\rangle.$$

Introducing  $\tilde{u}_2 = \varphi^{-1} u_2$  into the above expression, and integrating by parts,

(3.9) 
$$\langle L_{12}u_1, u_2 \rangle + \langle L_{21}u_2, u_1 \rangle = \frac{2\sigma}{\sigma+1} \left\langle \varphi^{2\sigma+1}\tilde{u}_{2y}, u_1 \right\rangle$$

Combining (3.6), (3.8) and (3.9),

$$\begin{split} \langle H_{\phi}u,u\rangle = &\langle \widetilde{L}_{11}u_1,u_1\rangle + \langle \varphi \widetilde{u}_{2y},\varphi \widetilde{u}_{2y}\rangle \\ &+ \left\langle \frac{\sigma}{\sigma+1}\varphi^{2\sigma}u_1,\frac{\sigma}{\sigma+1}\varphi^{2\sigma}u_1\right\rangle + \left\langle \frac{2\sigma}{\sigma+1}\varphi^{2\sigma}u_1,\varphi \widetilde{u}_{2y}\right\rangle \\ &= &\langle \widetilde{L}_{11}u_1,u_1\rangle + \int_{-\infty}^{\infty} \left[\varphi \widetilde{u}_{2y} + \frac{\sigma}{\sigma+1}\varphi^{2\sigma}u_1\right]^2 dy. \end{split}$$

3.1. The Negative Subspace. Next, we characterize the negative subspace, N. For that, we need the following lemma on  $\widetilde{L}_{11}$ .

**Lemma 3.4.** The spectrum of  $\widetilde{L}_{11}$  can be characterized as follows:

- $\widetilde{L}_{11}$  has exactly one negative eigenvalue, denoted  $-\lambda_{11}^2$ , with multiplicity one, and eigenfunction  $\chi_{11}$ ,
- $0 \in \sigma(\widetilde{L}_{11})$ , and the kernel is spanned by  $\varphi_y$ ,
- There exists  $\mu_{11} > 0$  such that

$$\sigma(\widetilde{L}_{11}) \setminus \left\{-\lambda_{11}^2, 0\right\} \subset [\mu_{11}, \infty).$$

*Proof.* First, we observe that since  $\varphi$  is exponentially localized,  $\widetilde{L}_{11}$  is a relatively compact perturbation of  $-\partial_y^2 + \omega - \frac{c^2}{4}$ . By Weyl's theorem,

the essential spectrum is then

$$\sigma_{\rm ess}(\widetilde{L}_{11}) = \sigma_{\rm ess}(-\partial_y^2 + \omega - \frac{c^2}{4}) = \left[\omega - \frac{c^2}{4}, \infty\right).$$

Consequently, all eigenvalues below the lower bound of the essential spectrum correspond to isolated eigenvalues of finite multiplicity. By differentiating (1.10) with respect to y, we see that

Hence,  $\widetilde{L}_{11}$  has a kernel. Viewed as a linear second order ordinary differential equation,  $\widetilde{L}_{11}f = 0$  has two linearly independent solutions. As  $y \to -\infty$ , one solution decays exponentially while the other grows exponentially. Thus, up to a multiplicative constant, there can be at most one spatially localized solution to  $\widetilde{L}_{11}f = 0$ . Therefore, the kernel is spanned by  $\varphi_y$ .

From Sturm-Liouville theory, this implies that zero is the second eigenvalue of  $\tilde{L}_{11}$ , and  $\tilde{L}_{11}$  has exactly one strictly negative eigenvalue,  $-\lambda_{11}^2$ , with a  $L^2$  normalized eigenfunction  $\chi_{11}$ :

(3.11) 
$$\widetilde{L}_{11}\chi_{11} = -\lambda_{11}^2\chi_{11}.$$

If we now let

(3.12) 
$$\mu_{11} \equiv \inf_{f \neq 0, f \perp \varphi_y, f \perp \chi_{11}} \frac{\left\langle \widetilde{L}_{11}f, f \right\rangle}{\left\langle f, f \right\rangle}$$

we see that  $\mu_{11} > 0$ , since if it were not, it would correspond to another discrete eigenvalue less than or equal to zero. It is either a discrete eigenvalue in the gap  $(0, \omega - \frac{c^2}{4})$  or the base of the essential spectrum. Regardless,  $\sigma(\tilde{L}_{11}) \setminus \{-\lambda_{11}^2, 0\}$  is bounded away from zero.

Using  $\chi_{11}$ , we construct the negative subspace N.

### Proposition 3.5. Let

$$(3.13) N \equiv \operatorname{span}\left\{\chi_{-}\right\}$$

where

(3.14a) 
$$\chi_{-} \equiv (\chi_{11} + i\chi_{12})e^{i\theta},$$
  
(3.14b) 
$$\chi_{12} \equiv \varphi \left[ -\frac{\sigma}{\sigma+1} \int_{-\infty}^{y} \varphi^{2\sigma-1}(s)\chi_{11}(s)ds + k_{12} \right]$$

and  $k_{12} \in \mathbb{R}$  is chosen such that

(3.15) 
$$\langle \chi_{12}, \varphi \rangle = 0.$$

For  $u \in N \setminus \{0\}$ ,

$$\langle H_{\phi}u, u \rangle < 0.$$

*Proof.* The function  $\chi_{12}$  is in  $L^2$ . Indeed, the integral in (3.14b) is well defined since, as  $|y| \to \infty$ ,

$$\begin{aligned} |\varphi(y)| &\lesssim \exp\left\{-\sqrt{\omega - c^2/4} \,|y|\right\},\\ |\chi_{11}(y)| &\lesssim \exp\left\{-\sqrt{\omega - c^2/4 + \lambda_{11}^2} \,|y|\right\}.\end{aligned}$$

Thus the integrand is bounded.

From (3.4) and (3.11),

$$\langle H_{\phi}\chi_{-},\chi_{-}\rangle = \langle \widetilde{L}_{11}\chi_{11},\chi_{11}\rangle = -\lambda_{11}^{2} < 0.$$

3.2. The Kernel. In this subsection, we give an explicit characterization of the kernel of  $H_{\phi}$ .

## Proposition 3.6. Let

$$(3.16) Z = \operatorname{span} \{\chi_1, \chi_2\}$$

where

(3.17a) 
$$\chi_1 = \left(\varphi_y + i(k_2 - \frac{1}{2\sigma + 2}\varphi^{2\sigma})\varphi\right)e^{i\theta},$$

(3.17b) 
$$\chi_2 = i\varphi e^{i\theta}$$

with  $k_2$  is a real constant such that

(3.18) 
$$\left\langle \left(k_2 - \frac{1}{2\sigma + 2}\varphi^{2\sigma}\right)\varphi, \varphi \right\rangle = 0.$$

Then  $Z = \ker H_{\phi}$ .

*Proof.* We first prove that  $\chi_1$  and  $\chi_2$  are linearly independent elements of the kernel, and then show that the kernel is at most two dimensional. Applying  $H_{\phi}$  (in the form (2.16)) to  $\chi_2$  and using that  $L_{21}\varphi = 0$  and (3.7), we get  $H_{\phi}\chi_2 = 0$ . For  $\chi_1$ , we compute

$$L_{11}\varphi_y + L_{21}(k_2 - \frac{1}{2\sigma + 2}\varphi^{2\sigma})\varphi$$
  
=  $\widetilde{L}_{11}\varphi_y + \frac{\sigma^2}{(\sigma + 1)^2}\varphi^{4\sigma}\varphi_y + k_2L_{21}\varphi - \frac{1}{2\sigma + 2}L_{21}\varphi^{2\sigma + 1}$   
=  $\frac{\sigma^2}{(\sigma + 1)^2}\varphi^{4\sigma}\varphi_y - \frac{1}{2\sigma + 2}\frac{2\sigma^2}{\sigma + 1}\varphi^{4\sigma}\varphi_y = 0$ 

and

$$L_{12}\varphi_y + L_{22}(k_2 - \frac{1}{2\sigma + 2}\varphi^{2\sigma})\varphi$$
  
=  $-\frac{2\sigma^2 + \sigma}{\sigma + 1}\varphi^{2\sigma - 1}\varphi_y^2 - \frac{\sigma}{\sigma + 1}\varphi^{2\sigma}\varphi_{yy}$   
 $-\frac{1}{2\sigma + 2}\left[-2\sigma(2\sigma + 1)\varphi^{2\sigma - 1}\varphi_y^2 - 2\sigma\varphi^{2\sigma}\varphi_{yy} + \varphi^{2\sigma}L_{22}\varphi\right] = 0.$ 

Thus,  $Z \subset \ker H_{\phi}$ , and the kernel is at least two dimensional.

We now show that it is exactly two dimensional. If we consider the problem

$$H_{\phi}f = 0$$

as a second order system of two real valued functions, we know there are four linearly independent solutions. As  $y \to -\infty$ , two of these solutions decay exponentially, while two grow exponentially. Thus, there are at most two linearly independent solutions which are spatially localized. Hence,  $Z = \ker H_{\phi}$ .

3.3. The Positive Subspace and Proof of the Spectral Decomposition. We define the subspace P and prove Theorem 3.1. For that, we need the following lemmas about  $\tilde{L}_{11}$  and  $L_{22}$ .

**Lemma 3.7.** For any real function  $f \in H^1(\mathbb{R})$  satisfying the orthogonality conditions

(3.19) 
$$\langle f, \varphi_y \rangle = \langle f, \chi_{11} \rangle = 0,$$

there exists a positive number  $\delta_{11} > 0$ , such that

(3.20) 
$$\langle \tilde{L}_{11}f, f \rangle \ge \delta_{11} ||f||_{H^1}^2$$

*Proof.* From Lemma 3.4, (3.12) holds on the subspace orthogonal to  $\varphi_y$  and  $\chi_{11}$ , so

$$\langle \widetilde{L}_{11}f, f \rangle \ge \mu_{11} \|f\|_{L_2}^2.$$

To get the  $H^1$  lower bound, let

$$V_1(y) = \omega - \frac{c^2}{4} + \frac{c(2\sigma+1)}{2}\varphi^{2\sigma} - \frac{8\sigma^2 + 6\sigma + 1}{4(\sigma+1)^2}\varphi^{4\sigma},$$

so that  $\widetilde{L}_{11} = -\partial_{yy} + V_1$ , with  $\|V_1\|_{L^{\infty}} < \infty$ . Thus,

$$\begin{split} \langle \widetilde{L}_{11}f, f \rangle &= \langle -\partial_{yy}f, f \rangle + \langle V_1f, f \rangle \\ &\geq \langle -\partial_{yy}f, f \rangle - \|V_1\|_{L^{\infty}} \|f\|_{L^2}^2 \\ &= \|\partial_y f\|_{L^2}^2 - \frac{1}{\mu_{11}} \|V_1\|_{L^{\infty}} \langle \widetilde{L}_{11}f, f \rangle \end{split}$$

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It follows that

$$\langle \widetilde{L}_{11}f, f \rangle \ge \frac{1}{1 + \mu_{11}^{-1} \|V_1\|_{L^{\infty}}} \|\partial_y f\|_{L^2}^2$$

Taking  $\delta_{11}$  sufficiently small, we have

$$\langle \widetilde{L}_{11}f, f \rangle \ge \delta_{11} ||f||_{H^1}^2.$$

**Lemma 3.8.** For any real function  $f \in H^1(\mathbb{R})$  satisfying

(3.21) 
$$\langle f, \varphi \rangle = 0,$$

there exists a positive number  $\delta_{22} > 0$ , such that

(3.22) 
$$\langle L_{22}f, f \rangle \ge \delta_{22} \|f\|_{H^1}^2.$$

*Proof.* As was the case for  $\widetilde{L}_{11}$ ,  $L_{22}$  is a relatively compact perturbation of  $-\partial_y^2 + \omega - c^2/4$ , so it also has

$$\sigma_{\rm ess}(L_{22}) = \left[\omega - \frac{c^2}{4}, \infty\right).$$

Thus, all points in the spectrum below  $\omega - c^2/4$  correspond to discrete eigenvalues. From (3.7) and  $\varphi$  is strictly positive, Sturm-Liouville theory tells us that zero is the lowest eigenvalue. Let

(3.23) 
$$\mu_{22} \equiv \inf_{f \neq 0, f \perp \varphi} \frac{\langle L_{22}f, f \rangle}{\langle f, f \rangle}.$$

We know that  $\mu_{22} > 0$ , otherwise this would contradict with 0 being the lowest eigenvalue. Therefore

$$\langle L_{22}f, f \rangle \ge \mu_{22} \|f\|_{L^2}$$
.

Using the same argument as in Lemma 3.7, we obtain (3.22).

We now prove Theorem 3.1.

*Proof.* Recall N, Z as defined define by (3.13) and (3.16). We define P as

(3.24)  

$$P = \{ p \in X \mid \langle \Re(e^{-i\theta}p), \chi_{11} \rangle = \langle \Re(e^{-i\theta}p), \varphi_y \rangle = \langle \Im(e^{-i\theta}p), \varphi \rangle = 0 \}$$
We express  $u \in X$  as

We express  $u \in X$  as

$$u = a_1 \chi_- + (b_1 \chi_1 + b_2 \chi_2) + p,$$

where

$$a_1 = \langle u_1, \chi_{11} \rangle, \quad b_1 = \frac{\langle u_1, \varphi_y \rangle}{\|\varphi_y\|_{L^2}^2}, \quad b_2 = \frac{\langle u_2, \varphi \rangle}{\|\varphi\|_{L^2}^2},$$

with  $u_1$  and  $u_2$  are real and imaginary part of  $e^{-i\theta}u$ . Clearly,  $a_1\chi_- \in N$ and  $b_1\chi_1 + b_2\chi_2 \in Z$ . It suffices to show  $p \in P$ . We write  $p = (p_1 + ip_2)e^{i\theta}$  with  $p_1$  and  $p_2$  real. Since  $\varphi_y$  is odd and  $\chi_{11}$  is even,  $\langle \varphi_y, \chi_{11} \rangle = 0$ , and we readily check that  $\langle p_1, \chi_{11} \rangle = \langle p_1, \varphi_y \rangle = 0$ . Furthermore, by (3.15) and (3.18), we also have  $\langle p_2, \varphi \rangle = 0$ . Thus,  $p \in P$ , and u is indeed decomposed into elements of N, Z and P.

Finally, we show that  $H_{\phi}$  is positive on P. Let  $\tilde{p}_2 = \varphi^{-1}p_2$ . By (3.4),

(3.25) 
$$\langle H_{\phi}p,p\rangle = \langle \widetilde{L}_{11}p_1,p_1\rangle + \int_{-\infty}^{\infty} (\varphi \partial_y \widetilde{p}_2 + \frac{\sigma}{\sigma+1} \varphi^{2\sigma} p_1)^2 dy.$$

Lemma 3.7 gives the desired lower bound on the first term. For the second term, we break it into two cases, depending on how  $\|\varphi \partial_y \tilde{p}_2\|_{L^2}$  and  $\|p_1\|_{L^2}$  compare. Let

(3.26) 
$$C_{\sigma} \equiv \frac{2\sigma}{\sigma+1} \|\varphi\|_{L^{\infty}}^{2\sigma}$$

(a) If  $\|\varphi \partial_y \tilde{p}_2\|_{L^2} \ge C_{\sigma} \|p_1\|_{L^2}$ , we estimate the second term in (3.25) as follows,

$$\begin{aligned} \left\| \varphi \partial_y \tilde{p}_2 + \frac{\sigma}{\sigma+1} \varphi^{2\sigma} p_1 \right\|_{L^2} &\geq \left\| \varphi \partial_y \tilde{p}_2 \right\|_{L^2} - \frac{\sigma}{\sigma+1} \left\| \varphi \right\|_{L^\infty}^{2\sigma} \left\| p_1 \right\|_{L^2} \\ &= \left\| \varphi \partial_y \tilde{p}_2 \right\|_{L^2} - \frac{1}{2} C_\sigma \left\| p_1 \right\|_{L^2} \geq \frac{1}{2} \left\| \varphi \partial_y \tilde{p}_2 \right\|_{L^2} \end{aligned}$$

By (3.8), we then have

$$\left\|\varphi\partial_y \tilde{p}_2 + \frac{\sigma}{\sigma+1}\varphi^{2\sigma}p_1\right\|_{L^2}^2 \ge \frac{1}{4} \left\langle L_{22}p_2, p_2 \right\rangle$$

By Lemmas 3.7 and 3.8, we get

$$\langle H_{\phi}p,p\rangle \geq \left\langle \widetilde{L}_{11}p_1,p_1\right\rangle + \frac{1}{4}\left\langle L_{22}p_2,p_2\right\rangle \geq \delta_a \left\|p\right\|_{H^1}^2,$$

for some small enough  $\delta_a$ .

(b) If instead,  $\|\varphi \partial_y \tilde{p}_2\|_{L^2} < C_{\sigma} \|p_1\|_{L^2}$ , then,

(3.27) 
$$\langle H_{\phi}p,p\rangle \geq \left\langle \widetilde{L}_{11}p_{1},p_{1}\right\rangle \geq \frac{\delta_{11}}{2} \|p_{1}\|_{H^{1}}^{2} + \frac{\delta_{11}}{2} \|p_{1}\|_{L^{2}}^{2} \\ \geq \frac{\delta_{11}}{2} \|p_{1}\|_{H^{1}}^{2} + \frac{\delta_{11}}{2C_{\sigma}^{2}} \|\varphi \partial_{y}\widetilde{p}_{2}\|_{L^{2}}^{2} \\ = \frac{\delta_{11}}{2} \|p_{1}\|_{H^{1}}^{2} + \frac{\delta_{11}}{2C_{\sigma}^{2}} \langle L_{22}p_{2},p_{2}\rangle \geq \delta_{b} \|p\|_{H^{1}}^{2}$$

Taking the smaller value of  $\delta_a$  and  $\delta_b$  as  $\delta$ , we have

(3.28) 
$$\langle H_{\phi}p,p\rangle \ge \delta \|p\|_{H^1}^2.$$

It follows that N, Z and P have trivial intersection amongst one another. Hence X = N + Z + P.

## 4. Analysis of the Hessian Matrix

In this section, we compute the number of the positive eigenvalues of the Hessian matrix of  $d(\omega, c)$ ,  $p(d''(\omega, c))$ . Since the number of negative eigenvalues of  $H_{\phi_{\omega,c}}$  is in all cases equal to one,  $p(d''(\omega, c))$  will determine whether or not the soliton is stable.

To make this assessment, we examine the determinant and the trace of  $d''(\omega, c)$ . From Lemmas 2.3 and A.3, the determinant can be expressed as

(4.1)  

$$\det[d''(\omega,c)] = \partial_{\omega}Q\partial_{c}P - \partial_{c}Q\partial_{\omega}P \\
= 2^{-\frac{2}{\sigma}-4}\sigma^{-2}(1+\sigma)^{\frac{2}{\sigma}}(4\omega-c^{2})^{\frac{2}{\sigma}-1}\omega^{-\frac{1}{\sigma}-2} \\
\times \left[4(\sigma-1)\omega\alpha_{0}-2\sqrt{\omega}c\alpha_{0}+(4\omega-c^{2})\alpha_{1}\right] \\
\times \left[4(\sigma-1)\omega\alpha_{0}+2\sqrt{\omega}c\alpha_{0}-(4\omega-c^{2})\alpha_{1}\right],$$

where

$$\alpha_n(\omega, c; \sigma) \equiv \int_0^\infty h^{-\frac{1}{\sigma} - n} dx > 0,$$
  
$$h(x; \sigma; \omega, c) \equiv \cosh(\sigma \sqrt{4\omega - c^2} x) - \frac{c}{2\sqrt{\omega}}.$$

Meanwhile, the trace is

(4.2) 
$$\operatorname{tr}[d''(\omega, c)] = \partial_{\omega}Q + \partial_{c}P = 2^{-\frac{1}{\sigma}-2}\sigma^{-1}(1+\sigma)^{\frac{1}{\sigma}}(4\omega-c^{2})^{\frac{1}{\sigma}-1}(1+\omega)\omega^{-\frac{1}{2\sigma}-\frac{3}{2}} \times (c(c^{2}-4\omega)\alpha_{1}+2\sqrt{\omega}(c^{2}-4(\sigma-1)\omega)\alpha_{0}).$$

**Theorem 4.1.** If  $\sigma \ge 2$ , and  $4\omega > c^2$ ,  $p(d''(\omega, c)) = 0$ .

*Proof.* We examine the terms appearing in (4.1). The first term is clearly positive. The second term is also positive,

$$4(\sigma - 1)\omega\alpha_0 - 2\sqrt{\omega}c\alpha_0 + (4\omega - c^2)\alpha_1$$
$$= 4\omega \left[ (\sigma - 1 - \frac{c}{2\sqrt{\omega}})\alpha_0 + (1 - \frac{c^2}{4\omega})\alpha_1 \right] > 0.$$

For the third term,

$$4(\sigma - 1)\omega\alpha_0 + 2\sqrt{\omega}c\alpha_0 - (4\omega - c^2)\alpha_1$$
  

$$\geq (4\omega + 2\sqrt{\omega}c)\alpha_0 - (4\omega - c^2)\alpha_1$$
  

$$= 4\omega(1 + \frac{c}{2\sqrt{\omega}})\int_0^\infty h^{-\frac{1}{\sigma} - 1}(\cosh(\sigma\sqrt{4\omega - c^2}x) - 1)dx > 0.$$

Thus det $[d''(\omega, c)] > 0$ , implying the eigenvalues of  $d''(\omega, c)$  have the same sign. Turning to the trace,  $c^2 - 4(\sigma - 1)\omega \leq c^2 - 4\omega < 0$  for  $\sigma \geq 2$ . By (4.2), tr $[d''(\omega, c)] < 0$ . Hence, the two eigenvalues of  $d''(\omega, c)$  are negative.

Closely related to det[d''] is the function

(4.3)  
$$F(z;\sigma) \equiv (\sigma-1)^{2} \left[ \int_{0}^{\infty} (\cosh y - z)^{-\frac{1}{\sigma}} dy \right]^{2} - \left[ \int_{0}^{\infty} (\cosh y - z)^{-\frac{1}{\sigma} - 1} (z \cosh y - 1) dy \right]^{2},$$

which helps count the number of positive and negative eigenvalues for  $\sigma \in (0, 2)$ . Indeed,

**Lemma 4.2.** For  $\sigma \in (0,2)$  and admissible  $(\omega, c)$ , det $[d''(\omega, c)]$  has the same sign as  $F(\frac{c}{2\sqrt{\omega}}; \sigma)$ .

*Proof.* We rewrite (4.1) as,

$$\frac{\det[d''(\omega,c)]}{2^{-\frac{2}{\sigma}-4}\sigma^{-2}(1+\sigma)^{\frac{2}{\sigma}}(4\omega-c^{2})^{\frac{2}{\sigma}-1}\omega^{-\frac{1}{\sigma}-2}} = 16(\sigma-1)^{2}\omega^{2}\alpha_{0}^{2} - \left[\alpha_{1}(c^{2}-4\omega)+2\sqrt{\omega}c\alpha_{0}\right]^{2} = 16\omega^{2}\left\{(\sigma-1)^{2}\alpha_{0}^{2} - \left[\int_{0}^{\infty}h^{-\frac{1}{\sigma}-1}\left(\frac{c}{2\sqrt{\omega}}\cosh(\sigma\sqrt{4\omega-c^{2}}x)-1\right)dx\right]^{2}\right\}.$$
etting  $u = \sigma\sqrt{4\omega-c^{2}}x$ 

Letting  $y = \sigma \sqrt{4\omega - c^2}x$ ,

$$\frac{\det[d''(\omega,c)]}{2^{-\frac{2}{\sigma}-4}\sigma^{-2}(1+\sigma)^{\frac{2}{\sigma}}(4\omega-c^2)^{\frac{2}{\sigma}-1}\omega^{-\frac{1}{\sigma}-2}} = \frac{16\omega^2}{\sigma^2(4\omega-c^2)}F\left(\frac{c}{2\sqrt{\omega}};\sigma\right).$$

When  $\sigma = 1$  and  $z \in (-1, 1)$ , we have  $F(z; \sigma) = -1$  and  $\det[d''(\omega, c)] = -1/\omega$ . For  $\sigma \in (0, 1) \cup (1, 2)$ , we can evaluate the function  $F(z; \sigma)$  numerically, as shown in Figures 1 and 2. For any fixed  $\sigma \in (1, 2)$ ,  $F(z; \sigma)$  is monotonically increasing in z and has exactly one root  $z_0$  in the interval (-1, 1). For fixed  $\sigma \in (0, 1)$ ,  $F(z; \sigma)$  is monotonically decreasing in z and strictly negative. It is this numerical computation of F which is used to complete the proofs of Theorems 1.3 and 1.4. In contrast, for  $\sigma \geq 2$ , we can prove that  $F(z; \sigma)$  is strictly positive without resorting to computation.

We thus have the following theorem about  $p(d''(\omega, c))$ : **Theorem 4.3** (Numerical). For admissible  $(\omega, c)$ ,

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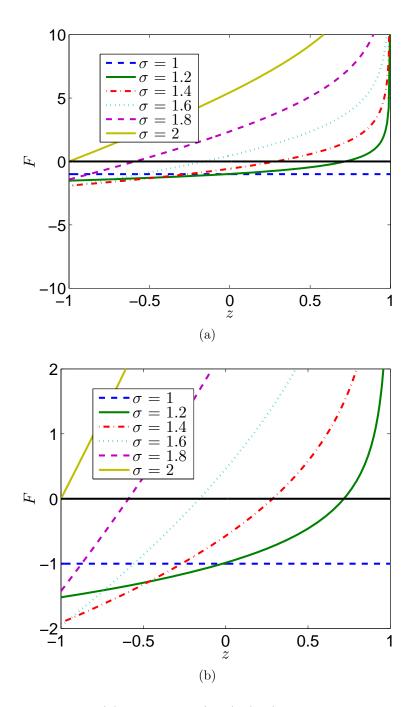


FIGURE 1. (a) Function  $F(z; \sigma)$ , (4.3), for several values of  $\sigma \in [1, 2]$ . (b) is a magnified plot near the z-axis.

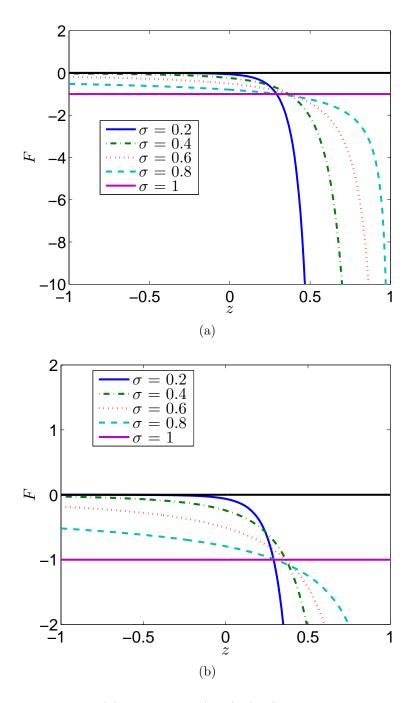


FIGURE 2. (a) Function  $F(z; \sigma)$ , (4.3), for several values of  $\sigma \in (0, 1]$ . (b) is a magnified plot near the z-axis.

- (i) when  $\sigma \in (1,2)$  and  $c = 2z_0\sqrt{\omega}$ ,  $\det[d''(\omega,c)] = 0$ ,
- (*ii*) when  $\sigma \in (1,2)$  and  $c < 2z_0\sqrt{\omega}$ , det $[d''(\omega,c)] < 0$ ;  $p(d''(\omega,c)) = 1$ ,
- (iii) when  $\sigma \in (1,2)$  and  $c > 2z_0 \sqrt{\omega}$ ,  $\det[d''(\omega,c)] > 0$ ;  $p(d''(\omega,c)) = 0$ or  $p(d''(\omega,c)) = 2$ .
- (iv) when  $\sigma \in (0, 1)$ , det $[d''(\omega, c)] < 0$ ;  $p(d''(\omega, c)) = 1$ .
- (v) when  $\sigma = 1$ , det $[d''(\omega, c)] = -1/\omega < 0$ ;  $p(d''(\omega, c)) = 1$ .

### 5. Orbital Stability and Instability

In this section, we complete the stability/instability proofs.

Proof of Theorem 1.2. From Theorem 3.1 and Theorem 4.1, n(H) = 1 for any  $\sigma > 0$ , and p(d'') = 0 for  $\sigma \ge 2$ . Thus n(H) - p(d'') = 1, is odd and all solitary waves are orbitally unstable by Theorem 1.5.

Proof of Theorem 1.3. By assumption, for each  $\sigma \in (1, 2)$ , there exists  $z_0$ , a unique zero crossing of (4.3), and this function is monotonically increasing. By Theorem 4.3,  $\det[d''(\omega, c)] < 0$  for admissible  $(\omega, c)$  satisfying  $c < 2z_0\sqrt{\omega}$ . It follows that there is one positive and one negative eigenvalues for  $d''(\omega, c)$ . Therefore, p(d'') = 1. Furthermore, from the theorem 3.1, n(H) = 1. Hence

$$n(H) - p(d'') = 0,$$

and we have the orbital stability of solitary waves.

When  $c > 2z_0\sqrt{\omega}$ , also by Theorem 4.3,  $\det[d''(\omega, c)] > 0$ . So the signs of the two eigenvalues of  $d''(\omega, c)$  are the same. If both of the eigenvalues were positive, then  $2 = p(d'') > n(H_{\phi}) = 1$ . This contradicts (1.17). Hence both of the eigenvalues are negative and p(d'') = 0. It follows that

$$n(H) - p(d'') = 1,$$

and we have the orbital instability of solitary waves.

Following the same argument, we can prove Theorem 1.4.

Proof of Theorem 1.4. When  $\sigma \in (0, 1]$ , from Theorem 4.3,  $\det[d''(\omega, c)] < 0$  for admissible  $(\omega, c)$ . Consequently,  $d''(\omega, c)$  has one positive eigenvalue and one negative eigenvalue; p(d'') = 1. By Theorem 3.1, n(H) = 1. Hence

$$n(H) - p(d'') = 0,$$

and the solitary waves are orbital stable.

### 6. Discussion and Numerical Illustration

We have explored the stability and instability of solitons for a generalized derivative nonlinear Schrödinger equation. We have found that for  $\sigma \geq 2$ , all solitons are orbital unstable. Using a numerical calculation of the function  $F(z;\sigma)$  defined in (4.3). we have also shown that for  $0 < \sigma \leq 1$ , all solitons are orbital stable. For  $1 < \sigma < 2$ , our computation of  $F(z;\sigma)$  indicates there exist both stable and unstable solitons, depending on the values of  $\omega$  and c. In particular, for fixed  $\omega$ and  $\sigma > 1$ , there are always both stable and unstable solitons for properly selected c. For  $\sigma$  near 1, the unstable solitons are always rightward moving, but, as Figure 1 shows, the root,  $z_0$ , becomes negative as  $\sigma$ approaches 2. Once  $z_0 < 0$ , unstable solitons can be both rightward and leftward moving.

Other dispersive PDEs possessing both stable and unstable solitons, such as NLS and KdV with saturating nonlinearities, [4, 5, 24, 28, 31], achieve this by introducing a nonlinearity that breaks scaling. In contrast, gDNLS always has a scaling symmetry, and throughout the regime  $1 < \sigma < 2$ , the scaling is  $L^2$  supercritical. This also implies the existence of an entire manifold of *critical* solitons, precisely when  $c = 2z_0\sqrt{\omega}$ . Along this curve, the standard stability results of [7, 8, 34, 35], break down, and a more detailed analysis is required. In [8], the stability can be demonstrated in this degenerate case provided  $d(\omega, c)$  remains convex. Given that within any neighborhood of a critical soliton there exist unstable solitons, we conjecture that it is unstable. While there has been recent work on critical one parameter solitons for NLS type equations, [4,5,24,28], to the best our knowledge, there has not been an analogous work on two parameter solitons.

While the equation retains the scaling symmetry, we observe that, in contrast to NLS solitons, not all gDNLS solitons can be obtained from scaling. Indeed, for (1.16), all solitons  $e^{i\lambda t}R(\mathbf{x};\lambda)$ , solving

$$-\Delta R + \lambda R - |R|^{2\sigma} R = 0$$

can be obtained from the  $\lambda = 1$  soliton via the transformation

$$e^{i\lambda t}R(\mathbf{x};\lambda) = e^{i\lambda t}\lambda^{\frac{1}{2\sigma}}R(\lambda^{\frac{1}{2}}\mathbf{x};1).$$

In contrast, while the gDNLS solitons also inherit the scaling symmetry of gDNLS, not all admissible  $(\omega, c)$  can be scaled to a particular soliton. Instead,

(6.1) 
$$\psi_{\omega,c}(x,t) = e^{i\omega t}\phi_{\omega,c}(x-ct) = e^{i\omega t}\phi_{1,c/\sqrt{\omega}}(\sqrt{\omega}(x-ct)).$$

Only solitons for which

$$\frac{c_1}{\sqrt{\omega_1}} = \frac{c_2}{\sqrt{\omega_2}}$$

can be scaled into one another.

Our results were based on the assumption that a weak solution existed. While we do not have an  $H^1$  theory in general, our results can, in part, be made rigorous as follows. For  $\sigma \geq 2$ , one should be able to apply the technique of [33] to obtain a local solution in  $H^s$ , with s > 1. Alternatively, for  $\sigma \geq 2$  and integer valued, [19, 20] can be invoked. Again, this yields a local solution in  $H^s$ , s > 1. For s sufficiently large, the solution will also conserve the invariants.

This is sufficient to fully justify the instability of the unstable solitons, since there is sufficient regularity such that if the solution leaves a neighborhood of the soliton in  $H^1$ , it also leaves in  $H^s$ , s > 1. However, this is insufficient to prove stability, because even if the solution stays close in the  $H^1$  norm, the norm of the solution could grow in a higher Sobolev norm.

There is also the question of the monotonicity of F, for which we relied on numerical computation for  $\sigma < 2$ . Looking at Figures 1 and 2, it would appear that the  $F(z; \sigma = 2)$  is an upper bound on  $F(z; \sigma)$  for  $1 < \sigma < 2$ . In addition, there appears to be a singularity at z = 1. Likewise, the line F = 0 appears to be an upper bound in the range  $\sigma \leq 1$ . A more subtle analysis may permit a rigorous justification of our work in this regime.

Lastly, we provide some numerical experiments of solitons on both the stable and unstable branches. We studied the stability near the turning point  $c = 2z_0\sqrt{\omega}$ . When  $\sigma = 1.5$ ,  $z_0 = 0.0618303$ . The initial condition is chosen as

(6.2) 
$$\psi_0(x,0) = \psi_{\omega,c}(x,0) + 0.0001e^{-2x^2}$$

We simulate (1.7) using the fourth order exponential time difference scheme of [17], and treat the nonlinearity pseudospectrally. Though the nonlinearity is not polynomial in its arguments,  $\psi$ ,  $\bar{\psi}$  and  $\psi_x$ ,

$$\left|\psi\right|^{3}\psi_{x}=\psi\bar{\psi}\left|\psi\right|\psi_{x},$$

we found that dealiasing as though it were a quintic problem proved robust.

Our results are as follows:

(1) When  $\omega = 1$  and c = 0 < 2z = 0.1236606, Figure 3 shows that the solitary wave retains its shape for a long time (t = 100).

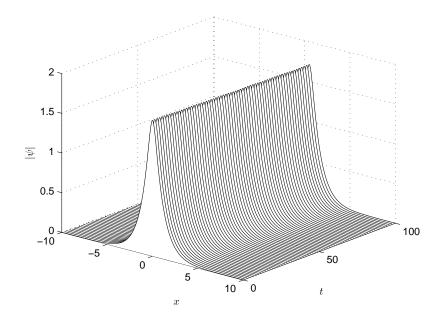


FIGURE 3. Evolution of a perturbed orbitally stable soliton,  $\omega = 1$  and c = 0, with initial condition (6.2).

(2) When  $\omega = 1$  and c = 0.2 > 2z = 0.1236606, Figure 4 shows that the amplitude of the solitary wave increases rapidly near t = 10 and it is not orbitally stable.

Our simulation of the unstable soliton suggests that, rather than disperse or converge to a stable soliton, gDNLS may result in a finite time singularity. We will explore the potential for singularity formation in the forthcoming work [23].

## APPENDIX A. AUXILIARY CALCULATIONS

In this section, we present certain integral relations that are helpful in studying the determinant and trace of  $d''(\omega, c)$ . In the following, we denote  $\kappa = \sqrt{4\omega - c^2} > 0$  and rewrite the solitary solution (1.9) as  $\varphi(x)^{2\sigma} = f(\omega, c)h(\omega, c; x)^{-1}$ , with

$$f(\omega, c) = \frac{(\sigma + 1)\kappa^2}{2\sqrt{\omega}}$$
,  $h(x; \sigma; \omega, c) = \cosh(\sigma \kappa x) - \frac{c}{2\sqrt{\omega}}$ .

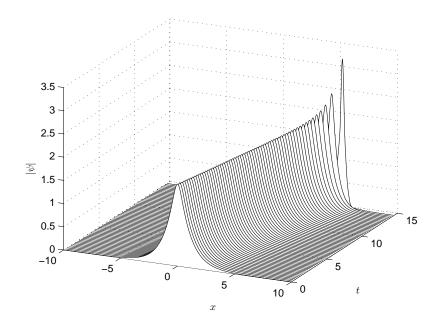


FIGURE 4. Evolution of a perturbed orbitally unstable soliton,  $\omega = 1$  and c = .2, with initial condition (6.2).

We also rewrite the functionals Q, P defined in (2.6)-(2.7) and their derivatives in terms of h and f:

(A.1) 
$$Q = \frac{1}{2} \int_0^\infty |\varphi|^2 dx = f^{\frac{1}{\sigma}} \int_0^\infty h^{-\frac{1}{\sigma}} dx,$$

(A.2) 
$$P = -\frac{c}{2} \int_{0}^{\infty} \varphi^{2} dx + \frac{1}{2\sigma + 2} \int_{0}^{\infty} \varphi^{2\sigma + 2} dx$$
$$= -\frac{c}{2} f^{\frac{1}{\sigma}} \int_{0}^{\infty} h^{-\frac{1}{\sigma}} dx + \frac{1}{2\sigma + 2} f^{\frac{\sigma+1}{\sigma}} \int_{0}^{\infty} h^{-\frac{\sigma+1}{\sigma}} dx.$$

(A.3) 
$$\partial_c Q = \frac{1}{\sigma} f^{\frac{1-\sigma}{\sigma}} f_c \int_0^\infty h^{-\frac{1}{\sigma}} dx - \frac{1}{\sigma} f^{\frac{1}{\sigma}} \int_0^\infty h^{-\frac{\sigma+1}{\sigma}} h_c dx,$$

(A.4) 
$$\partial_{\omega}Q = \frac{1}{\sigma}f^{\frac{1-\sigma}{\sigma}}f_{\omega}\int_{0}^{\infty}h^{-\frac{1}{\sigma}}dx - \frac{1}{\sigma}f^{\frac{1}{\sigma}}\int_{0}^{\infty}h^{-\frac{\sigma+1}{\sigma}}h_{\omega}dx,$$

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$$\partial_{c}P = -\frac{1}{2}f^{\frac{1}{\sigma}}\int_{0}^{\infty}h^{-\frac{1}{\sigma}}dx - \frac{c}{2\sigma}f^{\frac{1-\sigma}{\sigma}}f_{c}\int_{0}^{\infty}h^{-\frac{1}{\sigma}}dx + \frac{c}{2\sigma}f^{\frac{1}{\sigma}}\int_{0}^{\infty}h^{-\frac{\sigma+1}{\sigma}}h_{c}dx + \frac{1}{2\sigma}f^{\frac{1}{\sigma}}f_{c}\int_{0}^{\infty}h^{-\frac{1+\sigma}{\sigma}}dx - \frac{1}{2\sigma}f^{\frac{1+\sigma}{\sigma}}\int_{0}^{\infty}h^{-\frac{1+2\sigma}{\sigma}}h_{c}dx + \frac{1}{2\sigma}f^{\frac{1}{\sigma}}\int_{0}^{\infty}h^{-\frac{\sigma+1}{\sigma}}h_{\omega}dx + \frac{1}{2\sigma}f^{\frac{1}{\sigma}}f_{\omega}\int_{0}^{\infty}h^{-\frac{1+\sigma}{\sigma}}dx + \frac{c}{2\sigma}f^{\frac{1}{\sigma}}\int_{0}^{\infty}h^{-\frac{\sigma+1}{\sigma}}h_{\omega}dx + \frac{1}{2\sigma}f^{\frac{1}{\sigma}}f_{\omega}\int_{0}^{\infty}h^{-\frac{1+\sigma}{\sigma}}dx - \frac{1}{2\sigma}f^{\frac{1+\sigma}{\sigma}}\int_{0}^{\infty}h^{-\frac{1+2\sigma}{\sigma}}h_{\omega}dx,$$
(A.6)

where

$$f_c = -\frac{c(1+\sigma)}{\sqrt{\omega}}, \quad f_\omega = \frac{(1+\sigma)(4\omega+c^2)}{4\omega^{3/2}},$$
$$h_c = -\frac{\sigma c}{\kappa}x\sinh(\sigma\kappa x) - \frac{1}{2}\omega^{-1/2}, \quad h_\omega = \frac{2\sigma}{\kappa}x\sinh(\sigma\kappa x) + \frac{c}{4}\omega^{-\frac{3}{2}}.$$

The expressions in (A.3)-(A.6) involve various integrals. The next lemmas show that all of them can be expressed simply in terms of

$$\alpha_n = \int_0^\infty h^{-\frac{1}{\sigma} - n} dx.$$

First, we have

# Lemma A.1.

(A.7) 
$$\alpha_2 = \frac{4\omega}{(\sigma+1)\kappa^2}\alpha_0 + \frac{2c\sqrt{\omega}(2+\sigma)}{(\sigma+1)\kappa^2}\alpha_1$$

*Proof.* We first rewrite  $\alpha_0$ , and then integrate by parts:

$$\begin{aligned} \alpha_0 &= \int_0^\infty h^{-\frac{1}{\sigma}-1} h dx = \frac{1}{\sigma\kappa} \int_0^\infty h^{-\frac{1}{\sigma}-1} (\sinh(\sigma\kappa x))' dx - \alpha_1 \frac{c}{2\sqrt{\omega}} \\ &= \frac{(\sigma+1)}{\sigma^2} \int_0^\infty h^{-\frac{1}{\sigma}-2} (\sinh^2(\sigma\kappa x)) dx - \alpha_1 \frac{c}{2\sqrt{\omega}} \\ &= \frac{(\sigma+1)}{\sigma^2} \int_0^\infty h^{-\frac{1}{\sigma}-2} ((h+\frac{c}{2\sqrt{\omega}})^2 - 1) dx - \alpha_1 \frac{c}{2\sqrt{\omega}}. \end{aligned}$$

Regrouping the terms in this last expression, we obtain (A.7).  $\Box$ 

Lemma A.2. We have the following relations:

(A.8) 
$$\int_0^\infty h^{-\frac{1}{\sigma}-2}h_c dx = -\frac{1}{2\sqrt{\omega}}\alpha_2 - \frac{c\sigma}{(\sigma+1)\kappa^2}\alpha_1,$$

(A.9) 
$$\int_0^\infty h^{-\frac{1}{\sigma}-1} h_c dx = -\frac{1}{2\sqrt{\omega}} \alpha_1 - \frac{c\sigma}{\kappa^2} \alpha_0,$$

(A.10) 
$$\int_{0}^{\infty} h^{-\frac{1}{\sigma}-2} h_{\omega} dx = \frac{c}{4\omega^{3/2}} \alpha_{2} + \frac{2\sigma}{(\sigma+1)\kappa^{2}} \alpha_{1},$$

(A.11) 
$$\int_0^\infty h^{-\frac{1}{\sigma}-1} h_\omega dx = \frac{c}{4\omega^{3/2}} \alpha_1 + \frac{2\sigma}{\kappa^2} \alpha_0.$$

*Proof.* By integration by parts, and n integer,

$$\int_0^\infty h^{-\frac{1}{\sigma}-n} h_c dx = \frac{c}{\kappa^2(-\frac{1}{\sigma}-n+1)} \int_0^\infty h^{-\frac{1}{\sigma}-n+1} dx - \frac{1}{2\sqrt{\omega}} \int_0^\infty h^{-\frac{1}{\sigma}-n} dx$$

Choosing n = 2, 1, we get (A.8) and (A.9). The relations (A.10) and (A.11) are obtained from (A.8), (A.9) and  $h_{\omega} = -\frac{2}{c}h_c - \frac{\kappa^2}{4\omega^{3/2}c}$ .

Using Lemmas A.1 and A.2, we have:

Lemma A.3. Denoting 
$$\tilde{\kappa} = 2^{-\frac{1}{\sigma}-2}\sigma^{-1}(1+\sigma)^{\frac{1}{\sigma}}\kappa^{2(\frac{1}{\sigma}-1)}\omega^{-\frac{1}{2\sigma}-\frac{1}{2}}$$
, we have  
 $\partial_c Q = 2\tilde{\kappa} \left[2c(\sigma-2)\omega^{1/2}\alpha_0 + \kappa^2\alpha_1\right]$   
 $\partial_\omega Q = \tilde{\kappa}\omega^{-1} \left[\left(2c^2 - 8(\sigma-1)\omega\right)\omega^{1/2}\alpha_0 - \kappa^2c\alpha_1\right]$   
 $\partial_c P = \tilde{\kappa} \left[\left(2c^2 - 8(\sigma-1)\omega\right)\omega^{1/2}\alpha_0 - \kappa^2c\alpha_1\right]$   
 $\partial_\omega P = 2\tilde{\kappa} \left[2c(\sigma-2)\omega^{1/2}\alpha_0 + \kappa^2\alpha_1\right].$ 

This is used in Section 4 to obtain (4.1).

### References

- [1] Agrawal, G.P., Nonlinear Fiber Optics, Academic Press, San Diego, 2006.
- [2] Colin, M., Ohta, M., Stability of solitary waves for derivative nonlinear Schrödinger equation, Ann. Inst. H. Poincaré, Analyse Non Linéaire, 23 (2006), 753–764.
- [3] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T., A Refined Global Well-Posedness Result for Schrödinger Equations with Derivative, SIAM J. Math. Anal., 34 (2002), 64–86.
- [4] Comech, A., Cuccagna, S., Pelinovsky, D., Nonlinear instability of a critical traveling wave in the generalized Korteweg-de Vries equation, SIAM J. Math. Anal., 39 (2007), 1–33.
- [5] Comech, A., Pelinovsky, D., Purely nonlinear instability of standing waves with minimal energy, Commun. Pure Appl. Math., 56 (2003), 1565-1607.

- [6] DiFranco, J.C., Miller, P.D., The semiclassical modified nonlinear Schrödinger equation I: Modulation theory and spectral analysis, Physica D, 237 (2008), 947–997.
- [7] Grillakis, M., Shatah, J., Strauss, W., Stability theory of solitary waves in the presence of symmetry, I, J. Funct. Anal., 74 (1987), 160–197.
- [8] Grillakis, M., Shatah, J., Strauss, W., Stability theory of solitary waves in the presence of symmetry, II, J. Funct. Anal. 94 (1990), 308–348.
- [9] Grünrock, A., Bi-and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS, Int. Math. Res. Notices, 41 (2005), 2525–2558.
- [10] Grünrock, A., Herr, S., Low Regularity Local Well-Posedness of the Derivative Nonlinear Schrödinger Equation with Periodic Initial Data, SIAM J. Math. Anal., **39** (2008), 1890–1920.
- [11] Guo, B., Wu, Y., Orbital stability of solitary waves for the nonlinear derivative Schrödinger equation, J. Diff. Eqs., 123 (1995), 35–55.
- [12] Hao, C., Well-Posedness for One-Dimensional Derivative Nonlinear Schrödinger Equations, Commun. Pure Appl. Anal., 6 (2007), 997–1021.
- [13] Hayashi, N., The initial value problem for the derivative nonlinear Schrödinger equation in the energy space. Nonlinear Anal. 20(1993), 823–833.
- [14] Hayashi, N., Ozawa, T., On the derivative nonlinear Schrödinger equation. Physica D 55 (1992), 14–36.
- [15] Hayashi, N., Ozawa, T., Finite energy solutions of nonlinear Schrödinger equations of derivative type. SIAM J. Math. Anal. 25 (1994), 1488–150.
- [16] Hayashi, N. Ozawa, T., Remarks on nonlinear Schrödinger equations in one space dimension, Diff. Int. Eqs 7 (1994), 453–461.
- [17] Kassam, A.K., Trefethen, L.N., Fourth-Order Time-Stepping for Stiff PDEs, SIAM J. Sci. Comput. 26 (2005), 1214-1233.
- [18] Kaup, D.J., Newell, A.C., An exact solution for a derivative nonlinear Schrödinger equation, J. Math. Phys., 19 (1978), 798-801.
- [19] Kenig, C.E., Ponce, G., Vega, L., Small solutions to nonlinear Schrödinger equations, Ann. Inst. H. Poincaré, Analyse Non Linéaire, 10 (1993), 255-288.
- [20] Kenig, C.E., Ponce, G., Vega, L., Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations, Invent Math. 134 (1998), 489-545.
- [21] Lee, J., Global solvability of the derivative nonlinear Schrödinger equation, Trans. Am. Math. Soc. 314 (1989), 107–118.
- [22] Linares, F., Ponce, G., Introduction to Nonlinear Dispersive Equations, Springer, Berlin (2009).
- [23] Liu, X., Simpson, G., Sulem, C., Numerical simulations of a generelized Derviative Nonlinear Schrödinger Equation, In preparation.
- [24] Marzuola, J.L., Raynor, S., Simpson, G., A system of ODEs for a perturbation of a minimal mass soliton, J. Nonlinear. Sci., 20 (2010), 425-461.
- [25] Mio, K., Ogino, T., Minami, K. Takeda, S., Modified Nonlinear Schrödinger Equation for Alfvén Waves Propagating along the Magnetic Field in Cold Plasmas, J. Phys. Soc. 41(1976), 265–271.
- [26] Mjølhus, E., On the modulational instability of hydromagnetic waves parallel to the magnetic field, J.Plasma Phys., 16 (1976), 321–334.

- [27] Moses, J., Malomed, B., Wise, F., Self-steepening of ultrashort optical pulses without self-phase-modulation, Phys. Rev. A, 76 (2007), 1–4.
- [28] Ohta, M., Instability of bound states for abstract nonlinear Schrödinger equations, J. Funct. Anal., 261 (2011), 90–110.
- [29] Ozawa, T., On the nonlinear Schrödinger equations of derivative type, Indiana Univ. Math. J., 45 (1996), 137–163.
- [30] Passot, T., Sulem, P.L., Multidimensional modulation of Alfvén waves, Phys. Rev. E, 48 (1993), 2966–2974.
- [31] Sulem, C., Sulem, P.L., The nonlinear Schrödinger equation: self-focusing and wave collapse. Applied Mathematical Sciences, vol. 139, Springer, Berlin, 1999.
- [32] Tan, S.B., Zhang, L.H., On a weak solution of the mixed nonlinear Schrödinger equations, J. Math. Anal. Appl., 182 (1994), 409–421.
- [33] Tsutsumi, M., Fukuda, I., On solutions of the derivative nonlinear Schrödinger equation, Existence and uniqueness theorem, Funkcialaj Ekvacioj, 23 (1980), 259–277.
- [34] Weinstein, M.I., Modulational Stability of Ground States of Nonlinear Schrödinger Equations, SIAM J. Math. Anal., 16 (1985), 472–491.
- [35] Weinstein, M.I., Lyapunov Stability of Ground States of Nonlinear Dispersive Evolution Equations, Commun. Pure Appl. Math., XXXIX (1986), 51–68.

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