QUANTUM TORUS ALGEBRAS AND B(C) TYPE TODA SYSTEMS

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ABSTRACT. In this paper, we construct a new even constrained B(C) type Toda hierarchy and derive its B(C) type Block type additional symmetry. Also we generalize the B(C) type Toda hierarchy to the N-component B(C) type Toda hierarchy which is proved to have symmetries of a coupled $\bigotimes^N QT_+$ algebra (N-folds direct product of the positive half of the quantum torus algebra QT).

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1. Introduction

The Toda lattice hierarchy as a completely integrable system has many important applications in mathematics and physics including the representation theory of Lie algebras and random matrix models [1–3]. The Toda system has many kinds of reductions or extensions, for example the B and C type Toda hierarchies [2,4], extended Toda hierarchy (ETH) [5], bigraded Toda hierarchy (BTH) [6]- [11] and so on. There are some other generalizations called multi-component Toda systems [2,12] which are

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useful in the fields of multiple orthogonal polynomials and non-intersecting Brownian motions.

The multicomponent 2D Toda hierarchy was considered from the point of view of the Gauss-Borel factorization problem, the theory of multiple matrix orthogonal polynomials, non-intersecting Brownian motions and matrix Riemann-Hilbert problem [12]- [15]. In fact the multicomponent 2D Toda hierarchy in [13] is a periodic reduction of the bi-infinite matrix-formed two dimensional Toda hierarchy. In [16], we generalize the multicomponent Toda hierarchy to an extended multicomponent Toda hierarchy including extended logarithmic flow equations. Later by a commutative algebraic reduction on the extended multicomponent Toda hierarchy, we get an extended Z_N -Toda hierarchy [17] which might be useful in Gromov-Witten theory.

This paper is organized in the following way. In Section 2, we recall some basic knowledge about the B(C) type Toda hierarchy. We construct a new even constrained B(C) type Toda hierarchy and derive its Block type additional symmetry in Section 3. Next, in Section 4 we generalize the B(C) type Toda hierarchy to a new N-component B(C) type Toda hierarchy. In the last section, we construct the symmetry of the N-component B(C) type Toda hierarchy which constitutes a coupled $\bigotimes^N QT_+$ algebra (N-folds direct product of the positive half of the quantum torus algebra QT).

In this section, some basic facts about the B(C) type Toda hierarchy are reviewed. One can refer to [2, 4] for more details about the B(C) type Toda hierarchy (or BTH(CTH)).

Then the BTH hierarchy is defined in the Lax forms as

$$\partial_{x_{2n+1}}L_1 = [-(L_1^{2n+1})_-, L_1]$$
 and $\partial_{y_{2n+1}}L_1 = [-(L_2^{2n+1})_-, L_1], \quad n = 0, 1, 2, \cdots,$ (1)

$$\partial_{x_{2n+1}}L_2 = [(L_1^{2n+1})_+, L_2]$$
 and $\partial_{y_{2n+1}}L_2 = [(L_2^{2n+1})_+, L_2], \quad n = 0, 1, 2, \cdots, (2)$

where the Lax operator L_i is given by a pair of infinite matrices

$$L_1 = \sum_{-\infty < i \le 1} \operatorname{diag}[a_i^{(1)}(s)] \Lambda^i, \quad L_2 = \sum_{-1 \le i \le \infty} \operatorname{diag}[a_i^{(2)}(s)] \Lambda^i, \tag{3}$$

with $\Lambda = (\delta_{j-i,1})_{i,j\in\mathbb{Z}}$, and $a_i^{(k)}(s)$ and $a_i^{(k)}(s)$ depending on $x = (x_1, x_3, x_5, \cdots)$ and $y = (y_1, y_3, y_5, \cdots)$, such that

$$a_1^{(1)}(s) = 1$$
 and $a_{-1}^{(2)}(s) \neq 0 \quad \forall s$

and satisfies the BTH(CTH) constraint [2]

$$L_i^T = -JL_iJ^{-1}, \quad L_i^T = -KL_iK^{-1},$$
 (4)

where $J = ((-1)^i \delta_{i+j,0})_{i,j \in \mathbb{Z}}, K = \Lambda J$ and T refers to the matrix transpose. The BTH constraint is explicitly showed as

$$a_i^{(k)}(s) = (-1)^{i+1} a_i^{(k)}(-s-i), k = 1, 2.$$
 (5)

The CTH constraint means

$$a_i^{(k)}(s) = (-1)^{i+1} a_i^{(k)}(-s - i - 1).$$
(6)

The Lax equation for the BTH(CTH) can be expressed as a system of equations of the Zakharov-Shabat type:

$$\partial_{x_{2n+1}}(L_1^{2m+1})_+ - \partial_{x_{2m+1}}(L_1^{2n+1})_+ + [(L_1^{2m+1})_+, (L_1^{2n+1})_+] = 0, \tag{7}$$

$$\partial_{y_{2n+1}}(L_2^{2m+1})_- + \partial_{y_{2m+1}}(L_2^{2n+1})_- - [(L_2^{2m+1})_-, (L_2^{2n+1})_-] = 0, \tag{8}$$

$$\partial_{y_{2n+1}}(L_1^{2m+1})_+ + \partial_{x_{2m+1}}(L_2^{2n+1})_- - [(L_1^{2m+1})_+, (L_2^{2n+1})_-] = 0, \tag{9}$$

$$-\partial_{y_{2n+1}}(L_2^{2m+1})_- - \partial_{x_{2m+1}}(L_1^{2n+1})_+ - [(L_2^{2m+1})_-, (L_1^{2n+1})_+] = 0.$$
 (10)

When m = n = 0, one can get the B type Toda equation

$$\partial_{x_1} a_{-1}^{(2)}(1) = a_{-1}^{(2)}(1) a_0^{(1)}(1), \quad \partial_{x_1} a_{-1}^{(2)}(s) = a_{-1}^{(2)}(s) (a_0^{(1)}(s) - a_0^{(1)}(s - 1)) \quad (s \ge 2),$$

$$\partial_{y_1} a_0^{(1)}(s) = a_{-1}^{(2)}(s) - a_{-1}^{(2)}(s + 1) \quad (s \ge 1),$$
(11)

by considering the corresponding constraint (5). Also one can get the C type Toda equation

$$\partial_{x_1} a_{-1}^{(2)}(0) = 2a_{-1}^{(2)}(0)a_0^{(1)}(0), \quad \partial_{x_1} a_{-1}^{(2)}(s) = a_{-1}^{(2)}(s)(a_0^{(1)}(s) - a_0^{(1)}(s - 1)) \quad (s \ge 1),$$

$$\partial_{y_1} a_0^{(1)}(s) = a_{-1}^{(2)}(s) - a_{-1}^{(2)}(s + 1) \quad (s \ge 0),$$
(12)

The Lax operator of the BTH(CTH) (37) has the representation

$$L_1 = W_1 \Lambda W_1^{-1} = S_1 \Lambda S_1^{-1}, \tag{13}$$

$$L_2 = W_2 \Lambda^{-1} W_2^{-1} = S_2 \Lambda^{-1} S_2^{-1}, \tag{14}$$

where

$$S_1(x,y) = \sum_{i\geq 0} \operatorname{diag}[c_i(s;x,y)] \Lambda^{-i}, \quad S_2(x,y) = \sum_{i\geq 0} \operatorname{diag}[c_i'(s;x,y)] \Lambda^i$$
 (15)

and

$$W_1(x,y) = S_1(x,y)e^{\xi(x,\Lambda)}, \quad W_2(x,y) = S_2(x,y)e^{\xi(y,\Lambda^{-1})}$$
 (16)

with $c_0(s; x, y) = 1$ and $c'_0(s; x, y) \neq 0$ for any s, and $\xi(x, \Lambda^{\pm 1}) = \sum_{n \geq 0} x_{2n+1} \Lambda^{\pm 2n+1}$. For the B type Toda hierarchy, under an appropriate choice (W_1, W_2) satisfies

$$J^{-1}W_i^T J = W_i^{-1}, i = 1, 2. (17)$$

For the C type Toda hierarchy, under an appropriate choice (W_1, W_2) satisfies

$$K^{-1}W_i^T K = W_i^{-1}, i = 1, 2. (18)$$

The wave operators evolve as

$$\partial_{x_{2n+1}} S_1 = -(L_1^{2n+1})_- S_1, \quad \partial_{y_{2n+1}} S_1 = -(L_2^{2n+1})_- S_1,$$
 (19)

$$\partial_{x_{2n+1}}W_1 = (L_1^{2n+1})_+W_1, \quad \partial_{y_{2n+1}}W_1 = -(L_2^{2n+1})_-W_2,$$
 (20)

$$\partial_{x_{2n+1}} S_2 = (L_1^{2n+1})_+ S_2, \quad \partial_{y_{2n+1}} S_2 = (L_2^{2n+1})_+ S_2,$$
 (21)

$$\partial_{x_{2n+1}} W_2 = (L_1^{2n+1})_+ W_2, \quad \partial_{y_{2n+1}} W_2 = -(L_2^{2n+1})_- W_2.$$
 (22)

At last, we end this section with the introduction of the additional symmetries of the BTH(CTH). The Orlov-Shulman operator [4] is defined as

$$M_1 = W_1 \varepsilon W_1^{-1}, \quad M_2 = W_2 \varepsilon^* W_2^{-1},$$
 (23)

where

$$\varepsilon = \operatorname{diag}[s]\Lambda^{-1}, \quad \varepsilon^* = -J\varepsilon J^{-1},$$

satisfying

$$[L_i, M_i] = 1,$$

$$\partial_{x_{2n+1}} M_i = [(L_1^{2n+1})_+, M_i], \partial_{y_{2n+1}} M_i = [-(L_2^{2n+1})_-, M_i]. \tag{24}$$

To construct the Block symmetry of the BTH, the following lemma should be introduced.

Lemma 1. The following identities hold true

$$\Lambda^{-1}\varepsilon\Lambda = J^{-1}\varepsilon^T J, \qquad \Lambda\varepsilon^*\Lambda^{-1} = J^{-1}\varepsilon^{*T} J, \tag{25}$$

$$\varepsilon = K \varepsilon^T K^{-1}, \qquad \varepsilon^* = K \varepsilon^{*T} K^{-1}.$$
 (26)

For the BTH, using the above lemma, one can derive

$$M_i^T = JL_i^{-1}M_iL_iJ^{-1}. (27)$$

For the CTH, using the above lemma, one can derive

$$M_i^T = KM_iK^{-1}. (28)$$

The additional symmetry [4] of the BTH can be defined by introducing the additional independent variables $x_{m,l}$ and $y_{m,l}$,

$$\partial_{x_{m,l}} W_1 = -A_{ml}(M_1, L_1)_- W_1, \quad \partial_{y_{m,l}} W_1 = -A_{ml}(M_2, L_2)_- W_1,$$
 (29)

$$\partial_{x_{m,l}} W_2 = A_{ml}(M_1, L_1)_+ W_2, \quad \partial_{y_{m,l}} W_2 = A_{ml}(M_2, L_2)_+ W_2,$$
 (30)

where

$$A_{ml}(M_i, L_i) = M_i^m L_i^l - (-1)^l L_i^{l-1} M_i^m L_i.$$
(31)

For the case of the CTH, the operator A_{ml} will become

$$A_{ml}(M_i, L_i) = M_i^m L_i^l - (-1)^l L_i^l M_i^m.$$
(32)

These additional flows form a coupled W_{∞} Lie algebra [4].

3. The even constrained BTH(CTH)

In this section, for a new constrained BTH(CTH), the Lax operator L is given by an infinite matrices L as

$$L = L_1^{2N} = L_2^{2M} = \sum_{-2M < i < 2N} \operatorname{diag}[a_i(s)] \Lambda^i,$$
(33)

with $a_{2N}(s) = 1$, and for the BTH, it satisfies the B type constraint

$$L^T = JLJ^{-1}, (34)$$

and for the CTH, it satisfies the C type constraint

$$L^T = KLK^{-1}. (35)$$

Then the constrained BTH(CTH) hierarchy is defined in the Lax forms as

$$\partial_{x_{2n+1}}L = \left[-\left(L^{\frac{2n+1}{2N}}\right)_{-}, L\right] \quad \text{and} \quad \partial_{y_{2n+1}}L = \left[-\left(L^{\frac{2n+1}{2N}}\right)_{-}, L\right],$$
 (36)

$$\partial_{x_{2n+1}}L = [(L^{\frac{2n+1}{2M}})_+, L]$$
 and $\partial_{y_{2n+1}}L = [(L^{\frac{2n+1}{2M}})_+, L], \quad n = 0, 1, 2. \cdots$ (37)

The Lax operator of the constrained BTH(CTH) (37) has the representation

$$L = W_1 \Lambda^{2N} W_1^{-1} = W_2 \Lambda^{-2M} W_2^{-1}, \tag{38}$$

where

$$S_1(x,y) = \sum_{i>0} \operatorname{diag}[c_i(s;x,y)] \Lambda^{-i}, \qquad S_2(x,y) = \sum_{i>0} \operatorname{diag}[c'_i(s;x,y)] \Lambda^i$$
 (39)

and

$$W_1(x,y) = S_1(x,y)e^{\xi(x,\Lambda)}, \quad W_2(x,y) = S_2(x,y)e^{\xi(y,\Lambda^{-1})}$$
 (40)

with $c_0(s; x, y) = 1$ and $c'_o(s; x, y) \neq 0$ for any s, and $\xi(x, \Lambda^{\pm 1}) = \sum_{n \geq 0} x_{2n+1} \Lambda^{\pm 2n+1}$. Under an appropriate choice (W_1, W_2) of the constrained BTH(CTH) satisfies

$$J^{-1}W_i^T J = W_i^{-1}, (K^{-1}W_i^T K = W_i^{-1}), i = 1, 2.$$
(41)

The wave operators evolve according to

$$\partial_{x_{2n+1}} S_1 = -(L^{\frac{2n+1}{2N}})_- S_1, \quad \partial_{y_{2n+1}} S_1 = -(L^{\frac{2n+1}{2M}})_- S_1,$$
 (42)

$$\partial_{x_{2n+1}}W_1 = (L^{\frac{2n+1}{2N}})_+W_1, \quad \partial_{y_{2n+1}}W_1 = -(L^{\frac{2n+1}{2M}})_-W_1,$$
 (43)

$$\partial_{x_{2n+1}} S_2 = (L^{\frac{2n+1}{2N}})_+ S_2, \quad \partial_{y_{2n+1}} S_2 = (L^{\frac{2n+1}{2M}})_+ S_2,$$
 (44)

$$\partial_{x_{2n+1}} W_2 = (L^{\frac{2n+1}{2N}})_+ W_2, \quad \partial_{y_{2n+1}} W_2 = -(L^{\frac{2n+1}{2M}})_- W_2.$$
 (45)

The Orlov-Shulman operator \bar{M}_i will be defined as as

$$\bar{M}_1 = W_1 \varepsilon_{2N} W_1^{-1}, \quad \bar{M}_2 = W_2 \varepsilon_{-2M}^* W_2^{-1},$$
 (46)

where

$$\varepsilon_{2N} = \frac{1}{2N} \operatorname{diag}[s] \Lambda^{-2N}, \quad \varepsilon_{-2M}^* = -\frac{1}{2M} \varepsilon^{T} \Lambda^{2M},$$

satisfying

$$[L, \bar{M}_i] = 1,$$

$$\partial_{x_{2n+1}} \bar{M}_i = [(L^{\frac{2n+1}{2N}})_+, \bar{M}_i], \partial_{y_{2n+1}} \bar{M}_i = [-(L^{\frac{2n+1}{2M}})_-, \bar{M}_i]. \tag{47}$$

Lemma 2. The difference of two Orlov-Schulman operators M_i for constrained BTH hierarchy has following B type property:

$$L^{T}(\bar{M}_{1} - \bar{M}_{2})^{T} = J(L\bar{M}_{1} - L\bar{M}_{2})J^{-1}, \tag{48}$$

and for constrained CTH hierarchy has following C type property:

$$L^{T}(\bar{M}_{1} - \bar{M}_{2})^{T} = K(L\bar{M}_{1} - L\bar{M}_{2})K^{-1}.$$
(49)

Proof. It is easy to find the two Orlov-Schulman operators can be expressed as

$$\bar{M}_1 = \frac{M_1 L_1^{1-2N}}{2N}, \quad \bar{M}_2 = -\frac{M_2 L_2^{1-2M}}{2M}.$$
 (50)

Putting eq.(50) into $(\bar{M}_1 - \bar{M}_2)^T$ can lead to

$$(\bar{M}_1 - \bar{M}_2)^T = \frac{JL_1^{-2N}M_1L_1J^{-1}}{2N} + \frac{JL_2^{-2M}M_2L_2^{-1}J^{-1}}{2M}$$
(51)

$$=\frac{JL_1^{-2N}M_1L_1J^{-1}}{2N} + \frac{JL_2^{-2M}M_2L_2^{-1}J^{-1}}{2M}$$
(52)

$$=\frac{J(M_1L_1^{1-2N}-2NL_1^{-2N})J^{-1}}{2N}+\frac{J(M_2L_2^{1-2M}+2ML_2^{-2})J^{-1}}{2M},$$
 (53)

which can further lead to eq.(48). For the CTH, one can do the similar calculation as

$$(\bar{M}_1 - \bar{M}_2)^T = \frac{KL_1^{1-2N}M_1K^{-1}}{2N} + \frac{KL_2^{1-2M}M_2K^{-1}}{2M}$$
(54)

$$=\frac{KL_1^{1-2N}M_1K^{-1}}{2N} + \frac{KL_2^{1-2M}M_2K^{-1}}{2M}$$
 (55)

$$=\frac{K(M_1L_1^{1-2N}-2NL_1^{-2N})K^{-1}}{2N}+\frac{K(M_2L_2^{1-2M}+2ML_2^{-2})K^{-1}}{2M},$$
 (56)

which can further lead to eq.(49)

In above calculation, the commutativity between L and $\bar{M}_1 - \bar{M}_2$ is already used. Till now, the proof is finished.

For the constrained BTH(CTH), we need the following operator

$$\mathbb{B}_{m,l} = (\bar{M}_1 - \bar{M}_2)^m L^l, \quad m \in \mathbb{Z}_+^{\text{odd}}, l \in \mathbb{Z}_+.$$
 (57)

One can easily check that for the BTH

$$\mathbb{B}_{m,l}^T = J \mathbb{B}_{m,l} J^{-1}, \quad m \in \mathbb{Z}_+^{\text{odd}}, \tag{58}$$

and for the CTH

$$\mathbb{B}_{m,l}^T = K \mathbb{B}_{m,l} K^{-1}, \quad m \in \mathbb{Z}_+^{\text{odd}}. \tag{59}$$

That means it is reasonable to define additional flows of the constrained BTH(CTH) as

$$\frac{\partial L}{\partial c_{m,l}} = [-(\mathbb{B}_{m,l})_-, L], \quad m \in \mathbb{Z}_+^{\text{odd}}, l \in \mathbb{Z}_+. \tag{60}$$

Proposition 3. For the BTH(CTH), the flows (60) can commute with original flows of the BTH(CTH), namely,

$$\left[\frac{\partial}{\partial c_{m,l}}, \frac{\partial}{\partial x_k}\right] = 0, \quad \left[\frac{\partial}{\partial c_{m,l}}, \frac{\partial}{\partial y_k}\right] = 0, \quad l \in \mathbb{Z}_+, \ m, k \in \mathbb{Z}_+^{\text{odd}},$$

which hold in the sense of acting on W_i or L.

Theorem 4. The flows in eq.(60) about additional symmetries of constrained BTH(CTH) compose following Block type Lie algebra

$$[\partial_{c_{m,l}}, \partial_{c_{s,k}}] = (km - sl)\partial_{c_{m+s-1,k+l-1}}, \quad m, s \in \mathbb{Z}_+^{\text{odd}}, k, l \in \mathbb{Z}_+, \tag{61}$$

which holds in the sense of acting on W_i or L.

4. Multicomponent B(C) type Toda Hierarchy

In this section we will introduce the multicomponent B type Toda hierarchy (MBTH) and multicomponent C type Toda hierarchy (MCTH). In the following, we denote $E_{\mathbb{Z}\times\mathbb{Z}}$ as the bi-infinite identity matrix and $E_{N\times N}$ as the $N\times N$ identity matrix. We also denote E_{kk} as a $N\times N$ matrix which is 1 at the position of the k-th row and k-th column and 0 for other elements. The Lax operators $\mathcal{L}_1, \mathcal{L}_2$ of the MBTH(MCTH) are given by a pair of infinite matrices

$$\mathcal{L}_1 = \sum_{-\infty < i \le 1} \operatorname{diag}[b_i(s)] \bar{\Lambda}^i, \quad \mathcal{L}_2 = \sum_{-1 \le i < \infty} \operatorname{diag}[c_i(s)] \bar{\Lambda}^i, \tag{62}$$

where $b_i(s), c_i(s)$ are matrices of size $N \times N$ and $\bar{\Lambda}^i = \Lambda^i \otimes E_{N \times N}$ and they satisfy the B type(C Type) constraint [2]

$$\mathcal{L}_i^T = -\mathcal{J}\mathcal{L}_i \mathcal{J}^{-1}(\mathcal{L}_i^T = -\mathcal{K}\mathcal{L}_i \mathcal{K}^{-1}), \tag{63}$$

where $\mathcal{J} = ((-1)^i \delta_{i+j,0})_{i,j \in \mathbb{Z}} \otimes E_{N \times N}$, $\mathcal{K} = \Lambda J \otimes E_{N \times N}$. Here the product \otimes is the Kronecker product between a matrix of size $\mathbb{Z} \times \mathbb{Z}$ and a matrix of size $N \times N$. Let us first introduce some convenient notations as $\bar{E}_{kk} = E_{\mathbb{Z} \times \mathbb{Z}} \otimes E_{kk}$.

The Lax operators of the MBTH(MCTH) (37) can have the following dressing structure

$$\mathcal{L}_1 = \mathcal{W}_1 \bar{\Lambda} \mathcal{W}_1^{-1} = \mathcal{S}_1 \bar{\Lambda} \mathcal{S}_1^{-1}, \tag{64}$$

$$\mathcal{L}_2 = \mathcal{W}_2 \bar{\Lambda}^{-1} \mathcal{W}_2^{-1} = \mathcal{S}_2 \bar{\Lambda}^{-1} \mathcal{S}_2^{-1},$$
 (65)

where

$$\mathcal{W}_1(x,y) = \mathcal{S}_1(x,y)(e^{\xi(x,\Lambda)} \otimes E_{N\times N}), \quad \mathcal{W}_2(x,y) = \mathcal{S}_2(x,y)(e^{\xi(y,\Lambda^{-1})} \otimes E_{N\times N}).$$
 (66)

Now we define matrix operators C_{kk} , \bar{C}_{kk} , B_{jk} , \bar{B}_{jk} as follows

$$C_{kk} := W_1 \bar{E}_{kk} W_1^{-1}, \quad \bar{C}_{kk} := W_2 \bar{E}_{kk} W_2^{-1},$$

$$B_{jk} := W_1 \bar{E}_{kk} \bar{\Lambda}^j W_1^{-1}, \quad \bar{B}_{jk} := W_2 \bar{E}_{kk} \bar{\Lambda}^{-j} W_2^{-1}.$$
(67)

Now we give the definition of the multicomponent B(C) type Toda hierarchy(MBTH).

Definition 1. The multicomponent B(C) type Toda hierarchy is a hierarchy in which the dressing operators S_1, S_2 satisfy following Sato equations

$$\partial_{t_{jk}} \mathcal{S}_1 = -(B_{jk})_- \mathcal{S}_1, \qquad \partial_{t_{jk}} \mathcal{S}_2 = (B_{jk})_+ \mathcal{S}_2, \qquad (68)$$

$$\partial_{\bar{t}_{jk}} \mathcal{S}_1 = -(\bar{B}_{jk})_- \mathcal{S}_1, \qquad \partial_{\bar{t}_{jk}} \mathcal{S}_2 = (\bar{B}_{jk})_+ \mathcal{S}_2. \tag{69}$$

Then one can easily get the following proposition about W_1, W_2 .

Proposition 5. The matrix wave operators W_1, W_2 satisfy following Sato equations

$$\partial_{t_{jk}} \mathcal{W}_1 = (B_{jk})_+ \mathcal{W}_1, \qquad \partial_{t_{jk}} \mathcal{W}_2 = (B_{jk})_+ \mathcal{W}_2, \qquad (70)$$

$$\partial_{\bar{t}_{jk}} \mathcal{W}_1 = -(\bar{B}_{jk})_- \mathcal{W}_1, \qquad \partial_{\bar{t}_{jk}} \mathcal{W}_2 = -(\bar{B}_{jk})_- \mathcal{W}_2. \tag{71}$$

From the previous proposition we can derive the following Lax equations for the Lax operators.

Proposition 6. The Lax equations of the MBTH(MCTH) are as follows

$$\partial_{t_{jk}} \mathcal{L} = [(B_{jk})_+, \mathcal{L}], \quad \partial_{t_{jk}} C_{ss} = [(B_{jk})_+, C_{ss}], \quad \partial_{t_{jk}} \bar{C}_{ss} = [(B_{jk})_+, \bar{C}_{ss}], \quad (72)$$

$$\partial_{\bar{t}_{jk}} \mathcal{L} = [(\bar{B}_{jk})_+, \mathcal{L}], \quad \partial_{\bar{t}_{jk}} C_{ss} = [(\bar{B}_{jk})_+, C_{ss}], \quad \partial_{\bar{t}_{jk}} \bar{C}_{ss} = [(\bar{B}_{jk})_+, \bar{C}_{ss}]. \quad (73)$$

5. Symmetries of of MBTH(MCTH)

To introduce the additional symmetries of the MBTH(MCTH). The Orlov-Shulman operator of the MBTH(MCTH) will be defined as

$$\mathcal{M}_1 = \mathcal{W}_1(\varepsilon \otimes E_{N \times N}) \mathcal{W}_1^{-1}, \quad \mathcal{M}_2 = \mathcal{W}_2(\varepsilon^* \otimes E_{N \times N}) \mathcal{W}_2^{-1},$$
 (74)

$$R_{ij} = \mathcal{W}_i(E \otimes E_{jj})\mathcal{W}_i^{-1}. \tag{75}$$

To construct the additional quantum torus symmetry of the multicomponent BTH, firstly we define the operator $B_{mnj}^{(i)}$ as

$$B_{mnj}^{(i)} = \mathcal{M}_i^m \mathcal{L}_i^n R_{ij} - (-1)^n R_{ij} \mathcal{L}_i^{n-1} \mathcal{M}_i^m \mathcal{L}_i.$$
 (76)

For the multicomponent CTH, we define the operator $B_{mnj}^{(i)}$ as

$$B_{mnj}^{(i)} = \mathcal{M}_i^m \mathcal{L}_i^n R_{ij} - (-1)^n R_{ij} \mathcal{L}_i^n \mathcal{M}_i^m. \tag{77}$$

For any matrix operator $B_{mnj}^{(i)}$ in (77), one has

$$\frac{\partial B_{mnj}^{(i)}}{\partial t_{kj}} = [(\mathcal{L}_1^k R_{1j})_+, B_{mnj}^{(i)}], \ k \in \mathbb{Z}_+^{\text{odd}}.$$
 (78)

$$\frac{\partial B_{mnj}^{(i)}}{\partial \bar{t}_{kj}} = [(\mathcal{L}_2^k R_{2j})_+, B_{mnj}^{(i)}], \ k \in \mathbb{Z}_+^{\text{odd}}. \tag{79}$$

Then we can derive the following lemma.

Lemma 7. The following identities hold true

$$\bar{\Lambda}^{-1}(\varepsilon \otimes E_{N \times N})\bar{\Lambda} = \mathcal{J}^{-1}(\varepsilon^T \otimes E_{N \times N})\mathcal{J}, \qquad \bar{\Lambda}(\varepsilon^* \otimes E_{N \times N})\bar{\Lambda}^{-1} = \mathcal{J}^{-1}(\varepsilon^{*T} \otimes E_{N \times N})\mathcal{J}$$

$$\varepsilon(\varepsilon \otimes E_{N \times N}) = \mathcal{K}(\varepsilon^T \otimes E_{N \times N})\mathcal{K}^{-1}, \qquad \varepsilon^* \otimes E_{N \times N} = \mathcal{K}(\varepsilon^{*T} \otimes E_{N \times N})\mathcal{K}^{-1}.$$
(81)

Then for the MBTH, by (41) and (80), we can derive

$$\mathcal{M}_{1}^{T} = (\mathcal{W}_{1}(\varepsilon \otimes E_{N \times N})\mathcal{W}_{1}^{-1})^{T} = (\mathcal{W}_{1}^{-1})^{T}(\varepsilon^{T} \otimes E_{N \times N})\mathcal{W}_{1}^{T}$$

$$= \mathcal{J}\mathcal{W}_{1}\mathcal{J}^{-1}(\varepsilon^{T} \otimes E_{N \times N})\mathcal{J}\mathcal{W}_{1}^{-1}\mathcal{J}^{-1}$$

$$= \mathcal{J}\mathcal{W}_{1}\mathcal{J}^{-1}(\varepsilon^{T} \otimes E_{N \times N})\mathcal{J}\mathcal{W}_{1}^{-1}\mathcal{J}^{-1}$$

$$= \mathcal{J}\mathcal{W}_{1}\bar{\Lambda}^{-1}(\varepsilon \otimes E_{N \times N})\bar{\Lambda}\mathcal{W}_{1}^{-1}\mathcal{J}^{-1}$$

$$= \mathcal{J}\mathcal{W}_{1}\bar{\Lambda}^{-1}(\varepsilon \otimes E_{N \times N})\bar{\Lambda}\mathcal{W}_{1}^{-1}\mathcal{J}^{-1}$$

$$= \mathcal{J}\mathcal{W}_{1}\bar{\Lambda}^{-1}\mathcal{W}_{1}^{-1}\mathcal{W}_{1}\varepsilon\mathcal{W}_{1}^{-1}\mathcal{W}_{1}\bar{\Lambda}\mathcal{W}_{1}^{-1}\mathcal{J}^{-1}$$

$$= \mathcal{J}\mathcal{L}_{1}^{-1}\mathcal{M}_{1}\mathcal{L}_{1}\mathcal{J}^{-1}, \tag{82}$$

Using the second equation in eq.(80), we can also derive

$$\mathcal{M}_2^T = \mathcal{J}\mathcal{L}_2^{-1}\mathcal{M}_2\mathcal{L}_2\mathcal{J}^{-1}. \tag{83}$$

Similarly, for the CTH, we can derive

$$\mathcal{M}_i^T = \mathcal{K} \mathcal{M}_i \mathcal{K}^{-1}.$$
 (84)

Because of the constraints (63) on the Lax operators for the MBTH(MCTH), we can have the following proposition.

Proposition 8. For the MBTH, it is sufficient to ask for

$$B_{mnj}^{(i)T} = -\mathcal{J}B_{mnj}^{(i)}\mathcal{J}^{-1},\tag{85}$$

Proof. From (63) and (82), we have

$$(\mathcal{M}_{i}^{m}\mathcal{L}_{i}^{n}R_{ij})^{T} = R_{ij}^{T}(\mathcal{L}_{i}^{n})^{T}(\mathcal{M}_{i}^{m})^{T} = (-1)^{l}\mathcal{J}\mathcal{L}_{i}^{l}\mathcal{J}^{-1}\mathcal{J}\mathcal{L}_{i}^{-1}\mathcal{M}_{i}^{m}\mathcal{L}_{i}\mathcal{J}^{-1}$$
$$= \mathcal{J}(-1)^{n}R_{ij}\mathcal{L}_{i}^{n-1}\mathcal{M}_{i}^{m}\mathcal{L}_{i}\mathcal{J}^{-1}.$$

Since $J^T = J^{-1} = J$. Therefore $B_{mnj}^{(i)}$ will satisfy the B type condition.

Similarly, the following proposition can also be got.

Proposition 9. For the MCTH, the following C type condition must hold true

$$B_{mnj}^{(i)T} = \mathcal{K}B_{mnj}^{(i)}\mathcal{K}^{-1},\tag{86}$$

Now for the MBTH we will denote the matrix operator D_{mnj} as

$$D_{imnj} := e^{m\mathcal{M}_i} q^{n\mathcal{L}_i} R_{ij} - \mathcal{L}_i^{-1} R_{ij} q^{-n\mathcal{L}_i} e^{m\mathcal{M}_i} \mathcal{L}_i, \tag{87}$$

which further leads to

$$D_{imnj} = \sum_{p,s=0}^{\infty} \frac{m^p (n \log q)^s (\mathcal{M}_i^p \mathcal{L}_i^s R_{ij} - (-1)^s R_{ij} \mathcal{L}_i^{s-1} \mathcal{M}_i^p \mathcal{L}_i)}{p! s!} = \sum_{p,s=0}^{\infty} \frac{m^p (n \log q)^s B_{psj}^{(i)}}{p! s!}.$$
(88)

Then the following calculation will lead to the B(C) type anti-symmetry property of D_{imnj} as

$$D_{imnj}^{T} = \left(\sum_{p,s=0}^{\infty} \frac{m^{p} (n \log q)^{s} B_{psj}^{(i)}}{p! s!}\right)^{T}$$

$$= -\left(\sum_{p,s=0}^{\infty} \frac{m^{p} (n \log q)^{s} \mathcal{J} B_{psj}^{(i)} \mathcal{J}^{-1}}{p! s!}\right)$$

$$= -\mathcal{J}\left(\sum_{p,s=0}^{\infty} \frac{m^{p} (n \log q)^{s} B_{psj}^{(i)}}{p! s!}\right) \mathcal{J}^{-1}$$

$$= -\mathcal{J} D_{imnj} \mathcal{J}^{-1}.$$

Now for the MCTH we will denote the matrix operator D_{mnj} as

$$D_{imnj} := e^{m\mathcal{M}_i} q^{n\mathcal{L}_i} R_{ij} - R_{ij} q^{-n\mathcal{L}_i} e^{m\mathcal{M}_i}. \tag{89}$$

Therefore we get the following important B(C) type condition which the matrix operator D_{imnj} satisfies

$$D_{imnj}^{T} = -\mathcal{J}D_{imnj}\mathcal{J}^{-1}(D_{imnj}^{T} = -\mathcal{K}D_{imnj}\mathcal{K}^{-1}). \tag{90}$$

Then basing on a quantum parameter q, the additional flows for the time variable $t_{m,n}^{ij}, t_{m,n}^{*ij}$ are defined as follows

$$\frac{\partial \mathcal{S}_1}{\partial t_{m,n}^{ij}} = -(B_{mnj}^{(i)})_- \mathcal{S}_1, \quad \frac{\partial \mathcal{S}_1}{\partial t_{m,n}^{*ij}} = -(D_{imnj})_- \mathcal{S}_1, \tag{91}$$

$$\frac{\partial \mathcal{S}_2}{\partial t_{m,n}^{ij}} = (B_{mnj}^{(i)})_+ \mathcal{S}_2, \quad \frac{\partial \mathcal{S}_2}{\partial t_{m,n}^{*ij}} = (D_{imnj})_+ \mathcal{S}_2, \tag{92}$$

or equivalently rewritten as

$$\frac{\partial \mathcal{L}_1}{\partial t_{m,n}^{ij}} = -[(B_{mnj}^{(i)})_-, \mathcal{L}_1], \qquad \frac{\partial \mathcal{M}_1}{\partial t_{m,n}^{*ij}} = -[(D_{imnj})_-, \mathcal{M}_1], \tag{93}$$

$$\frac{\partial \mathcal{L}_2}{\partial t_{m,n}^{ij}} = [(B_{mnj}^{(i)})_+, \mathcal{L}_2], \qquad \frac{\partial \mathcal{M}_2}{\partial t_{m,n}^{*ij}} = [(D_{imnj})_+, \mathcal{M}_2]. \tag{94}$$

Generally, one can also derive

$$\partial_{t_{lk}^{*ip}}(D_{1mnj}) = [-(D_{ilkp})_{-}, D_{1mnj}], \tag{95}$$

$$\partial_{t_{lk}^{*ip}}(D_{2mnj}) = [(D_{ilkp})_{+}, D_{2mnj}]. \tag{96}$$

This further leads to the commutativity of the additional flow $\frac{\partial}{\partial t_{m,n}^{*ij}}$ with the flow $\partial_{t_{jn}}$, $\partial_{\bar{t}_{jn}}$ in the following theorem.

Theorem 10. The additional flows of $\partial_{t_{l,k}^{*is}}$ are symmetries of the multicomponent BTH(CTH), i.e. they commute with all $\partial_{t_{jn}}$, $\partial_{\bar{t}_{jn}}$ flows of the multicomponent BTH(CTH).

Comparing with the additional symmetry of the single-component BTH(CTH), the additional flows $\partial_{t_{l,k}^s}$ of the multicomponent BTH(CTH) form the following N-folds direct product of the \mathcal{W}_{∞} algebra as following

$$[\partial_{t_{p,s}^{ir}}, \partial_{t_{a,b}^{jc}}] \mathcal{L}_k = \delta_{ij} \delta_{rc} \sum_{\alpha\beta} C_{\alpha\beta}^{(ps)(ab)} \partial_{t_{\alpha,\beta}^{ic}} \mathcal{L}_k, \quad i, j, k = 1, 2; 1 \le r, c \le N.$$

Now it is time to identity the algebraic structure of the additional $t_{l,k}^{*j}$ flows of the multicomponent BTH(CTH).

Theorem 11. The additional flows $\partial_{t_{l,k}^{*dj}}$ of the multicomponent BTH(CTH) form the coupled $\bigotimes^{N} QT_{+}$ algebra (N-folds direct product of the positive half of the quantum torus algebra QT), i.e.,

$$[\partial_{t_{n,m}^{*cr}}, \partial_{t_{l,k}^{*dj}}] = \delta_{cd}\delta_{rj}(q^{ml} - q^{nk})\partial_{t_{n+l,m+k}^{*cr}}, \quad n, m, l, k \ge 0; \quad 1 \le r, j \le N; \quad c = d = 1, 2.$$
(97)

Proof. One can also prove this theorem as following by rewriting the quantum torus flow in terms of a combination of $\partial_{t_n^{ij}}$ flows

$$\begin{split} & [\partial_{t_{n,m}^{*cr}}, \partial_{t_{l,k}^{*dj}}] \mathcal{L}_{i} \\ & = \sum_{p,s=0}^{\infty} \frac{n^{p} (m \log q)^{s}}{p! s!} \partial_{t_{p,s}^{cr}}, \sum_{a,b=0}^{\infty} \frac{l^{a} (k \log q)^{b}}{a! b!} \partial_{t_{a,b}^{dj}}] \mathcal{L}_{i} \\ & = \sum_{p,s=0}^{\infty} \sum_{a,b=0}^{\infty} \frac{n^{p} (m \log q)^{s}}{p! s!} \frac{l^{a} (k \log q)^{b}}{a! b!} [\partial_{t_{p,s}^{cr}}, \partial_{t_{a,b}^{dj}}] \mathcal{L}_{i} \\ & = \sum_{p,s=0}^{\infty} \sum_{a,b=0}^{\infty} \frac{n^{p} (m \log q)^{s}}{p! s!} \frac{l^{a} (k \log q)^{b}}{a! b!} \sum_{\alpha \beta} C_{\alpha \beta}^{(ps)(ab)} \delta_{rj} \partial_{t_{\alpha,\beta}^{cr}} \mathcal{L}_{i} \\ & = (q^{ml} - q^{nk}) \sum_{\alpha,\beta=0}^{\infty} \frac{(n+l)^{\alpha} ((m+k) \log q)^{\beta}}{\alpha! \beta!} \delta_{cd} \delta_{rj} \partial_{t_{\alpha,\beta}^{cr}} \mathcal{L}_{i} \\ & = (q^{ml} - q^{nk}) \delta_{cd} \delta_{rj} \partial_{t_{n+l,m+k}^{*cr}} \mathcal{L}_{i}. \end{split}$$

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