ON THE STRIATED REGULARITY FOR THE 2D ANISOTROPIC BOUSSINESQ SYSTEM

MARIUS PAICU[1] AND NING ZHU[2]

ABSTRACT. In this paper, we investigate the global existence and uniqueness of strong solutions to 2D Boussinesq system with anisotropic thermal diffusion or anisotropic viscosity and with striated initial data. Using the key idea of Chemin to solve 2-D vortex patch of ideal fluid, namely the striated regularity can help to bound the gradient of the velocity, we can prove the global well-posedness of the Boussinesq system with anisotropic thermal diffusion with initial vorticity being discontinuous across some smooth interface. In the case of an anisotropic horizontal viscosity, we can study the propagation of the striated regularity for the smooth temperature patches problem.

1. INTRODUCTION

The Boussinesq system is a classical model in geophysical fluid dynamics which describes the large-scale atmospheric and oceanic flows and also play an important role in the study of Rayleigh-Bénard convection (see [46] for example). In the present paper, we investigate the 2D anisotropic Boussinesq equations with horizontal temperature diffusion or horizontal velocity dissipation. These are derivative models from the classical Boussinesq system where the vertical dimension of the domain is very small compared with the horizontal dimension of the domain. In this case, after rescaling the domain, the dissipation is not isotropic and we have to deal with the anisotropic problem. More precisely, we study the following system which is the Euler equations coupling with a transport-diffusion temperature equation with diffusion only in horizontal direction,

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \theta e_2, \ x \in \mathbb{R}^2, \ t > 0 \\ \partial_t \theta + u \cdot \nabla \theta - \kappa \partial_1^2 \theta = 0, \\ \nabla \cdot u = 0, \\ u(0, x) = u_0(x), \theta(0, x) = \theta_0(x), \end{cases} \tag{1.1}$$

and a system where the Navier-Stokes equations with no vertical viscosity coupling with a transport temperature equation,

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \partial_1^2 u = -\nabla p + \theta e_2, \ x \in \mathbb{R}^2, \ t > 0 \\ \partial_t \theta + u \cdot \nabla \theta = 0, \\ \nabla \cdot u = 0, \\ u(0, x) = u_0(x), \theta(0, x) = \theta_0(x). \end{cases} \tag{1.2}$$

Here $u = (u^1(x,t), u^2(x,t))$ denotes the velocity field, $p = p(x,t)$ is a scalar function denotes the pressure. $\theta = \theta(x,t)$ is a scalar representing the temperature in the content of thermal convection and the density in the modeling of geophysical fluids. $e_2 = (0,1)$ is the vertical unit vector field, and the forcing term θe_2 on behalf of the buoyancy force due to the gravity field. The parameters κ and ν denote the molecular diffusion and the viscosity respectively. These anisotropic system are important modeling dynamics of geophysical flows (see e.g. $[11, 12, 33, 44]$).

The general 2D anisotropic Boussinesq equations can be read as,

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu_1 \partial_1^2 u - \nu_2 \partial_2^2 u = -\nabla p + \theta e_2, \\ \partial_t \theta + u \cdot \nabla \theta - \kappa_1 \partial_1^2 \theta - \kappa_2 \partial_2^2 \theta = 0, \\ \nabla \cdot u = 0, \\ u(0,x) = u_0(x), \theta(0,x) = \theta_0(x). \end{cases} \tag{1.3}$$

where ν_1, ν_2, κ_1 and κ_2 are real parameters. Systems (1.1) and (1.2) are two special cases for (1.3). When $\nu_1 = \nu_2 > 0$, $\kappa_1 = \kappa_2 > 0$, the global well-posedness theory for (1.3) has been established in $[7, 27]$. On the contrary, when these four parameters are zero, whether (1.3) has an unique global solution is a challenging problem and still unsolved. This system has many similarities with the classical 3D incompressible Euler equations such as the vortex stretching mechanism (which will be explained later). So it has both physical motivation and mathematical significant to investigate the intermediate cases (only partial dissipation) and some improvements has been made in the past few years.

The global regularity for the case when $\nu_1 = \nu_2 > 0$ and $\kappa_1 = \kappa_2 = 0$ was proven by Chae in [8] and by Hou and Li in [32] with smooth initial data. Later, Abidi and Hmidi studied this system in the Besov space in [4]. The global weak solution with finite energy has been construct in [30] and has been proved to be unique later in [19]. For the case $\nu_1 = \nu_2 = 0$ and $\kappa_1 = \kappa_2 > 0$, Chae in [8] also studied the global regularity for smooth data. This result was improved by Hmidi and Keraani in [31], Danchin and the first author in [21] for rough initial data. The global well-posedness for (1.1) and (1.2) was considered by Danchin and the first author in [20], and they established the global existence and uniqueness theory. Then the global well-posedness for the anisotropic Boussinesq equations with vertical dissipation, namely (1.3) with only $\nu_2, \kappa_2 > 0$, was studied by Cao and Wu in [13]. Later, Adhikaria et. al. investigated other mixed dissipation cases [1]. Other interesting recent results on the 2D anisotropic Boussinesq equations and other related systems can be found in $[2, 3, 34, 38, 40, 41, 47]$.

Next we would like to introduce a quantity which is widely utilized in the literature we mention above. The quantity $\omega \triangleq \partial_1 u_2 - \partial_2 u_1$ which called vorticity measures how fast the fluid rotates. Taking *curl* operator to the first equation of (1.1) we obtain the corresponding vorticity equation,

$$\partial_t \omega + u \cdot \nabla \omega = \partial_1 \theta. \tag{1.4}$$

Similarly, the vorticity form of system (1.2),

$$\partial_t \omega + u \cdot \nabla \omega - \partial_1^2 \omega = \partial_1 \theta. \tag{1.5}$$

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The forcing term $\partial_1\theta$ is the so called "vortex-stretching" term which making this system become more complex than the 2D Euler system.

Another part of our paper is devoted to study the vortex (temperature) patches problem. Before we describe this problem, we need first to introduce some notations. Let us denote by $\psi(\cdot, t)$ the flow associated with the vector field u, that is

$$
\begin{cases}
\dfrac{d}{dt}\psi(x,t) = u(\psi(t,x),t), \\
\psi(0,x) = x.
\end{cases}
\tag{1.6}
$$

The classical vortex patch problem is associated to the 2D Euler equations. If the initial vorticity taking the characteristic function supported in some connected bounded domain, whether the regularity of the boundary can be preserved through the evolution of the flow ψ? It has been proved by Chemin that the regularity of the boundary can be persisted for all the time in some Hölder class (see [9, 10] for details). Other results about the vortex (temperature) patch problems corresponding to the Euler equations, homogeneous (inhomogeneous) Navier-Stokes equations and other fluid models can be found in [6, 14–18, 22–26, 28, 29, 42, 43, 45] and the references therein.

In order to understand the striated regularity clearly, we need first to introduce some notations and definitions which will be used to describe the boundary regularity. Let X_0 be a vector field defined on D_0 (a connected bounded domain), X is the evolution of X_0 along the flow ψ defining as follows,

$$
X(x,t) \triangleq \partial_{X_0}\psi(\psi^{-1}(x,t),t),
\tag{1.7}
$$

where $\partial_{X_0}f \triangleq X_0 \cdot \nabla f$ denoting the standard directional derivative.

Taking time derivative of (1.7), one can obtain X satisfies the following transport equation,

$$
\begin{cases}
\partial_t X + u \cdot \nabla X = \partial_X u, \\
X(0,x) = X_0(x).
\end{cases}
\tag{1.8}
$$

It is not hard to check that ∂_X satisfies,

$$
[\partial_X, D_t] = 0,
\tag{1.9}
$$

where $[A, B] \triangleq AB - BA$ represents the standard commutator, and $D_t \triangleq \partial_t + u \cdot \nabla$ denotes the material derivative.

We need also the following two definitions, which can be found in [5, 10].

Definition 1.1. *Let $s > 0$ and Ω be a bounded domain in $\mathbb{R}^d$. We say that Ω is of class C^s if there exists a compactly support function $f \in C^s(\mathbb{R}^2)$ and a neighborhood V of $\partial\Omega$ such that*

$$
\partial\Omega = f^{-1}(\{0\}) \cap V \quad and \quad \nabla f(x) \neq 0 \; \forall \; x \in V.
$$

Definition 1.2. *A family $(X_\lambda)_{\lambda\in\Lambda}$ of vector fields over $\mathbb{R}^2$ is said to be non-degenerate whenever*

$$
I(X) \triangleq \inf_{x\in\mathbb{R}^d} \sup_{\lambda\in\Lambda} |X_\lambda(x)| > 0.
$$

Let $r \in (0,1)$ and $(X_\lambda)_{\lambda \in \Lambda}$ be a non-degenerate family of C^r vector fields over $\mathbb{R}^2$. A bounded function f is said to be in the function space C_X^r if it satisfies

$$\|f\|_{C_X^r} \triangleq \sup_{\lambda \in \Lambda} \left(\frac{\|f\|_{L^\infty} \|X_\lambda\|_{C^r} + \|\nabla \cdot (X_\lambda f)\|_{C_X^r}}{I(X)} \right).$$

Next we present the main results for our paper. Since the concrete values of the constants κ in system (1.1) and ν in (1.2) play no role in our discussion, for this reason, we shall assume $\kappa = \nu = 1$ throughout this paper.

The main result pertaining to system (1.1) can be stated as follows.

Theorem 1.1. *Assume $u_0 \in L^2$ be a divergence-free vector field, the corresponding vorticity $\omega_0 \triangleq \partial_1 u_0^2 - \partial_2 u_0^1 \in L^\infty$. Let $(\omega_0, \theta_0) \in H^s \times H^{1+s}$ with $0 < s < 1$. Then system (1.1) exists a unique global solution (u, θ) satisfies*

$$u \in L^\infty([0,T]; H^{1+s}), \ \omega \in L^\infty([0,T]; L^\infty), \ \theta \in L^\infty([0,T]; H^{1+s}), \ \partial_1\theta \in L^\infty([0,T]; H^{1+s}).$$

Furthermore, for any non-degenerate vector field $X_0 \in C^s$ such that $\partial_{X_0}\omega_0 \in L^p$ ($2 < p < \infty$), there exists a unique global solution $X \in L^\infty([0,T]; C^s)$ to equation (1.8) and we have

$$\partial_X \omega \in L^\infty([0,T]; L^p), \ \nabla u \in L^1([0,T]; L^\infty).$$

As a direct application, this theorem can be used to deal with the so called "vortex patch" problem as follows.
For

$$\omega_0(x) = \chi_{D_0}(x) \triangleq \begin{cases} 1 & x \in D_0, \\ 0 & x \notin D_0, \end{cases} \tag{1.10}$$

where D_0 is a connected bounded domain, χ_{D_0} is the standard characteristic function of D_0. Let $\omega(x,t) = \omega^1(x,t) + \omega^2(x,t)$ where ω^1 is the solution of the system

$$\begin{cases} \partial_t \omega^1 + u \cdot \nabla \omega^1 = 0, \\ \omega^1(x,0) = \omega_0(x), \end{cases} \tag{1.11}$$

and ω^2 is the solution of the system

$$\begin{cases} \partial_t \omega^2 + u \cdot \nabla \omega^2 = \partial_1 \theta, \\ \omega^2(x,0) = 0. \end{cases} \tag{1.12}$$

Then the main result can be stated as follows.

Corollary 1.1. *Assume ω_0 defined as in (1.10) and D_0 be a connected bounded domain with its boundary ∂D_0 in Hölder class C^{1+s} ($0 < s < 1$). Then system (1.1) exists a unique global solution satisfies the properties shows in Theorem 1.1. Moreover, the solution of systems (1.11) and (1.12) satisfying*

$$\omega^1 = \chi_{D_t}, \qquad \omega^2 \in L^\infty([0,T]; C^s(X)),$$

with $D_t \triangleq \psi(D_0, t)$ and the boundary of the domain remains in the class C^{1+s}.

Then we present our main result pertaining to system (1.2).

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Theorem 1.2. *Assume $u_0 \in L^2$ be a divergence-free vector field, the corresponding vorticity $\omega_0 \triangleq \partial_1 u_0^2 - \partial_2 u_0^1 \in L^2 \cap L^\infty$. Let $(\omega_0, \theta_0) \in H^s \times H^\beta$ with $\frac{1}{2} < s < \beta$. Then system (1.2) exists a unique global solution (u, θ) satisfies*

$$u \in L^\infty([0,T]; H^{1+s}), \ \partial_1 u \in L^2([0,T]; H^{1+s}), \ \nabla u \in L^1([0,T]; L^\infty), \ \theta \in L^\infty([0,T]; H^s).$$

Furthermore, for any vector field $X_0 \in H^s$, there exists a unique global solution $X \in L^\infty([0,T]; H^s)$ to equation (1.8). Moreover, $X \in L^\infty([0,T]; H^{s'})$ for $s' > 1$ if provided $\omega_0 \in \dot{W}^{1,p} \cap H^{s'}, \theta_0 \in \dot{W}^{1,p} \cap H^{s'}$ with $2 < p < \infty$ and $X_0 \in H^{s'}$.

Remark 1. Here we obtain the velocity u is Lipschitz, which is more regular compared with the result of the paper of Danchin and the first author [20], where the velocity was only Log-Lipschitz.

Remark 2. In the critical case $s = 1/2$, we can prove the global well-posedness and Lipschitz information for velocity with $\omega_0 \in B_{2,1}^{\frac{1}{2}}$ and $\theta_0 \in H^\beta$, $1/2 < \beta$. The method is much similar to our proof of Theorem 1.2 but the process is more complicated. In order to make our paper easy to read, we only discuss the result in Sobolev space here.

Remark 3. We can even obtain the Lipschitz information of the velocity with initial vorticity ω_0 in anisotropic Besov space $\mathcal{B}^{0,\frac{1}{2}}$ through a similar idea. Here $\mathcal{B}^{0,\frac{1}{2}}$ is the space given by the norm

$$\|f\|_{\mathcal{B}^{0,\frac{1}{2}}} = \sum_{q \in Z} 2^{\frac{q}{2}} \|\Delta_q^v f\|_{L^2} \quad and \quad \Delta_q^v = \mathcal{F}^{-1}(\varphi(\xi_2/2^q)\hat{f}(\xi))$$

is the dyadic bloc in the vertical Fourier variable and the definition of $\varphi(\xi)$ will be given in the next section.

The above result can be used to solve the smooth "temperature patch" problem. Defining

$$\theta_0^\varepsilon(x) = \chi_{D_0} * \eta_\varepsilon(x) = \begin{cases} 1 & x \in D_\varepsilon^-, \\ 0 & x \in \mathbb{R}^2 \setminus D_\varepsilon^+, \end{cases} \tag{1.13}$$

where χ_{D_0} is the characteristic function of the domain D_0. η_ε is the standard mollified function. D_ε^- and D_ε^+ are two domains defined by

$$D_\varepsilon^- \triangleq \{x \in D : dist(x, \partial D_0) > \varepsilon\},$$
$$D_\varepsilon^+ \triangleq \{x \in \mathbb{R}^2 : dist(x, \partial D_0) > \varepsilon\}.$$

Along the evolution of the fluid, the distance of $\psi(D_\varepsilon^-, t)$ and $\psi(D_\varepsilon^+, t)$ denoted by $d(t)$ with $d(0) = 2\varepsilon$. Then the following result hold true.

Corollary 1.2. *Let $\frac{1}{2} < s < 1$, assume $\theta_0 = \theta_0^\varepsilon$ defined as in (1.13) with $\partial D_0 \in H^{1+s}$, $\omega_0 \in L^\infty \cap H^s$. Then there exists a unique solution (u, θ) to system (1.2) satisfying the properties listed in Theorem 1.2. Furthermore, $\theta(x, t)$ satisfies the same form as θ_0 that*

$$\theta(x, t) = \begin{cases} 1 & x \in \psi(D_\varepsilon^-, t), \\ 0 & x \in \mathbb{R}^2 \setminus \psi(D_\varepsilon^+, t), \end{cases}$$

and the distance $d(t)$ satisfies,

$$|d(t)| \leq 2\varepsilon e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau}. \tag{1.14}$$

Moreover, the flow $\psi(\cdot, t) \in H^{1+s}$ and the boundary ∂D_ε^-, $\partial D_\varepsilon^+ \in H^{1+s}$ for all $t \geq 0$.

Remark: We can propagate higher regularity of the boundary for the temperature patch if we improve the regularity condition of the initial data.

The rest of this paper is divided into three sections and an appendix. In section 2, we provide some definitions and lemmas which will be used in the next sections. Section 3 is devoted to the study of system (1.1) which divided into three subsections. The first one gives some regularity estimates, the second subsection shows the estimate for striated regularity and the last subsection gives the proof of Corollary 1.1. Section 4 deals with system (1.2) which is divided into five subsections unfolding similar as section 3. Finally, Appendix A provides the technical proof for some lemmas presented in the second section.

2. Preparations

In this section, we will give some definitions and lemmas which will be used in the next several sections. First we give some notations. Throughout this paper, C stands for some real positive constant which may vary from line to line. $\{b_q\}$ stands for the ℓ^1 sequence which may also different in each occurrence. $|D| \triangleq (-\Delta)^{\frac{1}{2}}$ denotes the Zygmund operator which is defined through the Fourier transform that

$$\widehat{|D|f} = |\xi|\widehat{f}, \tag{2.1}$$

where

$$\widehat{f} \triangleq \mathcal{F}(f) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x) \; dx.$$

Similarly, we can define

$$\widehat{|D|^s f} = |\xi|^s \widehat{f}, \qquad \widehat{|\partial_1|^s f} = |\xi_1|^s \widehat{f}. \tag{2.2}$$

Next we present the classical Littlewood-Paley theory in $\mathbb{R}^d$ which plays an important role in the proof of our result. Let χ be a smooth function support on the ball $\mathcal{B} \triangleq \{\xi \in \mathbb{R}^d : |\xi| \leq \frac{4}{3}\}$ and φ be a smooth function support on the ring $\mathcal{C} \triangleq \{\xi \in \mathbb{R}^d : \frac{3}{4} \leq \xi \leq \frac{8}{3}\}$ such that

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1, \text{ for all } \xi \in \mathbb{R}^d, \quad \sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1, \text{ for all } \xi \in \mathbb{R}^d \setminus \{0\}.$$

Then for every $u \in \mathcal{S}'$ (tempered distributions), we define the non-homogeneous Littlewood-Paley operators as follows,

$$\Delta_q u = 0 \text{ for } q \leq -2, \quad \Delta_{-1} u = \chi(D)u = \mathcal{F}^{-1}(\chi(\xi)\widehat{u}(\xi)),$$

$$\Delta_q u = \varphi(2^{-j}D)u = \mathcal{F}^{-1}(\varphi(2^{-j}\xi)\widehat{u}(\xi)), \; \forall \; q \geq 0, \quad S_q u = \sum_{j=-1}^{q-1} \Delta_j u.$$

Next we state the definition of non-homogeneous Besov spaces through the dyadic decomposition.

Definition 2.1. *For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the non-homogeneous Besov space $B^s_{p,r}$ is defined by*

$$B^s_{p,r} = \{f \in \mathcal{S}'; \|f\|_{B^s_{p,r}} < \infty\},$$

where

$$\|f\|_{B^s_{p,r}} = \begin{cases} \displaystyle\sum_{q \geq -1} (2^{qs}\|\Delta_q f\|^r_{L^p})^{\frac{1}{r}} & for \ r \leq \infty, \\ \displaystyle\sup_{q \geq -1} 2^{qs}\|\Delta_q f\|_{L^p} & for \ r = \infty. \end{cases}$$

We point out that when $p = r = 2$, for all $s \in \mathbb{R}$, we have $B^s_{2,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$.

Lemma 2.1. *(Bernstein inequality [5, 10]) Let $k \in \mathbb{N} \cup \{0\}$, $1 \leq a \leq b \leq \infty$. Assume that*

$$supp\widehat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \leq 2^q \mathcal{C}\},$$

for some integer q, then there exists a constant C_1 such that

$$\|\nabla^\alpha f\|_{L^b} \leq C_1 2^{q\left(k+d\left(\frac{1}{a}-\frac{1}{b}\right)\right)} \|f\|_{L^a}, \ k = |\alpha|.$$

If f satisfies

$$supp\widehat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| = 2^q \mathcal{C}\},$$

for some integer q, then

$$C_2 2^{qk}\|f\|_{L^b} \leq \|\nabla^\alpha f\|_{L^b} \leq C_3 2^{q\left(k+d\left(\frac{1}{a}-\frac{1}{b}\right)\right)} \|f\|_{L^a}, \ k = |\alpha|,$$

where C_2 and C_3 are constants depending on α, a and b only.

Noticing that if u is a divergence-free vector field in $\mathbb{R}^2$, then it can be recovered from the corresponding vorticity ω by means of the following Biot-Savart law

$$u = \nabla^\perp \Delta^{-1}\omega. \tag{2.3}$$

Combining the classical Calderón-Zygmund estimate and (2.3), it can lead to the following lemma [10].

Lemma 2.2. *For any smooth divergence-free vector field u with its vorticity $\omega \in L^p$ and $p \in (1,\infty)$, there exists a constant C such that*

$$\|\nabla u\|_{L^p} \leq C\frac{p^2}{p-1}\|\omega\|_{L^p}. \tag{2.4}$$

The next lemma shows the Hölder estimate for transport equation, which is useful in the estimate of the striated regularity. The proof can be found in [10].

Lemma 2.3. *Let v be a smooth divergence-free vector field, $r \in (-1, 1)$. Consider two functions $f \in L^\infty_{loc}(\mathbb{R}; C^r)$ and $g \in L^1_{loc}(\mathbb{R}; C^r)$ satisfy the transport equation*

$$\partial_t f + u \cdot \nabla f = g.$$

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Then we have

$$\|f(t)\|_{C^r} \le C\|f(0)\|_{C^r} e^{C\int_0^t \|\nabla u(\tau)\|_{L^\infty} \ d\tau} + C\int_0^t \|g(\tau)\|_{C^r} e^{C\int_\tau^t \|\nabla u(s)\|_{L^\infty} \ ds} \ d\tau,$$

and the constant C depends only on r.

The following logarithmic inequality plays an important role in the proof of the Lipschitz information for velocity of system (1.1). The proof of this lemma can be found in [5, 10].

Lemma 2.4. *Let $r \in (0,1)$ and $(X_\lambda)_{\lambda \in \Lambda}$ be a non-degenerate family of C^r vector fields over $\mathbb{R}^2$. Let u be a divergence-free vector field over $\mathbb{R}^2$ with vorticity $\omega \in C_X^r$. Assume, in addition that $u \in L^q$ for some $q \in [1,+\infty]$ or that $\nabla u \in L^p$ for some finite p. Then there exists a constant C depending on p and r such that*

$$\|\nabla u\|_{L^\infty} \le C\left(\min(\|u\|_{L^q}, \|\omega\|_{L^p}) + \|\omega\|_{L^\infty} \log\left(e + \frac{\|\omega\|_{C_X^r}}{\|\omega\|_{L^\infty}}\right)\right). \tag{2.5}$$

Then we give the definition of the space $\sqrt{L}$ and $LL^{\frac{1}{2}}$.

Definition 2.2. *The space $\sqrt{L}$ stands for the space of functions f in $\bigcap_{2 \le p < \infty} L^p$ such that*

$$\|f\|_{\sqrt{L}} \triangleq \sup_{p \ge 2} \frac{\|f\|_{L^p}}{\sqrt{p-1}} < \infty.$$

And the space $LL^{\frac{1}{2}}$ denotes by

$$LL^{\frac{1}{2}} \triangleq \left\{f \in \mathcal{S}' : \|f\|_{LL^{\frac{1}{2}}} \triangleq \sup_{j \ge 0} \frac{\|S_j f\|_{L^\infty}}{\sqrt{j+1}} < \infty\right\}.$$

Remark: It is not hard to check that $\sqrt{L} \hookrightarrow LL^{\frac{1}{2}}$.

The following lemma play a significant role in the estimate of the convection term. The proof of this lemma shall be shown in the Appendix.

Lemma 2.5. *Assume u is a smooth divergence free vector field with $u \in L^2$, $\nabla u \in L^\infty$, $f \in H^s$ with $s \in (0,1)$, then we have*

$$-\int_{\mathbb{R}^2} \Delta_q(u \cdot \nabla f)\Delta_q f \ dx \le Cb_q 2^{-2qs}\|\nabla u\|_{L^\infty}\|f\|_{H^s}^2, \tag{2.6}$$

with $b_q \in \ell^1$. Moreover, if $\omega, \partial_1\omega \in L^2$, $\partial_1 f \in H^s$, then we have

$$-\int_{\mathbb{R}^2} \Delta_q(u \cdot \nabla f)\Delta_q f \ dx \le Cb_q 2^{-2qs}(\|u\|_{L^2} + \|\omega\|_{L^2} + \|\partial_1\omega\|_{L^2})$$
$$\times (\|f\|_{H^s}^2 + \|f\|_{H^s}^{\frac{1}{2}}\|\partial_1 f\|_{H^s}^{\frac{1}{2}} + \|f\|_{H^s}^{\frac{3}{2}}\|\partial_1 f\|_{H^s}^{\frac{1}{2}}), \tag{2.7}$$

where ω is the corresponding vorticity of u.

Then we give a lemma which alerts the classical losing regularity estimate for the transport equation, and result can be found in [5, 20]. For the sake of completeness, we will give the proof in the Appendix.

Lemma 2.6 (Losing regularity estimate for transport equation). *Let ρ satisfies the transport equation*

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = f, \\ \rho(0, x) = \rho_0(x), \end{cases} \tag{2.8}$$

where $\rho_0 \in B_{2,r}^s$, $f \in L^1([0, T]; B_{2,r}^s)$ with $r \in [1, \infty]$. Here $v \in L^2$ is a divergence free vector field and for some $V(t) \in L^1([0, T])$, v satisfies

$$\sup_{N \geq 0} \frac{\|\nabla S_N v(t)\|_{L^\infty}}{\sqrt{1 + N}} \leq V(t).$$

Then for all $s > 0$, $\varepsilon \in (0, s)$ and $t \in [0, T]$, we have the following estimate,

$$\|\rho(t)\|_{B_{2,r}^{s-\varepsilon}} \leq C(T) \left(\|\rho_0\|_{B_{2,r}^s} + \int_0^T \|f(\tau)\|_{B_{2,r}^s} \, d\tau \right) e^{\frac{C}{\eta} \int_0^T V(\tau) \, d\tau},$$

The following Lemma gives the classical Kato-Ponce type inequality, which can be found in [35–37].

Lemma 2.7. *Assume $s > 0$ and $p \in (1, +\infty)$. Let f satisfies $f \in L^{p_1}$, $\nabla f \in L^{p_1}$, $|D|^s f \in L^{p_3}$, g satisfies $|D|^{s-1} g \in L^{p_2}$, $|D|^s g \in L^{p_2}$, $g \in L^{p_4}$, then we have*

$$\|[|D|^s, f]g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}} \||D|^{s-1} g\|_{L^{p_2}} + \||D|^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}), \tag{2.9}$$

$$\||D|^s(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}} \||D|^s g\|_{L^{p_2}} + \||D|^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}), \tag{2.10}$$

where $p_2, p_3 \in (1, +\infty)$ satisfy

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

3. The Case of Horizontal Diffusivity

This section is devoted to deal with the first model (1.1). At the beginning, we will give some regularity estimates for (ω, θ) in the first subsection. Then we will exam the Hölder estimate of X and prove Corollary 1.1.

3.1. A priori estimates for ω and θ.

Before we give the regularity estimate for (ω, θ), we need first recall the following existence and uniqueness result in [20] about system (1.1).

Theorem 3.1. *Let $1 < s < \frac{3}{2}$ and $\theta_0 \in H^1$ such that $|\partial_1|^s \theta_0 \in L^2$. Let $u_0 \in H^1$ be a divergence-free vector field and the corresponding vorticity ω_0 in L^∞. Then system (1.1) with initial data (θ_0, u_0) admits a global unique solution (θ, u) in $C_w(\mathbb{R}_+; H^1)$ such that*

$$\theta \in L^\infty(\mathbb{R}_+; H^1), \ \partial_1 \theta \in L^2(\mathbb{R}_+; H^1 \cap L^\infty), \ \omega \in L_{loc}^\infty(\mathbb{R}_+; L^\infty),$$

$$|\partial_1|^s \theta \in L^\infty(\mathbb{R}_+; L^2), \ |\partial_1|^{1+s} \theta \in L_{loc}^2(\mathbb{R}_+; L^2).$$

Then we give the proposition which showing the regularity estimate of (ω, θ).

Proposition 3.1. *Let $0 < s < 1$, assume the initial data $\omega_0 \in L^2 \cap L^\infty \cap H^s$ and $\theta_0 \in H^{1+s}$. Then the following estimate holds true,*

$$\|\omega(t)\|_{H^s}^2 + \|\theta(t)\|_{H^{1+s}}^2 + \int_0^t \|\partial_1 \theta(\tau)\|_{H^{1+s}}^2 \, d\tau \leq C(t) e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau}. \tag{3.1}$$

Proof. We first estimate ω. Applying Δ_q to (1.4), we get

$$\partial_t \Delta_q \omega + \Delta_q(u \cdot \nabla \omega) = \partial_1 \Delta_q \theta. \tag{3.2}$$

Taking L^2 inner product with $\Delta_q \omega$, one can deduce

$$\frac{1}{2}\frac{d}{dt}\|\Delta_q\omega(t)\|_{L^2}^2 = -\int_{\mathbb{R}^2} \Delta_q(u \cdot \nabla\omega)\Delta_q\omega \, dx + \int_{\mathbb{R}^2} \partial_1\Delta_q\theta\Delta_q\omega \, dx \tag{3.3}$$
$$\triangleq N_1 + N_2.$$

For N_1, making use of Lemma 2.5,

$$N_1 \leq Cb_q 2^{-2qs}\|\nabla u\|_{L^\infty}\|\omega\|_{H^s}^2. \tag{3.4}$$

Then we estimate N_2, by Hölder inequality and Young's inequality,

$$N_2 \leq C\|\partial_1\Delta_q\theta\|_{L^2}\|\Delta_q\omega\|_{L^2} \leq Cb_q 2^{-2qs}\|\partial_1\theta\|_{H^s}\|\omega\|_{H^s} \leq Cb_q 2^{-2qs}(\|\theta\|_{H^{1+s}}^2 + \|\omega\|_{H^s}^2). \tag{3.5}$$

Inserting the estimate (3.4) and (3.5) into (3.3), then multiplying both side by 2^{2qs} and summing up over $q \geq -1$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_{H^s}^2 \leq C(1 + \|\nabla u\|_{L^\infty}) \times (\|\omega\|_{H^s}^2 + \|\theta\|_{H^{1+s}}^2). \tag{3.6}$$

Then we estimate θ. Applying Δ_q to the second equation of (1.1), we obtain

$$\partial_t \Delta_q \theta + \Delta_q(u \cdot \nabla \theta) - \partial_1^2 \Delta_q \theta = 0. \tag{3.7}$$

Multiplying (3.7) by $\Delta_q\theta$ and integrating over $\mathbb{R}^2$ with respect to x, after integration by part, one can deduce

$$\frac{1}{2}\frac{d}{dt}\|\Delta_q\theta(t)\|_{L^2}^2 + \|\partial_1\Delta_q\theta\|_{L^2}^2 = -\int_{\mathbb{R}^2} \Delta_q(u \cdot \nabla\theta)\Delta_q\theta \, dx$$
$$= -\sum_{|k-q|\leq 2}\int_{\mathbb{R}^2} \Delta_q(S_{k-1}u \cdot \nabla\Delta_k\theta)\Delta_q\theta \, dx$$
$$- \sum_{|k-q|\leq 2}\int_{\mathbb{R}^2} \Delta_q(\Delta_k u \cdot \nabla S_{k-1}\theta)\Delta_q\theta \, dx \tag{3.8}$$
$$- \sum_{k\geq q-1}\sum_{|k-l|\leq 1}\int_{\mathbb{R}^2} \Delta_q(\Delta_k u \cdot \nabla\Delta_{k+l}\theta)\Delta_q\theta \, dx$$
$$\triangleq \Theta_1 + \Theta_2 + \Theta_3.$$

For Θ_1, along the same method as in the proof of Lemma 2.5 which showed in the Appendix, we can obtain

$$\Theta_1 \leq Cb_q 2^{-2q(1+s)}\|\nabla u\|_{L^\infty}\|\theta\|_{H^{1+s}}^2. \tag{3.9}$$

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For Θ_2, we can write it explicitly,

$$
\begin{aligned}
\Theta_2 &= - \sum_{|k-q|\leq 2} \int_{\mathbb{R}^2} \Delta_q(\Delta_k u \cdot \nabla S_{k-1}\theta)\Delta_q\theta \ dx \\
&= - \sum_{|k-q|\leq 2} \int_{\mathbb{R}^2} \Delta_q(\Delta_k u^1 \partial_1 S_{k-1}\theta)\Delta_q\theta \ dx \\
&\quad - \sum_{|k-q|\leq 2} \int_{\mathbb{R}^2} \Delta_q(\Delta_k u^2 \partial_2 S_{k-1}\theta)\Delta_q\theta \ dx \\
&\triangleq \Theta_{21} + \Theta_{22}.
\end{aligned}
\tag{3.10}
$$

Making use of Hölder inequality, Θ_{21} can be bounded by

$$
\begin{aligned}
\Theta_{21} &\leq C \sum_{|k-q|\leq 2} \|\Delta_k u^1\|_{L^2}\|\partial_1\theta\|_{L^\infty}\|\Delta_q\theta\|_{L^2} \\
&\leq C \sum_{|k-q|\leq 2} \|\Delta_k \partial_2 \Delta^{-1}\omega\|_{L^2}\|\partial_1\theta\|_{L^\infty}\|\Delta_q\theta\|_{L^2} \\
&\leq C \sum_{|k-q|\leq 2} 2^{-k}2^{-sk}2^{sk}\|\Delta_k\omega\|_{L^2}\|\partial_1\theta\|_{L^\infty}2^{-(1+s)k}2^{(1+s)k}\|\Delta_q\theta\|_{L^2} \\
&\leq C 2^{-2(1+s)q}b_q\|\partial_1\theta\|_{L^\infty}\|\omega\|_{H^s}\|\theta\|_{H^{1+s}},
\end{aligned}
$$

where we have used the Biot-Savart law (2.3).
Also making use of (2.3), combining with integration by part, we can write Θ_{22} as

$$
\begin{aligned}
\Theta_{22} &= - \sum_{|k-q|\leq 2} \int_{\mathbb{R}^2} \Delta_q(\Delta_k u^2 \partial_2 \Delta_{k+l}\theta)\Delta_q\theta \ dx \\
&= - \sum_{|k-q|\leq 2} \int_{\mathbb{R}^2} \Delta_q(\Delta_k \partial_1 \Delta^{-1}\omega \partial_2 S_{k-1}\theta)\Delta_q\theta \ dx \\
&= \sum_{|k-q|\leq 2} \int_{\mathbb{R}^2} \Delta_q(\Delta_k \Delta^{-1}\omega \partial_2 S_{k-1}\theta)\partial_1\Delta_q\theta \ dx \\
&\quad + \sum_{|k-q|\leq 2} \int_{\mathbb{R}^2} \Delta_q(\Delta_k \Delta^{-1}\omega \partial_1\partial_2 S_{k-1}\theta)\Delta_q\theta \ dx \\
&\triangleq \Theta_{221} + \Theta_{222}.
\end{aligned}
$$

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For Θ_{221}, by Hölder inequality and Bernstein inequality in Lemma 2.1,

$$
\begin{aligned}
\Theta_{221} &\le C \sum_{|k-q|\le 2} \|\Delta_k \Delta^{-1}\omega\|_{L^\infty} \|\partial_2 S_{k-1}\theta\|_{L^2} \|\partial_1 \Delta_q \theta\|_{L^2} \\
&\le C \sum_{|k-q|\le 2} 2^{-2k} \|\Delta_k \omega\|_{L^\infty} \|\theta\|_{H^1} 2^{-(1+s)q} 2^{(1+s)q} \|\Delta_q \theta\|_{L^2} \\
&\le C \sum_{|k-q|\le 2} 2^{-2k} 2^k \|\Delta_k \omega\|_{L^2} \|\theta\|_{H^1} 2^{-(1+s)q} 2^{(1+s)q} \|\partial_1 \Delta_q \theta\|_{L^2} \\
&\le C 2^{-2(1+s)q} b_q \|\theta\|_{H^1} \|\omega\|_{H^s} \|\partial_1 \theta\|_{H^{1+s}}.
\end{aligned}
$$

Next we bound Θ_{222}, by Hölder inequality and Bernstein inequality,

$$
\begin{aligned}
\Theta_{222} &\le C \sum_{|k-q|\le 2} \|\Delta_k \Delta^{-1}\omega\|_{L^2} \|\partial_1 \partial_2 S_{k-1}\theta\|_{L^\infty} \|\Delta_q \theta\|_{L^2} \\
&\le C \sum_{|k-q|\le 2} 2^{-2k} \|\Delta_k \omega\|_{L^2} \left(\sum_{k'\le k-2} \|\partial_1 \partial_2 \Delta_{k'}\theta\|_{L^\infty} \right) \|\Delta_q \theta\|_{L^2} \\
&\le C \sum_{|k-q|\le 2} 2^{-2k} \|\Delta_k \omega\|_{L^2} 2^{(1-s)q} \left(\sum_{k'\le k-2} 2^{(k'-k)(1-s)} 2^{sk'} \|\partial_1 \partial_2 \Delta_{k'}\theta\|_{L^2} \right) \|\Delta_q \theta\|_{L^2} \\
&\le C 2^{-2(1+s)q} b_q \|\omega\|_{L^2} \|\theta\|_{H^s} \|\partial_1 \theta\|_{H^{1+s}},
\end{aligned}
$$

where we have used the discrete Young's inequality in the last step.
Then inserting the estimates of Θ_{21}, Θ_{221} and Θ_{222} into (3.10), one can obtain

$$
\begin{aligned}
\Theta_2 &\le C 2^{-2(1+s)q} b_q \|\partial_1 \theta\|_{L^\infty} \|\omega\|_{H^s} \|\theta\|_{H^{1+s}} \\
&\quad + C 2^{-2(1+s)q} b_q (\|\omega\|_{H^s} + \|\theta\|_{H^{1+s}}) \|\partial_1 \theta\|_{H^{1+s}}.
\end{aligned}
\tag{3.11}
$$

Finally we estimate Θ_3, by Hölder inequality and Bernstein inequality,

$$
\begin{aligned}
\Theta_3 &= -\sum_{k\ge q-1} \sum_{|k-l|\le 1} \int_{\mathbb{R}^2} \Delta_q \nabla \cdot (\Delta_k u \Delta_l \theta) \Delta_q \theta \, dx \\
&\le C \sum_{k\ge q-1} 2^q \|\Delta_k u\|_{L^\infty} \|\Delta_k \theta\|_{L^2} \|\Delta_q \theta\|_{L^2} \\
&\le C \sum_{k\ge q-1,\ k\ge 0} 2^{q-k} \|\Delta_k \nabla u\|_{L^\infty} \|\Delta_k \theta\|_{L^2} \|\Delta_q \theta\|_{L^2} + C \|\Delta_{-1} u\|_{L^\infty} \|\Delta_{-1}\theta\|_{L^2}^2 \\
&\le C 2^{-2(1+s)q} b_q (1 + \|\nabla u\|_{L^\infty}) \|\theta\|_{H^{1+s}}^2.
\end{aligned}
\tag{3.12}
$$

Inserting the estimates (3.9), (3.11) and (3.12) into (3.8), and making use of Young's inequality, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta_q \theta(t)\|_{L^2}^2 + \|\partial_1 \Delta_q \theta\|_{L^2}^2 &\le C 2^{-2(1+s)q}(1 + \|\nabla u\|_{L^\infty} + \|\partial_1 \theta\|_{L^\infty}) \times (\|\omega\|_{H^s}^2 + \|\theta\|_{H^{1+s}}^2) \\
&\quad + \frac{\varepsilon}{\|b_q\|_{\ell^1}} b_q 2^{-2(1+s)q} \|\partial_1 \theta\|_{H^{1+s}}.
\end{aligned}
$$

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Multiplying both sides by $2^{(1+s)q}$ and summing up from -1 to ∞ with respect to q, choosing $\varepsilon = \frac{1}{2}$, one can deduce

$$\frac{d}{dt}\|\theta(t)\|_{H^{1+s}}^2 + \|\partial_1\theta\|_{H^{1+s}}^2 \leq C(1 + \|\nabla u\|_{L^\infty} + \|\partial_1\theta\|_{L^\infty}) \times (\|\omega\|_{H^s}^2 + \|\theta\|_{H^{1+s}}^2). \quad (3.13)$$

Combining (3.6) with (3.13) and by Grönwall's Lemma, because $\partial_1\theta \in L_t^2(L_x^\infty)$ (see Theorem 3.1), we obtain

$$\|\omega(t)\|_{H^s}^2 + \|\theta(t)\|_{H^{1+s}}^2 + \int_0^t |\partial_1\theta(\tau)\|_{H^{1+s}}^2 \, d\tau \leq C(t)e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau},$$

which complete the proof of this proposition. $\qquad \square$

3.2. **A priori estimates for the striated regularity.**

In this subsection, we will give the estimates of tangential derivatives of ω and the regularity estimates of X. The first lemma gives L^p ($p \in [1, \infty]$) estimate of X.

Lemma 3.1. *Let* $r \in [1, \infty]$, $X_0 \in L^r$ *and* (ω_0, θ_0) *satisfies the assumption in Lemma 4.1. Then the solution* X *of equation* (1.8) *satisfies*

$$\|X_0\|_{L^r}e^{-\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau} \leq \|X(t)\|_{L^r} \leq \|X_0\|_{L^r}e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau}. \quad (3.14)$$

Proof. Multiplying both side of equation (1.8) by $|X|^{r-2}X$ ($1 < r < \infty$) and integrating over $\mathbb{R}^2$ with respect to x, we can obtain

$$\frac{1}{r}\frac{d}{dt}\|X(t)\|_{L^r}^r \leq C\|\nabla u\|_{L^\infty}\|X\|_{L^r}^r, \quad (3.15)$$

which implies the right hand side inequality of (3.14). Using the time reversibility of this equation and the same L^r estimate, we can obtain the first inequality of (3.14). Then taking $r \to \infty$, we can deduct the result for the case $r = \infty$, which complete the proof of this lemma.

$\qquad \square$

Applying ∂_X to the vorticity equation, according to (1.9), we get $\partial_X\omega$ satisfies the following equation

$$\partial_t\partial_X\omega + u \cdot \nabla\partial_X\omega = \partial_X(\partial_1\theta) = X \cdot \nabla\partial_1\theta. \quad (3.16)$$

The next lemma deals with the L^p estimate of $\partial_X\omega$.

Lemma 3.2. *Let* $\partial_{X_0}\omega_0 \in L^p$ ($2 \leq p < \infty$), *and* (ω, θ) *satisfies the assumptions in Proposition 3.1, then we have*

$$\|\partial_X\omega(t)\|_{L^p} \leq \|\partial_{X_0}\omega_0\|_{L^p} + C(t)e^{2\int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau}.$$

Proof. Multiplying the equation (3.16) by $|\partial_X\omega|^{p-2}\partial_X\omega$ ($2 \leq p < \infty$), and integrating over $\mathbb{R}^2$ with respect to x, because u satisfies the divergence-free condition, by Hölder inequality,

$$\frac{1}{p}\frac{d}{dt}\|\partial_X\omega(t)\|_{L^p}^p \leq \|X\|_{L^\infty}\|\partial_1\nabla\theta\|_{L^p}\|\partial_X\omega\|_{L^p}^{p-1}.$$

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Because of the embedding $H^s \hookrightarrow L^p$ with $\frac{2}{p} = 1 - s$, we obtain

$$\frac{d}{dt} \|\partial_X \omega(t)\|_{L^p} \le \|X\|_{L^\infty} \|\partial_1 \nabla \theta\|_{H^s}.$$

Then integrating in time and combining with the result of Proposition 3.1,

$$\|\partial_X \omega(t)\|_{L^p} \le \|\partial_{X_0} \omega_0\|_{L^p} + \int_0^t \|X(\tau)\|_{L^\infty} \|\partial_1 \nabla \theta(\tau)\|_{H^s} \, d\tau$$

$$\le \|\partial_{X_0} \omega_0\|_{L^p} + \|X\|_{L_{t,x}^\infty} \int_0^t \|\partial_1 \nabla \theta(\tau)\|_{H^s} \, d\tau$$

$$\le \|\partial_{X_0} \omega_0\|_{L^p} + C(t) e^{2 \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau},$$

which complete the proof of this lemma. □

Then we give the Hölder estimate for X. The next proposition obtain the Lipschitz information of the velocity u and the C^s norm of X simultaneously.

Proposition 3.2. *Let $0 < s < 1$, assume $X_0 \in C^s$, $\partial_{X_0} \omega_0 \in L^p$ and (ω_0, θ_0) satisfies the assumptions in Proposition 3.1, the we have the velocity u satisfies*

$$\nabla u \in L^1([0,t]; L^\infty). \tag{3.17}$$

Moreover,

$$X \in L^\infty([0,t]; C^s), \ \omega \in L^\infty([0,t]; H^s), \ \partial_X \omega \in L^\infty([0,t]; L^p).$$
$$\theta \in L^\infty([0,t]; H^{1+s}), \ \partial_1 \theta \in L^2([0,t]; H^{1+s}). \tag{3.18}$$

Proof. Firstly, we compute the Hölder estimate of X. Applying Lemma 2.3 to (1.8), we obtain

$$\|X(t)\|_{C^s} \le C \|X_0\|_{C^s} e^{\tilde{C} \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau} + C \int_0^t \|\partial_X u(\tau)\|_{C^s} e^{\tilde{C} \int_\tau^t \|\nabla u(s)\|_{L^\infty} \, ds} \, d\tau$$

$$\le C e^{\tilde{C} \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau} \left(\|X_0\|_{C^s} + \int_0^t \|\partial_X u(\tau)\|_{C^s} e^{-\tilde{C} \int_0^\tau \|\nabla u(s)\|_{L^\infty} \, ds} \, d\tau \right), \tag{3.19}$$

where we can choose $\tilde{C} > 2$. In order to estimate Hölder norm of $\partial_X u$, we need the following estimate which proof can be found in [5, 10],

$$\|\partial_X u\|_{C^s} \le C(\|\nabla u\|_{L^\infty} \|X\|_{C^s} + \|\partial_X \omega\|_{C^{s-1}}). \tag{3.20}$$

By Sobolev embedding $L^p \hookrightarrow C^{s-1}$ $(1 - s = \frac{2}{p})$ and Lemma 3.2, we obtain

$$\|\partial_X \omega\|_{C^{s-1}} \le C \|\partial_X \omega\|_{L^p} \le C \|\partial_X \omega_0\|_{L^p} + C(t) e^{2 \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau}. \tag{3.21}$$

Inserting (3.20) and (3.21) into (3.19), one can deduce that

$$\|X(t)\|_{C^s} \le C e^{\tilde{C} \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau} \left(\|X_0\|_{C^s} + \int_0^t (C(\tau) \right.$$

$$\left. + \|\nabla u(\tau)\|_{L^\infty} \|X(\tau)\|_{C^s} e^{-\tilde{C} \int_0^\tau \|\nabla u(s)\|_{L^\infty} \, ds}) \, d\tau \right).$$

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Denoting
$$F(t) \triangleq \|X(t)\|_{C^s} e^{\tilde{C} \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau}.$$

Then according to the above estimates, we obtain
$$F(t) \le CF(0) + \int_0^t C(\tau)(\|\nabla u(\tau)\|_{L^\infty} + 1)(F(\tau) + 1) \, d\tau.$$

By Grönwall's Lemma,
$$F(t) \le C(F(0) + 1)e^{\int_0^t C(\tau)(\|\nabla u(\tau)\|_{L^\infty} + 1) \, d\tau}.$$

According to the definition of $F(t)$, we obtain the Hölder estimate of X that,
$$\|X(t)\|_{C^s} \le C(t)e^{C \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau}. \tag{3.22}$$

Recalling the logarithmic inequality in Lemma 2.4 that
$$\|\nabla v\|_{L^\infty} \le C\left(\|\omega\|_{L^2} + \|\omega\|_{L^\infty} \log\left(e + \frac{\|\omega\|_{C_X^s}}{\|\omega\|_{L^\infty}}\right)\right), \tag{3.23}$$

where $\|\omega\|_{C_X^s}$ is defined in Definition 1.2.

Because $\|\omega\|_{L^2 \cap L^\infty}$ is bounded, inserting the estimates (3.21), (3.22) into (3.23), we obtain
$$\|\nabla v\|_{L^\infty} \le C\left(1 + \log\left(e + C(t)e^{C \int_0^t \|\nabla u(\tau)\|_{L^\infty}}\right)\right)$$
$$\le C\left(1 + \int_0^t C(t)(1 + \|\nabla u(\tau)\|_{L^\infty}) \, d\tau\right).$$

Then by Grönwall's Lemma,
$$\|\nabla u(t)\|_{L^\infty} \le C(t), \ \forall t > 0. \tag{3.24}$$

Combining the estimates (3.22) and (3.24), we can obtain the desired Hölder norm of X, Then inserting the estimate (3.22) into Proposition 3.1 and Lemma 3.2, we can complete the proof of this proposition. $\qquad \square$

3.3. **The vortex patch problem.** In this subsection, we devote to prove Corollary 1.1, which solving the vortex patch problem. Because
$$\omega_0 = \chi_{D_0}(x) \triangleq \begin{cases} 1 & x \in D_0, \\ 0 & x \notin D_0, \end{cases}$$

where D_0 is a connected bounded domain with $\partial D_0 \in C^{1+s}$ for $0 < s < 1$. Then according to Definition 1.1, there exist a real function $f_0 \in C^{1+s}$ and a neighborhood V_0 such that $\partial D_0 = V_0 \cap f^{-1}(0)$ and $\nabla f_0 \neq 0$ on V_0. Noticing that at time t, the boundary $\partial D_t = \psi(D_0, t)$ is the level set of the function $f(\cdot, t) = f_0(\psi^{-1}(\cdot, t))$ with f being transported by the flow ψ:
$$\begin{cases} \partial_t f + u \cdot \nabla f = 0, \\ f(x, 0) = f_0(x). \end{cases} \tag{3.25}$$

Setting the vector field $X \triangleq \nabla^{\perp} f$ with initial data $X_0 \triangleq \nabla^{\perp} f_0$, it is not hard to verifies that X satisfying (1.7) and the corresponding system (1.8). Then we can parametrize ∂D_0 as

$$\gamma_0 : \mathbb{S}^1 \to \partial D_0, \text{ via } \sigma \mapsto \gamma_0(\sigma),$$

with

$$\begin{cases} \partial_\sigma \gamma_0 = X_0(\gamma_0(\sigma)), \ \forall \ \sigma \in \mathbb{S}^1, \\ \gamma_0(0) = x_0 \in \partial D_0. \end{cases} \tag{3.26}$$

In order to conclude the proof of Corollary 1.1, we observe that a parametrization for ∂D_t is given by $\gamma_t(\sigma) \triangleq \psi(\gamma_0(\sigma), t)$ and by differentiating with respect to the parameter σ, we get

$$\begin{cases} \partial_\sigma \gamma_t(\sigma) = X(\gamma_t(\sigma)), \ \forall \ \sigma \in \mathbb{S}^1, \\ \gamma_t(0) = \psi(x_0, t) \in \partial D_t. \end{cases} \tag{3.27}$$

According to Theorem 1.1, $X \in L^{\infty}([0, T]; C^s)$, thus $\gamma_t \in C^{1+s}(\mathbb{S}^1)$ for all $t \geq 0$. This completes the proof of Corollary 1.1.

4. The Case of Horizontal Viscosity

In this section, we focus on system (1.2). Before we begin to prove the result in Theorem 1.2, we need to review the following existence and uniqueness result for system (1.2) which can be found in [20].

Theorem 4.1. *Let $s \in (\frac{1}{2}, 1]$. For all function $\theta_0 \in H^s \cap L^{\infty}$ and divergence-free vector field $u_0 \in H^1$ with vorticity $\omega_0 \in \sqrt{L}$. System (1.2) with data (u_0, θ_0) admits a unique global solution (u, θ) such that $\theta \in C_w(\mathbb{R}_+; L^{\infty}) \cap C(\mathbb{R}_+; H^{s-\varepsilon})$ for all $\varepsilon > 0$ and*

$$u \in C_w(\mathbb{R}_+; H^1), \ \omega \in L^{\infty}_{loc}(\mathbb{R}_+; \sqrt{L}) \text{ and } \nabla u \in L^{\infty}_{loc}(\mathbb{R}_+; \sqrt{L}). \tag{4.1}$$

In the rest of this section, we will first show that the solution u of system (1.2) actually can be in $L^1([0, t]; L^{\infty})$ in the first subsection. Then we estimate the straited regularity in the second subsection. In subsections 4.3-4.4, we exam the higher regularity estimate of (ω, θ) and the vector field X. The proof of Corollary 1.2 will be given in the last subsection.

4.1. **A priori estimates for the Lipschitz norm of the velocity field.** In this subsection, we will give the estimates for the Lipschitz norm of the velocity field and H^s $(\frac{1}{2} < s < 1)$ norm of (ω, θ). Those estimate will be based on the following global existence theorem [20].

Then we give the estimate for $\|\nabla u\|_{L^{\infty}}$, which plays and important role in the estimate for striated regularity in the next subsections. The main results can be stated as follows.

Lemma 4.1. *Assume $\omega_0 \in H^s$ and $\theta_0 \in H^\beta$ with $\beta > s > \frac{1}{2}$, then the solution (ω, θ) satisfies*

$$\|\omega\|^2_{L^{\infty}_t(H^s)} + \|\partial_1 \nabla \omega\|^2_{L^2_t(H^s)} \leq C, \qquad \|\theta\|^2_{L^{\infty}_t(H^s)} \leq C,$$

moreover,

$$\|\nabla u\|_{L_t^2(L^\infty)} \leq C.$$

Proof. Because of Theorem 4.1, we have already know $\nabla u \in \sqrt{L}$. Then according to the definition of space $\sqrt{L}$ and Lemma 2.6, we know

$$\|\theta\|_{L^\infty([0,t];H^s)} \leq C(t). \tag{4.2}$$

Then we give the estimate of ω. Applying Δ_q to the vorticity equation (1.5) and taking L^2 inner product with $\Delta_q\omega$, one can obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_q\omega(t)\|_{L^2}^2 + \|\partial_1\Delta_q\omega\|_{L^2}^2 = \int_{\mathbb{R}^2} \partial_1\Delta_q\theta\Delta_q\omega \, dx - \int_{\mathbb{R}^2} \Delta_q(u\cdot\nabla\omega)\Delta_q\omega \, dx. \tag{4.3}$$

After integration by part, according to Hölder inequality and Young's inequality,

$$\frac{1}{2}\frac{d}{dt}\|\Delta_q\omega(t)\|_{L^2}^2 + \frac{1}{2}\|\partial_1\Delta_q\omega\|_{L^2}^2 \leq C\|\Delta_q\theta\|_{L^2}^2 - \int_{\mathbb{R}^2} \Delta_q(u\cdot\nabla\omega)\Delta_q\omega \, dx. \tag{4.4}$$

By Lemma 2.5 and Young's inequality,

$$
\begin{aligned}
-\int_{\mathbb{R}^2} \Delta_q(u\cdot\nabla\omega)\Delta_q\omega \, dx \leq & Cb_q 2^{-2qs}(\|u\|_{L^2} + \|\omega\|_{L^2} + \|\partial_1\omega\|_{L^2}) \\
& \times (\|\omega\|_{H^s}^2 + \|\omega\|_{H^s}^{\frac{1}{2}}\|\partial_1\omega\|_{H^s}^{\frac{1}{2}} + \|\omega\|_{H^s}^{\frac{3}{2}}\|\partial_1\omega\|_{H^s}^{\frac{1}{2}}) \\
\leq & Cb_q 2^{-2qs}\|\omega\|_{H^s}^2 + \frac{1}{4\|b_q\|_{\ell^1}}b_q 2^{-2qs}\|\partial_1\omega\|_{H^s}^2.
\end{aligned}
\tag{4.5}
$$

According to the bound (4.2),

$$\|\Delta_q\theta\|_{L^2}^2 \leq b_q 2^{-2qs}\|\theta\|_{H^s}^2. \tag{4.6}$$

Inserting (4.5), (4.6) into (4.4) and taking summation of q, after calculation we obtain

$$\frac{d}{dt}\|\omega(t)\|_{H^s}^2 + \|\partial_1\omega\|_{H^s}^2 \leq C(1 + \|\omega(t)\|_{H^s}^2).$$

Then by Grönwall's Lemma, we get

$$\|\omega(t)\|_{H^s}^2 + \int_0^t \|\partial_1\omega(\tau)\|_{H^s}^2 \, d\tau \leq C.$$

According to trace theory, we know

$$\|f(x_1, x_2)\|_{L_{x_2}^\infty(H_{x_1}^{\alpha-\frac{1}{2}})} \leq C\|f(x_1, x_2)\|_{H^\alpha}, \quad \text{for } \alpha > \frac{1}{2}.$$

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Thus by Sobolev embedding,

$$\int_0^t \|\omega(\tau)\|_{L^\infty}^2 \, d\tau \leq \int_0^t \|\omega(\tau)\|_{L^\infty_{x_2}(H^{s+\frac{1}{2}}_{x_1})}^2 \, d\tau$$

$$\leq \int_0^t \|\partial_1 \omega(\tau)\|_{L^\infty_{x_2}(H^{s-\frac{1}{2}}_{x_1})}^2 \, d\tau$$

$$\leq \int_0^t \|\partial_1 \omega(\tau)\|_{H^s}^2 \, d\tau$$

$$\leq C.$$

Noticing that $\partial_1 \omega = \Delta u^2$ and $\partial_1 u^1 + \partial_2 u^2 = 0$, we have

$$\int_0^t \|\partial_i u^j(\tau)\|_{H^{s+1}}^2 \, d\tau \leq C, \quad \text{for } i, j = 1, 2, \ (i, j) \neq (2, 1).$$

Then by Sobolev embedding,

$$\int_0^t \|\partial_i u^j(\tau)\|_{L^\infty}^2 \, d\tau \leq C, \quad \text{for } i, j = 1, 2, \ (i, j) \neq (2, 1).$$

As for $(i, j) = (2, 1)$, according to the definition of vorticity ω,

$$\partial_2 u^1 = \partial_1 u^2 - \omega,$$

so

$$\int_0^t \|\partial_2 u^1(\tau)\|_{L^\infty}^2 \, d\tau \leq \int_0^t \|\partial_1 u^2(\tau)\|_{L^\infty}^2 \, d\tau + \int_0^t \|\omega(\tau)\|_{L^\infty}^2 \, d\tau \leq C.$$

Thus we obtain $\|\nabla u\|_{L^2_t(L^\infty)}$ is bounded, which completes the proof of this lemma. $\qquad \square$

4.2. A priori estimates for striated regularity.

In this section, we will give some estimates about the vector field X. Along the same method of Lemma 3.14 and combining with Lemma 4.1, one can deduce for any $r \in [1, \infty]$,

$$\|X(t)\|_{L^r} \leq C\|X_0\|_{L^r} e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau} \leq C(t). \tag{4.7}$$

The next lemma shows the H^s ($\frac{1}{2} < s < 1$) estimate for X.

Lemma 4.2. *Let $s > \frac{1}{2}$, $X_0 \in H^s$ and $(\omega_0, \theta_0) \in H^s \times H^\beta$ with $\beta > s$. Then the solution X of (1.8) satisfies*

$$X \in L^\infty([0, t]; H^s),$$

for any $t > 0$.

Proof. Applying operator Δ_q to (1.8),

$$\partial_t \Delta_q X + \Delta_q(u \cdot \nabla X) = \Delta_q \partial_X u. \tag{4.8}$$

Taking the L^2 inner product of the above equality with $\Delta_q X$, we get

$$\frac{1}{2} \frac{d}{dt} \|\Delta_q X(t)\|_{L^2}^2 = -\int_{\mathbb{R}^2} \Delta_q(u \cdot \nabla X) \cdot \Delta_q X \, d\tau + \int_{\mathbb{R}^2} \Delta_q \partial_X u \cdot \Delta_q X \, d\tau. \tag{4.9}$$

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For the first term of the right hand side in (4.9). By Lemma 2.5, we have

$$-\int_{\mathbb{R}^2} \Delta_q(u \cdot \nabla X)\Delta_q X \, dx \leq Cb_q 2^{-2qs}\|\nabla u\|_{L^\infty}\|X\|_{H^s}^2. \tag{4.10}$$

Then we estimate the last term of (4.9), by Hölder inequality,

$$\int_{\mathbb{R}^2} \Delta_q\partial_X u \cdot \Delta_q X \, d\tau \leq \|\Delta_q\partial_X u\|_{L^2}\|\Delta_q X\|_{L^2} \leq Cb_q 2^{-2qs}\|\partial_X u\|_{H^s}\|X\|_{H^s}.$$

For H^s norm of $\partial_X u$, we can bound it by

$$\|\partial_X u\|_{H^s} = \|X \cdot \nabla u\|_{H^s} \leq C(\|X\|_{L^\infty}\|\nabla u\|_{H^s} + \|X\|_{H^s}\|\nabla u\|_{L^\infty}).$$

By Lemma 4.1 and Lemma 3.1, we have known

$$\|X\|_{L^\infty} \leq C, \qquad \|\nabla u\|_{H^s} \leq C\|\omega\|_{H^s} \leq C.$$

Thus we obtain

$$\int_0^t \Delta_q\partial_X u \cdot \Delta_q X \, d\tau \leq Cb_q 2^{-2qs}(\|X\|_{H^s} + \|\nabla u\|_{L^\infty}\|X\|_{H^s}^2). \tag{4.11}$$

Inserting the estimates (4.10) and (4.11) into (4.9) then multiplying both sides by 2^{2qs} and taking summation over $q \geq -1$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|X(t)\|_{H^s}^2 \leq C(\|X\|_{H^s} + \|\nabla u\|_{L^\infty}\|X\|_{H^s}^2). \tag{4.12}$$

Then by Grönwall's Lemma and combining with Lemma 4.1, we obtain

$$\|X\|_{H^s} \leq C(t),$$

which completes the proof of this lemma.

$\square$

4.3. **A priori estimates for ω and θ.**
In this subsection, we will give some regularity estimates for (ω, θ) based on the Lipschitz information $\|\nabla u\|_{L_t^1(L_x^\infty)}$. The following lemma gives the H^1 estimate of (ω, θ).

Lemma 4.3. *Assume $\omega_0 \in H^1$ and $\theta_0 \in H^1$, then the solution (ω, θ) satisfies*

$$\|\nabla\omega\|_{L_t^\infty(L^2)}^2 + \|\nabla\theta\|_{L_t^\infty(L^2)}^2 + \|\partial_1\nabla\omega\|_{L_t^2(L^2)}^2 \leq C. \tag{4.13}$$

Proof. Applying ∂_k $(k = 1, 2)$ to the vorticity equation (1.5), we can obtain $\partial_k\omega$ satisfies

$$\partial_t\partial_k\omega + u \cdot \nabla\partial_k\omega + \partial_k u \cdot \nabla\omega - \partial_1^2\partial_k\omega = \partial_1\partial_k\theta. \tag{4.14}$$

Multiplying $\partial_k\omega$ to (4.14) and integrating over $\mathbb{R}^2$ with respect to x, we have

$$\frac{1}{2}\frac{d}{dt}\|\partial_k\omega(t)\|_{L^2}^2 + \|\partial_1\partial_k\omega\|_{L^2}^2 = \int_{\mathbb{R}^2} \partial_1\partial_k\theta\partial_k\omega \, dx - \int_{\mathbb{R}^2} \partial_k u \cdot \nabla\omega\partial_k\omega \, dx$$
$$\triangleq N_1 + N_2. \tag{4.15}$$

After integration by part and using Hölder inequality and Young's inequality, one can deduce

$$N_1 \leq \frac{1}{2}\|\partial_1 \nabla \omega\|_{L^2}^2 + \frac{1}{2}\|\nabla \theta\|_{L^2}^2. \tag{4.16}$$

For N_2, by Hölder inequality,

$$N_2 \leq \|\nabla u\|_{L^\infty}\|\nabla \omega\|_{L^2}^2. \tag{4.17}$$

Applying ∂_k ($k = 1, 2$) to the temperature equation of (1.2), we can obtain $\partial_k \theta$ satisfies

$$\partial_t \partial_k \theta + u \cdot \nabla \partial_k \theta + \partial_k u \cdot \nabla \theta = 0. \tag{4.18}$$

Similarly, we can prove

$$\frac{1}{2}\frac{d}{dt}\|\partial_k \theta(t)\|_{L^2}^2 \leq C\|\nabla u\|_{L^\infty}\|\nabla \theta\|_{L^2}^2. \tag{4.19}$$

Inserting (4.16) and (4.17) into (4.15) and combining with (4.19), we can deduce

$$\frac{d}{dt}(\|\nabla \omega(t)\|_{L^2}^2 + \|\nabla \theta(t)\|_{L^2}^2) + \|\partial_1 \nabla \omega(t)\|_{L^2}^2 \leq C\|\nabla u\|_{L^\infty}(\|\nabla \omega\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2).$$

Then by virtue of the Grönwall's Lemma and Lemma 4.1,

$$\|\nabla \omega\|_{L_t^\infty(L^2)}^2 + \|\nabla \theta\|_{L_t^\infty(L^2)}^2 + \|\partial_1 \nabla \omega\|_{L_t^2(L^2)}^2 \leq C,$$

which complete the proof of this lemma. $\qquad\square$

The follow lemma shows the L^p estimate of $(\nabla \omega, \nabla \theta)$.

Lemma 4.4. *Assume $\nabla \omega_0 \in L^p$ and $\nabla \theta_0 \in L^p$ ($2 < p < \infty$), then the solution (ω, θ) satisfies*

$$\|\nabla \omega\|_{L_t^\infty(L^p)}^2 + \|\nabla \theta\|_{L_t^\infty(L^p)}^2 \leq C. \tag{4.20}$$

Proof. Multiplying $|\partial_k \omega|^{p-2}\partial_k \omega$ to (4.14) and integrating over $\mathbb{R}^2$ with respect to x, by Hölder inequality and Young's inequality,

$$\frac{1}{p}\frac{d}{dt}\|\partial_k \omega(t)\|_{L^p}^p + (p-1)\int_{\mathbb{R}^2}|\partial_1 \partial_k \omega|^2|\partial_k \omega|^{p-2}\,dx$$

$$= \int_{\mathbb{R}^2}\partial_1 \partial_k \theta|\partial_k \omega|^{p-2}\partial_k \omega\,dx - \int_{\mathbb{R}^2}\partial_k u \cdot \nabla \omega|\partial_k \omega|^{p-2}\partial_k \omega\,dx$$

$$= -(p-1)\int_{\mathbb{R}^2}\partial_k \theta|\partial_k \omega|^{p-2}\partial_1 \partial_k \omega\,dx - \int_{\mathbb{R}^2}\partial_k u \cdot \nabla \omega|\partial_k \omega|^{p-2}\partial_k \omega\,dx$$

$$\leq \frac{p-1}{2}\int_{\mathbb{R}^2}|\partial_1 \partial_k \omega|^2|\partial_k \omega|^{p-2}\,dx + C\int_{\mathbb{R}^2}|\partial_k \theta|^2|\partial_k \omega|^{p-2}\,dx$$

$$\quad + \|\nabla u\|_{L^\infty}\|\nabla \omega\|_{L^p}$$

$$\leq \frac{p-1}{2}\int_{\mathbb{R}^2}|\partial_1 \partial_k \omega|^2|\partial_k \omega|^{p-2}\,dx + C\|\nabla \theta\|_{L^p}^2\|\nabla \omega\|_{L^p}^{p-2} + \|\nabla u\|_{L^\infty}\|\nabla \omega\|_{L^p}^p.$$

Thus we obtain

$$\frac{d}{dt}\|\partial_k \omega(t)\|_{L^p}^2 \leq C\|\nabla\theta\|_{L^p}^2 + \|\nabla u\|_{L^\infty}\|\nabla\omega\|_{L^p}^2. \tag{4.21}$$

Similarly, we can prove

$$\frac{d}{dt}\|\partial_k \theta(t)\|_{L^p}^2 \leq C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^p}^2. \tag{4.22}$$

Combining (4.21) with (4.22), we can deduce

$$\frac{d}{dt}(\|\nabla\omega(t)\|_{L^p}^2 + \|\nabla\theta(t)\|_{L^p}^2) \leq C(1 + \|\nabla u\|_{L^\infty})(\|\nabla\omega\|_{L^p}^2 + \|\nabla\theta\|_{L^p}^2).$$

Then by virtue of the Grönwall's Lemma and Lemma 4.1,

$$\|\nabla\omega\|_{L^p}^2 + \|\nabla\theta\|_{L^p}^2 \leq C(t),$$

which completes the proof of this lemma. $\qquad\square$

Next we discuss the higher order regularity estimate for (ω, θ). Applying $|D|^s$ $(s > 0)$ to the vorticity equation (1.5) and temperature equation of (1.2), we can get $(|D|^s\omega, |D|^s\theta)$ satisfies the following system,

$$\begin{cases} \partial_t |D|^s\omega + u \cdot \nabla |D|^s\omega - \partial_1^2 |D|^s\omega = \partial_1 |D|^s\theta - [|D|^s, u \cdot \nabla]\omega, \\ \partial_t |D|^s\theta + u \cdot \nabla |D|^s\theta = -[|D|^s, u \cdot \nabla]\theta. \end{cases} \tag{4.23}$$

The follow lemma gives the H^s $(s > 1)$ estimate of (ω, θ).

Lemma 4.5. *Assume* $\omega_0 \in \dot{W}^{1,p} \cap H^s$ *and* $\theta_0 \in \dot{W}^{1,p} \cap H^s$ $(2 < p < \infty, s > 1)$, *then the solution* (ω, θ) *satisfies*

$$\||D|^s\omega\|_{L_t^\infty(L^2)}^2 + \||D|^s\theta\|_{L_t^\infty(L^2)}^2 + \|\partial_1 |D|^s\omega\|_{L_t^2(L^2)}^2 \leq C. \tag{4.24}$$

Proof. Taking L^2 inner product with $(|D|^s\omega, |D|^s\theta)$ and adding them up, we have

$$\frac{1}{2}\frac{d}{dt}(\||D|^s\omega(t)\|_{L^2}^2 + \||D|^s\theta(t)\|_{L^2}^2) + \|\partial_1 |D|^s\omega\|_{L^2}^2$$
$$= \int_{\mathbb{R}^2} \partial_1 |D|^s\theta |D|^s\omega \, dx - \int_{\mathbb{R}^2} [|D|^s, u \cdot \nabla]\omega |D|^s\omega \, dx - \int_{\mathbb{R}^2} [|D|^s, u \cdot \nabla]\theta |D|^s\theta \, dx \tag{4.25}$$
$$\triangleq K_1 + K_2 + K_3.$$

For K_1, after integration by part and Young's inequality,

$$K_1 \leq \frac{1}{2}\||D|^s\theta\|_{L^2} + \frac{1}{2}\|\partial_1 |D|^s\omega\|_{L^2}. \tag{4.26}$$

For K_2, by virtue of the Hölder inequality and (2.9) in Lemma 2.7,

$$K_2 \leq \|[|D|^s, u \cdot \nabla]\omega\|_{L^2}\||D|^s\omega\|_{L^2}$$
$$\leq C(\|\nabla u\|_{L^\infty}\||D|^{s-1}\nabla\omega\|_{L^2} + \||D|^s u\|_{L^p}\|\nabla\omega\|_{L^{p'}})\||D|^s\omega\|_{L^2},$$

with $\frac{1}{p} + \frac{1}{p'} = \frac{1}{2}$, $p \in (2, \infty)$. By interpolation

$$\|f\|_{L^p} \leq C\|f\|_{L^2}^{\frac{2}{p}}\|\nabla f\|_{L^2}^{1-\frac{2}{p}},$$

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then we can obtain

$$\||D|^s u\|_{L^p} \leq C\||D|^s u\|_{L^2}^{\frac{2}{p}} \|\nabla |D|^s u\|_{L^2}^{1-\frac{2}{p}} \leq C(\|u\|_{L^2} + \||D|^s \omega\|_{L^2}).$$

Thus we have

$$K_2 \leq C(\|\nabla u\|_{L^\infty} + \|\nabla \omega\|_{L^{p'}}) \times (\||D|^s \omega\|_{L^2}^2 + 1). \tag{4.27}$$

Similarly,

$$K_3 \leq C(\|\nabla u\|_{L^\infty} + \|\nabla \theta\|_{L^{p'}}) \times (\||D|^s \omega\|_{L^2}^2 + \||D|^s \theta\|_{L^2}^2 + 1). \tag{4.28}$$

Inserting (4.26), (4.27) and (4.28) into (4.25), making use of Lemma 4.4 and Grönwall's Lemma, we can deduce

$$\||D|^s \omega\|_{L_t^\infty(L^2)}^2 + \||D|^s \theta\|_{L_t^\infty(L^2)}^2 + \|\partial_1 |D|^s \omega\|_{L_t^2(L^2)}^2 \leq C,$$

which complete the proof of this lemma. $\qquad\square$

4.4. **A priori estimates for the higher order striated regularity.**
In this subsection, we will give the higher order estimates of the vector field X. The first lemma asserts the H^1 estimate of X.

Lemma 4.6. *Let $\omega_0 \in H^1, \theta_0 \in H^1$ and $X_0 \in H^1$, then we have*

$$\|\nabla X\|_{L_t^\infty(L^2)}^2 \leq C. \tag{4.29}$$

Proof. Applying ∂_k $(k = 1, 2)$ to the first equation of (1.8), we can obtain $\partial_k X$ satisfies

$$\partial_t \partial_k X + u \cdot \nabla \partial_k X + \partial_k u \cdot \nabla X = \partial_k \partial_X u \tag{4.30}$$

Multiplying $\partial_k X$ to (4.30) and integrating over $\mathbb{R}^2$ with respect to x, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_k X(t)\|_{L^2}^2 &= \int_{\mathbb{R}^2} \partial_k \partial_X u \cdot \partial_k X \, dx - \int_{\mathbb{R}^2} \partial_k u \cdot \nabla X \cdot \partial_k X \, dx \\
&= \int_{\mathbb{R}^2} \partial_k X \cdot \nabla u \cdot \partial_k X \, dx + \int_{\mathbb{R}^2} X \cdot \nabla \partial_k u \cdot \partial_k X \, dx \\
&\quad - \int_{\mathbb{R}^2} \partial_k u \cdot \nabla X \cdot \partial_k X \, dx \\
&\triangleq B_1 + B_2 + B_3.
\end{aligned} \tag{4.31}$$

By Hölder inequality, B_1 can be bounded by

$$B_1 = \int_{\mathbb{R}^2} \partial_k X \cdot \nabla u \cdot \partial_k X \, dx \leq \|\nabla u\|_{L^\infty} \|\nabla X\|_{L^2}^2. \tag{4.32}$$

Similarly,

$$B_3 = -\int_{\mathbb{R}^2} \partial_k u \cdot \nabla X \cdot \partial_k X \, dx \leq \|\nabla u\|_{L^\infty} \|\nabla X\|_{L^2}^2. \tag{4.33}$$

Then by virtue of anisotropic Hölder inequality,

$$
\begin{aligned}
B_2 &= \int_{\mathbb{R}^2} X \cdot \nabla \partial_k u \cdot \partial_k X \ dx \\
&\leq C\|X\|_{L_{x_2}^\infty(L_{x_1}^2)}\|\partial_k \nabla u\|_{L_{x_2}^2(L_{x_1}^\infty)}\|\partial_k X\|_{L^2(\mathbb{R}^2)} \\
&\leq C\|X\|_{L^2}^{\frac{1}{2}}\|\partial_2 X\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1 \nabla\omega\|_{L^2}^{\frac{1}{2}}\|\partial_k X\|_{L^2} \\
&\leq C(\|\nabla\omega\|_{L^2} + \|\partial_1 \nabla\omega\|_{L^2}) \times (\|X\|_{L^2}^2 + \|\nabla X\|_{L^2}^2).
\end{aligned}
\tag{4.34}
$$

After substituting (4.32), (4.34) and (4.33) into (4.31), we find that

$$
\frac{d}{dt}\|\nabla X(t)\|_{L^2}^2 \leq C(\|\nabla u\|_{L^\infty} + \|\nabla\omega\|_{L^2} + \|\partial_1 \nabla\omega\|_{L^2}) \times (\|X\|_{L^2}^2 + \|\nabla X\|_{L^2}^2).
\tag{4.35}
$$

Combining the estimates (3.14) and (4.35), using Gronwall's Lemma and by Lemma 4.1 and Lemma 4.15, we can deduce

$$
\|\partial_X u\|_{L_t^\infty(L^2)}^2 + \|\partial_1 \partial_X u\|_{L_t^2(L^2)}^2 + \|\nabla X\|_{L_t^\infty(L^2)}^2 \leq C,
$$

which complete the proof of this lemma. $\qquad\square$

The next lemma shows the H^s $(s > 1)$ estimate for X.

Lemma 4.7. *Assume $\omega_0 \in \dot{W}^{1,p} \cap H^s, \theta_0 \in \dot{W}^{1,p} \cap H^s, X_0 \in H^s$ $(2 < p < \infty, s > 1)$, then we have*

$$
\||D|^s X\|_{L_t^\infty(L^2)}^2 \leq C.
\tag{4.36}
$$

Proof. Applying $|D|^s$ to the first equation of (1.8), making use of the definition of commutator, we can obtain $|D|^s X$ satisfies the follow equation

$$
\partial_t |D|^s X + u \cdot \nabla |D|^s X = -[|D|^s, u \cdot \nabla]X + |D|^s(X \cdot \nabla u).
\tag{4.37}
$$

Taking L^2 inner product with $|D|^s X$,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\||D|^s X(t)\|_{L^2}^2 &= -\int_{\mathbb{R}^2}[|D|^s, u \cdot \nabla]X \cdot |D|^s X \ dx + \int_{\mathbb{R}^2}|D|^s(X \cdot \nabla u) \cdot |D|^s X \ dx \\
&\triangleq M_1 + M_2.
\end{aligned}
\tag{4.38}
$$

For M_1, by Hölder inequality and (2.9) in Lemma 2.7,

$$
\begin{aligned}
M_1 &\leq \|[|D|^s, u \cdot \nabla]X\|_{L^2}\||D|^s X\|_{L^2} \\
&\leq C(\|\nabla u\|_{L^\infty}\||D|^{s-1}\nabla X\|_{L^2} + \||D|^s u\|_{L^p}\|\nabla X\|_{L^{p'}})\||D|^s X\|_{L^2},
\end{aligned}
$$

with $\frac{1}{p} + \frac{1}{p'} = \frac{1}{2}$, $p \in (2, \infty)$. Choosing p such that

$$
\|\nabla X\|_{L^{p'}} \leq C\||D|^s X\|_{L^2},
$$

and noticing that by interpolation

$$
\||D|^s u\|_{L^p} \leq C\||D|^s u\|_{L^2}^{\frac{2}{p}}\||D|^{s+1}u\|_{L^2}^{1-\frac{2}{p}} \leq C(\|u\|_{L^2} + \||D|^s \omega\|_{L^2}),
$$

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then we have

$$M_1 \leq C(\|\nabla u\|_{L^\infty} + \||D|^s\omega\|_{L^2} + 1)\||D|^sX\|_{L^2}^2. \tag{4.39}$$

Next we estimate M_2, making use of the Hölder inequality and inequality (2.10) in Lemma 2.7,

$$M_2 \leq \||D|^s(X \cdot \nabla u)\|_{L^2}\||D|^sX\|_{L^2}$$
$$\leq C(\|X\|_{L^\infty}\||D|^s\nabla u\|_{L^2} + \||D|^sX\|_{L^2}\|\nabla u\|_{L^\infty})\||D|^sX\|_{L^2}.$$

By Sobolev embedding,

$$\|X\|_{L^\infty} \leq C\||D|^sX\|_{L^2}, \text{ for } s > 1.$$

Thus we have

$$M_2 \leq C(\||D|^s\omega\|_{L^2} + \|\nabla u\|_{L^\infty})\||D|^sX\|_{L^2}^2. \tag{4.40}$$

Inserting (4.39) and (4.40) into (4.38), using Grönwall's Lemma, we can deduce

$$\||D|^sX\|_{L_t^\infty(L^2)}^2 \leq C,$$

which complete the proof of this lemma. $\qquad\square$

4.5. **The temperature patch problem.** This subsection is devoted to the proof of Corollary 1.2. Because most of the proof is the same to Corollary 1.1, here we just need to verify the inequality (1.14). Choosing arbitrary two points that $x_1 \in D_\varepsilon^-$, $x_2 \in D_\varepsilon^+$, consider the difference

$$|\psi(x_1, t) - \psi(x_2, t)| \leq \|\nabla\psi\|_{L^\infty}|x_1 - x_2|, \quad \text{for any } t > 0. \tag{4.41}$$

Noticing that from (1.6), we have

$$\|\nabla\psi\|_{L^\infty} \leq e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau}. \tag{4.42}$$

Then inserting the estimate (4.42) into (4.41) and taking infimum of x_1, x_2, we can obtain

$$|d(t)| \leq 2\varepsilon e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau}.$$

which is the desired bounded (1.14).

A. APPENDIX

The goal of this appendix is to give the proof of Lemma 2.5 and Lemma 2.8.

Proof of Lemma 2.5. The proof of (2.6) can be found in [20] which used the standard Bony's decomposition (see [5,10]). Here we focus on proving (2.7) using the anisotropic idea. Firstly, we divide the first term of (2.7) into two terms,

$$-\int_{\mathbb{R}^2} \Delta_q(u \cdot \nabla f)\Delta_q f \, dx = -\int_{\mathbb{R}^2} \Delta_q(u^1\partial_1 f)\Delta_q f \, dx - \int_{\mathbb{R}^2} \Delta_q(u^2\partial_2 f)\Delta_q f \, dx$$
$$\triangleq P + Q.$$

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For P, by Bony's decomposition, we can divide it into the following three terms,

$$-\int_{\mathbb{R}^2} \Delta_q(u^1 \partial_1 f) \Delta_q f \, dx$$

$$= -\sum_{|k-q| \le 2} \int_{\mathbb{R}^2} \Delta_q(S_{k-1} u^1 \Delta_k \partial_1 f) \Delta_q f \, dx$$

$$- \sum_{|k-q| \le 2} \int_{\mathbb{R}^2} \Delta_q(\Delta_k u^1 S_{k-1} \partial_1 f) \Delta_q f \, dx \qquad \text{(A.1)}$$

$$- \sum_{k \ge q-1} \sum_{|k-l| \le 1} \int_{\mathbb{R}^2} \Delta_q(\Delta_k u^1 \Delta_l \partial_1 f) \Delta_q f \, dx$$

$$\triangleq P_1 + P_2 + P_3.$$

For P_1, we can rewrite it as

$$P_1 = -\sum_{|q-k| \le 2} \int_{\mathbb{R}^2} \Delta_q(S_{k-1} u^1 \partial_1 \Delta_k f) \Delta_q f \, dx$$

$$= -\sum_{|q-k| \le 2} \int_{\mathbb{R}^2} [\Delta_q, S_{k-1} u^1 \partial_1] \Delta_k f \Delta_q f \, dx$$

$$- \sum_{|q-k| \le 2} \int_{\mathbb{R}^2} S_{k-1} u^1 \partial_1 \Delta_q \Delta_k f \Delta_q f \, dx$$

$$= -\sum_{|q-k| \le 2} \int_{\mathbb{R}^2} [\Delta_q, S_{k-1} u^1 \partial_1] \Delta_k f \Delta_q f \, dx$$

$$- \sum_{|q-k| \le 2} \int_{\mathbb{R}^2} (S_{k-1} u^1 - S_q u^1) \partial_1 \Delta_q \Delta_k f \Delta_q f \, dx$$

$$- \int_{\mathbb{R}^2} S_q u^1 \partial_1 \Delta_q f \Delta_q f \, dx$$

$$\triangleq P_{11} + P_{12} + P_{13},$$

where we have used the fact $\sum_{|q-k| \le 2} \partial_1 \Delta_q \Delta_k f = \Delta_q f$. For P_{11}, by Hölder inequality,

$$|P_{11}| \le \sum_{|q-k| \le 2} \left| \int_{\mathbb{R}^2} [\Delta_q, S_{k-1} u^1 \partial_1] \Delta_k f \Delta_k f \, dx \right|$$

$$\le C \sum_{|q-k| \le 2} \| [\Delta_q, S_{k-1} u^1 \partial_1] \Delta_k f \|_{L^2} \| \Delta_q f \|_{L^2}.$$

According to the definition of Δ_q,

$$[\Delta_q, S_{k-1}u^1\partial_1]\Delta_k f = \int_{\mathbb{R}^2} \phi_q(x-y)(S_{k-1}u^1(y)\partial_1\Delta_k f(y)) \, dy$$

$$- S_{k-1}u^1(x)\int_{\mathbb{R}^d} \phi_q(x-y)\partial_1\Delta_k f(y) \, dy$$

$$= \int_{\mathbb{R}^2} \phi_q(x-y)(S_{k-1}u^1(y) - S_{k-1}u^1(x))\partial_1\Delta_k f(y) \, dy$$

$$= \int_{\mathbb{R}^2} \phi_q(x-y)\int_0^1 (y-x)\cdot\nabla S_{k-1}u^1(sy+(1-s)x) \, ds\partial_1\Delta_k f(y) \, dy$$

$$= \int_{\mathbb{R}^2}\int_0^1 \phi_q(z)z\cdot\nabla S_{k-1}u^1(x-sz)\partial_1\Delta_k f(x-z) \, dsdz,$$

where $\phi_j(x) \triangleq 2^{jd}\mathcal{F}^{-1}(\phi)(2^j x)$. Thus we have by Hölder inequality and Bernstein inequality,

$$\|[\Delta_q, S_{k-1}u^1\partial_1]\Delta_k f\|_{L^2} = \left\| \int_{\mathbb{R}^2}\int_0^1 \phi_q(z)z\cdot\nabla S_{k-1}u^1(x-sz)\partial_1\Delta_k f(x-z) \, dsdz \right\|_{L^2}$$

$$\leq C\int_{\mathbb{R}^2} |\phi_q(z)||z| \, dz\|\nabla S_{k-1}u^1(x-sz)\|_{L^\infty}\|\partial_1\Delta_k f(x-z)\|_{L^2}$$

$$\leq C\int_{\mathbb{R}^2} |\phi_q(z)||z| \, dz\|\nabla S_{k-1}u^1\|_{L^\infty}\|\partial_1\Delta_k f\|_{L^2}$$

$$\leq C2^{-q}2^k\|\nabla S_{k-1}u^1\|_{L^2}\|\partial_1\Delta_k f\|_{L^2}$$

$$\leq C2^{k-q}\|\omega\|_{L^2}\|\partial_1\Delta_k f\|_{L^2}.$$

Then we obtain

$$|P_{11}| \leq C\sum_{|q-k|\leq 2} \|[\Delta_q, S_{k-1}u^1\partial_1]\Delta_k f\|_{L^2}\|\Delta_q f\|_{L^2}$$

$$\leq C\sum_{|q-k|\leq 2} 2^{k-q}\|\omega\|_{L^2}\|\partial_1\Delta_k f\|_{L^2}\|\Delta_q f\|_{L^2}$$

$$\leq Cb_q 2^{2qs}\|\omega\|_{L^2}\|f\|_{H^s}\|\partial_1 f\|_{H^s}.$$

For P_{12}, by Hölder inequality and Bernstein inequality,

$$|P_{12}| = \sum_{|q-k|\leq 2} \left| \int_{\mathbb{R}^2} ((S_{k-1}u^1 - S_q u^1)\partial_1\Delta_q\Delta_k f)\Delta_q f \, dx \right|$$

$$\leq C\sum_{|q-k|\leq 2} \|(S_{k-1}u^1 - S_q u^1)\partial_1\Delta_q\Delta_k f\|_{L^1}\|\Delta_q f\|_{L^\infty}$$

$$\leq C\sum_{|q-k|\leq 2} \|\Delta_k u^1\|_{L^2}\|\Delta_q\Delta_k\partial_1 f\|_{L^2}2^q\|\Delta_q f\|_{L^2}.$$

For the case $k = -1$, by Bernstein inequality,

$$|P_{12}| \leq C\|\Delta_{-1}u^1\|_{L^2}2^{-1}\|\Delta_q\Delta_{-1}f\|_{L^2}2^{-1}\|\Delta_q f\|_{L^2}$$

$$\leq Cb_q 2^{2qs}\|u^1\|_{L^2}\|f\|_{H^s}^2.$$

For the case $k \geq 0$, by Bernstein inequality,

$$
\begin{aligned}
|P_{12}| &\leq C \sum_{|q-k|\leq 2} 2^{-k} \|\nabla \Delta_k u^1\|_{L^2} \|\Delta_q \Delta_k \partial_1 f\|_{L^2} 2^q \|\Delta_q f\|_{L^2} \\
&\leq C \sum_{|q-k|\leq 2} 2^{-k} \|\omega\|_{L^2} 2^q \|\Delta_q \partial_1 f\|_{L^2} \|\Delta_q f\|_{L^2} \\
&\leq C b_q 2^{2qs} \|\omega\|_{L^2} \|f\|_{H^s} \|\partial_1 f\|_{H^s}.
\end{aligned}
$$

Thus,

$$
|P_1| \leq C b_q 2^{2qs} (\|u\|_{L^2} + \|\omega\|_{L^2})(\|f\|_{H^s}^2 + \|f\|_{H^s} \|\partial_1 f\|_{H^s}). \tag{A.2}
$$

For P_2, we can bound it by Hölder inequality that

$$
|P_2| \leq C \sum_{|q-k|\leq 2} \|\Delta_k u^1\|_{L^2} \|\partial_1 S_{k-1} f\|_{L^\infty} \|\Delta_q f\|_{L^2}.
$$

Applying Bernstein inequality, similar as P_{12},

$$
\begin{aligned}
|P_2| &\leq C \sum_{|q-k|\leq 2} \|\Delta_k u^1\|_{L^2} \sum_{m\leq k-2} 2^m \|\Delta_m \partial_1 f\|_{L^2} \|\Delta_q f\|_{L^2} \\
&\leq C \sum_{|q-k|\leq 2} \|\Delta_k u^1\|_{L^2} \sum_{m\leq q-2} 2^m \|\Delta_m f\|_{L^2} \|\Delta_q f\|_{L^2} \\
&\leq C \sum_{|q-k|\leq 2} 2^q \|\Delta_k u^1\|_{L^2} \sum_{m\leq q-2} 2^{m-q} \|\Delta_m f\|_{L^2} \|\Delta_q f\|_{L^2} \\
&\leq C 2^{-qs} \sum_{|q-k|\leq 2} 2^{q-k} 2^k \|\Delta_k u^1\|_{L^2} \sum_{m\leq q-2} 2^{(m-q)(1-s)} 2^{ms} \|\Delta_m f\|_{L^2} \|\Delta_q f\|_{L^2} \\
&\leq C b_q 2^{-2qs} (\|u^1\|_{L^2} + \|\omega\|_{L^2}) \|f\|_{H^s} \|\partial_1 f\|_{H^s},
\end{aligned} \tag{A.3}
$$

where we have used discrete Young's inequality in the last step.

Next we estimate P_3. By Hölder inequality and Bernstein inequality,

$$
\begin{aligned}
|P_3| &\leq \left| \sum_{k\geq q-1} \sum_{|k-l|\leq 1} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u^1 \partial_1 \Delta_q f) \Delta_q f \, dx \right| \\
&\leq C \sum_{k\geq q-1} \sum_{|k-l|\leq 1} \|\Delta_q (\Delta_k u^1 \Delta_l \partial_1 f)\|_{L^1} \|\Delta_q f\|_{L^\infty} \\
&\leq C 2^q \sum_{k\geq q-1} 2^{-k} 2^k \|\Delta_k u^1\|_{L^2} \|\Delta_k \partial_1 f\|_{L^2} \|\Delta_q f\|_{L^2} \\
&\leq C 2^{-qs} \sum_{k\geq q-1} 2^{(q-k)(1+s)} 2^{ks} \|\Delta_k \partial_1 f\|_{L^2} (\|u^1\|_{L^2} + \|\omega\|_{L^2}) \|\Delta_q f\|_{L^2} \\
&\leq C b_q 2^{-2qs} (\|u^1\|_{L^2} + \|\omega\|_{L^2}) \|f\|_{H^s} \|\partial_1 f\|_{H^s},
\end{aligned} \tag{A.4}
$$

where discrete Young's inequality have been used in the last two line.

For Q, we can also divide it into three parts,

$$
-\int_{\mathbb{R}^2} \Delta_q (u^2 \partial_2 f) \Delta_q f \, dx = Q_1 + Q_2 + Q_3, \tag{A.5}
$$

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with

$$Q_1 = -\sum_{|k-q|\leq 2}\int_{\mathbb{R}^2}\Delta_q(S_{k-1}u^2\Delta_k\partial_2 f)\Delta_q f\ dx,$$

$$Q_2 = -\sum_{|k-q|\leq 2}\int_{\mathbb{R}^2}\Delta_q(\Delta_k u^2 S_{k-1}\partial_2 f)\Delta_q f\ dx$$

and

$$Q_3 - \sum_{k\geq q-1}\sum_{|k-l|\leq 1}\int_{\mathbb{R}^2}\Delta_q(\Delta_k u^2\Delta_l\partial_2 f)\Delta_q f\ dx.$$

Similar as P_1, we can rewrite Q_1 as

$$\begin{aligned}
Q_1 &= -\sum_{|q-k|\leq 2}\int_{\mathbb{R}^2}[\Delta_q, S_{k-1}u^2\partial_2]\Delta_k f\Delta_q f\ dx \\
&\quad -\sum_{|q-k|\leq 2}\int_{\mathbb{R}^2}(S_{k-2}u^1 - S_q u^1)\partial_2\Delta_q\Delta_k f\Delta_q f\ dx \\
&\quad -\int_{\mathbb{R}^2}S_q u^2\partial_2\Delta_q f\Delta_q f\ dx \\
&\triangleq Q_{11} + Q_{12} + Q_{13}.
\end{aligned}$$

Here we should notice that $P_{13} + Q_{13} = 0$ because of the divergence free condition of u, so we do not need to estimate these two terms.

For Q_{11}, by Hölder inequality,

$$\begin{aligned}
|Q_{11}| &\leq \sum_{|q-k|\leq 2}\left|\int_{\mathbb{R}^2}[\Delta_q, S_{k-1}u^1\partial_1]\Delta_k f\Delta_k f\ dx\right| \\
&\leq C\sum_{|q-k|\leq 2}\|[\Delta_q, S_{k-1}u^1\partial_1]\Delta_k f\|_{L^2}\|\Delta_q f\|_{L^2}.
\end{aligned}$$

According to the definition of Δ_q and similar as P_{11},

$$[\Delta_q, S_{k-1}u^1\partial_1]\Delta_k f = \int_{\mathbb{R}^2}\int_0^1\phi_q(z)z\cdot\nabla S_{k-1}u^2(x-sz)\partial_2\Delta_k f(x-z)\ dsdz.$$

Making use of the anisotropic Hölder inequality and Bernstein inequality,

$$\begin{aligned}
&\|[\Delta_q, S_{k-1}u^2\partial_2]\Delta_k f\|_{L^2} \\
&= \left\|\int_{\mathbb{R}^2}\int_0^1\varphi_q(z)z\cdot\nabla S_{k-1}u^2(x-sz)\partial_2\Delta_k f(x-z)\ dsdz\right\|_{L^2} \\
&\leq C\int_{\mathbb{R}^2}|\varphi_q(z)||z|\ dz\|\nabla S_{k-1}u^2(x-sz)\|_{L^\infty_{x_2}(L^2_{x_1})}\|\partial_2\Delta_k f(x-z)\|_{L^2_{x_2}(L^\infty_{x_1})} \\
&\leq C2^{-q}\|\nabla S_{k-1}u^2\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla S_{k-1}u^2\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta_k f\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\Delta_k f\|_{L^2}^{\frac{1}{2}}.
\end{aligned}$$

Noticing that by Biot-Savart law $u^2 = \partial_1 \Delta^{-1} \omega$, and combining with the boundedness of Riesz transform in L^2,

$$\|[\Delta_q, S_{k-1}u^2\partial_2]\Delta_k f\|_{L^2} \le C2^{k-q}\|\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla\partial_1\Delta^{-1}\omega\|_{L^2}^{\frac{1}{2}}\|\Delta_k f\|_{L^2}^{\frac{1}{2}}\|\partial_1\Delta_k f\|_{L^2}^{\frac{1}{2}}$$
$$\le C2^{k-q}\|\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\Delta_k f\|_{L^2}^{\frac{1}{2}}\|\partial_1\Delta_k f\|_{L^2}^{\frac{1}{2}}.$$

Then Q_{11} is bounded by

$$|Q_{11}| \le C\sum_{|q-k|\le 2}\|[\Delta_q, S_{k-1}u^2\partial_2]\Delta_k f\|_{L^2}\|\Delta_q f\|_{L^2}$$
$$\le C\sum_{|q-k|\le 2}2^{k-q}\|\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\Delta_k f\|_{L^2}^{\frac{1}{2}}\|\partial_1\Delta_k f\|_{L^2}^{\frac{1}{2}}\|\Delta_q f\|_{L^2}$$
$$\le Cb_q 2^{-2qs}\|\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|f\|_{H^s}^{\frac{3}{2}}\|\partial_1 f\|_{H^s}^{\frac{1}{2}}.$$

For Q_{12}, by the anisotropic Hölder inequality and interpolation inequality,

$$|Q_{12}| = \sum_{|q-k|\le 2}\left|\int_{\mathbb{R}^2}((S_{k-1}u^2 - S_q u^2)\partial_2\Delta_q\Delta_k f)\Delta_q f\ dx\right|$$
$$\le C\sum_{|q-k|\le 2}\|(S_{k-1}u^2 - S_q u^2)\partial_2\Delta_q\Delta_k f\|_{L^2}\|\Delta_q f\|_{L^2}$$
$$\le C\sum_{|q-k|\le 2}\|\Delta_k u^2\|_{L_{x_2}^\infty(L_{x_1}^2)}\|\Delta_q\Delta_k\partial_2 f\|_{L_{x_2}^2(L_{x_1}^\infty)}\|\Delta_q f\|_{L^2}$$
$$\le C\sum_{|q-k|\le 2}\|\Delta_k u^2\|_{L^2}^{\frac{1}{2}}\|\Delta_k\partial_2 u^2\|_{L^2}^{\frac{1}{2}}\|\Delta_q\Delta_k\partial_2 f\|_{L^2}^{\frac{1}{2}}\|\Delta_q\Delta_k\partial_1\partial_2 f\|_{L^2}^{\frac{1}{2}}\|\Delta_q f\|_{L^2}$$

For the case $k = -1$, by Bernstein inequality,

$$|Q_{12}| \le C\|\Delta_{-1}u^2\|_{L^2}\|\Delta_q\Delta_{-1} f\|_{L^2}\|\Delta_q f\|_{L^2}$$
$$\le Cb_q 2^{-2qs}\|u^2\|_{L^2}\|f\|_{H^s}^2.$$

For the case $k \ge 0$, by Bernstein inequality and the relation $u^2 = \partial_1\Delta^{-1}\omega$,

$$|Q_{12}| \le C\sum_{|q-k|\le 2}2^{q-k}\|\nabla\Delta_k u^2\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta_k\partial_1\Delta^{-1}\omega\|_{L^2}^{\frac{1}{2}}\|\Delta_q\partial_1 f\|_{L^2}^{\frac{1}{2}}\|\Delta_q f\|_{L^2}^{\frac{3}{2}}$$
$$\le C\sum_{|q-k|\le 2}2^{q-k}\|\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\Delta_q\partial_1 f\|_{L^2}^{\frac{1}{2}}\|\Delta_q f\|_{L^2}^{\frac{3}{2}}$$
$$\le Cb_q 2^{-2qs}\|\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|f\|_{H^s}\|\partial_1 f\|_{H^s}.$$

Thus,

$$|Q_1| \le Cb_q 2^{-2qs}(\|u\|_{L^2} + \|\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}})(\|f\|_{H^s}^2 + \|f\|_{H^s}\|\partial_1 f\|_{H^s}). \tag{A.6}$$

Similar as Q_{12}, applying anisotropic Hölder inequality and Bernstein inequality, Q_2 can be bounded by

$$|Q_2| \leq C \sum_{|q-k|\leq 2} \|\Delta_k u^2\|_{L^\infty_{x_2}(L^2_{x_1})} \|\partial_2 S_{k-1} f\|_{L^2_{x_2}(L^\infty_{x_1})} \|\Delta_q f\|_{L^2}$$

$$\leq C \sum_{|q-k|\leq 2} \|\Delta_k u^2\|_{L^2}^{\frac{1}{2}} \|\Delta_k \partial_2 u^2\|_{L^2}^{\frac{1}{2}} \|\partial_2 S_{k-1} f\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 S_{k-1} f\|_{L^2}^{\frac{1}{2}} \|\Delta_q f\|_{L^2}$$

$$\leq C b_q 2^{-2qs} \|u\|_{L^2} \|f\|_{H^s}^2 + C \|\omega\|_{L^2}^{\frac{1}{2}} \|\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \left(\sum_{m\leq q-2} 2^{m-q} \|\Delta_m f\|_{L^2} \right)^{\frac{1}{2}} \qquad \text{(A.7)}$$

$$\times \left(\sum_{n\leq q-2} 2^{n-q} \|\Delta_n \partial_1 f\|_{L^2} \right)^{\frac{1}{2}} \|\Delta_q f\|_{L^2}$$

$$\leq C b_q 2^{-2qs} (\|u\|_{L^2} + \|\omega\|_{L^2}^{\frac{1}{2}} \|\partial_1 \omega\|_{L^2}^{\frac{1}{2}})(\|f\|_{H^s}^2 + \|f\|_{H^s}^{\frac{3}{2}} \|\partial_1 f\|_{H^s}^{\frac{1}{2}}).$$

Finally we estimate Q_3. By Hölder inequality and Bernstein inequality,

$$|Q_3| \leq \left| \sum_{k\geq q-1} \sum_{|k-l|\leq 1} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u^2 \partial_2 \Delta_q f) \Delta_q f \; dx \right|$$

$$\leq C \sum_{k\geq q-1} \sum_{|k-l|\leq 1} \|\Delta_q (\Delta_k u^2 \Delta_l \partial_2 f)\|_{L^1} \|\Delta_q f\|_{L^\infty}$$

$$\leq C 2^q \sum_{k\geq q-1} \|\Delta_k u^2\|_{L^2} \|\Delta_k \partial_2 f\|_{L^2} \|\Delta_q f\|_{L^2} \qquad \text{(A.8)}$$

$$\leq C 2^q \sum_{k\geq q-1} (\|u\|_{L^2} + \|\partial_1 \omega\|_{L^2}) 2^{-2k} 2^k \|\Delta_k f\|_{L^2} \|\Delta_q f\|_{L^2}$$

$$\leq C b_q 2^{-2qs} (\|u\|_{L^2} + \|\partial_1 \omega\|_{L^2}) \|f\|_{H^s}^2.$$

Taking all these estimates into account, we can obtain

$$-\int_{\mathbb{R}^2} \Delta_q (u \cdot \nabla f) \Delta_q f \; dx \leq C b_q 2^{-2qs} (\|u\|_{L^2} + \|\omega\|_{L^2} + \|\partial_1 \omega\|_{L^2})$$

$$\times (\|f\|_{H^s}^2 + \|f\|_{H^s}^{\frac{1}{2}} \|\partial_1 f\|_{H^s}^{\frac{1}{2}} + \|f\|_{H^s}^{\frac{3}{2}} \|\partial_1 f\|_{H^s}^{\frac{1}{2}}),$$

which complete the proof of this lemma.

$\square$

Lemma A.1 (Losing regularity estimate for transport equation). *Let ρ satisfies the transport equation*

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = f, \\ \rho(0, x) = \rho_0(x), \end{cases} \qquad \text{(A.9)}$$

where $\rho_0 \in B^s_{2,r}$, $f \in L^1([0,T]; B^s_{2,r})$ with $r \in [1, \infty]$. Here $v \in L^2$ is a divergence free vector field and for some $V(t) \in L^1([0,T])$, v satisfies

$$\sup_{N\geq 0} \frac{\|\nabla S_N v(t)\|_{L^\infty}}{\sqrt{1+N}} \leq V(t).$$

Then for all $s > 0$, $\varepsilon \in (0, s)$ and $t \in [0, T]$, we have the following estimate,

$$\|\rho(t)\|_{B_{2,r}^{s-\varepsilon}} \le C(T) \left(\|\rho_0\|_{B_{2,r}^s} + \int_0^T \|f(\tau)\|_{B_{2,r}^s} \, d\tau \right) e^{\frac{C}{\varepsilon} \left(\int_0^T V(\tau) \, d\tau \right)^2}.$$

Proof. The case $r = \infty$ has been shown in [20], here we just discuss $1 \le r < \infty$. Applying Δ_q to (2.8), we obtain

$$\partial_t \Delta_q \rho + \Delta_q (v \cdot \nabla \rho) = \Delta f. \tag{A.10}$$

Taking L^2 inner product with $\Delta_q \rho$,

$$\frac{1}{2} \frac{d}{dt} \|\Delta_q \rho\|_{L^2}^2 = - \int_{\mathbb{R}^2} \Delta_q (v \cdot \nabla \rho) \Delta_q \rho \, dx + \int_{\mathbb{R}^2} \Delta_q f \Delta_q \rho \, dx \triangleq I + II. \tag{A.11}$$

For II, by Hölder inequality,

$$II = \int_{\mathbb{R}^2} \Delta_q f \Delta_q \rho \le \|\Delta_q f\|_{L^2} \|\Delta_q \rho\|_{L^2}. \tag{A.12}$$

For I, along a similar argument as Lemma 2.5, we can divide it as

$$I = - \int_{\mathbb{R}^2} \Delta_q (u \cdot \nabla \rho) \Delta_q \rho \, dx$$

$$= - \sum_{|k-q| \le 2} \int_{\mathbb{R}^2} \Delta_q (S_{k-1} u \cdot \Delta_k \nabla \rho) \Delta_q \rho \, dx$$

$$- \sum_{|k-q| \le 2} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u \cdot \nabla S_{k-1} \rho) \Delta_q \rho \, dx$$

$$- \sum_{k \ge q-1} \sum_{|k-l| \le 1} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u \cdot \nabla \Delta_l \rho) \Delta_q \rho \, dx$$

$$\triangleq L_1 + L_2 + L_3.$$

For L_1, we can rewrite it as

$$L_1 = - \sum_{|q-k| \le 2} \int_{\mathbb{R}^2} [\Delta_q, S_{k-1} u \cdot \nabla] \Delta_k \rho \Delta_q \rho \, dx$$

$$- \sum_{|q-k| \le 2} \int_{\mathbb{R}^2} (S_{k-1} u - S_q u) \cdot \nabla \Delta_q \Delta_k \rho \Delta_q \rho \, dx$$

$$- \int_{\mathbb{R}^2} S_q u \cdot \nabla \Delta_q \rho \Delta_q f \, dx$$

$$\triangleq L_{11} + L_{12} + L_{13},$$

According to divergence-free condition of u, it is not difficult to find that $L_{13} = 0$. For L_{11}, by Hölder inequality,

$$|L_{11}| \le \sum_{|q-k| \le 2} \left| \int_{\mathbb{R}^2} [\Delta_q, S_{k-1} u \cdot \nabla] \Delta_k \rho \Delta_k \rho \, dx \right|$$

$$\le C \sum_{|q-k| \le 2} \|[\Delta_q, S_{k-1} u \cdot \nabla] \Delta_k \rho\|_{L^2} \|\Delta_q \rho\|_{L^2}.$$

According to the definition of Δ_q,

$$
[\Delta_q, S_{k-1}u \cdot \nabla]\Delta_k\rho = \int_{\mathbb{R}^2} \phi_q(x-y)(S_{k-1}u(y) \cdot \nabla\Delta_k\rho(y)) \, dy
$$

$$
- S_{k-1}u(x) \cdot \int_{\mathbb{R}^d} \phi_q(x-y)\nabla\Delta_k\rho(y) \, dy
$$

$$
= \int_{\mathbb{R}^2} \phi_q(x-y)(S_{k-1}u(y) - S_{k-1}u(x)) \cdot \nabla\Delta_k\rho(y) \, dy
$$

$$
= \int_{\mathbb{R}^2} \phi_q(x-y) \int_0^1 (y-x) \cdot \nabla S_{k-1}u(sy + (1-s)x) \, ds \cdot \nabla\Delta_k\rho(y) \, dy
$$

$$
= \int_{\mathbb{R}^2} \int_0^1 \phi_q(z)z \cdot \nabla S_{k-1}u(x-sz) \cdot \nabla\Delta_k\rho(x-z) \, dsdz.
$$

Thus we have

$$
\|[\Delta_q, S_{k-1}u \cdot \nabla]\Delta_k\rho\|_{L^2} = \left\| \int_{\mathbb{R}^2} \int_0^1 \phi_q(z)z \cdot \nabla S_{k-1}u(x-sz) \cdot \nabla\Delta_k\rho(x-z) \, dsdz \right\|_{L^2}
$$

$$
\leq C \int_{\mathbb{R}^2} |\phi_q(z)||z| \, dz \|\nabla S_{k-1}u(x-sz)\|_{L^\infty} \|\nabla\Delta_k\rho(x-z)\|_{L^2}
$$

$$
\leq C 2^{-q} \int_{\mathbb{R}^2} |\phi_q(z)||z| \, dz \|\nabla S_{k-1}u\|_{L^\infty} \|\nabla\Delta_k\rho\|_{L^2}
$$

$$
\leq C 2^{-q} \|\nabla S_{k-1}u\|_{L^\infty} 2^k \|\Delta_k\rho\|_{L^2}.
$$

Then we obtain

$$
|L_{11}| \leq C \sum_{|q-k|\leq 2} \|[\Delta_q, S_{k-1}u \cdot \nabla]\Delta_k\rho\|_{L^2} \|\Delta_q\rho\|_{L^2}
$$

$$
\leq C \sum_{|q-k|\leq 2} 2^{k-q} \|\nabla S_{k-1}u\|_{L^\infty} \|\Delta_q\rho\|_{L^2}^2
$$

$$
\leq C\sqrt{q}V(t)\|\Delta_q\rho\|_{L^2}^2
$$

$$
\leq C d_q 2^{-\sigma q} \sqrt{q}V(t)\|\rho\|_{B^\sigma_{2,r}}\|\Delta_q\rho\|_{L^2},
$$

where $d_q \in \ell^r$.

For L_{12}, by Hölder inequality,

$$
|L_{12}| = \sum_{|q-k|\leq 2} \left| \int_{\mathbb{R}^2} ((S_{k-1}u - S_q u) \cdot \nabla\Delta_q\Delta_k\rho)\Delta_q\rho \, dx \right|
$$

$$
\leq C \sum_{|q-k|\leq 2} 2^{q-k} \|\nabla\Delta_k u\|_{L^\infty} \|\Delta_q\rho\|_{L^2}^2 + \|u\|_{L^2}\|\Delta_q\rho\|_{L^2}^2
$$

$$
\leq C(\sqrt{q+2}V(t) + \|u\|_{L^2})\|\Delta_q\rho\|_{L^2}^2
$$

$$
\leq C d_q 2^{-\sigma q}(\sqrt{q+2}V(t) + \|u\|_{L^2})\|\rho\|_{B^\sigma_{2,r}}\|\Delta_q\rho\|_{L^2}.
$$

For L_2, we can bound it by Hölder inequality that

$$
|L_2| \leq C \sum_{|q-k|\leq 2} \|\Delta_k u\|_{L^\infty} \|\nabla S_{k-1}\rho\|_{L^2} \|\Delta_q\rho\|_{L^2}.
$$

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According to Bernstein inequality,

$$\begin{aligned}
|L_2| &\leq C \sum_{|q-k|\leq 2} \|\Delta_k u\|_{L^\infty} \sum_{m\leq q-2} 2^m \|\Delta_m \rho\|_{L^2} \|\Delta_q \rho\|_{L^2} \\
&\leq C \sum_{|q-k|\leq 2} 2^q \|\Delta_k u\|_{L^\infty} \sum_{m\leq q-2} 2^{m-q} \|\Delta_m \rho\|_{L^2} \|\Delta_q \rho\|_{L^2} \\
&\leq C \sum_{|q-k|\leq 2} 2^q \|\Delta_k u\|_{L^\infty} \sum_{m\leq q-2} 2^{m-q} \|\Delta_m \rho\|_{L^2} \|\Delta_q \rho\|_{L^2} \\
&\leq C(\sqrt{q+2} V(t) + \|u\|_{L^2}) \sum_{m\leq q-2} 2^{m-q} \|\Delta_m \rho\|_{L^2} \|\Delta_q \rho\|_{L^2} \\
&\leq C d_q 2^{-\sigma q} (\sqrt{q+2} V(t) + \|u\|_{L^2}) \|\rho\|_{B_{2,r}^\sigma} \|\Delta_q \rho\|_{L^2}.
\end{aligned}$$

Then we bound L_3. By Hölder inequality and Bernstein inequality,

$$\begin{aligned}
|L_3| &\leq \left| \sum_{k\geq q-1} \sum_{|k-l|\leq 1} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u \cdot \nabla \Delta_q \rho) \|\Delta_q \rho\|_{L^2} \, dx \right| \\
&\leq C \sum_{k\geq q-1} \sum_{|k-l|\leq 1} \|\Delta_q \nabla \cdot (\Delta_k u \Delta_l \rho)\|_{L^2} \|\Delta_q \rho\|_{L^2} \\
&\leq C 2^q \sum_{k\geq q-1} \|\Delta_k u\|_{L^\infty} \|\Delta_k \rho\|_{L^2} \|\Delta_q \rho\|_{L^2} \\
&\leq C(\sqrt{q+2} V(t) + \|u\|_{L^2}) \sum_{k\geq q-1} 2^{q-k} \|\Delta_k \rho\|_{L^2} \|\Delta_q \rho\|_{L^2} \\
&\leq C d_q 2^{-\sigma q} (\sqrt{q+2} V(t) + \|u\|_{L^2}) \|\rho\|_{B_{2,r}^\sigma} \|\Delta_q \rho\|_{L^2}.
\end{aligned}$$

Thus we obtain I can be bounded by

$$I \leq C d_q 2^{-\sigma q} (\sqrt{q+2} V(t) + 1) \|\rho\|_{B_{2,r}^\sigma} \|\Delta_q \rho\|_{L^2}. \tag{A.13}$$

Inserting (A.12) and (A.13) into (A.11), one can obtain

$$\frac{d}{dt} \|\Delta_q \rho(t)\|_{L^2} \leq \|\Delta_q f\|_{L^2} + C d_q 2^{-\sigma q} (\sqrt{q+2} V(t) + 1) \|\rho\|_{B_{2,r}^\sigma}. \tag{A.14}$$

Denoting $s_t \triangleq s - \eta \int_0^t V(\tau) \, d\tau$ for $t \in [0,T]$ with $\eta = \varepsilon \left(\int_0^T V(\tau) \, d\tau \right)^{-1}$. Choosing $\sigma = s_t$ and integrating (A.14) from 0 to t with respect to time variable and then multiplying by $2^{s_t q}$,

$$\begin{aligned}
2^{s_t q} \|\Delta_q \rho(t)\|_{L^2} &\leq d_q \|\rho_0\|_{B_{2,r}^{s_t}} + d_q \int_0^t \|f(\tau)\|_{B_{2,1}^{s_t}} \, d\tau \\
&\quad + C d_q \int_0^t 2^{\left(-\eta \int_\tau^t V(s) ds \right) q} (\sqrt{q+2} V(\tau) + 1) \|\rho\|_{B_{2,r}^{s_\tau}} \, d\tau.
\end{aligned} \tag{A.15}$$

Choosing $q_0 > 0$ is the smallest integer such that

$$\frac{4 C^2 \|d_q\|_{\ell^r}^2}{(\log 2)^2 \eta^2} \leq q_0 + 2.$$

Then for $q \geq q_0$, we have

$$C \int_0^t 2^{\left(-\eta \int_\tau^t V(s)ds\right)q} \sqrt{q+2} V(\tau) \, d\tau \leq \frac{1}{2\|b_q\|_{\ell^r}}. \tag{A.16}$$

Inserting these result into (A.15) and taking ℓ^r norm of q, on can deduce

$$
\begin{aligned}
\|\rho(t)\|_{B_{2,r}^{s_t}} &\leq C\|\rho_0\|_{B_{2,r}^s} + C \int_0^t \|f(\tau)\|_{B_{2,r}^s} \, d\tau \\
&\quad + C\left(\sum_{q \geq q_0} \left(d_q \int_0^t 2^{\left(-\eta \int_\tau^t V(s)ds\right)q} \sqrt{q+2} V(\tau) \|\rho\|_{B_{2,r}^{s_\tau}} \, d\tau \right)^r \right)^{\frac{1}{r}} \\
&\quad + C\left(\sum_{1 \leq q < q_0} \left(d_q \int_0^t 2^{\left(-\eta \int_\tau^t V(s)ds\right)q} \sqrt{q+2} V(\tau) \|\rho\|_{B_{2,r}^{s_\tau}} \, d\tau \right)^r \right)^{\frac{1}{r}} \\
&\leq C\|\rho_0\|_{B_{2,r}^s} + C \int_0^t \|f(\tau)\|_{B_{2,r}^s} \, d\tau \\
&\quad + \frac{1}{2} \sup_{t \in [0,T]} \|\rho\|_{B_{2,r}^{s_t}} + C\sqrt{q_0+1} \int_0^t V(\tau) \|\rho\|_{B_{2,r}^{s_\tau}} \, d\tau.
\end{aligned}
\tag{A.17}
$$

Taking supremum of time t from 0 to T and applying the Grönwall's Lemma, we deduce

$$\sup_{t \in [0,T]} \|\rho(t)\|_{B_{2,r}^{s_t}} \leq C(T) \left(\|\rho_0\|_{B_{2,r}^s} + \int_0^T \|f(\tau)\|_{B_{2,r}^s} \, d\tau \right) e^{\sqrt{q_0+1} \int_0^T V(\tau) \, d\tau}.$$

According to the definition of q_0, finally we obtain

$$\sup_{t \in [0,T]} \|\rho(t)\|_{B_{2,r}^{s_t}} \leq C(T) \left(\|\rho_0\|_{B_{2,r}^s} + \int_0^T \|f(\tau)\|_{B_{2,r}^s} \, d\tau \right) e^{\frac{C}{\eta} \int_0^T V(\tau) \, d\tau},$$

which entails the desired inequality given that $s \geq s_t \geq s - \varepsilon$ for all $t \in [0, T]$.

$\square$

Acknowledgements

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[1] Université de Bordeaux, Institut de Mathématiques de Bordeaux, F-33405 Talence Cedex, France

E-mail address: marius.paicu@math.u-bordeaux.fr

[2] School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China

E-mail address: mathzhu1@163.com