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Implications of Kunita–Itô–Wentzell Formula for *k*-Forms in Stochastic Fluid Dynamics

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Abstract
We extend the Itô–Wentzell formula for the evolution of a time-dependent stochastic field along a semimartingale to k-form-valued stochastic processes. The result is the Kunita–Itô–Wentzell (KIW) formula for k-forms. We also establish a correspondence between the KIW formula for k-forms derived here and a certain class of stochastic fluid dynamics models which preserve the geometric structure of deterministic ideal fluid dynamics. This geometric structure includes Eulerian and Lagrangian variational principles, Lie–Poisson Hamiltonian formulations and natural analogues of the Kelvin circulation theorem, all derived in the stochastic setting.

Keywords Stochastic geometric mechanics · Lie derivatives with respect to stochastic vector fields · Pull-back by smooth maps with stochastic time parameterization

Mathematics Subject Classification 70H33 · 70H25 · 70S05 · 70S10 · 70S20 · 70S99

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1 Introduction

Purpose of this paper. This paper aims to derive stochastic partial differential equations (SPDEs) for continuum dynamics with stochastic advective Lie transport (SALT). These derivations require stochastic counterparts of the deterministic approaches for deriving fluid equations (PDEs). The approach we follow is the stochastic counterpart of the Euler–Poincaré variational principle as in Holm et al. (1998) which reveals the geometric structure of deterministic ideal fluid dynamics. Our goal is to create stochastic counterparts which preserve this geometric structure. Such variational formulations of stochastic fluid PDEs will also possess auxiliary SPDEs for stochastic advection of material properties which will correspond to various differential k-forms. The auxiliary SPDE for advective transport of a given material property by a stochastic fluid flow corresponds to a type of stochastic "chain rule" in which a k-form-valued semimartingale $K(t, x)$ is evaluated (via pullback) along a stochastic flow ϕ_t. The stochastic aspect of the flow map ϕ_t represents uncertainty in the Lagrange-to-Euler map for the fluid. Examples of the k-form-valued process $K(t, x)$ advected by the pullback of the stochastic flow map ϕ_t include mass density, regarded as a volume form, and magnetic field, interpreted as a two-form for ideal magnetohydrodynamics (MHD) (Holm et al. 1998). We denote pullback of a k-form K by $\phi_t^* K$, which for scalar functions f takes the simple form $\phi_t^* f := f \circ \phi_t$. We will refer to this "chain rule" that gives us the auxiliary SPDE for SALT satisfied by the k-form-valued process $\phi_t^* K$ as the *Kunita–Itô–Wentzell (KIW) formula* for k-forms.

In Kunita (1981, 1997), Kunita derived an extension of Itô's formula showing that if K is a sufficiently regular (l, m)-tensor and ϕ_t is the flow of the SDE

$$\mathrm{d}\phi_t(x) = b(t, \phi_t(x))\,\mathrm{d}t + \xi(t, \phi_t(x)) \circ \mathrm{d}B_t\,, \tag{1.1}$$

with sufficiently regular coefficients, then an analogue of Itô's formula holds for tensor fields, namely

$$\phi_t^* K(t, x) - K(0, x) = \int_0^t \phi_s^*(\mathcal{L}_b K)(s, x)\,\mathrm{d}s + \int_0^t \phi_s^*(\mathcal{L}_\xi K)(s, x) \circ \mathrm{d}B_s\,. \tag{1.2}$$

Here, $\circ\,\mathrm{d}B_s$ denotes Stratonovich integration with respect to the Brownian motion B_s. Throughout the paper, we assume that we are working with a stochastic basis of the form $(\Omega, (\mathcal{F})_t, \mathcal{F}, \mathbb{P}, (B_t))_{t\in[0,T]}$ with the usual conditions, where B_t is an n-dimensional Brownian motion. $(\mathcal{F})_{s,t}$ represents the completed σ-algebra generated by $B_r - B_u$, $s \le u \le r \le t$, for $0 \le s < t$.

In addition, $\phi_t^*(\,\cdot\,)$ denotes the pullback with respect to the map ϕ_t and $\mathcal{L}_b$ denotes the Lie derivative with respect to the vector field b. In Krylov (2011), Krylov considered an approach using mollifiers to provide a general proof of the classical Itô–Wentzell formula (Kunita 1981; Bismut 1981; Kunita 1997). This classical formula states that for a sufficiently smooth scalar function-valued semimartingale f, represented as

$$\mathrm{d}f(t, x) = g(t, x)\,\mathrm{d}t + h(t, x) \circ \mathrm{d}W_t\,, \tag{1.3}$$

the equation describing the evolution of the function f via pullback by ϕ_t reads

$$\begin{aligned}\phi_t^* f(t,x) = f(0,x) + \int_0^t \phi_s^* g(s,x)\ \mathrm{d}s + \int_0^t \phi_s^* h(s,x) \circ \mathrm{d}W_s \\ + \int_0^t \phi_s^* \big(b \cdot \nabla f\big)(s,x)\ \mathrm{d}s + \int_0^t \phi_s^* \big(\xi \cdot \nabla f\big)(s,x) \circ \mathrm{d}B_s\,,\end{aligned} \tag{1.4}$$

where ϕ_t is the flow of the SDE in (1.1). Here W_t denotes Brownian motion defined with respect to the same stochastic basis as B_t, but is not assumed to be independent of B_t. For more a precise statement of the regularity conditions, see (Krylov 2011). In the present paper, we derive the Kunita–Itô–Wentzell Theorem which establishes the formula for the evolution of a k-form-valued process $\phi_t^* K$. This result generalises Kunita's formula (1.2) and the Itô–Wentzell formula for a scalar function (1.4) by allowing K to be any smooth-in-space, stochastic-in-time k-form on $\mathbb{R}^n$. Omitting the technical regularity assumptions provided in the more detailed statement of the theorem in Sect. 3, we now state a simplified version of our main theorem, as follows.

Theorem (Kunita–Itô–Wentzell formula for k-forms, simplified version) *Consider a sufficiently smooth k-form $K(t,x)$ in space which is a semimartingale in time*

$$\mathrm{d}K(t,x) = G(t,x)\,\mathrm{d}t + \sum_{i=1}^{M} H_i(t,x) \circ \mathrm{d}W_t^i, \tag{1.5}$$

where W_t^i are i.i.d. Brownian motions. Let ϕ_t be a sufficiently smooth flow satisfying the SDE

$$\mathrm{d}\phi_t(x) = b(t,\phi_t(x))\,\mathrm{d}t + \sum_{i=1}^{N} \xi_i(t,\phi_t(x)) \circ \mathrm{d}B_t^i\,,$$

in which B_t^i are i.i.d. Brownian motions. Then the pullback $\phi_t^* K$ satisfies the formula

$$\begin{aligned}\mathrm{d}(\phi_t^* K)(t,x) = &\,\phi_t^* G(t,x)\,\mathrm{d}t + \sum_{i=1}^{M} \phi_t^* H_i(t,x) \circ \mathrm{d}W_t^i \\ &+ \phi_t^* \mathcal{L}_b K(t,x)\,\mathrm{d}t + \sum_{i=1}^{N} \phi_t^* \mathcal{L}_{\xi_i} K(t,x) \circ \mathrm{d}B_t^i.\end{aligned} \tag{1.6}$$

Formulas (1.5) and (1.6) are compact forms of Eqs. (3.5) and (3.8) in Sect. 3. The latter equations are written in integral notation to make the stochastic processes more explicit.

Remark 1.1 In applications, we will sometimes express (1.6) using the differential notation

$$\mathrm{d}(\phi_t^* K)(t,x) = \phi_t^* \left(\mathrm{d}K + \mathcal{L}_{\mathrm{d}x_t} K\right)(t,x),$$

where $\mathrm{d}x_t$ is the stochastic vector field $\mathrm{d}x_t(x) = b(t,x)\,\mathrm{d}t + \sum_{i=1}^N \xi_i(t,x) \circ \mathrm{d}B_t^i$. This formula is also valid when K is a vector field rather than a k-form.

A quick comparison of the Itô–Wentzell formulas in (1.4) and (1.6) shows the parallels and differences between the scalar and k-form cases. Our proof of this theorem relies on a slight extension of Krylov's mollifier approach in Krylov (2011). Our proof uses mollifiers to evaluate the time-dependent k-form $K(t,x)$ along the flow ϕ_t without having to discretise the time and take limits, as is usually done. The result for deterministic, smooth-in-time K is already available in Kunita (1984), and for the particular case in which K is a deterministic k-form-valued process, some consequences in fluid dynamics have also been discussed previously in Catuogno and Stelmastchuk (2016), Rezakhanlou (2016). In a related work, Drivas and Holm (2018) prove the KIW theorem for one-forms in the course of proving Kelvin's circulation theorem rigorously for stochastic fluids. The approach of Drivas and Holm (2018) converts the line integral of a one-form along a closed circulation loop to a Riemann integral by parametrising the loop, and then it applies the standard Itô–Wentzell formula. The method employed in the present work does not depend on parametrising the surface over which the integral is taken. Consequently, our mollifier approach based on Krylov (2011) allows for a natural coordinate-free generalisation of the Itô–Wentzell formula to k-forms.

The KIW Theorem 3.4 confirms *a posteriori* a well known rule of thumb in stochastic differential geometry, called by Malliavin the *transfer principle* (Émery 1990). The transfer principle allows one to replace classical calculus with Stratonovich stochastic calculus in certain circumstances. In finite dimensions, this is admitted via Stratonovich differentials obeying the ordinary chain rule and product rule, and also via certain approximation results, although various regularity conditions would need to be added for SPDE. See Émery (1990) for an extensive discussion of the transfer principle in the finite-dimensional situation. No mention of a transfer principle occurs in Krylov's detailed technical proof of (1.4) for the scalar case in Krylov (2011).

Background

Stochastic geometric mechanics for continuum dynamics has recently had a sequence of developments, which we now briefly sketch.

Stochastic geometric mechanics. In Holm (2015), the extension of geometric mechanics to include stochasticity in nonlinear fluid theories was accomplished by applying Hamilton's variational principle, constrained by using the Clebsch approach to enforce stochastic Lagrangian fluid trajectories arising from the stochastic Eulerian vector field

$$\mathrm{d}x_t(x,t) := u(x,t)\ \mathrm{d}t + \sum_{i=1}^N \xi_i(x) \circ \mathrm{d}W^i(t)\,, \tag{1.7}$$

regarded as a decomposition into a drift velocity $u(x,t)$ and a sum over independent stochastic terms. Imposing this decomposition as a constraint on the variations in Hamilton's principle for fluid dynamics (Holm et al. 1998), led in Holm (2015) to new stochastic partial differential equation (SPDE) models which serve to represent the effects of unknown, rapidly fluctuating scales of motion on slower resolvable timescales in a variety of fluid theories, and particularly in geophysical fluid dynamics (GFD).

Analytical properties of stochastic fluid equations. One should expect that the properties of the fluid equations with stochastic transport noise as formulated in Holm (2015) should closely track the properties of the unapproximated solutions of the fluid equations. For example, if the unapproximated model equations are Hamiltonian, then the model equations with stochastic transport noise should also be Hamiltonian, as shown in Holm (2015). In addition, local well-posedness in regular Sobolev spaces and a Beale–Kato–Majda blowup criterion were proved in Crisan et al. (2018) for the stochastic model of the 3D Euler fluid equation for incompressible flow derived in Holm (2015).

Fluid flow velocity decomposition. The same decomposition of the fluid flow velocity into a sum of drift and stochastic parts derived in Holm (2015) was also discovered in Cotter et al. (2017) to arise in a multi-scale decomposition of the deterministic Lagrange-to-Euler flow map into a slow large-scale mean and a rapidly fluctuating small-scale map. Homogenisation theory was used to derive effective slow stochastic particle dynamics for the resolved mean part, thereby justifying the stochastic fluid partial differential equations in the Eulerian formulation. The results of Cotter et al. (2017) justified regarding the Eulerian vector field in (1.7) as a genuine decomposition of the fluid velocity into a sum of drift and stochastic parts, rather than simply as a perturbation of the dynamics meant to model unknown effects in uncertainty quantification. This result implied that the velocity decomposition (1.7) could be used in parallel with data assimilation for the purpose of reduction in uncertainty.

The main content of this paper.

- Section 2 uses the Kunita–Itô–Wentzell (KIW) formula for k-forms as a crucial element in proving the Euler–Poincaré theorem and the Clebsch Hamilton's principle for deriving the equations of stochastic continuum dynamics. These two stochastic variational approaches each recover the stochastic transport versions of all of the deterministic continuum dynamics models with advected quantities derived in Holm et al. (1998). They also confirm the stochastic continuum dynamics equations derived in Holm (2015). The case of stochastic compressible adiabatic magnetohydrodynamics (MHD) is presented as a new illustrative example of the power of this method for continuum dynamics with a variety of forces depending on several advected k-forms.
- Section 3 summarises our main theorem, which derives the KIW formula, thereby extending the Itô–Wentzell formula to stochastic k-form-valued processes. A brief sketch of the proof is also outlined.
- Section 4 explains some implications of the KIW formula for stochastic fluid dynamics. These implications include stochastic advection by Lie transport of k-forms, including details of the derivations of the continuity equation and Kelvin circulation theorem for stochastic fluid flows.
- Section 5 carries out the detailed proof of the KIW formula in the Itô representation.
- Section 6 concludes the paper with a brief summary and some outlook for further research.

2 Stochastic Continuum Euler–Poincaré Theorem

Following Arnold (1966), we consider the Lagrangian trajectories of ideal continuum flows as time-dependent curves x_t on a manifold without boundary M. These curves are generated by the action $x_t = \phi_t X$ of a curve on the manifold of diffeomorphisms ϕ_t parameterised by time t such that $X = \phi_0 X$ at time $t = 0$. Inspired by related results of Arnaudon et al. (2014), Holm (2015), Chen et al. (2015), we examine a family of stochastic paths (Lagrangian trajectories) generated by the action $x_t = \phi_t X$ of the diffeomorphism ϕ_t on the manifold M where ϕ_t is stochastic and given by the Stratonovich stochastic process

$$\mathrm{d}\phi_t(X) = u(t, \phi_t(X))\,\mathrm{d}t + \xi(t, \phi_t(X)) \circ \mathrm{d}W_t\,, \tag{2.1}$$

in which notation for the probability variable ω has been suppressed, and the subscript t in ϕ_t, for example, denotes explicit time dependence, not partial time derivative. Equation (2.1) is written in differential notation for a Stratonovich stochastic process, as explained, e.g., in Lázaro-Camí and Ortega (2007), Cruzeiro et al. (2018). Namely, Eq. (2.1) is short notation for the sum of Stratonovich stochastic integrals:

$$x_t - X = \phi_t(X) - \phi_0(X) = \int_0^t \circ\, \mathrm{d}\phi_s(X) \;= \int_0^t u_s(x_s)\,\mathrm{d}s + \int_0^t \xi(x_s) \circ \mathrm{d}W_s\,. \tag{2.2}$$

Conversely, given any stochastic flow of diffeomorphism ϕ_t with *local characteristics* (a, b), where

$$b(t, x) := \lim_{h\to 0} \frac{1}{h}\,(\mathbb{E}[\phi_{t+h}(x)] - x)\,, \tag{2.3}$$

$$a(t, x, y) := \lim_{h\to 0} \frac{1}{h}\left[(\mathbb{E}[\phi_{t+h}(x)] - x)(\mathbb{E}[\phi_{t+h}(y)] - y)^T\right], \tag{2.4}$$

and we took $\phi_t(x) = x$, one can represent ϕ_t as a solution to an SDE driven by a stochastic vector field $\mathrm{d}x_t(x) := u(t, x)\,\mathrm{d}t + \xi(t, x) \circ \mathrm{d}W_t$, where $a(t, x, y) = \xi(t, x)\xi(t, y)^T$ and $b(t, x) = u(t, x) + \frac{1}{2}\xi(t, x)\cdot\nabla\xi(t, x)$. This describes the infinitesimal mean and covariance of the flow, respectively (see Kunita and Ghosh 1986 for more details).

2.1 Stochastic Continuum Euler–Poincaré Theorem with Advected Quantities

In preparation for introducing a stochastic version of the Euler–Poincaré variational principle for deterministic continuum dynamics established in Holm et al. (1998), we consider next a family of smooth pathwise deformations of the action $x_t = \phi_t X$, by a *second family of diffeomorphisms*, where ϕ_t is a stochastic flow of diffeomorphism with local characteristics (a, b). The second family of diffeomorphisms is deterministic and is parameterised by ε, with $\varepsilon = 0$ at the identity. We take the combined action of the two diffeomorphisms to be a single two-parameter family, whose action on the flow manifold M is denoted as $x_{t,\varepsilon} = \phi_{t,\varepsilon} X$, and is stochastic in time t and deterministic in the parameter ε. We also impose that the deformation under ε fixes the infinitesimal covariance $a(t, x, y)$ of the flow so that $\phi_{t,\varepsilon}$ has local characteristics (a, b_ε), where the ε dependence only appears in the infinitesimal mean b_ε. Since the two parameters t and ε are independent, we may compute the partial derivative of either parameter, while holding the other one fixed. Moreover, since t and ε are independent parameters, we may take partial derivatives with respect to these parameters in either order and equate their cross derivatives.

In this situation of two-parameter diffeomorphisms, the family of Lagrangian trajectories (2.1) has been extended to include their deterministic deformations. This extension is expressed as:

$$\mathrm{d}x_{t,\varepsilon} = u_{t,\varepsilon}(x_{t,\varepsilon})\,\mathrm{d}t + \xi(x_{t,\varepsilon}) \circ \mathrm{d}W_t\,, \tag{2.5}$$

where $u_{t,\varepsilon}(x) = b_{t,\varepsilon}(x) - \frac{1}{2}\xi(t,x)\cdot\nabla\xi(t,x)$ and $b_{t,\varepsilon}$ is given by (2.3) with $\phi_{t,\varepsilon}$ instead of ϕ_t.

We define two time-dependent vector fields $w_t(x_t)$ and $\delta u_t(x_t)$ in terms of the following two different types of tangents of the perturbed trajectories at the identity, $\varepsilon = 0$,

$$\begin{aligned} w_t(x_t) &:= \frac{\partial}{\partial\varepsilon}(\phi_{t,\varepsilon}X)\bigg|_{\varepsilon=0} =: \frac{\partial x_{t,\varepsilon}}{\partial\varepsilon}\bigg|_{\varepsilon=0}\,,\\ \delta u_t(x_t) &:= \frac{\partial\, u_{t,\varepsilon}(x_{t,\varepsilon})}{\partial\varepsilon}\bigg|_{\varepsilon=0}\,. \end{aligned} \tag{2.6}$$

The definitions of the path (2.5) and its tangent vectors at the identity with respect to ε in (2.6) lead to the following Lemma, which will be useful in proving the Euler–Poincaré theorem for stochastic continuum dynamics in the next subsection.

Lemma 2.1 (Velocity variations) *The variational vector fields $\delta u_t(x_t)$ and $w_t(x_t)$ defined in (2.6) satisfy the following advective transport relation:*

$$\delta u_t(x_t)\,\mathrm{d}t = \mathrm{d}w_t + \pounds_{\mathrm{d}x_t} w_t = \mathrm{d}w_t + \big[\,\mathrm{d}x_t\,,\ w_t\big] =: \mathrm{d}w_t - \mathrm{ad}_{\mathrm{d}x_t} w_t\,. \tag{2.7}$$

Remark 2.2 The advective transport relation (2.7) in Lemma 2.1 implies that the variation of the velocity vector field $\delta u_t(x_t)$ is determined by integrating the pullback of the stochastic flow process ϕ_t acting on the infinitesimal deformation vector field, w_t, as

$$\delta u_t(x_t)\,\mathrm{d}t = \mathrm{d}(\phi_t^* w_t) = \phi_t^*(\mathrm{d}w_t + \mathcal{L}_{\mathrm{d}x_t} w_t)\,, \tag{2.8}$$

where $x_t = \phi_t X$ and $\mathrm{d}x_t$ is given in Eq. (2.1). Equation (2.8) is an example of the type of result which is obtained from the KIW formula in (1.6), which clearly also applies for vector fields.

Proof The proof of Lemma 2.1 follows from equality of cross derivatives of the smooth map $\phi_{t,\varepsilon}$ with respect to its two independent parameters, t and ε, when the latter parameter is evaluated at the identity $\varepsilon = 0$. One calculates directly that

$$\begin{aligned} \mathrm{d}\left[\frac{\partial}{\partial\varepsilon}(\phi_{t,\varepsilon}X)\right]_{\varepsilon=0} &= \mathrm{d}w_t(x_t) + \left[\frac{\partial w_{t,\varepsilon}}{\partial x_{t,\varepsilon}}\,\mathrm{d}x_{t,\varepsilon}\right]_{\varepsilon=0} = \mathrm{d}w_t(x_t) + \frac{\partial w_t}{\partial x_t}\cdot \mathrm{d}x_t\,,\\ \left[\frac{\partial}{\partial\varepsilon}\,\mathrm{d}(\phi_{t,\varepsilon}X)\right]_{\varepsilon=0} &:= \delta u_t(x_t)\mathrm{d}t + \left[\frac{\partial(\mathrm{d}x_{t,\varepsilon})}{\partial x_{t,\varepsilon}}\,\frac{\partial}{\partial\varepsilon}(\phi_{t,\varepsilon}X)\right]_{\varepsilon=0} = \delta u_t(x_t)\mathrm{d}t + \frac{\partial(\mathrm{d}x_t)}{\partial x_t}\cdot w_t\,. \end{aligned} \tag{2.9}$$

Taking the difference between these two equalities then yields Eq. (2.7) of Lemma 2.1. □

Definition 2.1 The operation $\diamond : V \times V^* \to \mathfrak{X}^*$ between tensor space elements $a \in V^*$ and $b \in V$ produces an element of $\mathfrak{X}(M)^*$, a one-form density, given by

$$\Big\langle b\diamond a\,,\,u\Big\rangle_{\mathfrak{X}} = -\int_{\mathcal{D}} b\cdot \pounds_u\, a =: \Big\langle b\,,\,-\mathcal{L}_u\, a\Big\rangle_V\,, \tag{2.10}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{X}}$ denotes the symmetric, non-degenerate L^2 pairing between vector fields and one-form densities, which are dual with respect to this pairing. Likewise, $\langle\,\cdot\,,\,\cdot\,\rangle_V$ represents the corresponding L^2 pairing between elements of V and V^*. Also, $\mathcal{L}_u a$ stands for the Lie derivative of an element $a \in V^*$ with respect to a vector field $u \in \mathfrak{X}(M)$, and $b\cdot\mathcal{L}_u\, a$ denotes the contraction between elements of V and elements of V^*.

For a stochastic Stratonovich path $x_t = \phi_t X$ with $\phi_t \in \mathrm{Diff}(M)$, let

$$\mathrm{d}x_t = u_t(x_t)\,\mathrm{d}t + \xi(x_t)\circ \mathrm{d}W_t \tag{2.11}$$

be its corresponding process and consider the curve a_t with initial condition a_0 determined by the *stochastic transport equation*

$$\mathrm{d}\big(\phi_t^* a_t\big) = \phi_t^*\Big(\,\mathrm{d}a_t + \mathcal{L}_{\mathrm{d}x_t} a_t\Big) = 0, \tag{2.12}$$

which is another application of the dynamical KIW formula in (1.6).
We can now state the Stochastic Euler–Poincaré Theorem for Continua.

Theorem 2.3 (Stochastic Euler–Poincaré Theorem for Continua) *Consider a stochastic Stratonovich path $x_t = \phi_t X$ with $\phi_t \in \mathrm{Diff}(M)$. The following two statements are equivalent:*

(i) *Hamilton's variational principle in Eulerian coordinates*

$$\delta S := \delta\int_{t_1}^{t_2} l(u,a)\;dt = 0 \tag{2.13}$$

holds on $\mathfrak{X}(M) \times V^*$*, using variations of the form given in Eq.* (2.7)

$$\delta u \, \mathrm{d}t = \mathrm{d}w - \mathrm{ad}_{dx_t} w \,, \qquad \delta a = -\mathcal{L}_w \, a, \tag{2.14}$$

where the vector field w *vanishes at the endpoints in time,* t_1 *and* t_2*.*

(ii) *The Euler–Poincaré equations for continua, namely the auxiliary advection Eq.* (2.12) *and the following equation of motion,*

$$\mathrm{d}\frac{\delta l}{\delta u} = - \,\mathrm{ad}^*_{\mathrm{d}x_t} \frac{\delta l}{\delta u} + \frac{\delta l}{\delta a} \diamond a \, \mathrm{d}t = - \,\mathcal{L}_{\mathrm{d}x_t} \frac{\delta l}{\delta u} + \frac{\delta l}{\delta a} \diamond a \, \mathrm{d}t \,, \tag{2.15}$$

hold, where the $\diamond$ *operation given by* (2.10) *needs to be determined on a case by case basis, since it depends on the nature of the tensor* a. (Recall that $\delta l/\delta u$ is a one-form density).

Proof The following string of equalities shows that (i) is equivalent to (ii):

$$\begin{aligned} 0 = \delta \int_{t_1}^{t_2} l(u, a) \mathrm{d}t &= \int_{t_1}^{t_2} \left(\frac{\delta l}{\delta u} \cdot \delta u + \frac{\delta l}{\delta a} \cdot \delta a \right) \mathrm{d}t \\ &= \int_{t_1}^{t_2} \left[\frac{\delta l}{\delta u} \cdot \big(\mathrm{d}w - \mathrm{ad}_{\mathrm{d}x_t} w \big) - \frac{\delta l}{\delta a} \cdot \mathcal{L}_w \, a \, \mathrm{d}t \right] \\ &= \int_{t_1}^{t_2} w \cdot \left[-\, \mathrm{d}\frac{\delta l}{\delta u} - \mathrm{ad}^*_{\mathrm{d}x_t} \frac{\delta l}{\delta u} + \frac{\delta l}{\delta a} \diamond a \, \mathrm{d}t \right] \\ &= \int_{t_1}^{t_2} w \cdot \left[-\, \mathrm{d}\frac{\delta l}{\delta u} - \mathcal{L}_{\mathrm{d}x_t} \frac{\delta l}{\delta u} + \frac{\delta l}{\delta a} \diamond a \, \mathrm{d}t \right] . \end{aligned} \tag{2.16}$$

In the second line of the calculation in (2.16), we have substituted the constrained variations in Eq. (2.14). In the third line, we have used the product rule for the stochastic differential (d) and applied homogeneous endpoint conditions for w under integration by parts using (2.2). □

The KIW formula is also essential in formulating and proving the following theorem for the Lagrange-to-Euler pullback of the Clebsch variational principle for stochastic fluids appearing in Holm (2015).

Theorem 2.4 (Lagrange–Clebsch variational principle for stochastic continuum dynamics) *Consider a cylindrically stochastic Stratonovich path* $x_t = \phi_t X$ *with* $\phi_t \in \mathrm{Diff}(M)$. *The following two statements are equivalent:*

(i) *The Clebsch-constrained Hamilton's variational principle*

$$\delta S := \delta \int_{t_1}^{t_2} l(\phi_t^* u, \phi_t^* a) + \Big\langle \phi_t^* b \,,\, \mathrm{d}(\phi_t^* a) \Big\rangle_V \, dt = 0 \,, \tag{2.17}$$

holds on $\mathfrak{X}(M) \times V^*$.

(ii) *The Euler–Poincaré equations for continua hold, in the form*

$$
\begin{aligned}
\mathrm{d}\Big(\phi_t^*\frac{\delta l}{\delta u}\Big) &= \phi_t^*\left(\mathrm{d}\frac{\delta l}{\delta u}+\mathcal{L}_{\mathrm{d}x_t}\frac{\delta l}{\delta u}\right)=\phi_t^*\left(\frac{\delta l}{\delta a}\diamond a\right)\mathrm{d}t\,,\\
\mathrm{d}\big(\phi_t^* a_t\big) &= \phi_t^*\Big(\,\mathrm{d}a_t+\mathcal{L}_{\mathrm{d}x_t}a_t\Big)=0\,.
\end{aligned}
\tag{2.18}
$$

Proof Evaluating the variational derivatives at fixed time t and coordinate X yields the following relations:

$$
\begin{aligned}
\delta(\phi_t^* b)\ &: 0=\mathrm{d}(\phi_t^* a)=\phi_t^*\Big(\,\mathrm{d}a_t+\mathcal{L}_{\mathrm{d}x_t}a_t\Big),\\
\delta(\phi_t^* a)\ &: 0=-\,\mathrm{d}(\phi_t^* b)+\phi_t^*\left(\frac{\delta l}{\delta a}\right)\mathrm{d}t\,,\\
\delta(\phi_t^* u)\ &: 0=\frac{\delta l}{\delta(\phi_t^* u)}-(\phi_t^* b)\diamond(\phi_t^* a)\,.
\end{aligned}
\tag{2.19}
$$

Stationarity under the variations associated with the quantity on the left-hand side of the colon implies the equations on the right-hand side of the colon. One then computes the motion equation to be

$$
\begin{aligned}
\mathrm{d}\frac{\delta l}{\delta(\phi_t^* u)} &= \mathrm{d}(\phi_t^* b)\diamond(\phi_t^* a)+(\phi_t^* b)\diamond\mathrm{d}(\phi_t^* a)\\
\phi_t^*\left(\mathrm{d}\frac{\delta l}{\delta u}+\mathcal{L}_{\mathrm{d}x_t}\frac{\delta l}{\delta u}\right) &= \phi_t^*\left(\frac{\delta l}{\delta a}\diamond a\right)\mathrm{d}t\,.
\end{aligned}
\tag{2.20}
$$

Then, assembling the results of this computation yields the equations in (2.18). □

Remark 2.5 Note that the stochastic equations for continuum dynamics in Theorems 2.3 and 2.4 are equivalent, since the second set of resulting equations is the pullback of the first one by the Lagrange-to-Euler map.

Remark 2.6 At this point, one may proceed to recover stochastic transport versions of all of the deterministic continuum dynamics models with advected quantities derived in Holm et al. (1998). In doing so, one would also obtain the corresponding stochastic versions of all of their Kelvin–Noether theorems. In each case, given the Lagrangian $l:\mathfrak{X}(M)\times V^*\to\mathbb{R}$, the Kelvin–Noether quantity is given by the circulation integral

$$
I(\gamma_t,u,a)=\oint_{\gamma_t}\frac{1}{\rho}\frac{\delta l}{\delta u}\,,
\tag{2.21}
$$

around a material loop γ_t moving with the stochastic velocity $\mathrm{d}x_t=u\,\mathrm{d}t+\xi(x)\circ\mathrm{d}W_t$, in which the quantity $\rho^{-1}(\delta l/\delta u)$ is the circulation one-form integrand and the circulation integral evolves according to

$$
\mathrm{d}I(\gamma_t,u,a)=\oint_{\gamma_t}\Big(\mathrm{d}+\mathcal{L}_{\mathrm{d}x_t}\Big)\left(\frac{1}{\rho}\frac{\delta l}{\delta u}\right)=\oint_{\gamma_t}\frac{1}{\rho}\frac{\delta l}{\delta a}\diamond a\,,
\tag{2.22}
$$

which in the deterministic case becomes the classical Kelvin's circulation theorem. As we will see in Sect. 4, the proof of Kelvin's circulation theorem for stochastic fluid dynamics is yet another application of the Kunita–Itô–Wentzell formula in (1.6). The Itô versions of the Stratonovich stochastic equations in (2.12) and (2.15) follow by standard methods.

The differences between the deterministic and stochastic continuum dynamics equations will always be that, while the geometric structure in each case will be preserved (including Lie–Poisson brackets and Casimirs) the stochastic versions will introduce stochastic advection by Lie Transport (SALT). In the Lie–Poisson Hamiltonian formulations of these equations, the Hamiltonian function will be stochastic in the form

$$\mathrm{d}H = H(m,a)\,\mathrm{d}t + \langle m\,,\,\xi(x)\rangle\,\circ \mathrm{d}W_t\,, \tag{2.23}$$

where, as a result of the Legendre transformation,

$$m = \frac{\delta l(u,a)}{\delta u} \quad\text{and}\quad \frac{\delta\,\mathrm{d}H(m,a)}{\delta m} = u\,\mathrm{d}t + \xi(x)\circ \mathrm{d}W_t\,, \tag{2.24}$$

the equations of motion will adopt the semidirect-product Lie Poisson form,

$$\mathrm{d}F(m,a) = \Big\{F\,,\,\mathrm{d}H\Big\} = -\Big\langle (m,a), \Big[\frac{\delta F}{\delta(m,a)}\,,\,\frac{\delta\,\mathrm{d}H}{\delta(m,a)}\Big]\Big\rangle, \tag{2.25}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{g}^*\times\mathfrak{g}\to\mathbb{R}$ is the L^2 pairing of the semidirect-product Lie algebra $\mathfrak{g}$ with its dual $\mathfrak{g}^*$, and $[\,\cdot\,,\,\cdot\,] : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is the Lie algebra bracket. This semidirect-product Lie Poisson Hamiltonian form of the equations is given in a more explicit matrix operator form as

$$\mathrm{d}\begin{bmatrix} m \\ a \end{bmatrix} = -\begin{bmatrix} \big(\partial_j m_i + m_j\partial_i\big)\Box & \Box\diamond a \\ \mathcal{L}_\Box\, a & 0 \end{bmatrix}\begin{bmatrix} \delta\,\mathrm{d}H/\delta m_j \\ \delta\,\mathrm{d}H/\delta a \end{bmatrix}, \tag{2.26}$$

where $\Box$ denotes where the Lie Poisson bracket operations in (2.25) are applied. Note that the deterministic energy Hamiltonian $H(m,a)$ is not preserved, because in general $\{H,\mathrm{d}H\}\neq 0$. The Itô versions of the Stratonovich these stochastic fluid equations follow by standard methods.

Lie–Poisson Hamiltonian formulations of stochastic fluid dynamics extend the finite-dimensional theory of stochastic Hamiltonian systems introduced for symplectic manifolds in Bismut (1982) and then generalised to Poisson manifolds in Lázaro-Camí and Ortega (2007). Variational integrators for stochastic motion on the Lie group $SO(3)$ were developed in Bou-Rabee and Owhadi (2009). Stochastic coadjoint motion for geometric mechanics in finite dimensions also discussed in detail in Arnaudon et al. (2018), Cruzeiro et al. (2018).

Example: Adiabatic compressible stochastic MHD In the case of adiabatic compressible stochastic magnetohydrodynamics (MHD), the action in Hamilton's principle (2.13) is given by

$$S = \int l(\mathbf{u}, D, s, \mathbf{B})\,\mathrm{d}t = \int \left(\frac{D}{2}\,|\mathbf{u}|^2 - De(D, s) - \frac{1}{2}|\mathbf{B}|^2\right) d^3x\,\mathrm{d}t\,. \tag{2.27}$$

Here, $\mathbf{u}$ is the fluid velocity vector in Eq. (2.11) and $\mathbf{B}$ is the flux of the magnetic field. Geometrically, the vector $\mathbf{B}$ comprises the components of an exact two-form

$$\mathbf{B}\cdot d\mathbf{S} = d(\mathbf{A}\cdot d\mathbf{x}) = \mathrm{curl}\mathbf{A}\cdot d\mathbf{S}, \tag{2.28}$$

so that $\nabla\cdot\mathbf{B} = 0$. The fluid's internal energy per unit mass is denoted as $e(D, s)$, and its dependence on the mass density D and entropy per unit mass s is provided by the "equation of state", which for an isotropic medium satisfies the Thermodynamic First Law, in the form

$$de = -\,p\,d(1/D) + T ds\,, \tag{2.29}$$

with pressure $p(D, s)$ and temperature $T(D, s)$. The variations of the Lagrangian l in (2.27) yield Hamilton's principle for stochastic MHD as

$$0 = \delta S = \int D\mathbf{u}\cdot\delta\mathbf{u} - DT\delta s + \left(\frac{1}{2}|\mathbf{u}|^2 - h(p, s)\right)\delta D - \mathbf{B}\cdot\delta\mathbf{B}\,\mathrm{d}t\,d^3x\,. \tag{2.30}$$

The quantity $h = e + p/D$ denotes the enthalpy per unit mass, which satisfies the thermodynamic relation

$$dh = (1/D)dp + T\mathrm{d}s\,, \tag{2.31}$$

as a result of the First Law (2.29). The Euler–Poincaré formula in Kelvin–Noether form (2.15) yields the stochastic MHD motion equation as

$$\left(\mathrm{d}+\mathcal{L}_{\mathrm{d}x_t}\right)(\mathbf{u}\cdot d\mathbf{x}) - (T\,ds)\mathrm{d}t + \left(\frac{1}{D}\mathbf{B}\times\mathrm{curl}\,\mathbf{B}\cdot d\mathbf{x}\right)\mathrm{d}t - \left(d\left(\frac{1}{2}|\mathbf{u}|^2 - h\right)\right)\mathrm{d}t = 0\,, \tag{2.32}$$

or, in three-dimensional vector form,

$$\mathrm{d}\mathbf{u} + (\mathrm{d}\mathbf{x}_t\cdot\nabla)\mathbf{u} + (\nabla\mathbf{u})^T\cdot\mathrm{d}\mathbf{x}_t + \left(\frac{1}{D}\nabla p\right)\mathrm{d}t + \left(\frac{1}{D}\mathbf{B}\times\mathrm{curl}\,\mathbf{B}\right)\mathrm{d}t = 0\,. \tag{2.33}$$

where

$$\mathrm{d}\mathbf{x}_t := \mathbf{u}(t, \mathbf{x}_t)\,\mathrm{d}t + \boldsymbol{\xi}(\mathbf{x}_t)\circ\mathrm{d}W_t \tag{2.34}$$

is the stochastic Lagrangian trajectory.

By definition, the advected variables $\{s, \mathbf{B}, D\}$ satisfy the following Lie derivative relations which close the ideal MHD system, by applying the KIW formula for the advective dynamics,

$$\begin{aligned} \left(\mathrm{d}+\mathcal{L}_{\mathrm{d}x_t}\right)s &= 0, \quad \text{or} \quad \mathrm{d}s = -\,\mathrm{d}\mathbf{x}_t\cdot\nabla s\,,\\ \left(\mathrm{d}+\mathcal{L}_{\mathrm{d}x_t}\right)(\mathbf{B}\cdot d\mathbf{S}) &= 0, \quad \text{or} \quad \mathrm{d}\mathbf{B} = \mathrm{curl}\,(\mathrm{d}\mathbf{x}_t\times\mathbf{B}),\\ \left(\mathrm{d}+\mathcal{L}_{\mathrm{d}x_t}\right)(D\,d^3x) &= 0\,, \quad \text{or} \quad \mathrm{d}D = -\,\nabla\cdot(D\,\mathrm{d}\mathbf{x}_t)\,, \end{aligned} \tag{2.35}$$

and the pressure is a function $p(D, s) = D^2 \partial e/\partial D$ specified by giving the equation of state of the fluid, $e = e(D, s)$. If the divergence-free condition $\nabla \cdot \mathbf{B} = 0$ holds initially, then it holds for all time; since this constraint is preserved by the stochastic advection equation for $\mathbf{B}$.

The Stratonovich Eqs. (2.32)–(2.35) for stochastic MHD preserve several integral quantities, provided $\mathrm{d}\mathbf{x}_t$ and $\mathbf{B}$ both have no normal components on the boundary. Two of these are magnetic helicity and entropy

$$\Lambda_{mag} = \int \mathbf{B} \cdot \mathrm{curl}^{-1}\mathbf{B}\, d^3x, \qquad \mathcal{S} = \int D\Phi(s)\, d^3x\,. \tag{2.36}$$

The two preserved integral quantities Λ_{mag} and $\mathcal{S}$ in (2.37) are Casimir functions for the Lie–Poisson bracket in (2.26). This means that they are preserved for every Hamiltonian. An additional conservation law exists in the special cases of isentropic ($\nabla s = 0$) and isothermal flow ($\nabla T = 0$) stochastic MHD. The additional conserved quantity $\Lambda_X = \int \mathbf{u}\cdot\mathbf{B}\, d^3x$ is the called the "cross-helicity," and its stochastic evolution satisfies

$$\mathrm{d}\Lambda_X = \mathrm{d}\int \mathbf{u} \cdot \mathbf{B}\, d^3x = \int T\, \mathbf{B} \cdot \nabla s\, d^3x\,, \tag{2.37}$$

for the stochastic Hamiltonian

$$\mathrm{d}H = \int \left(\frac{1}{2D}|\mathbf{m}|^2 + De(D, s) + \frac{1}{2}|\mathbf{B}|^2\right) d^3x\, \mathrm{d}t + \int \mathbf{m} \cdot \boldsymbol{\xi}(\mathbf{x}) \circ \mathrm{d}W_t,$$

subject to the First Law (2.29).

Distinctions of the Present Approach from Other Approaches

The results in this section are distinct from the related results of Arnaudon et al. (2014), Holm (2015) and Chen et al. (2015) in many ways. The closest relation of the present work is with (Holm 2015), since Theorem 2.3 does in fact recover all of the equations derived in Holm (2015) from the Clebsch-constrained variational approach in the Eulerian representation. However, the variational approach in Holm (2015) is purely Eulerian, while the present approach deals directly with stochastic Lagrangian trajectories. The results of Arnaudon et al. (2014) and Chen et al. (2015) may be regarded as similar in spirit to the present work, because of their variational basis in the Lagrangian fluid description. However, (i) the objectives of the latter two papers differ from the present work, (ii) they use different variational procedures, and (iii) they use different Lagrangians. All three of these differences lead to different dynamical equations from those derived in the present approach. First, the objectives of Chen et al. (2015) are to derive Navier–Stokes PDE, while we are deriving SPDE which preserve the geometric properties of deterministic ideal fluid dynamics. Second, the variation $x_{t,\epsilon}$ in Chen et al. (2015) is given by a composition of maps $e_t^{\epsilon}(x_t)$ which does not solve a stochastic differential equation (SDE), while the variations here are defined as two-parameter smooth maps $x_{t,\varepsilon} = \phi_{t,\varepsilon}X$ which satisfy the SDE in Eq. (2.5). Moreover, the Lagrangian trajectories in Chen et al. (2015) have fixed amplitude which would correspond to the special case $\partial_x \xi = 0$ for the velocity vector field decomposition in our Eq. (1.7). In addition, the drift velocity u is not determined in Chen et al. (2015) from stationarity under arbitrary variations, δu. Third, the paper (Chen et al. 2015) and the present work choose different Lagrangians. Namely, paper (Chen et al. 2015) chooses stochastic Lagrangian functionals whose variations lead to stochastic momenta, while the Lagrangian functionals in the present work are the same as in the deterministic case and the variations of the Lagrangian particle trajectories in Eq. (2.5) are stochastic. The result is that in Chen et al. (2015) the variational derivatives produce stochastic momenta, whereas for us the Lagrangian paths which produce advective transport are stochastic. Thus, the difference in the choice of Lagrangians also leads to different dynamics.

Two other prominant recent approaches to stochastic fluid dynamics which differ from the present work include that of Mikulevicius and Rozovskii (2004, 2005) and those of Mémin (2014), Resseguier et al. (2017a, b, c), Resseguier et al. (2017). These two separate approaches each start with Newton's Law of particle motion and introduce a stochastic Lagrangian trajectory as in (2.34). However, they then take different approaches, do not invoke either Hamilton's principle, or the KIW formula. Moreover, the equations derived in these two Newtonian approaches differ both from each other and from those derived in the present approach.

Outlook for the Rest of the Paper

The present section has demonstrated that the variational derivation of the class of stochastic fluid dynamics equations considered here depends vitally on the KIW formula (1.6). Indeed, when the Lagrangian in Eq. (2.15) is chosen to be the kinetic energy of Euler's fluid equations for incompressible flow, one recovers the 3D SPDE stochastic Euler fluid equations which have been shown in Crisan et al. (2018) to preserve the corresponding analytical properties of their deterministic counterparts.

The purpose of the remainder of the present paper will be to investigate some additional implications of the Kunita–Itô–Wentzell formula for stochastic fluid dynamics and to characterise the analytical requirements under which this KIW formula is valid.

3 Extension of Kunita–Itô–Wentzell Formula to *k*-Forms

We say that a k-form $K(x)$ is of differentiability class $C^r\left(\bigwedge^k(\mathbb{R}^n)\right)$ if every component $K_{i_1,\ldots,i_k}(x)$ is r-times differentiable. We also define the L^p norm of k-forms by

$$\|K\|_{L^p} := \left(\int_{\mathbb{R}^n} |K(x)|^p \, d^n x\right)^{\frac{1}{p}},$$

for $p < \infty$ and

$$\|K\|_{L^\infty} := \sup_{x\in\mathbb{R}^n} |K(x)|,$$

if $p = \infty$, where the norm $|\cdot|$ is given by

$$|K(x)| := \sqrt{\delta^{i_1 j_1} \cdots \delta^{i_k j_k} K_{i_1,\ldots,i_k}(x) K_{j_1,\ldots,j_k}(x)},$$

where $\{\delta^{ij}\}_{i,j=1,\ldots,k}$ are the components of the Euclidean cometric tensor, which is one if $i = j$ and zero if $i \neq j$. Moreover, sum over repeated indices is assumed. We say that a k-form $K(x)$ is of integrability class $L^p\left(\bigwedge^k(\mathbb{R}^n)\right)$ if $\|K\|_{L^p} < \infty$.

We now give the statement of our main theorem. Namely, we determine precise conditions under which the Kunita–Itô–Wentzell formula in (3.3) below holds for k-form-valued diffusion processes on $\mathbb{R}^n$.

Theorem 3.1 (Kunita–Itô–Wentzell (KIW) formula for k-forms: Itô version) *Let $K(t,x) \in L^\infty\left([0,T]; C^2\left(\bigwedge^k(\mathbb{R}^n)\right)\right)$ be a continuous adapted semimartingale taking values in the k-forms*

$$K(t,x) = K(0,x) + \int_0^t G(s,x)\,\mathrm{d}s + \sum_{i=1}^M \int_0^t H_i(s,x)\,\mathrm{d}W_s^i, \quad t \in [0,T], \tag{3.1}$$

where $W_t^1, \ldots, W_t^M$ are i.i.d. Brownian motions, $G \in L^1\left([0,T]; C^2\left(\bigwedge^k(\mathbb{R}^n)\right)\right)$ and $H_i \in L^2\left([0,T]; C^2\left(\bigwedge^k(\mathbb{R}^n)\right)\right), i = 1,\ldots,M$ are k-form-valued continuous adapted semimartingales. Let $\{\phi_t\}_{t\in[0,T]}$ be a continuous adapted solution of the diffusion process

$$\mathrm{d}\phi_t(x) = b(t,\phi_t(x))\,\mathrm{d}t + \sum_{i=1}^N \xi_i(t,\phi_t(x)) \circ \mathrm{d}B_t^i, \quad \phi_0(x) = x, \tag{3.2}$$

which is assumed to be a C^1-diffeomorphism, where $B_t^1, \ldots, B_t^N$ are i.i.d. Brownian motions, $b(t,\cdot) \in W^{1,1}_{loc}(\mathbb{R}^n,\mathbb{R}^n)$, $\xi_i(t,\cdot) \in C^2(\mathbb{R}^n,\mathbb{R}^n), i = 1,\ldots,N$ for all $t \in [0,T]$ and $\int_0^T |b(s,\phi_s(x)) + \frac{1}{2}\sum_i \xi_i \cdot \nabla\xi_i(s,\phi_s(x))| + \sum_i |\xi_i(s,\phi_s(x))|^2\,\mathrm{d}s < \infty$ for all $x \in \mathbb{R}^n$. Then, the following formula holds.

$$\begin{aligned}\phi_t^* K(t,x) = K(0,x) &+ \int_0^t \phi_s^* G(s,x)\,\mathrm{d}s + \sum_{i=1}^M \int_0^t \phi_s^* H_i(s,x)\,\mathrm{d}W_s^i \\ &+ \int_0^t \phi_s^* \mathcal{L}_b K(s,x)\,\mathrm{d}s + \sum_{j=1}^N \int_0^t \phi_s^* \mathcal{L}_{\xi_j} K(s,x)\,\mathrm{d}B_s^j\,,\end{aligned}$$

$$+\sum_{i=1}^{M}\sum_{j=1}^{N}\int_0^t \phi_s^*\mathcal{L}_{\xi_j}H_i(s,x)\,\mathrm{d}[W^i,B^j]_s$$
$$+\sum_{j=1}^{N}\frac{1}{2}\int_0^t \phi_s^*\mathcal{L}_{\xi_j}\mathcal{L}_{\xi_j}K(s,x)\,\mathrm{d}s. \tag{3.3}$$

Remark 3.2 We note that in Theorem 3.1, the two families of i.i.d. Brownian motions $W_t^1,\ldots,W_t^M$ and $B_t^1,\ldots,B_t^N$ are defined with respect to the same stochastic basis, but are not assumed to be independent between each other.

Remark 3.3 We note that Eq. (3.1) is defined as an Itô integral but the flow Eq. (3.2) is given by a Stratonovich equation. The Stratonovich form in (3.2) is taken merely to simplify the expressions. Of course, one can obtain similar expressions using the flow equation in Itô form by simply replacing $b\to b-\frac{1}{2}\xi\xi'$.

We will defer the full proof of Theorem 3.1 to Sect. 5. Here, we will sketch the proof in the case of scalar fields, following (Krylov 2011) and illustrate how we could extend it to k-forms and vector fields, which we will discuss in more details in the full proof in Sect. 5.

Sketch proof of Theorem 3.1 for scalar fields. We will only prove the case $N=M=1$ here for simplicity. Extension to more noise terms is straightforward. Let $K:\mathbb{R}^n\to\mathbb{R}$ be a scalar function satisfying the assumptions in Theorem 3.1 and let ρ^ϵ be a sequence of mollifiers. For any $t\in[0,T]$ and $x,y\in\mathbb{R}^n$, consider the following process

$$F^\epsilon(t,x,y)=\rho^\epsilon(y-\phi_t(x))K(t,y), \tag{3.4}$$

where ϕ_t is the flow of the SDE (3.2). By Itô's lemma, we have

$$\mathrm{d}\rho^\epsilon(y-\phi_t(x))=-\frac{\partial\rho^\epsilon}{\partial y^k}\circ\mathrm{d}\phi_t^k(x)=-\left(b^k\frac{\partial\rho^\epsilon}{\partial y^k}+\frac{1}{2}\xi^k\frac{\partial}{\partial x^k}\left(\xi^l\frac{\partial\rho^\epsilon}{\partial y^l}\right)\right)\mathrm{d}t-\xi^k\frac{\partial\rho^\epsilon}{\partial y^k}\,\mathrm{d}B_t$$

and by the stochastic product rule, we get

$$\begin{aligned}
\mathrm{d}F^\epsilon(t,x,y)=&\rho^\epsilon(y-\phi_t(x))\,\mathrm{d}K(t,y)\\
&+K(t,y)\,\mathrm{d}\rho^\epsilon(y-\phi_t(x))+\mathrm{d}\big[\rho^\epsilon(y-\phi_\cdot(x)),K(\cdot,y)\big]_t\\
=&\left(\rho^\epsilon(y-\phi_t(x))G(t,y)-K(t,y)\left(b^k\frac{\partial\rho^\epsilon}{\partial y^k}\right.\right.\\
&\left.\left.+\frac{1}{2}\xi^k\frac{\partial}{\partial x^k}\left(\xi^l\frac{\partial\rho^\epsilon}{\partial y^l}\right)\right)(t,\phi_t(x))\right)\mathrm{d}t\\
&+\rho^\epsilon(y-\phi_t(x))H(t,y)\,\mathrm{d}W_t-K(t,y)\xi^k\frac{\partial\rho^\epsilon}{\partial y^k}(t,\phi_t(x))\,\mathrm{d}B_t\\
&-H(t,y)\xi^k\frac{\partial\rho^\epsilon}{\partial y^k}\,\mathrm{d}[W,B]_t\,.
\end{aligned}$$

Next, we integrate the F^ϵ equation with respect to y and remove the derivatives on ρ^ϵ by integrating by parts. This step gives us

$$\begin{aligned}
&\int_{\mathbb{R}^n} F^\epsilon(t,x,y)\,\mathrm{d}^n y - \int_{\mathbb{R}^n} F^\epsilon(0,x,y)\,\mathrm{d}^n y \\
&= \int_0^t \int_{\mathbb{R}^n} \rho^\epsilon(y-\phi_s(x)) \left(G(s,y) + b^k \frac{\partial K}{\partial y^k} + \frac{1}{2}\xi^k \frac{\partial \xi^l}{\partial x^k}\frac{\partial K}{\partial y^l} \right. \\
&\quad \left. + \frac{1}{2}\xi^k \xi^l \frac{\partial^2 K}{\partial y^k \partial y^l} \right) \mathrm{d}^n y\,\mathrm{d}s \\
&\quad + \int_0^t \int_{\mathbb{R}^n} \rho^\epsilon(y-\phi_s(x)) \left(H(s,y)\,\mathrm{d}W_s + \xi^k \frac{\partial K}{\partial y^k}\,\mathrm{d}B_t \right) \mathrm{d}^n y \\
&\quad + \int_0^t \int_{\mathbb{R}^n} \rho^\epsilon(y-\phi_s(x)) \xi^k \frac{\partial H}{\partial y^k}\,\mathrm{d}^n y\,\mathrm{d}[W,B]_s\,,
\end{aligned}$$

where we have assumed that Fubini's theorem can be applied. Now, taking the limit $\epsilon \to 0$ on both sides and using the dominated convergence theorem for Itô integrals, we obtain

$$\begin{aligned}
K(t,\phi_t(x)) - K(0,x) &= \int_0^t \left(G(s,\phi_s(x)) + \mathcal{L}_b K(s,\phi_s(x)) + \frac{1}{2}\mathcal{L}_\xi \mathcal{L}_\xi K(s,\phi_s(x)) \right) \mathrm{d}s \\
&\quad + \int_0^t H(s,\phi_s(x))\,\mathrm{d}W_s + \int_0^t \mathcal{L}_\xi K(s,\phi_s(x))\,\mathrm{d}B_s \\
&\quad + \int_0^t \mathcal{L}_\xi H(s,\phi_s(x))\,\mathrm{d}[W,B]_s\,,
\end{aligned}$$

as expected, where $\mathcal{L}_b K = b \cdot \nabla K$ is the Lie derivative for scalar fields.

To prove (3.3) for k-forms (see full proof in Sect. 5), we follow through a similar argument, except we adapt (3.4) by setting

$$F^\epsilon(t,x,y) := \rho^\epsilon(y-\phi_t(x))\,\langle K(t,y), (\phi_t)_* \boldsymbol{u}(\phi_t(x))\rangle\,,$$

where $\boldsymbol{u} = (u_1,\ldots,u_k) \in \mathfrak{X}(\mathbb{R}^n)^k$ are k arbitrary vector fields and $\langle\cdot,\cdot\rangle$ denotes the contraction of tensors. By contracting with arbitrary vector fields, one can keep track of how the basis vectors for $K(t,x)$ transform under the pullback. Using a slight modification, we can also show that (3.3) holds for vector fields $K \in \mathfrak{X}(\mathbb{R}^n)$, by setting

$$F^\epsilon(t,x,y) := \rho^\epsilon(y-\phi_t(x))\,\langle K(t,y), (\phi_t)_* \alpha(\phi_t(x))\rangle\,,$$

where $\alpha \in \Gamma(\bigwedge^1(\mathbb{R}^n))$ is an arbitrary one-form. □

Next, we show that the Stratonovich version of the Kunita–Itô–Wentzell formula for k-forms follows as a corollary of the previous theorem.

Theorem 3.4 (Kunita–Itô–Wentzell (KIW) formula for k -forms: Stratonovich version) *Let $K(t,x) \in L^\infty\left([0,T]; C^3\left(\bigwedge^k(\mathbb{R}^n)\right)\right)$ be a k-form-valued continuous adapted semimartingale satisfying the Stratonovich SPDE*

$$K(t,x) = K(0,x) + \int_0^t G(s,x)\ \mathrm{d}s + \sum_{i=1}^M \int_0^t H_i(s,x) \circ \mathrm{d}W_s^i, \quad t \in [0,T], \tag{3.5}$$

where W_t^i are i.i.d. Brownian motions, $G \in L^1\left([0,T]; C^3\left(\bigwedge^k(\mathbb{R}^n)\right)\right)$ and $H_i \in L^\infty\left([0,T]; C^3\left(\bigwedge^k(\mathbb{R}^n)\right)\right)$, $i = 1,\dots,M$ are k-form-valued continuous adapted semimartingales such that

$$H_i(t,x) = H_i(0,x) + \int_0^t g(s,x)\ \mathrm{d}s + \sum_{j=1}^S \int_0^t h_{ij}(s,x)\,\mathrm{d}N_s^{ij}, \quad t \in [0,T], \quad i = 1,\dots,M \tag{3.6}$$

satisfies the assumptions in Theorem 3.1 and N_s^{ij} are i.i.d. Brownian motions. Let $\{\phi_t\}_{t\in[0,T]}$ be a continuous adapted solution of the diffusion process

$$\mathrm{d}\phi_t(x) = b(t,\phi_t(x))\ \mathrm{d}t + \sum_{i=1}^N \xi_i(t,\phi_t(x)) \circ \mathrm{d}B_t^i, \quad \phi_0(x) = x, \tag{3.7}$$

which is assumed to be a C^1-diffeomorphism, where B_t^i are i.i.d. Brownian motion, $b(t,\cdot) \in W^{1,1}_{loc}(\mathbb{R}^n,\mathbb{R}^n)$ for all $t \in [0,T]$, $\xi_i \in L^\infty\left([0,T]; C^3(\mathbb{R}^n,\mathbb{R}^n)\right)$ and $\int_0^T |b(s,\phi_s(x)) + \frac{1}{2}\sum_i \xi_i \cdot \nabla\xi_i(s,\phi_s(x))| + \sum_i |\xi_i(s,\phi_s(x))|^2\,\mathrm{d}s < \infty$ for all $x \in \mathbb{R}^n$. Then, the following holds.

$$\begin{aligned}\phi_t^* K(t,x) = K(0,x) &+ \int_0^t \phi_s^* G(s,x)\ \mathrm{d}s + \sum_{i=1}^M \int_0^t \phi_s^* H_i(s,x) \circ \mathrm{d}W_s^i \\ &+ \int_0^t \phi_s^* \mathcal{L}_b K(s,x)\ \mathrm{d}s + \sum_{i=1}^N \int_0^t \phi_s^* \mathcal{L}_{\xi_i} K(s,x) \circ \mathrm{d}B_s^i .\end{aligned} \tag{3.8}$$

We refer to Remark 3.2 for details about the two families of Brownian motions.

Proof of Theorem 3.4 In Itô form, (3.5) is given by

$$\begin{aligned}K(t,x) = &K(0,x) + \int_0^t G(s,x)\ \mathrm{d}s + \sum_i \int_0^t H_i(s,x)\,\mathrm{d}W_s^i \\ &+ \frac{1}{2}\sum_{i,j}\int_0^t h_{ij}(s,x)\,\mathrm{d}[W^i, N^{ij}]_s.\end{aligned}$$

Now, applying (3.3) on K, we get

$$\begin{aligned}\phi_t^* K(t,x) &= K(0,x) + \int_0^t \phi_s^* G(s,x)\,\mathrm{d}s + \frac{1}{2}\sum_{i,j}\int_0^t \phi_s^* h_{ij}(s,x)\,\mathrm{d}[W^i, N^{ij}]_s \\ &\quad + \sum_i \int_0^t \phi_s^* H_i(s,x)\,\mathrm{d}W_s^i \\ &\quad + \int_0^t \phi_s^* \mathcal{L}_b K(s,x)\,\mathrm{d}s + \sum_i \int_0^t \phi_s^* \mathcal{L}_{\xi_i} K(s,x)\,\mathrm{d}B_s^i, \\ &\quad + \sum_{i,j}\int_0^t \phi_s^* \mathcal{L}_{\xi_j} H_i(s,x)\,\mathrm{d}[W^i, B^j]_s + \frac{1}{2}\sum_i \int_0^t \phi_s^* \mathcal{L}_{\xi_i}\mathcal{L}_{\xi_i} K(s,x)\,\mathrm{d}s.\end{aligned} \tag{3.9}$$

On the other hand, since H_i and $\mathcal{L}_{\xi_j} K$ satisfy the assumptions in Theorem 3.1 for $i = 1, \ldots, M$ and $j = 1, \ldots, N$, applying (3.3) to H_i and $\mathcal{L}_{\xi_i} K$ respectively gives us

$$\begin{aligned}\phi_t^* H_i(t,x) &= H_i(0,x) + \int_0^t \phi_s^* g(s,x)\,\mathrm{d}s + \sum_j \int_0^t \phi_s^* h_{ij}(s,x)\,\mathrm{d}N_s^{ij} \\ &\quad + \int_0^t \phi_s^* \mathcal{L}_b H_i(s,x)\,\mathrm{d}s + \sum_j \int_0^t \phi_s^* \mathcal{L}_{\xi_j} H_i(s,x)\,\mathrm{d}B_s^j, \\ &\quad + \frac{1}{2}\sum_{j,k}\int_0^t \phi_s^* \mathcal{L}_{\xi_j} h_{ik}(s,x)\,\mathrm{d}[N^{ik}, B^j]_s \\ &\quad + \frac{1}{2}\sum_j \int_0^t \phi_s^* \mathcal{L}_{\xi_j}\mathcal{L}_{\xi_j} H_i(s,x)\,\mathrm{d}s \\ \phi_t^* \mathcal{L}_{\xi_i} K(t,x) &= \mathcal{L}_{\xi_i} K(0,x) + \int_0^t \phi_s^* \mathcal{L}_{\xi_i} G(s,x)\,\mathrm{d}s \\ &\quad + \frac{1}{2}\sum_{j,k}\int_0^t \mathcal{L}_{\xi_i} h_{jk}(s,x)\,\mathrm{d}[W^j, N^{jk}]_s \\ &\quad + \sum_j \int_0^t \phi_s^* \mathcal{L}_{\xi_i} H_j(s,x)\,\mathrm{d}W_s^j \\ &\quad + \int_0^t \phi_s^* \mathcal{L}_b \mathcal{L}_{\xi_i} K(s,x)\,\mathrm{d}s + \sum_j \int_0^t \phi_s^* \mathcal{L}_{\xi_j}\mathcal{L}_{\xi_i} K(s,x)\,\mathrm{d}B_s^j, \\ &\quad + \sum_{j,k}\int_0^t \phi_s^* \mathcal{L}_{\xi_j}\mathcal{L}_{\xi_i} H_k(s,x)\,\mathrm{d}[W^k, B^j]_s \\ &\quad + \frac{1}{2}\sum_j \int_0^t \phi_s^* \mathcal{L}_{\xi_j}\mathcal{L}_{\xi_j}\mathcal{L}_{\xi_i} K(s,x)\,\mathrm{d}s.\end{aligned}$$

Therefore, we get

$$
\begin{aligned}
\left[\phi_t^* H_i(\cdot, x), W^i\right]_t &= \sum_j \int_0^t \phi_s^* h_{ij}(s, x)\, \mathrm{d}[W^i, N^{ij}]_s \\
&\quad + \sum_j \int_0^t \phi_s^* \mathcal{L}_{\xi_j} H_i(s, x)\, \mathrm{d}[W^i, B^j]_s \\
\left[\phi_t^* \mathcal{L}_{\xi_i} K(\cdot, x), B^i\right]_t &= \sum_j \int_0^t \phi_s^* \mathcal{L}_{\xi_i} H_j(s, x)\, \mathrm{d}[W^j, B^i]_s \\
&\quad + \int_0^t \phi_s^* \mathcal{L}_{\xi_i} \mathcal{L}_{\xi_i} K(s, x)\, \mathrm{d}s.
\end{aligned}
$$

Now, it is easy to check that (3.9) is identical to (3.8). □

4 Implications of KIW in Stochastic Fluid Dynamics

In stochastic fluid dynamics, besides the momentum one-form density, one encounters several types of advected k-forms such as mass density and magnetic field. Consider a fluid in a domain $D \subset \mathbb{R}^n$ and an arbitrary control volume $\Omega_0 \subset D$ with spatial coordinates X. We assume the fluid particles which are initially at point X evolve under the flow ϕ_t, determined by the stochastic differential equation

$$
\mathrm{d}\phi_t(X) = b(t, \phi_t(X))\, \mathrm{d}t + \xi(t, \phi_t(X)) \circ \mathrm{d}W_t\, . \tag{4.1}
$$

The control volume at time t is given by $\Omega_t = \phi_t(\Omega_0)$, and the position at time t of the fluid particle initially at X is denoted by $x(t; X) = \phi_t(X)$. Assuming that a k-form α satisfies an advection equation of the form (3.5) and its integral over the control volume Ω_t is conserved with time, i.e.,

$$
\int_{\Omega_t} \alpha(t, x) - \int_{\Omega_0} \alpha(0, X) = 0, \tag{4.2}
$$

then the previous equation can be rewritten as

$$
\int_{\Omega_0} \big((\phi_t^* \alpha)(t, X) - \alpha(0, X)\big) = 0, \tag{4.3}
$$

after changing variables. We can then apply the KIW formula for k-forms (1.6) in the integrand of (4.3) to obtain

$$
\int_{\Omega_0} \big((\phi_t^* \alpha)(t, X) - \alpha(0, X)\big) = \int_0^t \int_{\Omega_0} \phi_s^* \Big(\mathrm{d}\alpha + \mathcal{L}_b \alpha\, \mathrm{d}s + \mathcal{L}_\xi \alpha \circ \mathrm{d}W_s\Big)(s, X) = 0.
$$

Transforming back the coordinates yields

$$\int_0^t \int_{\Omega_s} \big(\mathrm{d}\alpha(s,x) + \mathcal{L}_b\alpha(s,x)\,\mathrm{d}s + \mathcal{L}_\xi\alpha(s,x)\circ \mathrm{d}W_s\big) = 0,$$

which implies that α satisfies the SPDE

$$\mathrm{d}\alpha(s,x) + \mathcal{L}_b\alpha(s,x)\,\mathrm{d}s + \mathcal{L}_\xi\alpha(s,x)\circ \mathrm{d}W_s = 0, \tag{4.4}$$

since the control volume Ω_0 was chosen arbitrarily,

Example: Conservation of fluid mass. As a common application of the above, we use the conservation of mass to derive a stochastic counterpart of the continuity equation in $\mathbb{R}^3$. Let $D(0, \boldsymbol{X})$ be the mass density in the reference configuration and let $D(t, \boldsymbol{x})$ be the mass density at time t, where we employ the bold font notation "$\boldsymbol{x}$" and "$\boldsymbol{X}$" to denote the coordinate expression of the points x and X, respectively. Conservation of mass reads

$$\int_{\Omega_t} D(t,\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x} - \int_{\Omega_0} D(0,\boldsymbol{X})\,\mathrm{d}^3\boldsymbol{X} = 0.$$

By applying the argument above, we obtain (4.4) for the particular case of $\alpha(t,x) = D(t,\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x}$, whose Lie derivative is expressed in coordinates as $\mathcal{L}_b(D(t,\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x}) = \nabla\cdot(D(t,\boldsymbol{x})\boldsymbol{b}(t,\boldsymbol{x}))\,\mathrm{d}^3\boldsymbol{x}$. Thus, we arrive at the stochastic continuity equation

$$\mathrm{d}D(s,\boldsymbol{x}) + \nabla\cdot(D(s,\boldsymbol{x})\boldsymbol{b}(s,\boldsymbol{x}))\,\mathrm{d}s + \nabla\cdot(D(s,\boldsymbol{x})\boldsymbol{\xi}(s,\boldsymbol{x}))\circ\mathrm{d}W_s = 0. \tag{4.5}$$

Similar equations can be derived for the advection of entropy per unit mass, which is a scalar function, and magnetic field, which is a two-form, thus recovering all the equations given in (2.35).

Kelvin's Circulation Theorem. Here, we will use the KIW theorem for k-forms to prove that the stochastic Euler–Poincaré equation (2.15) obtained in Sect. 2 satisfies a stochastic Kelvin's circulation theorem.

Theorem 4.1 (Stochastic Kelvin's circulation theorem) *Suppose that an arbitrary material loop c_0 is advected by the stochastic flow ϕ_t solving the SDE (4.1), and denote by $c_t = \phi_t(c_0)$ the loop at time t. Assume that we have a stochastic Euler–Poincaré equation (2.15) for the Lagrangian $l(u, D, \alpha) = \int_{\mathbb{R}^n}\left(\frac{1}{2}|u|^2 - V(\alpha)\right) D\,\mathrm{d}^n x$, such that*

- *α is a k-form satisfying the advection Eq. (4.4),*
- *$V(\alpha)$ is a smooth function usually representing the potential energy, and*
- *The mass density $\rho := D(t,x)d^n x$ solves the continuity Eq. (4.5).*

Then the Kelvin–Noether quantity $v(t,x) := \rho^{-1}(t,x)\,(\delta\ell/\delta u)(t,x)$ satisfies the SPDE

$$\mathrm{d}v(t,x) = -\mathcal{L}_{\mathrm{d}x_t} v(t,x) + \rho^{-1}F(t,x)\,\mathrm{d}t, \tag{4.6}$$

where $\rho^{-1}F := \rho^{-1}(\delta\ell/\delta\alpha) \diamond \alpha + \rho^{-1}(\delta\ell/\delta\rho) \diamond \rho$ is the force per unit mass, which is a one-form, and the stochastic Kelvin's circulation theorem reads

$$\oint_{c_t} v(t,x) - \oint_{c_0} v(0,X) = \int_0^t \oint_{c_s} \rho^{-1}(s,x) F(s,x)\, \mathrm{d}s\,. \tag{4.7}$$

Proof From the Euler–Poincaré equation (2.15), we have

$$\begin{aligned}\mathrm{d}(\rho v) &= \rho \circ \mathrm{d}v + v \circ \mathrm{d}\rho = -\mathcal{L}_{\mathrm{d}x_t}(\rho v) + F(t,x)\,\mathrm{d}t\\ &= \rho\left(-\mathcal{L}_{\mathrm{d}x_t} v + \rho^{-1} F(t,x)\,\mathrm{d}t\right) - v\mathcal{L}_{\mathrm{d}x_t}\rho\,\mathrm{d}t,\end{aligned}$$

from which we deduce (4.6) since ρ satisfies the continuity Eq. (4.5). By changing variables, we can express the LHS of (4.7) as

$$\oint_{c_t} v(t,x) - \oint_{c_0} v(0,X) = \oint_{c_0} \left(\phi_t^* v(t,X) - v(0,X)\right)\,.$$

By applying the KIW formula (1.6) under the integral sign, we obtain

$$\begin{aligned}\oint_{c_0} \left(\phi_t^* v(t,X) - v(0,X)\right) &= \oint_{c_0} \int_0^t \phi_s^* \left(\mathrm{d}v + \mathcal{L}_b v\ \mathrm{d}s + \mathcal{L}_\xi v \circ \mathrm{d}W_s\right)(s,X)\\ &= \int_0^t \oint_{c_s} \left(\mathrm{d}v + \mathcal{L}_b v\ \mathrm{d}s + \mathcal{L}_\xi v \circ \mathrm{d}W_s\right)(s,x),\end{aligned}$$

where we changed back the coordinates and applied Fubini theorem. Finally, from the momentum Eq. (4.6), we conclude

$$\oint_{c_t} v(t,x) - \oint_{c_0} v(0,X) = \int_0^t \oint_{c_s} \rho^{-1} F(s,\mathbf{X})\,\mathrm{d}s.$$ □

Remark 4.2 From the stochastic Kelvin's circulation Theorem (4.7), we can deduce that if the fluid is not acted on by external forces, then the circulation $I(t) = \oint_{c_t} v$ is preserved.

5 Proof of the KIW Theorem

In this section, we will prove the KIW theorem (in Itô formulation) in full detail. We start by introducing some preparatory results that are used in the proof.

Theorem 5.1 (Stochastic Fubini Theorem, Krylov 2011) *Let $T \in \mathbb{R}^+$, $\mathcal{P}$ be the predictable sigma algebra on $[0,\infty) \times \Omega$, and $G_t(x)$, $H_t(x)$ be real functions defined on $[0,T] \times \mathbb{R}^n \times \Omega$, satisfying the following properties:*

(1) *$G_t(x)$, $H_t(x)$ are $\mathcal{P}_T-$measurable, where $\mathcal{P}_T$ is the restriction of $\mathcal{P}$ to $[0,T]\times\Omega$.*

(2) *$G_t(x)$ and $H_t(x)$ satisfy*

$$\int_0^T (|G_t(x)| + |H_t(x)|^2)\,\mathrm{d}t < \infty, \quad (x,\omega) \in \mathbb{R}^n \times \Omega \backslash A,$$

for A a set of measure zero.

(3) *$G_t(x)$ and $H_t(x)$ also satisfy*

$$\int_0^T \int_{\mathbb{R}^n} |G_t(x)|\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \left(\int_{\mathbb{R}^n} |H_t(x)|^2\,\mathrm{d}x\right)^{1/2} \mathrm{d}t < \infty, \quad a.s.$$

Then the stochastic process

$$\int_0^t G_s(x)\,\mathrm{d}s + \int_0^t H_s(x)\,\mathrm{d}B_s, \quad t \in [0,T], \tag{5.1}$$

is well-defined, $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R}^n)$-measurable, and can be modified into a continuous stochastic process by only changing its values in a set of measure zero. Moreover, the stochastic integral

$$\int_0^t \int_{\mathbb{R}^n} H_s(x)\,\mathrm{d}x\,\mathrm{d}B_s$$

is well-defined, and the following equality holds

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_0^t G_s(x)\,\mathrm{d}s\,\mathrm{d}x + \int_{\mathbb{R}^n} \int_0^t H_s(x)\,\mathrm{d}B_s\,\mathrm{d}x \\ &= \int_0^t \int_{\mathbb{R}^n} G_s(x)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}^n} H_s(x)\,\mathrm{d}x\,\mathrm{d}B_s, \quad a.s. \quad t \in [0,T]. \end{aligned}$$

The full proof of this result is provided in Krylov (2011).

Lemma 5.2 (Itô's product rule) *Let $X_t^1, \ldots, X_t^k$ be semimartingales. Then, we have the following:*

$$\begin{aligned} \mathrm{d}\left(X_t^1 \cdots X_t^k\right) &= \sum_{j=1}^k \left(\prod_{\alpha \neq j}^k X_t^\alpha\right) \mathrm{d}X_t^j \\ &\quad + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^k \left(\prod_{\alpha \neq i,j}^k X_t^\alpha\right) \mathrm{d}\left[X^i, X^j\right]_t. \end{aligned} \tag{5.2}$$

Proof This can be proved straight-forwardly by induction. □

Lemma 5.3 (Lie derivative of k-forms) *Given a differentiable k-form $K \in C^1\left(\bigwedge^k(\mathbb{R}^n)\right)$, and a C^1-vector field u, we have the following*

$$\begin{aligned}\mathcal{L}_u K(x)(v_1,\ldots,v_k) &= u^l(x)\frac{\partial K_{i_1,\ldots,i_k}}{\partial x^l}(x)v_1^{i_1}\cdots v_k^{i_k}\\ &+\sum_{p=1}^k K_{i_1,\ldots,i_k}(x)\frac{\partial u^{i_p}}{\partial x^l}(x)v_1^{i_1}\cdots v_p^l\cdots v_k^{i_k},\end{aligned} \tag{5.3}$$

$$\begin{aligned}\mathcal{L}_u\mathcal{L}_u K(x)(v_1,\ldots,v_k) &= u^l(x)\frac{\partial}{\partial x^l}\left(u^m(x)\frac{\partial K_{i_1,\ldots,i_k}}{\partial x^m}(x)\right)v_1^{i_1}\cdots v_k^{i_k}\\ &+\sum_{p=1}^k\left(u^l(x)\frac{\partial}{\partial x^l}\left(K_{i_1,\ldots,i_k}(x)\frac{\partial u^{i_p}}{\partial x^m}(x)\right)\right.\\ &\left.+u^l(x)\frac{\partial u^{i_p}}{\partial x^m}(x)\frac{\partial K_{i_1,\ldots,i_k}}{\partial x^l}(x)\right)v_1^{i_1}\cdots v_p^m\cdots v_k^{i_k}\\ &+\sum_{\substack{p,q=1\\p\neq q}}^k K_{i_1,\ldots,i_k}(x)\frac{\partial u^{i_p}}{\partial x^l}(x)\frac{\partial u^{i_q}}{\partial x^m}v_1^{i_1}\cdots v_p^l v_q^m\cdots v_k^{i_k},\end{aligned} \tag{5.4}$$

for arbitrary vector fields $v_1,\ldots,v_k$.

Proof The explicit formula (5.3) for Lie derivatives can be found in Marsden and Ratiu (2013), Chapter 4.4, and the double Lie derivative formula (5.4) can be deduced directly from (5.3) applied twice. □

Now, we are ready to prove the Kunita–Itô–Wentzell formula for k-forms (3.3) in Itô form.

5.1 Proof of Theorem 3.1

For convenience, we denote the drift term of the Stratonovich-to-Itô corrected version of (3.2) by $\hat{b}^i := b^i + \frac{1}{2}\xi^j D_j\xi^i$ and set $N = M = 1$ (the more general case can be proved similarly).

Step 1: For fixed $x \in \mathbb{R}^n$ and any $y \in \mathbb{R}^n$, we consider the following (real-valued) process

$$\begin{aligned}F_t^\epsilon(x,y) &:= \rho^\epsilon(y-\phi_t(x))\,\langle K(t,y), (\phi_t)_*\boldsymbol{u}(\phi_t(x))\rangle\\ &= \rho^\epsilon(y-\phi_t(x))K_{i_1,\ldots,i_k}(t,y)\prod_{\alpha=1}^k J^{i_\alpha}_{j_\alpha}(t,x)u_\alpha^{j_\alpha}(x),\end{aligned} \tag{5.5}$$

where $K(t,y) = K_{i_1,\ldots,i_k}(t,y)\,\mathrm{d}y^{i_1}\wedge\cdots\wedge\mathrm{d}y^{i_k}$ in coordinate expression, $u_\alpha(x) = u_\alpha^i(x)\partial/\partial x^i$, $\alpha = 1,\ldots,k$ are k arbitrary smooth vector fields, $\rho^\epsilon(y) := \epsilon^{-n}\rho(y/\epsilon)$ is a family of mollifiers with $\mathrm{Supp}(\rho) \subset B_\gamma(0)$ for some $\gamma > 0$, and $J^i_j(t,x) :=$

$D_j\phi_t^i(x)$ is a shorthand notation for the Jacobian matrix. The philosophy behind considering this process will become clearer later, but the main idea is that when we integrate F_t^ϵ with respect to the y variable and take the limit $\epsilon \to 0$, we obtain the process of $K(t,\cdot)$ pulled back by the flow ϕ_t.

Let τ_1 be the first exit time of the flow $\phi_t(x)$ leaving the ball $B_{R_1}(0)$ for some $|x| < R_1 < \infty$, and let τ_2 be the first exit time of $D\phi_t(x)$ leaving the ball $B_{R_2}(0)$ with respect to the supremum norm $\|\cdot\|_\infty$, for some $1 < R_2 < \infty$. Setting $\tau = \tau_1 \wedge \tau_2$, we have $|\phi_t(x)| < R_1$ and $|J_j^i(t,x)| < R_2$ for all $t < \tau$. Once we prove that Eq. (3.3) holds for all $t \in [0, \tau \wedge T]$, then we can take $R_1, R_2 \to \infty$ to show that it holds for any $t \in [0, T]$.

By Itô's product rule, F_t^ϵ satisfies the following equation

$$
\begin{aligned}
&\mathrm{d}F_t^\epsilon(x,y)\\
&\quad= K_{i_1,\ldots,i_k}(t,y)\left(\prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(t,x)u_\alpha^{j_\alpha}(x)\right)\mathrm{d}\rho^\epsilon(y-\phi_t(x))\\
&\qquad+\rho^\epsilon(y-\phi_t(x))\left(\prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(t,x)u_\alpha^{j_\alpha}(x)\right)\mathrm{d}K_{i_1,\ldots,i_k}(t,y)\\
&\qquad+\rho^\epsilon(y-\phi_t(x))K_{i_1,\ldots,i_k}(t,y)\,\mathrm{d}\left(\prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(t,x)u_\alpha^{j_\alpha}(x)\right)\\
&\qquad+K_{i_1,\ldots,i_k}(t,y)\,\mathrm{d}\left[\rho^\epsilon(y-\phi_\cdot(x)),\prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(\cdot,x)u_\alpha^{j_\alpha}(x)\right]_t\\
&\qquad+\rho^\epsilon(y-\phi_t(x))\,\mathrm{d}\left[K_{i_1,\ldots,i_k}(\cdot,y),\prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(\cdot,x)u_\alpha^{j_\alpha}(x)\right]_t\\
&\qquad+\prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(t,x)u_\alpha^{j_\alpha}(x)\,\mathrm{d}\big[\rho^\epsilon(y-\phi_\cdot(x)),K_{i_1,\ldots,i_k}(\cdot,y)\big]_t.
\end{aligned}
$$

Applying Itô's product rule (5.2), we get

$$
\begin{aligned}
\mathrm{d}\left(\prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(t,x)u_\alpha^{j_\alpha}(x)\right) &= \sum_{p=1}^k\left(\prod_{\alpha\neq p}^k J_{j_\alpha}^{i_\alpha}(t,x)u_\alpha^{j_\alpha}(x)\right)u_p^{j_p}(x)\,\mathrm{d}J_{j_p}^{i_p}(t,x)\\
&+\frac{1}{2}\sum_{\substack{p,q=1\\p\neq q}}^k\left(\prod_{\alpha\neq p,q}^k J_{j_\alpha}^{i_\alpha}(t,x)u_\alpha^{j_\alpha}(x)\right)u_p^{j_p}(x)u_q^{j_q}(x)\,\mathrm{d}\Big[J_{j_p}^{i_p}(\cdot,x),J_{j_q}^{i_q}(\cdot,x)\Big]_t,
\end{aligned}
$$

and by Itô's lemma, we obtain

$$\begin{aligned}\mathrm{d}\rho^\epsilon(y-\phi_t(x)) = \Big(&-\hat{b}^l(t,\phi_t(x))D_l\rho^\epsilon(y-\phi_t(x))\\ &+\frac{1}{2}\xi^l(t,\phi_t(x))\xi^m(t,\phi_t(x))D^2_{lm}\rho^\epsilon(y-\phi_t(x))\Big)\,\mathrm{d}t\\ &+\xi^i(t,\phi_t(x))D_i\rho^\epsilon(y-\phi_t(x))\,\mathrm{d}B_t,\end{aligned}$$

where $D\rho^\epsilon(y-\phi_t(x))$ denotes the derivative with respect to the y variable. We differentiate (3.2) with respect to x to derive (recall that ϕ_t is a C^1-diffeomorphism)

$$\mathrm{d}J^i_j(t,x) = D_l\hat{b}^i(t,\phi_t(x))J^l_j(t,x)\,\mathrm{d}t + D_l\xi^i(t,\phi_t(x))J^l_j(t,x)\,\mathrm{d}B_t.$$

This naturally imposes the condition $\int_0^T \left(\|D_x\hat{b}(t,\phi_t(x))\| + \|D_x\xi(t,\phi_t(x))\|^2\right)\mathrm{d}t < \infty$. By direct calculation, one can show that $F^\epsilon_t(x,y)$ can be expressed as

$$\begin{aligned}F^\epsilon_t(x,y)-F^\epsilon_0(x,y) = &\int_0^t \hat{G}^{1,\epsilon}_s(x,y)\,\mathrm{d}s + \int_0^t \hat{G}^{2,\epsilon}_s(x,y)\,\mathrm{d}[W,B]_s\\ &+\int_0^t \hat{H}^{1,\epsilon}_s(x,y)\,\mathrm{d}W_s + \int_0^t \hat{H}^{2,\epsilon}_s(x,y)\,\mathrm{d}B_s,\end{aligned} \tag{5.6}$$

for all $t\in[0,\tau\wedge T]$, where

$$\begin{aligned}\hat{G}^{1,\epsilon}_s(x,y) := &\Big[\rho^\epsilon(y-\phi_s(x))G_{i_1,\ldots,i_k}(s,y) + \Big(-\hat{b}^l(s,\phi_s(x))D_l\rho^\epsilon(y-\phi_s(x))\\ &+\frac{1}{2}\xi^l(s,\phi_s(x))\xi^m(s,\phi_s(x))D^2_{lm}\rho^\epsilon(y-\phi_s(x))\Big)K_{i_1,\ldots,i_k}(s,y)\Big]\prod_{\alpha=1}^k J^{i_\alpha}_{j_\alpha}(s,x)u^{j_\alpha}_\alpha(x)\\ &+\sum_{p=1}^k K_{i_1,\ldots,i_k}(s,y)\Big(\rho^\epsilon(y-\phi_s(x))D_l\hat{b}^{i_p}(s,\phi_s(x))\\ &-\xi^m(s,\phi_s(x))D_m\rho^\epsilon(y-\phi_s(x))D_l\xi^{i_p}(s,\phi_s(x))\Big)\\ &\times\left(\prod_{\alpha\neq p}^k J^{i_\alpha}_{j_\alpha}(s,x)u^{j_\alpha}_\alpha(x)\right)J^l_{j_p}(s,x)u^{j_p}_p(x)\\ &+\frac{1}{2}\sum_{\substack{p,q=1\\ p\neq q}}^k \rho^\epsilon(y-\phi_s(s))K_{i_1,\ldots,i_k}(s,y)D_l\xi^{i_p}(s,\phi_s(x))D_m\xi^{i_q}(s,\phi_s(x))\\ &\left(\prod_{\alpha\neq p,q}^k J^{i_\alpha}_{j_\alpha}(s,x)u^{j_\alpha}_\alpha(x)\right)\times J^l_{j_p}(s,x)J^m_{j_q}(s,x)u^{j_p}_p(x)u^{j_q}_q(x),\end{aligned}$$

$$\hat{G}^{2,\epsilon}_s(x,y) := -\xi^l(s,\phi_s(x))D_l\rho^\epsilon(y-\phi_s(x))H_{i_1,\ldots,i_k}(s,y)\prod_{\alpha=1}^k J^{i_\alpha}_{j_\alpha}(s,x)u^{j_\alpha}_\alpha(x)$$

$$
\begin{aligned}
&+ \sum_{p=1}^{k} \rho^\epsilon(y-\phi_s(x)) H_{i_1,\ldots,i_k}(s,y) D_l \xi^{i_p}(s,\phi_s(x)) \\
&\left(\prod_{\alpha \neq p}^{k} J^{i_\alpha}_{j_\alpha}(s,x) u^{j_\alpha}_{\alpha}(x)\right) J^{l}_{j_p}(s,x) u^{j_p}_{p}(x),
\end{aligned}
$$

$$
\hat{H}^{1,\epsilon}_s(x,y) := \rho^\epsilon(y-\phi_s(x)) H_{i_1,\ldots,i_k}(s,y) \prod_{\alpha=1}^{k} J^{i_\alpha}_{j_\alpha}(s,x) u^{j_\alpha}_{\alpha}(x),
$$

$$
\begin{aligned}
\hat{H}^{2,\epsilon}_s(x,y) &:= -\xi^l(s,\phi_s(x)) D_l \rho^\epsilon(y-\phi_s(x)) K_{i_1,\ldots,i_k}(s,y) \prod_{\alpha=1}^{k} J^{i_\alpha}_{j_\alpha}(s,x) u^{j_\alpha}_{\alpha}(x) \\
&+ \sum_{p=1}^{k} \rho^\epsilon(y-\phi_s(x)) K_{i_1,\ldots,i_k}(s,y) D_l \xi^{i_p}(s,\phi_s(x)) \\
&\left(\prod_{\alpha \neq p}^{k} J^{i_\alpha}_{j_\alpha}(s,x) u^{j_\alpha}_{\alpha}(x)\right) J^{l}_{j_p}(s,x) u^{j_p}_{p}(x).
\end{aligned}
$$

Step 2: We integrate (5.6) with respect to the variable y on both sides and switch the order of the integrals using the stochastic Fubini theorem (Theorem 5.1). To check that the conditions in the stochastic Fubini theorem are satisfied, first note that $\hat{G}^{i,\epsilon}$, $\hat{H}^{i,\epsilon}$, $i = 1, 2$ are predictable, owing to the measurability conditions imposed in the assumptions. We also have

$$
\begin{aligned}
\int_0^{\tau \wedge T} |\hat{G}^{1,\epsilon}(x,y)| \, \mathrm{d}t &\lesssim \|\rho^\epsilon\|_{L^\infty_y} \|G(\cdot,y)\|_{L^1_t} + \|K(\cdot,y)\|_{L^\infty_t} \Big[\|D\rho^\epsilon\|_{L^\infty_y} \|\hat{b}(\cdot,\phi_\cdot(x))\|_{L^1_t} \\
&\quad + \|D^2\rho^\epsilon\|_{L^\infty_y} \|\xi(\cdot,\phi_\cdot(x))\|^2_{L^2_t} + \|\rho^\epsilon\|_{L^\infty_y} \|D_x \hat{b}(\cdot,\phi_\cdot(x))\|_{L^1_t} \\
&\quad + \|D\rho^\epsilon\|_{L^\infty_y} \|\xi(\cdot,\phi_\cdot(x))\|_{L^2_t} \|D_x\xi(\cdot,\phi_\cdot(x))\|_{L^2_t} \\
&\quad + \|\rho^\epsilon\|_{L^\infty_y} \|D_x\xi(\cdot,\phi_\cdot(x))\|_{L^2_t} \Big], \\
\int_0^{\tau \wedge T} |\hat{G}^{2,\epsilon}(x,y)| \, \mathrm{d}t &\lesssim \|H(\cdot,y)\|_{L^2_t} \Big(\|\rho^\epsilon\|_{L^\infty_y} \|D_x\xi(\cdot,\phi_\cdot(x))\|_{L^2_t} \\
&\quad + \|D\rho^\epsilon\|_{L^\infty_y} \|\xi(\cdot,\phi_\cdot(x))\|_{L^2_t} \Big), \\
\int_0^{\tau \wedge T} |\hat{H}^{1,\epsilon}(x,y)|^2 \, \mathrm{d}t &\lesssim \|\rho^\epsilon\|_{L^\infty_y} \|H(\cdot,y)\|^2_{L^2_t}, \\
\int_0^{\tau \wedge T} |\hat{H}^{2,\epsilon}(x,y)|^2 \, \mathrm{d}t &\lesssim \|K(\cdot,y)\|_{L^\infty_t} \Big(\|\rho^\epsilon\|_{L^\infty_y} \|D_x\xi(\cdot,\phi_\cdot(x))\|^2_{L^2_t} \\
&\quad + \|D\rho^\epsilon\|_{L^\infty_y} \|\xi(\cdot,\phi_\cdot(x))\|^2_{L^2_t} \Big),
\end{aligned}
$$

where $\|\cdot\|_{L^p_t}$ denotes the L^p norm with respect to time for $t \in [0, \tau \wedge T]$, $\|\cdot\|_{L^p_y}$ denotes the L^p norm with respect to space, and we used that $J^i_j(t,x) < R_2$ for all $t \in [0, \tau \wedge T]$. So for every $y \in \mathbb{R}^n$, the second condition is satisfied. Next, taking $D := B_{R_1+\epsilon\gamma}(0)$, we check that

$$
\begin{aligned}
\int_0^{\tau\wedge T}\left(\int_{\mathbb{R}^n}|\hat{G}^{1,\epsilon}(x,y)|\,\mathrm{d}y\right)\mathrm{d}t \lesssim\ & \sup_{y\in D}\|G(\cdot,y)\|_{L^1_t}\|\rho^\epsilon\|_{L^\infty_y} \\
&+ \sup_{y\in D}\|K(\cdot,y)\|_{L^\infty_t}\Big[\|D\rho^\epsilon\|_{L^\infty_y}\|\hat{b}(\cdot,\phi_\cdot(x))\|_{L^1_t} \\
&+\|D^2\rho^\epsilon\|_{L^\infty_y}\|\xi(\cdot,\phi_\cdot(x))\|^2_{L^2_t} \\
&+\|\rho^\epsilon\|_{L^\infty_y}\|D_x\hat{b}(\cdot,\phi_\cdot(x))\|_{L^1_t} \\
&+\|D\rho^\epsilon\|_{L^\infty_y}\|\xi(\cdot,\phi_\cdot(x))\|_{L^2_t}\|D_x\xi(\cdot,\phi_\cdot(x))\|_{L^2_t} \\
&+\|\rho^\epsilon\|_{L^\infty_y}\|D_x\xi(\cdot,\phi_\cdot(x))\|_{L^2_t}\Big], \\
\int_0^{\tau\wedge T}\left(\int_{\mathbb{R}^n}|\hat{G}^{2,\epsilon}_t(x,y)|\,\mathrm{d}y\right)\mathrm{d}t \lesssim\ & \sup_{y\in D}\|H(\cdot,y)\|_{L^2_t}\Big(\|\rho^\epsilon\|_{L^\infty_y}\|D_x\xi(\cdot,\phi_\cdot(x))\|_{L^2_t} \\
&+\|D\rho^\epsilon\|_{L^\infty_y}\|\xi(\cdot,\phi_\cdot(x))\|_{L^2_t}\Big), \\
\int_0^{\tau\wedge T}\left(\int_{\mathbb{R}^n}|\hat{H}^{1,\epsilon}_t(x,y)|^2\,\mathrm{d}y\right)^{\frac{1}{2}}\mathrm{d}t \lesssim\ & \sup_{y\in D}\|H(\cdot,y)\|_{L^2_t}\|\rho^\epsilon\|_{L^\infty_y}, \\
\int_0^{\tau\wedge T}\left(\int_{\mathbb{R}^n}|\hat{H}^{2,\epsilon}_t(x,y)|^2\,\mathrm{d}y\right)^{\frac{1}{2}}\mathrm{d}t \lesssim\ & \sup_{y\in D}\|K(\cdot,y)\|_{L^\infty_t}\Big(\|\rho^\epsilon\|_{L^\infty_y}\|D_x\xi(\cdot,\phi_\cdot(x))\|_{L^2_t} \\
&+\|D\rho^\epsilon\|_{L^\infty_y}\|\xi(\cdot,\phi_\cdot(x))\|_{L^2_t}\Big),
\end{aligned}
$$

so the third condition is also satisfied. Hence, applying the stochastic Fubini theorem and integrating by parts in y, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^n}F^\epsilon_t(x,y)\,\mathrm{d}y-\int_{\mathbb{R}^n}F^\epsilon_0(x,y)\,\mathrm{d}y = & \int_0^t\int_{\mathbb{R}^n}\widetilde{G}^{1,\epsilon}_s(x,y)\,\mathrm{d}y\,\mathrm{d}s \\
&+\int_0^t\int_{\mathbb{R}^n}\widetilde{G}^{2,\epsilon}_s(x,y)\,\mathrm{d}y\,\mathrm{d}[W,B]_s \\
&+\int_0^t\int_{\mathbb{R}^n}\hat{H}^{1,\epsilon}_s(x,y)\,\mathrm{d}y\,\mathrm{d}W_s \\
&+\int_0^t\int_{\mathbb{R}^n}\widetilde{H}^{2,\epsilon}_s(x,y)\,\mathrm{d}y\,\mathrm{d}B_s,
\end{aligned}
\tag{5.7}
$$

where

$$
\begin{aligned}
\widetilde{G}^{1,\epsilon}_s(x,y) :=\ & \rho^\epsilon(y-\phi_s(x))G_{i_1,\ldots,i_k}(s,y)\prod_{\alpha=1}^k J^{i_\alpha}_{j_\alpha}(s,x)u^{j_\alpha}_\alpha(x) \\
&+\rho^\epsilon(y-\phi_s(x))\Bigg[b^l(s,\phi_s(x))D_lK_{i_1,\ldots,i_k}(s,y)\prod_{\alpha=1}^k J^{i_\alpha}_{j_\alpha}(s,x)u^{j_\alpha}_\alpha(x) \\
&+\sum_{p=1}^k K_{i_1,\ldots,i_k}(s,y)D_lb^{i_p}(s,\phi_s(x))\left(\prod_{\alpha\neq p}^k J^{i_\alpha}_{j_\alpha}(s,x)u^{j_\alpha}_\alpha(x)\right)J^l_{j_p}(s,x)u^{j_p}_p(x)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+ \rho^\epsilon(y-\phi_s(x)) \Bigg[\frac{1}{2}\xi^l(s,\phi_s(x))\xi^m(s,\phi_s(x)) D^2_{lm} K_{i_1,\ldots,i_k}(s,y) \prod_{\alpha=1}^k J^{i_\alpha}_{j_\alpha}(s,x) u^{j_\alpha}_\alpha(x) \\
&+ \frac{1}{2}\xi^m(s,\phi_s(x)) D_m \xi^l(s,\phi_s(x)) D_l K_{i_1,\ldots,i_k}(s,y) \prod_{\alpha=1}^k J^{i_\alpha}_{j_\alpha}(s,x) u^{j_\alpha}_\alpha(x) \\
&+ \sum_{p=1}^k \Big[\xi^m(s,\phi_s(x)) D_l \xi^{i_p}(s,\phi_s(x)) D_m K_{i_1,\ldots,i_k}(s,y) \\
&+ \frac{1}{2} D_l \Big[\xi^m(s,\phi_s(x)) D_m \xi^{i_p}(s,\phi_s(x)) \Big] K_{i_1,\ldots,i_k}(s,y) \Big] \\
&\left(\prod_{\alpha\neq p}^k J^{i_\alpha}_{j_\alpha}(s,x) u^{j_\alpha}_\alpha(x) \right) J^l_{j_p}(s,x) u^{j_p}_p(x) \\
&+ \frac{1}{2} \sum_{\substack{p,q=1 \\ p\neq q}}^k K_{i_1,\ldots,i_k}(s,y) D_l \xi^{i_p}(s,\phi_s(x)) D_m \xi^{i_q}(s,\phi_s(x)) \\
&\left(\prod_{\alpha\neq p,q}^k J^{i_\alpha}_{j_\alpha}(s,x) u^{j_\alpha}_\alpha(x) \right) \\
&\times J^l_{j_p}(s,x) J^m_{j_q}(s,x) u^{j_p}_p(x) u^{j_q}_q(x) \Bigg], \\
&\widetilde{G}^{2,\epsilon}_s(x,y) := \rho^\epsilon(y-\phi_s(x)) \Bigg[\xi^l(s,\phi_s(x)) D_l H_{i_1,\ldots,i_k}(s,y) \prod_{\alpha=1}^k J^{i_\alpha}_{j_\alpha}(s,x) u^{j_\alpha}_\alpha(x) \\
&\quad + \sum_{p=1}^k H_{i_1,\ldots,i_k}(s,y) D_l \xi^{i_p}(s,\phi_s(x)) \left(\prod_{\alpha\neq p}^k J^{i_\alpha}_{j_\alpha}(s,x) u^{j_\alpha}_\alpha(x) \right) J^l_{j_p}(s,x) u^{j_p}_p(x) \Bigg], \\
&\widetilde{H}^{2,\epsilon}_s(x,y) := \rho^\epsilon(y-\phi_s(x)) \Bigg[\xi^l(s,\phi_s(x)) D_l K_{i_1,\ldots,i_k}(s,y) \prod_{\alpha=1}^k J^{i_\alpha}_{j_\alpha}(s,x) u^{j_\alpha}_\alpha(x) \\
&\quad + \sum_{p=1}^k K_{i_1,\ldots,i_k}(s,y) D_l \xi^{i_p}(s,\phi_s(x)) \left(\prod_{\alpha\neq p}^k J^{i_\alpha}_{j_\alpha}(s,x) u^{j_\alpha}_\alpha(x) \right) J^l_{j_p}(s,x) u^{j_p}_p(x) \Bigg].
\end{aligned}
$$

Step 3: Finally, we investigate the convergence of each term in the limit $\epsilon \to 0$. First, since K is continuous in y, we obtain the following limit on the LHS of (5.7):

$$
\int_{\mathbb{R}^n} \left(F^\epsilon_t(x,y) - F^\epsilon_0(x,y) \right) \mathrm{d}y \to \left\langle \phi^*_t K(t,x), \boldsymbol{u}(x) \right\rangle - \langle K(0,x), \boldsymbol{u}(x) \rangle ,
$$

as $\epsilon \to 0$, where $\boldsymbol{u}(x) = (u_1(x), \ldots, u_k(x))$, and $\langle K(x), \boldsymbol{u}(x) \rangle := K(x)(u_1(x), \ldots, u_k(x))$ denotes the contraction of tensors. For the terms on the RHS, we apply the dominated convergence theorem to obtain the limit. Using Hölder's inequality and noting that $\|\rho^\epsilon\|_{L^1} = 1$, we derive

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^n} \widetilde{G}_s^{1,\epsilon}(x, y)\,\mathrm{d}y\right| \le \lambda_1^1(s, x)\|G(s,\cdot)\|_{L^\infty_{R_1+\gamma}} \\
&\quad + \lambda_2^1(s, x)\|DK(s,\cdot)\|_{L^\infty_{R_1+\gamma}} + \lambda_3^1(s, x)\|D^2K(s,\cdot)\|_{L^\infty_{R_1+\gamma}}, \\
&\left|\int_{\mathbb{R}^n} \widetilde{G}_s^{2,\epsilon}(x, y)\,\mathrm{d}y\right| \le \lambda_1^2(s, x)\|H(s,\cdot)\|_{L^\infty_{R_1+\gamma}} + \lambda_2^2(s, x)\|DH(s,\cdot)\|_{L^\infty_{R_1+\gamma}},
\end{aligned}
$$

for all $\epsilon < 1$, where $\lambda_i^j(s, x)$ are locally integrable in time for $s \in [0, \tau \wedge T]$. Hence, by the dominated convergence theorem, one can show that the bounded variation parts converge as follows

- $$\begin{aligned}&\int_0^t \int_{\mathbb{R}^n} \widetilde{G}_s^{1,\epsilon}(x, y)\,\mathrm{d}y\,\mathrm{d}s \\ &\to \int_0^t \int_{\mathbb{R}^n} \left\langle \phi_s^* G(s, x), \boldsymbol{u}(x)\right\rangle \mathrm{d}s + \int_0^t \left\langle \phi_s^* \mathcal{L}_b K(s, x), \boldsymbol{u}(x)\right\rangle \mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t \left\langle \phi_s^* \mathcal{L}_\xi \mathcal{L}_\xi K(s, x), \boldsymbol{u}(x)\right\rangle \mathrm{d}s,\end{aligned}$$
- $$\int_0^t \int_{\mathbb{R}^n} \widetilde{G}_s^{2,\epsilon}(x, y)\,\mathrm{d}y\,\mathrm{d}[W, B]_s \to \int_0^t \left\langle \phi_s^* \mathcal{L}_\xi H(s, x), \boldsymbol{u}(x)\right\rangle \mathrm{d}[W, B]_s,$$

where we have taken into account the explicit formulae for Lie derivatives in Lemma 5.3. Similarly, we can show that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^n} \hat{H}_s^{1,\epsilon}(x, y)\,\mathrm{d}y\right| \le \mu_1^1(s, x)\|H(s,\cdot)\|_{L^\infty_{R_1+\gamma}}, \\
&\left|\int_{\mathbb{R}^n} \widetilde{H}_s^{2,\epsilon}(x, y)\,\mathrm{d}y\right| \le \mu_1^2(s, x)\|K(s,\cdot)\|_{L^\infty_{R_1+\gamma}} + \mu_2^2(s, x)\|DK(s,\cdot)\|_{L^\infty_{R_1+\gamma}},
\end{aligned}
$$

where $\mu_i^j(s, x)$ are locally square integrable in time for $s \in [0, \tau \wedge T]$, so by the dominated convergence theorem for Itô integrals, the martingale terms converge to

- $$\int_0^t \int_{\mathbb{R}^n} \hat{H}_s^{1,\epsilon}(x, y)\,\mathrm{d}y\,\mathrm{d}W_s \to \int_0^t \left\langle \phi_s^* H(s, x), \boldsymbol{u}(x)\right\rangle \mathrm{d}W_s,$$
- $$\int_0^t \int_{\mathbb{R}^n} \widetilde{H}_s^{2,\epsilon}(x, y)\,\mathrm{d}y\,\mathrm{d}B_s \to \int_0^t \left\langle \phi_s^* \mathcal{L}_\xi K(s, x), \boldsymbol{u}(x)\right\rangle \mathrm{d}B_s,$$

in probability. Since $\boldsymbol{u}$ was chosen arbitrarily, this proves (3.3) for $t \in [0, \tau \wedge T]$.

6 Conclusions and Outlook for Further Research

In this paper we have:

- Proved the Kunita–Itô–Wentzell (KIW) formula for evaluation of stochastic k-forms along stochastic flows. This formula generalises the classic Itô–Wentzell formula (see Kunita 1981, 1984, 1997), as well as the Kunita's Itô lemma for k-forms on $\mathbb{R}^n$ (shown in Kunita 1997).

- Employed the KIW formula in deriving an Euler–Poincaré variational principle and a Clebsch-constrained Hamilton's principle which each introduce stochastic advection by Lie transport (SALT) into the semidirect-product continuum equations derived in Holm et al. (1998) while preserving their Kelvin–Noether theorem and Lie–Poisson Hamiltonian structure.
- Applied the KIW formula to provide a rigorous derivation of stochastic advection by Lie transport (SALT) equations, continuity equations in fluid dynamics, and Kelvin's circulation Theorem. SALT has been found to be a valuable tool in the modelling of geophysical fluid dynamics, where it enables uncertainty quantification (Cotter et al. 2018a, b) and is expected to lead to uncertainty reduction via data assimilation. It also has been shown to play a similar important role in shape analysis (Arnaudon et al. 2018a, b). All of these results have been developed within the context of Holm (2015), Cotter et al. (2017), Crisan et al. (2018), where the geometric approach for adding SALT to deterministic fluid equations was first introduced, understood and applied.

Some near-term future research directions may include:

- *Tensor fields.* We already know that the KIW formula is valid for vector fields, as well as k-forms. Extending the KIW formula to stochastic time-dependent (r, s)-tensor fields would provide a basis for deriving the stochastic counterparts of the deterministic transport formulas appearing in Holm et al. (1998), e.g., for nonlinear elasticity.
- *Stochastic transport on manifolds.* One would expect that the KIW formula for k-forms (and more generally, for (r, s) tensor fields) could naturally be extended to manifolds. An extensive literature about stochastic flows on manifolds exists, see. e.g., David Elworthy et al. (2007, 2010). The obstacle in this direction for us is that in our proof, first, one would have to make sense of (5.5), where we evaluate the k-form and vector fields at different points in space, which may be justified for instance by introducing a connection and taking the parallel transport to the same point. Secondly, our proof is not local since we consider mollifiers and integrate by parts, which may cause difficulty in the manifold case where we can only work locally on charts, unless we have a coordinate-free proof. However, we have good reasons to conjecture that our KIW formula does hold on manifolds, since our final expression (1.6) is coordinate free, and it would also recover Kunita's Itô-lemma (Kunita 1981, 1984) for k-forms on manifolds in the deterministic case.
- *A new methodology for uncertainty quantification and reduction.* The stochastic fluid velocity decomposition results of Holm (2015) and Cotter et al. (2017) show that the principles of transformation theory and multi-time homogenisation comprise the foundations for a physically meaningful, data-driven and mathematically based approach for decomposing the fluid transport velocity into its drift and stochastic parts. This approach can be applied immediately to the class of continuum flows whose deterministic motion is based on fundamental variational principles.
  Two related papers (Cotter et al. 2018a, b) have recently employed this approach

to develop a new methodology to implement the velocity decomposition of Holm (2015) and Cotter et al. (2017) for uncertainty quantification in computational simulations of fluid dynamics. The new methodology was tested numerically in these papers and found to be suitable for coarse graining in two separate types of problems based on discretisations using either finite elements, or finite differences. Preliminary results of work in progress show that combining stochastic uncertainty quantification with data assimilation can be very effective in reduction of uncertainty.

We expect that the stochastic modelling approach developed using the KIW formula in the present paper will be tenable whenever a body of hydrodynamic transport data shows the characteristic signal of high power at low frequencies. This characteristic signal is often seen in flows in Nature, such as atmospheric and oceanic geophysical flows. In such flows, the opportunity arises to decompose the corresponding Lagrangian trajectories into fast and slow, or resolvable and unresolvable, components and apply the stochastic modelling approach described here as a basis for quantifying *a priori* uncertainty and then using data assimilation methods (e.g., particle filtering) for reducing uncertainty.

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