# On the geometry of discrete contact mechanics 

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#### Abstract

In this paper, we continue the construction of variational integrators adapted to contact geometry started in Vermeeren et al., 2019, in particular, we introduce a discrete Herglotz Principle and the corresponding discrete Herglotz Equations for a discrete Lagrangian in the contact setting. This allows us to develop convenient numerical integrators for contact Lagrangian systems that are conformally contact by construction. The existence of an exact Lagrangian function is also discussed.


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## 1 Introduction

Contact Hamiltonian and Lagrangian systems have deserved a lot of attention in recent years Bravetti, 2017, Bravetti, 2018 or [de León and Lainz Valcázar, 2019b . One of the most relevant features of contact dynamics is the absence of conservative properties contrarily to the conservative character of the energy in symplectic dynamics; indeed, we have a dissipative behavior. This fact suggests that contact geometry may be the appropriate framework to model many physical and mathematical problems with dissipation we find in thermodynamics, statistical physics, quantum mechanics, gravity or control theory, among many others. Consequently, it becomes an important necessity to develop numerical methods adapted to the contact setting for applications in the above mentioned subjects. The idea is to develop geometric integrators, that is, numerical methods for differential equations which preserve geometric properties like contact structure, symmetries, configuration space... This preservation of structural properties is often desirable to achieve correct qualitative behavior and long time stability Hairer et al., 2010, Sanz-Serna and Calvo, 1994, Blanes and Casas, 2016.

As far as we know, the first attempt to develop geometric integrators for the contact case is in the paper Vermeeren et al., 2019] (see also [Bravetti et al., 2020]), where the authors present geometric numerical integrators for contact flows that stem from a discretization of Herglotz variational principle.

Our goal in the current paper is to go further in the discrete description of contact dynamics, so we will mention some of the new and relevant results that the reader can find in the next pages. Instead of deriving the discrete Herglotz equations by an heuristic argument, they are directly obtained from a clear discrete variational principle. In addition, to develop the discrete algorithm we use the natural discretization $Q \times Q \times \mathbb{R}$, which preserves all the contact geometry flavor.

Another relevant point is the discussion of the existence of an exact discrete Lagrangian function Marsden and West, 2001, Patrick and Cuell, 2009], which will lead us to define the contact exponential map and prove its existence. This construction is essential to develop a complete theory of variational error analysis for contact Lagrangian systems.

Finally, we consider a discrete version of the infinitesimal symmetries discussed in Gaset et al., 2019, jointly with the corresponding dissipated quantities.

The paper is structured as follows. Section 2 is devoted to a quick review
of contact Hamiltonian and Lagrangian systems in the continuous setting. In particular, we recall the Herglotz variational principle, since it will be the motivation to develop the corresponding discrete version. Section 3 is devoted to construct the discrete version of contact Lagrangian dynamics for a discrete Lagrangian $L_{d}: Q \times Q \times \mathbb{R} \rightarrow \mathbb{R}$, where $Q$ is the configuration manifold. We consider the discrete Herglotz principle to obtain the so-called discrete Herglotz equations. The Legendre transformations $F^{-} L_{d}$ and $F^{+} L_{d}$ are defined, and consequently the discrete flow (at the Lagrangian and Hamiltonian levels); the main result is that the discrete flow is a conformal contactomorphism. In Section 4 we define the contact exponential map for the Herglotz vector field and prove that it is a local diffeomorphism. This result permits to study the existence of an exact Lagrangian function. Finally, we consider several examples to illustrate our theoretical developments.

## 2 Continuous contact mechanics

### 2.1 Contact manifolds and Hamiltonian systems

In this section we will recall the main definitions and results on the theory of contact manifolds and Hamiltonian system. See [de León and Lainz Valcázar, 2019a for a more detailed overview.

A contact manifold $(M, \eta)$ is an $(2 n+1)$-dimensional manifold with a contact form $\eta$ Libermann and Marle, 1987. That is, $\eta$ is a 1-form on $M$ such that $\eta \wedge \mathrm{d} \eta^{n}$ is a volume form. This type of manifolds have a distinguished vector field: the so-called Reeb vector field $\mathcal{R}$, which is the unique vector field that satisfies:

$$
\begin{equation*}
i_{\mathcal{R}} \mathrm{d} \eta=0, \quad \eta(\mathcal{R})=1 \tag{1}
\end{equation*}
$$

On a contact manifold $(M, \eta)$, we define the following isomorphism of vector bundles:

$$
\begin{align*}
b: T M & \longrightarrow T^{*} M, \\
v & \longmapsto i_{v} \mathrm{~d} \eta+\eta(v) \eta . \tag{2}
\end{align*}
$$

Notice that $b(\mathcal{R})=\eta$.
There is a Darboux theorem for contact manifolds. In a neighborhood of each point in $M$ one can find local coordinates $\left(q^{i}, p_{i}, z\right)$ such that

$$
\begin{equation*}
\eta=\mathrm{d} z-p_{i} \mathrm{~d} q^{i} . \tag{3}
\end{equation*}
$$

In these coordinates, we have

$$
\begin{equation*}
\mathcal{R}=\frac{\partial}{\partial z} \tag{4}
\end{equation*}
$$

An example of a contact manifold is $T^{*} Q \times \mathbb{R}$. Here, the contact form is given by

$$
\begin{equation*}
\eta_{Q}=\mathrm{d} z-\theta_{Q}=\mathrm{d} z-p_{i} \mathrm{~d} q^{i} \tag{5}
\end{equation*}
$$

where $\theta_{Q}$ is pullback the tautological 1-form of $T^{*} Q,\left(q^{i}, p_{i}\right)$ are natural coordinates on $T^{*} Q$ and $z$ is the $\mathbb{R}$-coordinate.

We say that a (local) diffeomorphism between two contact manifolds $F:(M, \eta) \rightarrow(N, \tau)$ is a (local) contactomorphism if $F^{*} \tau=\eta$. We say that $F$ is a (local) conformal contactomorphism if $F^{*} \operatorname{ker} \tau=\operatorname{ker} \eta$ or, equivalently, $F^{*} \tau=\sigma \eta$, where $\sigma: M \rightarrow \mathbb{R} \backslash\{0\}$ is the conformal factor.

We say that a vector field $X$ on $M$ is an infinitesimal (conformal) contactomorphism if its flow $F_{t}$ consists of (conformal) contactomorphisms.

From the general identify, where $F_{t}$ is a flow and $X$ is its infinitesimal generator

$$
\begin{equation*}
\frac{\partial}{\partial t} F_{t}^{*} \eta=F_{t}^{*} \mathcal{L}_{X} \eta \tag{6}
\end{equation*}
$$

we deduce that $X$ is infinitesimal contactomorphism if and only if

$$
\begin{equation*}
\mathcal{L}_{X} \eta=0 \tag{7}
\end{equation*}
$$

Furthermore, $X$ is a conformal contactomorphism if and only if

$$
\begin{equation*}
\mathcal{L}_{X} \eta=a \eta \tag{8}
\end{equation*}
$$

for some $a: M \rightarrow \mathbb{R}$. The function $a$ is related to the conformal factors $\sigma_{t}$ of the conformal contactomorphisms $F_{t}$ by

$$
\begin{equation*}
\sigma_{t}(x)=\int_{0}^{t} \exp \left(a\left(F_{\tau}(x)\right)\right) \mathrm{d} \tau \tag{9}
\end{equation*}
$$

Given a smooth function $f: M \rightarrow \mathbb{R}$, its Hamiltonian vector field $X_{f}$ is given by

$$
\begin{equation*}
b\left(X_{f}\right)=\mathrm{d} f-(f+\mathcal{R}(f)) \eta \tag{10}
\end{equation*}
$$

A vector field $X$ is the Hamiltonian vector field of some function $f$ if and only if it is an infinitesimal conformal contactomorphism. In that case $X=X_{f}$ for $f=-\eta(X)$. Moreover, $\mathcal{L}_{X} \eta=-\mathcal{R}(f) \eta$. Hence $X$ is an infinitesimal contactomorphism if and only if $X=X_{f}$ for some function $f$ such that $\mathcal{R}(f)=0$.

We call the triple $(M, \eta, H)$ a contact Hamiltonian system, where $(M, \eta)$ is a contact manifold and $H: M \rightarrow \mathbb{R}$ is the Hamiltonian function.

In contrast to their symplectic counterpart, contact Hamiltonian vector fields do not preserve the Hamiltonian. In fact

$$
\begin{equation*}
X_{H}(H)=-\mathcal{R}(H) H . \tag{11}
\end{equation*}
$$

### 2.2 Contact Lagrangian systems

Now we review the Lagrangian picture of contact systems. In [de León and Lainz Valcázar, 2019b we give a more comprehensive description which also covers the case of singular Lagrangians.

Let $Q$ be an $n$-dimensional configuration manifold and consider the extended phase space $T Q \times \mathbb{R}$ and a Lagrangian function $L: T Q \times \mathbb{R} \rightarrow \mathbb{R}$. In this paper, we will assume that the Lagrangian is regular, that is, the Hessian matrix with respect to the velocities $\left(W_{i j}\right)$ is regular where

$$
\begin{equation*}
W_{i j}=\frac{\partial^{2} L}{\partial \dot{q}^{i} \partial \dot{q}^{j}}, \tag{12}
\end{equation*}
$$

and $\left(q^{i}, \dot{q}^{i}, z\right)$ are bundle coordinates for $T Q \times \mathbb{R}$. Equivalently, $L$ is regular if and only if the one-form

$$
\begin{equation*}
\eta_{L}=\mathrm{d} z-\theta_{L} \tag{13}
\end{equation*}
$$

is a contact form. Here,

$$
\begin{equation*}
\theta_{L}=S^{*}(\mathrm{~d} L)=\frac{\partial L}{\partial \dot{q}^{i}} \mathrm{~d} q^{i}, \tag{14}
\end{equation*}
$$

where $S$ is the canonical vertical endomorphism on $T Q$ extended to $T Q \times \mathbb{R}$, that is, in local $T Q \times \mathbb{R}$ bundle coordinates,

$$
\begin{equation*}
S=\mathrm{d} q^{i} \otimes \frac{\partial}{\partial \dot{q}^{i}} \tag{15}
\end{equation*}
$$

The energy of the system is defined by

$$
\begin{equation*}
E_{L}=\Delta(L)-L=\dot{q}^{i} \frac{\partial L}{\partial \dot{q}^{i}}-L, \tag{16}
\end{equation*}
$$

where $\Delta$ is the Liouville vector field on $T Q$ extended to $T Q \times \mathbb{R}$ in the natural way.

The Reeb vector field of $\eta_{L}$, which we will denoted by $\mathcal{R}_{L}$ is given by

$$
\begin{equation*}
\mathcal{R}_{L}=\frac{\partial}{\partial z}-W^{i j} \frac{\partial^{2} L}{\partial \dot{q}^{i} \partial z} \frac{\partial}{\partial \dot{q}^{j}}, \tag{17}
\end{equation*}
$$

where $\left(W^{i j}\right)$ is the inverse of the Hessian matrix with respect to the velocities $W^{i j}$ (Equation (12)).

The Hamiltonian vector field of the energy $E_{L}$ will be denoted $\xi_{L}=X_{E_{L}}$, hence

$$
\begin{equation*}
b_{L}\left(\xi_{L}\right)=\mathrm{d} E_{L}-\left(\mathcal{R}_{L}\left(E_{L}\right)+E_{L}\right) \eta_{L}, \tag{18}
\end{equation*}
$$

where $b_{L}(v)=i_{v} \mathrm{~d} \eta_{L}+\eta_{L}(v) \eta_{L}$ is the isomorphism defined in Equation (17) for this particular contact structure.
$\xi_{L}$ is a second order differential equation (SODE) (that is, $S\left(\xi_{L}\right)=\Delta$ ) and its solutions are just the ones of the Herglotz equations (also called generalized Euler-Lagrange equations) for $L$ (see [de León and Lainz Valcázar, 2019b]:

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t}\left(\frac{\partial L}{\partial \dot{q}^{i}}\right)-\frac{\partial L}{\partial q^{i}}=\frac{\partial L}{\partial \dot{q}^{i}} \frac{\partial L}{\partial z} . \tag{19}
\end{equation*}
$$

There exists a Legendre transformation for contact Lagrangian systems. Given the vector bundle $T Q \times \mathbb{R} \rightarrow Q \times \mathbb{R}$, one can consider the fiber derivative $\mathbb{F} L$ of $L: T Q \times \mathbb{R} \rightarrow \mathbb{R}$, which has the following coordinate expression in natural coordinates:

$$
\begin{align*}
\mathbb{F} L: T Q \times \mathbb{R} & \rightarrow T^{*} Q \times \mathbb{R} \\
\left(q^{i}, \dot{q}^{i}, z\right) & \mapsto\left(q^{i}, \frac{\partial L}{\partial \dot{q}^{i}}, z\right) . \tag{20}
\end{align*}
$$

If we consider the contact structure $\eta_{Q}$ (5) on $T^{*} Q \times \mathbb{R}$, and $\eta_{L}$ on $T Q \times \mathbb{R}$ then $\mathbb{F} L$ is a local contactomorphism.

In the case that $\mathbb{F} L$ is a global contactomorphism, then we say that $L$ is hyperregular. In this situation, we can define a Hamiltonian $H: T^{*} Q \times \mathbb{R} \rightarrow$ $\mathbb{R}$ such that $E_{L}=H \circ \mathbb{F} L$ and the Lagrangian and Hamiltonian dynamics are $\mathbb{F} L$-related, that is, $\mathbb{F} L_{*} \xi_{L}=X_{H}$.

### 2.2.1 Herglotz variational principle

Equations (19) can be derived from a modified variational principle [Herglotz, 1930]. In contrast to the symplectic case, the action is not a definite integral. The contact action is the value at the endpoint of solution to a non-autonomous ODE.

In de León and Lainz Valcázar, 2019b we defined the action on the space of curves with fixed endpoints. However, for our purposes here it is more convenient to define the action on the space of all curves and all initial conditions and then restrict it to the appropriate submanifold.

Let $\Omega$ be the (infinite dimensional) manifold of curves on $Q, c:[0,1] \rightarrow$ $Q$. We denote by $\Omega\left(q_{0}, q_{1}\right) \subseteq \Omega$, where $q_{0}, q_{1} \in Q$, the submanifold whose elements are the smooth curves $c \in \Omega$ such that $c(0)=q_{0}, c(1)=q_{1}$. The tangent space of $\Omega$ at a curve $c$ is given by vector fields over $c$. In the case of $T_{c} \Omega\left(q_{0}, q_{1}\right)$, the vector fields over $c$ vanish a the endpoints. That is,

$$
\begin{align*}
T_{c} \Omega & =\left\{\delta v \in \mathcal{C}^{\infty}([0,1] \rightarrow T Q) \mid \tau_{Q} \circ \delta v=c\right\}  \tag{21}\\
T_{c} \Omega\left(q_{0}, q_{1}\right) & =\left\{\delta c \in T_{c} \Omega \mid \delta c(0)=0, \delta c(1)=0\right\} \tag{22}
\end{align*}
$$

We define the operator

$$
\begin{equation*}
\mathcal{Z}: \Omega \times \mathbb{R} \rightarrow \mathcal{C}^{\infty}([0,1] \rightarrow \mathbb{R}) \tag{23}
\end{equation*}
$$

which assigns to each curve and initial condition $\left(c, z_{0}\right)$ the curve $\mathcal{Z}_{z_{0}}(c)$ that solves the following ODE:

$$
\begin{cases}\frac{\mathrm{d} \mathcal{Z}_{z_{0}}(c)}{\mathrm{d} t} & =L\left(c, \dot{c}, \mathcal{Z}_{z_{0}}(c)\right)  \tag{24}\\ \mathcal{Z}_{z_{0}}(c)(0) & =z_{0}\end{cases}
$$

Now we define the contact action functional as the map which assigns to each curve $c$ and initial condition $z_{0}$, the solution to the previous ODE evaluated at the endpoint:

$$
\begin{align*}
\mathcal{A}: \Omega \times \mathbb{R} & \rightarrow \mathbb{R} \\
\left(c, z_{0}\right) & \mapsto \mathcal{Z}_{z_{0}}(c)(1) \tag{25}
\end{align*}
$$

When restricted to $\Omega\left(q_{0}, q_{1}\right) \times\left\{z_{0}\right\}$, the critical points of $\mathcal{A}$ are the solutions to Herglotz equation. More precisely,

Theorem 2.1 (Herglotz variational principle). Let $L: T Q \times \mathbb{R} \rightarrow \mathbb{R}$ be a Lagrangian function and let $c \in \Omega\left(q_{0}, q_{1}\right)$ and $z_{0} \in \mathbb{R}$. Then, $\left(c, \dot{c}, \mathcal{Z}_{z_{0}}(c)\right)$ satisfies the Herglotz equations (19) if and only if $c$ is a critical point of $\left.\mathcal{A}_{z_{0}}\right|_{\Omega\left(q_{0}, q_{1}\right)}$.
Proof. We will compute $T \mathcal{Z}$. Let $c \in \Omega\left(q_{0}, q_{1}\right)$ be a curve and consider some tangent vector $\delta c \in T_{c} \Omega$. We will first compute the partial derivative with respect to $c$ by fixing $z_{0} \in \mathbb{R}$, and then we will fix the curve and compute the partial derivative with respect to the initial condition $z_{0}$. In order to simplify the notation, let $\chi=\left(c, \dot{c}, \mathcal{Z}_{z_{0}}(c)\right)$ and let $\psi=T_{c} \mathcal{Z}_{z_{0}}(\delta v)$.

Consider a curve $c_{\lambda} \in \Omega$ (that is, a smoothly parametrized family of curves) such that

$$
\delta c=\left.\frac{\mathrm{d} c_{\lambda}}{\mathrm{d} \lambda}\right|_{\lambda=0}
$$

Since $\mathcal{Z}_{z_{0}}\left(c_{\lambda}\right)(0)=z_{0}$ for all $\lambda$, then $\psi(0)=0$.
We compute the derivative of $\psi$ by interchanging the order of the derivatives using the ODE defining $\mathcal{Z}$ :

$$
\begin{aligned}
\dot{\psi}(t) & =\left.\frac{\mathrm{d}}{\mathrm{~d} \lambda} \frac{\mathrm{~d}}{\mathrm{~d} t} \mathcal{Z}_{z_{0}}\left(c_{\lambda}(t)\right)\right|_{\lambda=0} \\
& =\left.\frac{\mathrm{d}}{\mathrm{~d} \lambda} L\left(c_{\lambda}(t), \dot{c}_{\lambda}(t), \mathcal{Z}\left(c_{\lambda}\right)(t)\right)\right|_{\lambda=0} \\
& =\frac{\partial L}{\partial q^{i}}(\chi(t)) \delta c^{i}(t)+\frac{\partial L}{\partial \dot{q}^{i}}(\chi(t)) \delta \dot{c}^{i}(t)+\frac{\partial L}{\partial z}(\chi(t)) \psi(t) .
\end{aligned}
$$

Hence, the function $\psi$ is the solution to the ODE above. Explicitly

$$
\begin{equation*}
\psi(t)=\frac{1}{\sigma(t)} \int_{0}^{t} \sigma(\tau)\left(\frac{\partial L}{\partial q^{i}}(\chi(\tau)) \delta c^{i}(\tau)+\frac{\partial L}{\partial \dot{q}^{i}}(\chi(\tau)) \delta \dot{c}^{i}(\tau)\right) \mathrm{d} \tau, \tag{26}
\end{equation*}
$$

where

$$
\begin{equation*}
\sigma(t)=\exp \left(-\int_{0}^{t} \frac{\partial L}{\partial z}(\chi(\tau)) \mathrm{d} \tau\right)>0 \tag{27}
\end{equation*}
$$

Integrating by parts and using that $\psi(0)=0$, we get the following expression

$$
\begin{aligned}
\psi(t) & =\delta c^{i}(t) \frac{\partial L}{\partial \dot{q}^{i}}(\chi(t)) \\
& +\frac{1}{\sigma(t)} \int_{0}^{t} \delta c^{i}(t)\left(\sigma(t) \frac{\partial L}{\partial q^{i}}(\chi(t))-\frac{\mathrm{d}}{\mathrm{~d} t}\left(\sigma(t) \frac{\partial L}{\partial \dot{q}^{i}}(\chi(t))\right)\right) \mathrm{d} t \\
& =\delta c^{i}(t) \frac{\partial L}{\partial \dot{q}^{i}}(\chi(t)) \\
& +\delta c^{i}(t) \frac{\partial L}{\partial \dot{q}^{i}}(\chi(t)) \\
& +\frac{1}{\sigma(t)} \int_{0}^{t} \delta c^{i}(\tau) \sigma(\tau)\left(\frac{\partial L}{\partial q^{i}}(\chi(\tau))-\frac{\mathrm{d}}{\mathrm{~d} \tau} \frac{\partial L}{\partial \dot{q}^{i}}(\chi(\tau))+\frac{\partial L}{\partial \dot{q}^{i}}(\chi(\tau)) \frac{\partial L}{\partial z}(\chi(\tau))\right) \mathrm{d} \tau .
\end{aligned}
$$

Now we compute the partial derivative with respect to the initial condition $z_{0}$. We interchange the order of the derivatives

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \frac{\partial \mathcal{Z}_{z_{0}}(c)}{\partial z_{0}}=\frac{\partial L}{\partial z}(c, \dot{c}, \mathcal{Z}(c)) \frac{\partial \mathcal{Z}_{z_{0}}(c)}{\partial z_{0}} \tag{28}
\end{equation*}
$$

If we solve for $\frac{\partial \mathcal{Z}_{z_{0}}(c)}{\partial z_{0}}$ the ODE above using that $\frac{\partial \mathcal{Z}_{z_{0}}(c)}{\partial z_{0}}(0)=1$, we notice that

$$
\begin{equation*}
\frac{\partial \mathcal{Z}_{z_{0}}(c)}{\partial z_{0}}(t)=\exp \left(\int_{0}^{t} \frac{\partial L}{\partial z}(\chi(\tau)) \mathrm{d} \tau\right)=\frac{1}{\sigma(t)}, \tag{29}
\end{equation*}
$$

where $\sigma$ is defined in (27).

### 2.2.2 Symmetries and dissipated quantities on contact Lagrangian systems

As explained in Gaset et al., 2019, de León and Valcázar, 2020, given a symmetry on a contact system, one does not obtain a conserved quantity, but a quantity $f$ that dissipates at the same rate as the Lagrangian.

Given a contact Hamiltonian system $(M, \eta, H)$, we say that a quantity $f: M \rightarrow \mathbb{R}$ is dissipated if

$$
\begin{equation*}
\mathcal{L}_{X_{H}} f=-\mathcal{R}(H) f, \tag{30}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
\phi_{t}^{*}(f)=\sigma_{t}, \tag{31}
\end{equation*}
$$

where $\phi$ is the flow of $X_{H}$ and $\sigma_{t}$, its conformal factor.
Notice that the quotient of two dissipated quantities (if it is well defined) is a conserved quantity.

We end this section by stating a Noether theorem in this setting, which provides a link between symmetries of the Lagrangian and conserved quantities.

Let $L: T Q \times \mathbb{R} \rightarrow \mathbb{R}$ be a regular Lagrangian. Let $G$ be a Lie group acting on $Q$

$$
\begin{equation*}
\Phi: G \times Q \rightarrow Q . \tag{32}
\end{equation*}
$$

We defined the lifted action as

$$
\begin{equation*}
\tilde{\Phi}: G \times T Q \times \mathbb{R} \rightarrow T Q \times \mathbb{R}, \tag{33}
\end{equation*}
$$

given by $\tilde{\Phi}\left(g, v_{q}, z\right)=\left(T_{q} \Phi\left(v_{q}\right), z\right)$ where $v_{q} \in T_{q} Q$. We denote by $\xi_{T Q \times \mathbb{R}}$ to the vector field on $T Q \times \mathbb{R}$ which is the infinitesimal generator by the lifted action of an element $\xi$ of the Lie algebra $\mathfrak{g}$ of $G$.

We define the momentum map $J_{L}$ :

$$
\begin{align*}
& J_{L}: T Q \times \mathbb{R} \rightarrow \mathfrak{g}^{*}  \tag{34}\\
& \left\langle J_{L}\left(v_{q}, z\right), \xi\right\rangle=-\eta_{L}\left(\xi_{T Q \times \mathbb{R}}\right) .
\end{align*}
$$

and we define $\hat{J}(\xi): T Q \times \mathbb{R} \rightarrow \mathbb{R}$ by $\hat{J}(\xi)\left(v_{q}, z\right)=\left\langle J_{L}\left(v_{q}, z\right), \xi\right\rangle$.
Then we have the following de León and Valcázar, 2020, Section 4.1]

Theorem 2.2. Let the lifted action $\tilde{\Phi}$ preserve the Lagrangian L, then $\tilde{\Phi}$ acts by contactomorphisms on $\left(T Q \times \mathbb{R}, \eta_{L}, E_{L}\right)$ and $\hat{J}(\xi)$ is a dissipated quantity for every $\xi \in \mathfrak{g}$.

## 3 Discrete contact mechanics

In this section, we will extend the approach to discrete mechanics as in Marsden and West, 2001 to the case of contct dynamics (see also [Vermeeren et al., 2019]).

Let $L_{d}: Q \times Q \times \mathbb{R} \rightarrow \mathbb{R}$ be a discrete Lagrangian function. In our point of view $Q \times Q \times \mathbb{R}$ will be the discrete space corresponding to the manifold $T Q \times \mathbb{R}$, where continuous contact Lagrangian mechanics takes place. We fix a time-step $h>0$, on which $L_{d}$ depends, though we will omit this explicit dependence.

For each $N \in \mathbb{N}$, let us define the discrete path space as the space containing sequences on $Q$ with length $N+1$, i.e.,

$$
\mathcal{C}_{d}^{N}(Q)=\left\{\left(q_{0}, q_{1}, \ldots, q_{N}\right) \mid q_{k} \in Q, k=0, \ldots, N\right\}
$$

The set $\mathcal{C}_{d}^{N}(Q)$ is a manifold and it is canonically identified with the product space $Q^{N+1}$.

To each $q_{d} \in \mathcal{C}_{d}^{N}(Q)$ and each $z_{0} \in \mathbb{R}$ we will associate another sequence $\left(z_{k}\right) \in \mathbb{R}^{N+1}$ defined by

$$
\begin{equation*}
z_{k+1}-z_{k}=L_{d}\left(q_{k}, q_{k+1}, z_{k}\right), \quad k=0, \ldots, N-1 . \tag{35}
\end{equation*}
$$

In the sequel, for each $1 \leqslant k \leqslant N$, we will denote by $\mathcal{Z}_{k}$ the function $\mathcal{Z}_{k}: Q \times Q \times \mathbb{R} \longrightarrow \mathbb{R}$

$$
\mathcal{Z}_{k}\left(q_{k-1}, q_{k}, z_{k-1}\right)=z_{k-1}+L_{d}\left(q_{k-1}, q_{k}, z_{k-1}\right) .
$$

We define the contact discrete action to be the functional that for each point $q_{d} \in \mathcal{C}_{d}^{N}(Q)$ and each real number $z_{0}$ returns as output the real number $z_{N}$ obtained recursively from (35), i.e.,

$$
\begin{align*}
\mathcal{A}_{d}: \mathcal{C}_{d}^{N}(Q) \times \mathbb{R} & \longrightarrow \mathbb{R}  \tag{36}\\
\left(q_{d}, z_{0}\right) & \mapsto z_{N} .
\end{align*}
$$

A variation of a sequence $q_{d} \in \mathcal{C}_{d}^{N}(Q)$ is a curve $\widetilde{q}_{d}:(-\epsilon, \epsilon) \rightarrow \mathcal{C}_{d}^{N}(Q)$ satisfying $\widetilde{q}_{d}(0)=q_{d}$. Given such a variation, we will define its infinitesimal variation by

$$
\delta q_{d}:=\left.\frac{\mathrm{d}}{\mathrm{~d} \epsilon}\right|_{\epsilon=0} \widetilde{q}_{d}(\epsilon)=\left(\delta q_{0}, \ldots, \delta q_{N}\right)
$$

where $\delta q_{k}:=\left.\frac{\mathrm{d}}{\mathrm{d} \epsilon}\right|_{\epsilon=0} \widetilde{q}_{k}(\epsilon)$.
Proposition 3.1. Let $L_{d}$ be a smooth discrete Lagrangian. Then, if we fix $z_{0} \in \mathbb{R}$, we obtain the functional

$$
\begin{aligned}
\mathcal{A}_{d, z_{0}}: \mathcal{C}_{d}^{N}(Q) & \longrightarrow \mathbb{R} \\
q_{d} & \mapsto \mathcal{A}_{d}\left(q_{d}, z_{0}\right) .
\end{aligned}
$$

The differential of the functional $\mathcal{A}_{d, z_{0}}$ is the following

$$
\begin{align*}
\mathrm{d} \mathcal{A}_{d, z_{0}}\left(q_{d}\right)= & \sigma_{N} \cdots \sigma_{2} \frac{\partial \mathcal{Z}_{1}}{\partial q_{0}}\left(q_{0}, q_{1}, z_{0}\right) \mathrm{d} q_{0} \\
& +\sum_{k=1}^{N-1} \prod_{j=k+2}^{N} \sigma_{j} \cdot\left(\frac{\partial \mathcal{Z}_{k+1}}{\partial q_{k}}+\frac{\partial \mathcal{Z}_{k+1}}{\partial z_{k}} \frac{\partial \mathcal{Z}_{k}}{\partial q_{k}}\right) \mathrm{d} q_{k}  \tag{37}\\
& +\frac{\partial \mathcal{Z}_{N}}{\partial q_{N}}\left(q_{N-1}, q_{N}, z_{N-1}\right) \mathrm{d} q_{N},
\end{align*}
$$

where we are using the identification of $\mathcal{C}_{d}^{N}(Q)$ with $Q^{N+1}$ and for each $1 \leqslant j \leqslant N$

$$
\sigma_{j}=\frac{\partial \mathcal{Z}_{j}}{\partial z_{j-1}}\left(q_{j-1}, q_{j}, z_{j-1}\right) .
$$

Proof. Using the identification of $\mathcal{C}_{d}^{N}(Q)$ with $Q^{N+1}$, note that the discrete action may be rewritten as

$$
\mathcal{A}_{d, z_{0}}\left(q_{d}\right)=\mathcal{Z}_{N}\left(q_{N-1}, q_{N}, \mathcal{Z}_{N-1}\left(q_{N-2}, q_{N-1}, \mathcal{Z}_{N-2}\left(\ldots \mathcal{Z}_{1}\left(q_{0}, q_{1}, z_{0}\right) \ldots\right)\right)\right.
$$

Using that

$$
\mathrm{d} \mathcal{A}_{d, z_{0}}\left(q_{d}\right)=\frac{\partial \mathcal{A}_{d, z_{0}}}{\partial q_{0}} \mathrm{~d} q_{0}+\sum_{k=1}^{N-1} \frac{\partial \mathcal{A}_{d, z_{0}}}{\partial q_{k}} \mathrm{~d} q_{k}+\frac{\partial \mathcal{A}_{d, z_{0}}}{\partial q_{N}} \mathrm{~d} q_{N} .
$$

and applying the chain rule, we deduce that

$$
\frac{\partial \mathcal{A}_{d, z_{0}}}{\partial q_{0}}=\frac{\partial \mathcal{Z}_{N}}{\partial z_{N-1}} \cdots \frac{\partial \mathcal{Z}_{2}}{\partial z_{1}} \frac{\partial \mathcal{Z}_{1}}{\partial q_{0}}
$$

since the function $\mathcal{Z}_{1}$ is the only one that depends on $q_{0}$ among all the $N$ functions $\mathcal{Z}_{k}$. It is also clear that

$$
\frac{\partial \mathcal{A}_{d, z_{0}}}{\partial q_{N}}=\frac{\partial \mathcal{Z}_{N}}{\partial q_{N}},
$$

since none of the functions $\mathcal{Z}_{k}$ depend on $q_{N}$ except the function $\mathcal{Z}_{N}$. Finally if $1 \leqslant k \leqslant N-1$ we have that

$$
\frac{\partial \mathcal{A}_{d, z_{0}}}{\partial q_{k}}=\frac{\partial \mathcal{Z}_{N}}{\partial z_{N-1}} \cdots \frac{\partial \mathcal{Z}_{k+2}}{\partial z_{k+1}}\left(\frac{\partial \mathcal{Z}_{k+1}}{\partial q_{k}}+\frac{\partial \mathcal{Z}_{k+1}}{\partial z_{k}} \frac{\partial \mathcal{Z}_{k}}{\partial q_{k}}\right)
$$

where we applied the chain rule and the fact that the functions $\mathcal{Z}_{k+1}$ and $\mathcal{Z}_{k}$ are the only ones that depend on $q_{k}$. Hence, we finished the proof.

Remark 3.2. Let us see the special case $N=2$, where we can directly compute the differential of the action:

Let $L_{d}$ be a smooth discrete Lagrangian. In the case where $N=2$, the differential of the discrete action function satisfies:

$$
\begin{align*}
& \mathrm{d} \mathcal{A}_{d, z_{0}}=\left(D_{1} L_{d}\left(q_{1}, q_{2}, z_{1}\right)+\left(1+D_{z} L_{d}\left(q_{1}, q_{2}, z_{1}\right) D_{2} L_{d}\left(q_{0}, q_{1}, z_{0}\right)\right) \mathrm{d} q_{1}\right.  \tag{38}\\
& \quad+D_{2} L_{d}\left(q_{1}, q_{2}, z_{1}\right) \mathrm{d} q_{2}+\left(1+D_{z} L_{d}\left(q_{1}, q_{2}, z_{1}\right)\right) D_{1} L_{d}\left(q_{0}, q_{1}, z_{0}\right) \mathrm{d} q_{0} .
\end{align*}
$$

Definition 3.3 (Discrete Herglotz Principle). Given $z_{0} \in \mathbb{R}$, a discrete path $q_{d}=\left(q_{0}, \ldots, q_{N}\right)$ in $\mathcal{C}_{d}^{N}(Q)$ is said to satisfy the Discrete Herglotz Principle if $q_{d}$ is a critical value of the discrete action functional $\mathcal{A}_{d, z_{0}}$ among all paths in $\mathcal{C}_{d}^{N}(Q)$ with fixed end points $q_{0}, q_{N}$.

We will now obtain as a sufficient and necessary condition for a path to satisfy the discrete Herglotz principle, a set of equations called Discrete Herglotz equations Vermeeren et al., 2019.

Theorem 3.4. Let $L_{d}$ be a discrete Lagrangian function such that $1+D_{z} L_{d}$ is non-vanishing everywhere. Given $z_{0} \in \mathbb{R}$, a discrete path $q_{d} \in \mathcal{C}_{d}^{N}(Q)$ satisfies the discrete Herglotz principle if and only if it satisfies

$$
\begin{align*}
& D_{1} L_{d}\left(q_{k}, q_{k+1}, z_{k}\right)+\left(1+D_{z} L_{d}\left(q_{k}, q_{k+1}, z_{k}\right)\right) D_{2} L_{d}\left(q_{k-1}, q_{k}, z_{k-1}\right)=0, \\
& \quad z_{k}-z_{k-1}=L_{d}\left(q_{k-1}, q_{k}, z_{k-1}\right), \tag{39}
\end{align*}
$$

for $k=1, \ldots, N-1$.
Proof. Let $q_{d}(\epsilon)$ be a variation of $q_{d} \in \mathcal{C}_{d}^{N}(Q)$ with fixed end-points $q_{0}$ and $q_{N}$. Then $q_{d}$ is a critical value of the discrete action functional if and only if

$$
\left.\frac{\mathrm{d}}{\mathrm{~d} \epsilon}\right|_{\epsilon=0}\left(\mathcal{A}_{d, z_{0}}\left(q_{d}(\epsilon)\right)\right)=\mathrm{d} \mathcal{A}_{d, z_{0}}\left(\delta q_{d}\right)=0 .
$$

By (37) the last expression is equivalent to

$$
\sum_{k=1}^{N-1} \prod_{j=k+2}^{N} \sigma_{j} \cdot\left(\frac{\partial \mathcal{Z}_{k+1}}{\partial q_{k}}+\frac{\partial \mathcal{Z}_{k+1}}{\partial z_{k}} \frac{\partial \mathcal{Z}_{k}}{\partial q_{k}}\right) \delta q_{k}=0
$$

Since the infinitesimal variations $\delta q_{k}, 1 \leq k \leq N-1$, are arbitrary we deduce

$$
\prod_{j=k+2}^{N} \sigma_{j} \cdot\left(\frac{\partial \mathcal{Z}_{k+1}}{\partial q_{k}}+\frac{\partial \mathcal{Z}_{k+1}}{\partial z_{k}} \frac{\partial \mathcal{Z}_{k}}{\partial q_{k}}\right)=0
$$

Note that,

$$
\sigma_{j}=\frac{\partial \mathcal{Z}_{j}}{\partial z_{j-1}}\left(q_{j-1}, q_{j}, z_{j-1}\right)=1+D_{z} L_{d}\left(q_{j-1}, q_{j}, z_{j-1}\right)
$$

is non-vanishing by hypothesis and

$$
\frac{\partial \mathcal{Z}_{k+1}}{\partial q_{k}}+\frac{\partial \mathcal{Z}_{k+1}}{\partial z_{k}} \frac{\partial \mathcal{Z}_{k}}{\partial q_{k}}=D_{1} L_{d}\left(q_{k}, q_{k+1}, z_{k}\right)+\sigma_{k+1} D_{2} L_{d}\left(q_{k-1}, q_{k}, z_{k-1}\right)
$$

from where the result follows.
Remark 3.5. The discrete principle introduced in Vermeeren et al., 2019 is just the condition

$$
\frac{\partial \mathcal{Z}_{k+1}}{\partial q_{k}}+\frac{\partial \mathcal{Z}_{k+1}}{\partial z_{k}} \frac{\partial \mathcal{Z}_{k}}{\partial q_{k}}=0,
$$

afer rewriting it in our notation. For discrete Lagrangian functions where $1+D_{z} L_{d}$ is non-vanishing, the condition above is equivalent to the Herglotz discrete principle.

### 3.1 Discrete Lagrangian flows and discrete Legendre transforms

Given a discrete contact Lagrangian $L_{d}$, if $1+D_{z} L_{d}\left(q_{0}, q_{1}, z_{0}\right)$ does not vanish, we can define two maps called discrete Legendre transforms: $\mathbb{F}^{ \pm} L_{d}$ : $Q \times Q \times \mathbb{R} \rightarrow T^{*} Q \times \mathbb{R}$

$$
\begin{align*}
& \mathbb{F}^{+} L_{d}\left(q_{0}, q_{1}, z_{0}\right)=\left(q_{1}, D_{2} L_{d}\left(q_{0}, q_{1}, z_{0}\right), z_{0}+L_{d}\left(q_{0}, q_{1}, z_{0}\right)\right) \\
& \mathbb{F}^{-} L_{d}\left(q_{0}, q_{1}, z_{0}\right)=\left(q_{0},-\frac{D_{1} L_{d}\left(q_{0}, q_{1}, z_{0}\right)}{1+D_{z} L_{d}\left(q_{0}, q_{1}, z_{0}\right)}, z_{0}\right) \tag{40}
\end{align*}
$$

Lemma 3.6. $\mathbb{F}^{+} L_{d}$ is a local diffeomorphism if and only if $\mathbb{F}^{-} L_{d}$ is a local diffeomorphism.

Proof. It is a direct consequence of the implicit function theorem.

The Legendre transforms allow us to rewrite discrete Herglotz equations (39) as a momentum matching equations as in Marsden and West, 2001. Indeed, provided $1+D_{z} L_{d}\left(q_{0}, q_{1}, z_{0}\right)$ is not zero, we may write

$$
\begin{equation*}
\mathbb{F}^{+} L_{d}\left(q_{0}, q_{1}, z_{0}\right)=\mathbb{F}^{-} L_{d}\left(q_{1}, q_{2}, z_{1}\right) \tag{41}
\end{equation*}
$$

We have the following theorem
Theorem 3.7. Suppose $1+D_{z} L_{d}\left(q_{0}, q_{1}, z_{0}\right)$ does not vanish. Given a discrete Lagrangian $L_{d}: Q \times Q \times \mathbb{R} \rightarrow \mathbb{R}$, there is a well-defined discrete Lagrangian flow $\Phi_{d}: Q \times Q \times \mathbb{R} \rightarrow Q \times Q \times \mathbb{R}$ if and only if $\mathbb{F}^{-} L_{d}$ is a local diffeomorphism where

$$
\Phi_{d}=\left(\mathbb{F}^{-} L_{d}\right)^{-1} \circ \mathbb{F}^{+} L_{d}
$$

Proof. Given $\left(q_{0}, q_{1}, z_{0}\right) \in Q \times Q \times \mathbb{R}$, from the definition $\Phi_{d}=\left(\mathbb{F}^{-} L_{d}\right)^{-1} \circ$ $\mathbb{F}^{+} L_{d}$ it is easy to show that the point $\left(q_{2}, z_{1}\right) \in Q \times \mathbb{R}$ required to satisfy

$$
\Phi_{d}\left(q_{0}, q_{1}, z_{0}\right)=\left(q_{1}, q_{2}, z_{1}\right)
$$

is defined by Equations (39), and so $q_{2}$ and $z_{1}$ is uniquely defined as a function of $\left(q_{0}, q_{1}, z_{0}\right)$ if and only if $\mathbb{F}^{-} L_{d}$ is locally an isomorphism.

Inspired by the last theorem, we say that a discrete contact Lagrangian is regular if $1+D_{z} L_{d}\left(q_{0}, q_{1}, z_{0}\right)$ does not vanish and its negative discrete Legendre transform $\mathbb{F}^{-} L_{d}$ is a local diffeomorphism.

The discrete Legendre transforms also allow us to define an associated discrete Hamiltonian flow on $T^{*} Q \times \mathbb{R}$. Indeed, considering a regular discrete Lagrangian function $L_{d}$, let $\widetilde{\Phi_{d}}: T^{*} Q \times \mathbb{R} \rightarrow T^{*} Q \times \mathbb{R}$ be defined by

$$
\begin{equation*}
\widetilde{\Phi_{d}}=\mathbb{F}^{+} L_{d} \circ \Phi_{d} \circ\left(\mathbb{F}^{+} L_{d}\right)^{-1} \tag{42}
\end{equation*}
$$

It is not difficult to show that the discrete Hamiltonian flow admits the alternative expressions

$$
\begin{equation*}
\widetilde{\Phi_{d}}=\mathbb{F}^{-} L_{d} \circ \Phi_{d} \circ\left(\mathbb{F}^{-} L_{d}\right)^{-1} \quad \text { or } \quad \widetilde{\Phi_{d}}=\mathbb{F}^{+} L_{d} \circ\left(\mathbb{F}^{-} L_{d}\right)^{-1} \tag{43}
\end{equation*}
$$



We may define the one-forms

$$
\begin{equation*}
\eta^{+}=\left(\mathbb{F}^{+} L_{d}\right)^{*} \eta, \quad \eta^{-}=\left(\mathbb{F}^{-} L_{d}\right)^{*} \eta, \tag{45}
\end{equation*}
$$

where $\eta$ is the canonical contact form on $T^{*} Q \times \mathbb{R}$. These are contact forms on $Q \times Q \times \mathbb{R}$. If we chose natural coordinates $\left(q^{i}, p_{i}, z\right)$ on $T^{*} Q \times \mathbb{R}$ where $\eta=\mathrm{d} z-p_{i} \mathrm{~d} q^{i}$, the discrete 1 -forms may be locally written as the pullback

$$
\begin{align*}
& \eta^{+}=\mathrm{d} z_{0}+\mathrm{d} L_{d}\left(q_{0}, q_{1}, z_{0}\right)-D_{2} L_{d}\left(q_{0}, q_{1}, z_{0}\right) \mathrm{d} q_{1}, \\
& \eta^{-}=\mathrm{d} z_{0}+\frac{D_{1} L_{d}\left(q_{0}, q_{1}, z_{0}\right)}{1+D_{z} L_{d}\left(q_{0}, q_{1}, z_{0}\right)} \mathrm{d} q_{0} \tag{46}
\end{align*}
$$

by the corresponding discrete Legendre transform. The one-form $\eta^{+}$is further simplified to

$$
\begin{equation*}
\eta^{+}=\left(1+D_{z} L_{d}\left(q_{0}, q_{1}, z_{0}\right)\right) \mathrm{d} z_{0}+D_{1} L_{d}\left(q_{0}, q_{1}, z_{0}\right) \mathrm{d} q_{0} \tag{47}
\end{equation*}
$$

Given a discrete Lagrangian $L_{d}$, let $\sigma_{d}: Q \times Q \times \mathbb{R} \rightarrow \mathbb{R}$ be the smooth function given by

$$
\sigma_{d}\left(q_{0}, q_{1}, z_{0}\right)=1+D_{z} L_{d}\left(q_{0}, q_{1}, z_{0}\right)
$$

then we have that:
Lemma 3.8. The discrete contact forms $\eta^{ \pm}$satisfy
(i) $\eta^{+}=\sigma_{d} \cdot \eta^{-}$;
(ii) $\left(\Phi_{d}\right)^{*} \eta^{-}=\eta^{+}$.

Proof. For the first item, observe that (47) is equivalent to

$$
\eta^{+}=\left(1+D_{z} L_{d}\left(q_{0}, q_{1}, z_{0}\right)\right) \eta^{-}
$$

For the second one, note that

$$
\left(\Phi_{d}\right)^{*} \eta^{-}=\left(\Phi_{d}\right)^{*} \circ\left(\mathbb{F}^{-} L_{d}\right)^{*} \eta=\left(\mathbb{F}^{-} L_{d} \circ \Phi_{d}\right)^{*} \eta=\left(\mathbb{F}^{+} L_{d}\right)^{*} \eta
$$

by applying Theorem 3.7.
As a consequence of the last Lemma we have the following theorem:

Theorem 3.9. Let $L_{d}$ be a regular discrete Lagrangian function. The discrete flow $\Phi_{d}$ associated to $L_{d}$ is a conformal contactomorphism with respect to both contact structures $\eta^{ \pm}$. In particular, it satisfies

$$
\begin{equation*}
\left(\Phi_{d}\right)^{*} \eta^{+}=\left(\sigma_{d} \circ \Phi_{d}\right) \cdot \eta^{+}, \quad\left(\Phi_{d}\right)^{*} \eta^{-}=\sigma_{d} \cdot \eta^{-} \tag{48}
\end{equation*}
$$

Likewise, the discrete Hamiltonian flow $\widetilde{\Phi_{d}}$ is also a conformal contactomorphism satisfying

$$
\begin{equation*}
\left(\widetilde{\Phi_{d}}\right)^{*} \eta=\left(\sigma_{d} \circ\left(\mathbb{F}^{-} L_{d}\right)^{-1}\right) \cdot \eta \tag{49}
\end{equation*}
$$

Proof. These are trivial consequences of Lemma 3.8. Combining the two statements of the Lemma we get

$$
\left(\Phi_{d}\right)^{*} \eta^{-}=\sigma_{d} \cdot \eta^{-}
$$

Then, also

$$
\left(\Phi_{d}\right)^{*} \eta^{+}=\left(\Phi_{d}\right)^{*}\left(\sigma_{d} \cdot \eta^{-}\right)=\left(\sigma_{d} \circ \Phi_{d}\right) \cdot\left(\Phi_{d}\right)^{*} \eta^{-}=\left(\sigma_{d} \circ \Phi_{d}\right) \cdot \eta^{+}
$$

Observing that the discrete Hamiltonian flow satisfies $\widetilde{\Phi_{d}}=\mathbb{F}^{+} L_{d} \circ \Phi_{d} \circ$ $\left(\mathbb{F}^{+} L_{d}\right)^{-1}$ by definition, then

$$
\begin{aligned}
\left(\widetilde{\Phi_{d}}\right)^{*} \eta & =\left(\left(\mathbb{F}^{+} L_{d}\right)^{-1}\right)^{*} \circ\left(\Phi_{d}\right)^{*} \eta^{+}=\left(\left(\mathbb{F}^{+} L_{d}\right)^{-1}\right)^{*}\left(\left(\sigma_{d} \circ \Phi_{d}\right) \cdot \eta^{+}\right) \\
& =\left(\sigma_{d} \circ \Phi_{d} \circ\left(\mathbb{F}^{+} L_{d}\right)^{-1}\right) \cdot\left(\left(\mathbb{F}^{+} L_{d}\right)^{-1}\right)^{*} \eta^{+}
\end{aligned}
$$

where the last equality comes from the properties of the pullback. Since we have that

$$
\Phi_{d} \circ\left(\mathbb{F}^{+} L_{d}\right)^{-1}=\left(\mathbb{F}^{-} L_{d}\right)^{-1} \quad \text { and } \quad\left(\left(\mathbb{F}^{+} L_{d}\right)^{-1}\right)^{*} \eta^{+}=\eta
$$

the desired result follows.
Moreover, since the discrete Lagrangian function $L_{d}$ is regular, the function $\sigma_{d}$ does not vanish. Hence, the discrete flows $\Phi_{d}$ and $\Phi_{d}$ are conformal contact.

### 3.2 Discrete symmetries and dissipated quantities

Let $G$ be a Lie group acting on $Q$ through the map $\Phi: G \times Q \rightarrow Q$. We define the lifted action on $Q \times Q \times \mathbb{R}$ to be the diagonal action on $Q \times Q$ and the identity on $\mathbb{R}$, so that

$$
\widetilde{\Phi}: G \times Q \times Q \times \mathbb{R} \rightarrow Q \times Q \times \mathbb{R}, \quad \widetilde{\Phi}_{g}\left(q_{0}, q_{1}, z_{0}\right)=\left(\Phi_{g}\left(q_{0}\right), \Phi_{g}\left(q_{1}\right), z_{0}\right)
$$

Let us denote by $\xi_{Q} \in \mathfrak{X}(Q)$ the infinitesimal generator associated to a Lie algebra element $\xi \in \mathfrak{g}$ and by $\widetilde{\xi} \in \mathfrak{X}(Q \times Q \times \mathbb{R})$ the corresponding infinitesimal generator on $Q \times Q \times \mathbb{R}$.

Notice that, $\operatorname{since} \operatorname{pr}_{3}\left(\Phi_{g}\left(q_{0}, q_{1}, z_{0}\right)\right)=z_{0}$ is constant for all $g \in G$, where $\operatorname{pr}_{3}: Q \times Q \times \mathbb{R} \rightarrow \mathbb{R}$ is the projection onto the third factor, then we have that

$$
T_{\left(q_{0}, q_{1}, z_{0}\right)} \operatorname{pr}_{3}\left(\widetilde{\xi}\left(q_{0}, q_{1}, z_{0}\right)\right)=0 .
$$

In fact, the infinitesimal generator may be identified with

$$
\begin{equation*}
\widetilde{\xi}\left(q_{0}, q_{1}, z_{0}\right)=\left(\xi_{Q}\left(q_{0}\right), \xi_{Q}\left(q_{1}\right), 0_{z_{0}}\right) \in T_{q_{0}} Q \times T_{q_{1}} Q \times T_{z_{0}} \mathbb{R}, \tag{50}
\end{equation*}
$$

where $0: \mathbb{R} \rightarrow T \mathbb{R}$ is the zero section of $T \mathbb{R}$.
Lemma 3.10. If $L_{d}: Q \times Q \times \mathbb{R} \rightarrow \mathbb{R}$ is an invariant discrete Lagrangian function, i.e., $L_{d} \circ \widetilde{\Phi}_{g}=L_{d}$ for all $g \in G$, then it satisfies the equation

$$
\begin{equation*}
D_{1} L_{d}\left(q_{0}, q_{1}, z_{0}\right) \xi_{Q}\left(q_{0}\right)+D_{2} L_{d}\left(q_{0}, q_{1}, z_{0}\right) \xi_{Q}\left(q_{1}\right)=0 \tag{51}
\end{equation*}
$$

Proof. Since the discrete Lagrangian function is invariant for the lifted action, it satisfies

$$
\left\langle\mathrm{d} L_{d}\left(q_{0}, q_{1}, z_{0}\right), \widetilde{\xi}\left(q_{0}, q_{1}, z_{0}\right)\right\rangle=0, \forall\left(q_{0}, q_{1}, z_{0}\right) \in Q \times Q \times \mathbb{R} .
$$

Then using equation (50), one immediately gets the desired expression.
Now consider the discrete momentum map $J_{d}$ given by

$$
\begin{align*}
J_{d}: Q \times Q \times \mathbb{R} & \rightarrow \mathfrak{g}^{*}, \\
\left\langle J_{d}\left(q_{0}, q_{1}, z_{0}\right), \xi\right\rangle & =\left\langle\eta^{-}, \tilde{\xi}\left(q_{0}, q_{1}, z_{0}\right)\right\rangle . \tag{52}
\end{align*}
$$

Theorem 3.11. Let $L_{d}$ be an invariant discrete Lagrangian function for the lifted action $\widetilde{\Phi}$. Then $\widetilde{\Phi}$ acts by contactomorphisms on $Q \times Q \times \mathbb{R}$ and the function $\hat{J}_{d}(\xi): Q \times Q \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$
\hat{J}_{d}(\xi)\left(q_{0}, q_{1}, z_{0}\right)=\left\langle J_{d}\left(q_{0}, q_{1}, z_{0}\right), \xi\right\rangle
$$

is dissipated along the discrete flow of Herglotz equations in the sense that

$$
\hat{J}_{d}(\xi)\left(\Phi_{d}\left(q_{0}, q_{1}, z_{0}\right)\right)=\sigma_{d}\left(q_{0}, q_{1}, z_{0}\right) \hat{J}_{d}(\xi)\left(q_{0}, q_{1}, z_{0}\right)
$$

where $\sigma_{d}\left(q_{0}, q_{1}, z_{0}\right)=1+D_{z} L_{d}\left(q_{0}, q_{1}, z_{0}\right)$.

Proof. The fact that $\widetilde{\Phi}$ acts by contactomorphisms is immediately checked by computing the pullback of either the 1 -forms $\eta^{ \pm}$:

$$
\left(\widetilde{\Phi}_{g}\right)^{*} \eta^{ \pm}=\eta^{ \pm}
$$

as a direct consequence of the $G$-invariance of $L_{d}$ (see Subsection 1.3.3 in Marsden and West, 2001).

In order to simplify the notation, let $P_{0}=\left(q_{0}, q_{1}, z_{0}\right)$ and $P_{1}=\Phi_{d}\left(q_{0}, q_{1}, z_{0}\right)$. By definition we have that

$$
\hat{J}_{d}(\xi)\left(P_{1}\right)=\left\langle\eta^{-}\left(P_{1}\right), \widetilde{\xi}\left(P_{1}\right)\right\rangle
$$

Now applying the definition of $\eta^{-}$and equation (50) we get

$$
\hat{J}_{d}(\xi)\left(P_{1}\right)=\frac{1}{\sigma_{d}\left(P_{1}\right)}\left\langle D_{1} L_{d}\left(P_{1}\right), \xi_{Q}\left(q_{1}\right)\right\rangle .
$$

Using the discrete Herglotz equations, the right-hand side reduces to

$$
\hat{J}_{d}(\xi)\left(P_{1}\right)=-\left\langle D_{2} L_{d}\left(P_{0}\right), \xi_{Q}\left(q_{1}\right)\right\rangle
$$

From the infinitesimal symmetry formula in equation (51), we deduce

$$
\hat{J}_{d}(\xi)\left(P_{1}\right)=\left\langle D_{1} L_{d}\left(P_{0}\right), \xi_{Q}\left(q_{0}\right)\right\rangle
$$

Now inserting $\sigma_{d}\left(P_{0}\right)$ so that

$$
\hat{J}_{d}(\xi)\left(P_{1}\right)=\sigma_{d}\left(P_{0}\right)\left\langle\frac{D_{1} L_{d}\left(P_{0}\right)}{\sigma_{d}\left(P_{0}\right)}, \xi_{Q}\left(q_{0}\right)\right\rangle,
$$

we deduce

$$
\hat{J}_{d}(\xi)\left(P_{1}\right)=\sigma_{d}\left(P_{0}\right)\left\langle\eta^{-}\left(P_{0}\right)\left(\widetilde{\xi}\left(P_{0}\right)\right\rangle\right.
$$

and so we have proved that

$$
\hat{J}_{d}(\xi)\left(P_{1}\right)=\sigma_{d}\left(P_{0}\right) \hat{J}_{d}(\xi)\left(P_{0}\right)
$$

## 4 Exact discrete Lagrangian for contact systems

### 4.1 The contact exponential map

Given a contact regular Lagrangian $L: T Q \times \mathbb{R} \rightarrow \mathbb{R}$, consider the corresponding Lagrangian vector field $\xi_{L}$ and denote its flow by $\phi_{t}^{\xi_{L}}$.

Define the open subset $U_{h}$ of $T Q \times \mathbb{R}$ given by

$$
U_{h}=\left\{\left(q_{0}, \dot{q}_{0}, z_{0}\right) \in T Q \times \mathbb{R} \mid \phi_{t}^{\xi_{L}} \text { is defined for } t \in[0, h]\right\}
$$

and let the contact exponential map be defined by

$$
\begin{gather*}
\exp _{h}^{\xi_{L}}: U_{h} \subseteq T Q \times \mathbb{R} \rightarrow Q \times Q \times \mathbb{R}  \tag{53}\\
\left(q_{0}, \dot{q}_{0}, z_{0}\right) \mapsto\left(q_{0}, q_{1}, z_{0}\right),
\end{gather*}
$$

where $q_{1}=p_{Q} \circ \phi_{h}^{\xi_{L}}\left(q_{0}, \dot{q}_{0}, z_{0}\right)$ and $p_{Q}: T Q \times \mathbb{R} \rightarrow Q$ is the projection onto $Q$ given by $p_{Q}\left(v_{q}, z\right)=q$ for $v_{q} \in T_{q} Q$.

We will prove that the contact exponential map is a local diffeomorphism, using the fact that the non-holonomic exponential map, i.e., the exponential map of a non-holonomic system is a local embedding (see [Simoes A. and de Diego D., 2020, Marrero et al., 2016 ). In fact, to every regular contact system, one can associate a non-holonomic Lagrangian system on $T(Q \times \mathbb{R})$ with nonlinear constraints.

Consider the singular Lagrangian function

$$
\begin{equation*}
\widetilde{L}: T(Q \times \mathbb{R}) \rightarrow \mathbb{R}, \quad \widetilde{L}=L \circ \pi, \tag{54}
\end{equation*}
$$

where $\pi: T(Q \times \mathbb{R}) \rightarrow T Q \times \mathbb{R}$ is a projection onto $T Q \times \mathbb{R}$. Also, we take the non-linear constraints

$$
\begin{equation*}
M_{L}=\{(q, z, \dot{q}, \dot{z}) \in T(Q \times \mathbb{R}) \mid \dot{z}=L(q, \dot{q}, z)\} \tag{55}
\end{equation*}
$$

Observe that $M_{L}$ is the zero level set of the real-valued function $\Phi: T(Q \times$ $\mathbb{R}) \rightarrow \mathbb{R}$ given by $\Phi(q, z, \dot{q}, \dot{z})=\dot{z}-L(q, \dot{q}, z)$.

The pair ( $\widetilde{L}, M_{L}$ ) forms a Lagrangian non-holonomic system with nonlinear constraints determined by the submanifold $M_{L}$ and dynamics given by Chetaev's principle (see Bloch, 2015, de León and de Diego, 1996 and references therein). According to this principle the equations of motion are

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} t}\left(\frac{\partial \widetilde{L}}{\partial \dot{q}^{i}}\right)-\frac{\partial \widetilde{L}}{\partial q^{i}}=\lambda \frac{\partial \Phi}{\partial \dot{q}^{i}} \\
& \frac{\mathrm{~d}}{\mathrm{~d} t}\left(\frac{\partial \widetilde{L}}{\partial \dot{z}}\right)-\frac{\partial \widetilde{L}}{\partial z}=\lambda \frac{\partial \Phi}{\partial \dot{z}}  \tag{56}\\
& \Phi\left(q^{i}, z, \dot{q}^{i}, \dot{z}\right)=0
\end{align*}
$$

with Lagrange multiplier $\lambda$. As $\widetilde{L}$ does not depend on $\dot{z}$ it is straightforward to check that the Lagrange multiplier is just

$$
\lambda=-\frac{\partial L}{\partial z}
$$

and that equations (56) are equivalent to the Herglotz equations for $L$.
Moreover, since $L$ is regular, we can define a SODE vector field $\Gamma_{\left(\widetilde{L}, M_{L}\right)}$ $\in \mathfrak{X}\left(M_{L}\right)$ as the unique vector field on $M_{L}$ whose integral curves satisfy equations (56). Hence, we deduce

$$
\begin{equation*}
T \pi\left(\Gamma_{\left(\widetilde{L}, M_{L}\right)}\right)=\xi_{L} \circ \pi . \tag{57}
\end{equation*}
$$

Let us denote the flow of the vector field $\Gamma_{\left(\widetilde{L}, M_{L}\right)}$ by $\phi_{t}^{\Gamma_{\left(\widetilde{L}, M_{L}\right)}}: M_{L} \rightarrow M_{L}$. Consider now the submanifold of $\mathcal{M}_{L}$ given by

$$
M_{L, h}=\left\{\left(q_{0}, \dot{q}_{0}, z_{0}, \dot{z}_{0}\right) \in T(Q \times \mathbb{R}) \mid \phi_{t}^{\Gamma_{\left(\widetilde{L}, M_{L}\right)}} \text { is defined for } t \in[0, h]\right\} .
$$

We define the non-holonomic exponential map to be

$$
\begin{align*}
\exp _{h}^{\Gamma_{\left(\tilde{L}, M_{L}\right)}}: & M_{L, h} \subseteq M_{L} \longrightarrow(Q \times \mathbb{R}) \times(Q \times \mathbb{R})  \tag{58}\\
& \left(q_{0}, z_{0}, \dot{q}_{0}, \dot{z}_{0}\right) \mapsto\left(q_{0}, z_{0}, q_{1}, z_{1}\right),
\end{align*}
$$

where $\left(q_{1}, z_{1}\right)=\tau_{Q \times \mathbb{R}} \circ \phi_{h}^{\Gamma\left(\widetilde{L}, M_{L}\right)}\left(q_{0}, z_{0}, \dot{q}_{0}, \dot{z}_{0}\right)$, with $\tau_{Q \times \mathbb{R}}: T(Q \times \mathbb{R}) \rightarrow Q \times \mathbb{R}$ the tangent bundle projection.

In Simoes A. and de Diego D., 2020 the authors prove that there is an open subset $N_{h} \subseteq M_{L, h}$ such that the non-holonomic exponential map
 image, which we will denote by $M_{d}$.

Theorem 4.1. There exists an open set $V_{h} \subseteq U_{h}$ such that the contact exponential map $\left.\exp _{h}^{\xi_{L}}\right|_{V_{h}}$ is a diffeomorphism.
Proof. Let us consider the non-holonomic system ( $\widetilde{L}, M_{L}$ ) defined previously.
According to equation (57), the vector fields $\xi_{L}$ and $\Gamma_{\left(\widetilde{L}, M_{L}\right)}$ are $\pi$-related therefore, its flows satisfy

$$
\pi \circ \phi_{t}^{\Gamma_{\left(\widetilde{L}, M_{L}\right)}}=\phi_{t}^{\xi_{L}} \circ \pi .
$$

We remark that $\left.\pi\right|_{M_{L}}$ is a diffeomorphism, since $M_{L}$ is diffeomorphic to the graph of the Lagrangian function $L$. As such, we can also write

$$
\phi_{t}^{\Gamma_{\left(\widetilde{L}, M_{L}\right)}}=\left.\left(\left.\pi\right|_{M_{L}}\right)^{-1} \circ \phi_{t}^{\xi_{L}} \circ \pi\right|_{M_{L}} .
$$

Thus, we can write the non-holonomic exponential map in terms of the contact dynamics in the following way

$$
\exp _{h}^{\Gamma_{\left(\tilde{L}, M_{L}\right)}}\left(q_{0}, z_{0}, \dot{q}_{0}, \dot{z}_{0}\right)=\left(q_{0}, z_{0}, q_{1}, z_{1}\right),
$$

with $\left(q_{1}, z_{1}\right)=\left.\tau_{Q \times \mathbb{R}^{\circ}} \circ\left(\left.\pi\right|_{M_{L}}\right)^{-1} \circ \phi_{h}^{\xi_{L}} \circ \pi\right|_{M_{L}}\left(q_{0}, z_{0}, \dot{q}_{0}, \dot{z}_{0}\right)$ where $\dot{z}_{0}=L\left(q_{0}, \dot{q}_{0}, z_{0}\right)$.
Also note that $\tau_{Q \times \mathbb{R}} \circ\left(\left.\pi\right|_{M_{L}}\right)^{-1}=p_{Q \times \mathbb{R}}$, where

$$
p_{Q \times \mathbb{R}}: T Q \times \mathbb{R} \rightarrow Q \times \mathbb{R}, \quad p_{Q \times \mathbb{R}}\left(v_{q}, z\right)=(q, z) .
$$

In Diagram (59) we show the different projections we can define on the manifolds involved in this section.


With these projections we can also write the contact exponential map as

$$
\exp _{h}^{\xi_{L}}\left(q_{0}, \dot{q}_{0}, z_{0}\right)=\left(q_{0}, q_{1}, z_{0}\right)
$$

with $q_{1}=\operatorname{pr}_{1} \circ p_{Q \times \mathbb{R}} \circ \phi_{h}^{\xi_{L}}\left(q_{0}, \dot{q}_{0}, z_{0}\right)$. Hence, we can write it as

$$
\begin{equation*}
\exp _{h}^{\xi_{L}}=\widetilde{\operatorname{pr}}_{1} \circ \exp _{h}^{\Gamma_{\left(\widetilde{L}, M_{L}\right)}} \circ\left(\left.\pi\right|_{M_{L}}\right)^{-1}, \tag{60}
\end{equation*}
$$

with

$$
\begin{aligned}
\widetilde{\operatorname{pr}}_{1}:(Q \times \mathbb{R}) \times(Q \times \mathbb{R}) & \longrightarrow Q \times Q \times \mathbb{R} \\
\left(q_{0}, z_{0}, q_{1}, z_{1}\right) & \mapsto\left(q_{0}, \operatorname{pr}_{1}\left(q_{1}, z_{1}\right), z_{0}\right) .
\end{aligned}
$$

Therefore, if $\left.\widetilde{\mathrm{pr}}_{1}\right|_{M_{d}}$ is a local diffeomorphism then, by equation (60), the contact exponential map $\left.\exp _{h}^{\xi_{L}}\right|_{V_{h}}$ is a diffeomorphism if we choose

$$
V_{h}=\left.\pi\right|_{M_{L}}\left(N_{h}\right),
$$


We are going to prove in the next Lemma that $\left.\widetilde{\mathrm{pr}}_{1}\right|_{M_{d}}$ is a local diffeomorphism.

Lemma 4.2. Using the same notation as in the previous theorem, $\widetilde{p} r_{1} \mid M_{d}$ is a local diffeomorphism.

Proof. All we must prove is that $\left.\widetilde{\mathrm{pr}}_{1}\right|_{M_{d}}$ is a local submersion (immersion) since, by dimensional reasons, this forces $\left.\widetilde{\mathrm{pr}}_{1}\right|_{M_{d}}$ to be also a local immersion (submersion).

Let $x \in M_{d}$. Any vector in the kernel of $\left.T_{x} \widetilde{\mathrm{pr}}_{1}\right|_{M_{d}}$ must be the tangent vector of a curve of the form

$$
Z(s)=\left(q_{0}, z_{0}, q_{1}, w \cdot s\right) \in M_{d}, \quad w \in \mathbb{R} .
$$

Let $\gamma_{s}(t)=\phi_{t}^{\Gamma_{\left(\tilde{L}, M_{L}\right)}} \circ\left(\exp _{h}^{\Gamma_{\left(\tilde{L}, M_{L}\right)}}\right)^{-1}(Z(s))$. For each fixed value of $s$, this is an integral curve of $\Gamma_{\left(\widetilde{L}, M_{L}\right)}$ satisfying

$$
\tau_{Q \times \mathbb{R}} \circ \gamma_{s}(0)=\left(q_{0}, z_{0}\right), \quad \tau_{Q \times \mathbb{R}} \circ \gamma_{s}(h)=\left(q_{1}, w \cdot s\right) .
$$

Moreover, note that the projection of $\gamma_{s}(t)$ to $T Q \times \mathbb{R}$, i.e., the curve $\pi \circ \gamma_{s}(t)$ is an integral curve of $\xi_{L}$ with endpoints $q_{0}$ and $q_{1}$ for each fixed value of $s$ and so $\pi \circ \gamma_{0}(t)$ must satisfy Herglotz' principle. Note that the action over the curves $\pi \circ \gamma_{s}(t)$ is given by

$$
\mathcal{A}\left(p_{Q} \circ \pi \circ \gamma_{s}(t)\right)=p_{\mathbb{R}} \circ \pi \circ \gamma_{s}(h)=w \cdot s
$$

where $p_{\mathbb{R}}: T Q \times \mathbb{R} \rightarrow \mathbb{R}$ is the projection onto the second factor.
Therefore, $p_{Q} \circ \pi \circ \gamma_{0}(t)$ is a critical value of the action if and only if $w=0$. Therefore, $\left.T_{x} \widetilde{\mathrm{pr}}_{1}\right|_{M_{d}}$ is trivial and $\left.\widetilde{\mathrm{pr}}_{1}\right|_{M_{d}}$ must be a local diffeomorphism in a neighbourhood of each point.

Since the contact exponential map is a local diffeomorphism we can define a local inverse called the exact retraction and denote it by $R_{h}^{e-}: Q \times Q \times \mathbb{R} \rightarrow$ $T Q \times \mathbb{R}$. We will also use its translation by the flow

$$
R_{h}^{e+}: Q \times Q \times \mathbb{R} \rightarrow T Q \times \mathbb{R}, \quad R_{h}^{e+}:=\phi_{h}^{\xi_{L}} \circ R_{h}^{e-} .
$$

### 4.2 The exact discrete Lagrangian function

Let $L_{h}^{e}: Q \times Q \times \mathbb{R} \rightarrow \mathbb{R}$ and defined by

$$
\begin{equation*}
L_{h}^{e}\left(q_{0}, q_{1}, z_{0}\right)=\int_{0}^{h} L \circ \phi_{t}^{\xi_{L}} \circ R_{h}^{e-}\left(q_{0}, q_{1}, z_{0}\right) \mathrm{d} t \tag{61}
\end{equation*}
$$

is called the exact discrete Lagrangian function.
We will need the following classical result in the proof of the next theorem: the solution of the first order linear equation $\dot{y}=a(t)+\frac{\mathrm{d} b}{\mathrm{~d} t}(t) y$ is

$$
\begin{equation*}
y(t)=e^{b(t)}\left(\int_{0}^{t} a(s) e^{-b(s)} \mathrm{d} s+y(0)\right) . \tag{62}
\end{equation*}
$$

Theorem 4.3. The Legendre transforms of a regular Lagrangian $L: T Q \times$ $\mathbb{R} \rightarrow \mathbb{R}$ are related to the discrete Legendre transforms of the corresponding exact discrete Lagrangian $L_{h}^{e}: Q \times Q \times \mathbb{R} \rightarrow \mathbb{R}$ in the following way

$$
\begin{equation*}
\mathbb{F}^{+} L_{h}^{e}=\mathbb{F} L \circ R_{h}^{e+}, \quad \mathbb{F}^{-} L_{h}^{e}=\mathbb{F} L \circ R_{h}^{e-} \tag{63}
\end{equation*}
$$

Proof. We will prove in local computations that the derivatives of the exact discrete Lagrangian function satisfy

$$
\begin{align*}
D_{1} L_{h}^{e}\left(q_{0}, q_{1}, z_{0}\right) & =-\frac{\partial L}{\partial \dot{q}}\left(q_{0}, \dot{q}_{0}, z_{0}\right) e^{b(h)} \\
D_{2} L_{h}^{e}\left(q_{0}, q_{1}, z_{0}\right) & =\frac{\partial L}{\partial \dot{q}}\left(q_{1}, \dot{q}_{1}, z_{1}\right)  \tag{64}\\
D_{z} L_{h}^{e}\left(q_{0}, q_{1}, z_{0}\right) & =e^{b(h)}-1
\end{align*}
$$

where

$$
\begin{align*}
& \left(q_{0}, \dot{q}_{0}, z_{0}\right)=R_{h}^{e-}\left(q_{0}, q_{1}, z_{0}\right), \quad\left(q_{1}, \dot{q}_{1}, z_{1}\right)=\phi_{h}^{\xi_{L}} \circ R_{h}^{e-}\left(q_{0}, q_{1}, z_{0}\right) \\
& \text { and } \quad b(t)=\int_{0}^{t} \frac{\partial L}{\partial z}\left(\phi_{s}^{\xi_{L}} \circ R_{h}^{e-}\left(q_{0}, q_{1}, z_{0}\right)\right) \mathrm{d} s \tag{65}
\end{align*}
$$

To simplify the notation in the proof we will use the notation $\gamma_{0,1}(t)=$ $\left(q_{0,1}(t), \dot{q}_{0,1}(t), z_{0,1}(t)\right):=\phi_{t}^{\xi_{L}} \circ R_{h}^{e-}\left(q_{0}, q_{1}, z_{0}\right)$. Under this convention we will have

$$
L_{h}^{e}\left(q_{0}, q_{1}, z_{0}\right)=\int_{0}^{h} L\left(\gamma_{0,1}(t)\right) \mathrm{d} t
$$

Note first that any variation of the exact discrete Lagrangian will take the form

$$
\begin{align*}
& \delta L_{h}^{e}\left(q_{0}, q_{1}, z_{0}\right)=\left.\frac{\mathrm{d}}{\mathrm{~d} s}\right|_{s=0} L_{h}^{e}\left(q_{0}(s), q_{1}(s), z_{0}(s)\right) \\
& =\int_{0}^{h} \frac{\partial L}{\partial q}\left(\gamma_{0,1}(t)\right) \delta q_{0,1}+\frac{\partial L}{\partial \dot{q}}\left(\gamma_{0,1}(t)\right) \delta \dot{q}_{0,1}+\frac{\partial L}{\partial z}\left(\gamma_{0,1}(t)\right) \delta z_{0,1} \mathrm{~d} t \tag{66}
\end{align*}
$$

Since $\gamma_{0,1}(t)$ is a solution of Euler-Lagrange equations, it satisfies

$$
\dot{z}_{0,1}=L\left(q_{0,1}(t), \dot{q}_{0,1}(t), z_{0,1}(t)\right)
$$

Therefore, any variation of $z_{0,1}$ satisfies the variational equation

$$
\begin{equation*}
\delta \dot{z}_{0,1}=\frac{\partial L}{\partial q}\left(\gamma_{0,1}(t)\right) \delta q_{0,1}+\frac{\partial L}{\partial \dot{q}}\left(\gamma_{0,1}(t)\right) \delta \dot{q}_{0,1}+\frac{\partial L}{\partial z}\left(\gamma_{0,1}(t)\right) \delta z_{0,1} \tag{67}
\end{equation*}
$$

Hence, any variation of the exact discrete Lagrangian reduces to

$$
\begin{equation*}
\delta L_{h}^{e}\left(q_{0}, q_{1}, z_{0}\right)=\delta z_{0,1}(h)-\delta z_{0,1}(0) . \tag{68}
\end{equation*}
$$

Moreover, we can solve the function $\delta z_{0,1}$ explicitly, by solving the differential equation (67)

$$
\begin{equation*}
\delta z_{0,1}(h)=e^{b(h)}\left(\int_{0}^{h} a(s) e^{-b(s)} \mathrm{d} s+\delta z_{0,1}(0)\right) \tag{69}
\end{equation*}
$$

with

$$
\begin{aligned}
& b(t)=\int_{0}^{t} \frac{\partial L}{\partial z}\left(\gamma_{0,1}(s)\right) \mathrm{d} s \\
& a(t)=\frac{\partial L}{\partial q}\left(\gamma_{0,1}(t)\right) \delta q_{0,1}+\frac{\partial L}{\partial \dot{q}}\left(\gamma_{0,1}(t)\right) \delta \dot{q}_{0,1}
\end{aligned}
$$

Let us compute the integration in the expression of $\delta z_{0,1}$ :

$$
\begin{aligned}
\int_{0}^{h} a(s) e^{-b(s)} \mathrm{d} s & =\int_{0}^{h}\left(\frac{\partial L}{\partial q} \delta q_{0,1}+\frac{\partial L}{\partial \dot{q}} \delta \dot{q}_{0,1}\right) e^{-b(t)} \mathrm{d} t \\
& =\int_{0}^{h}\left(\frac{\partial L}{\partial q}-\frac{d}{d t} \frac{\partial L}{\partial \dot{q}}+\frac{\partial L}{\partial \dot{q}} \frac{\partial L}{\partial z}\right) \delta q_{0,1} e^{-b(t)} \mathrm{d} t \\
& +\frac{\partial L}{\partial \dot{q}}\left(\gamma_{0,1}(h)\right) e^{-b(h)} \delta q_{0,1}(h)-\frac{\partial L}{\partial \dot{q}}\left(\gamma_{0,1}(0)\right) \delta q_{0,1}(0),
\end{aligned}
$$

where we are using integration by parts. Note that the term between brackets is zero, since we are over solutions of Euler-Lagrange equations. Therefore,

$$
\begin{equation*}
\delta z_{0,1}(h)=\frac{\partial L}{\partial \dot{q}}\left(\gamma_{0,1}(h)\right) \delta q_{0,1}(h)-\frac{\partial L}{\partial \dot{q}}\left(\gamma_{0,1}(0)\right) e^{b(h)} \delta q_{0,1}(0)+e^{b(h)} \delta z_{0,1}(0) . \tag{70}
\end{equation*}
$$

Note that the differentials of the discrete Lagrangian $D_{1} L_{h}^{e}, D_{2} L_{h}^{e}$ and $D_{z} L_{h}^{e}$ are instances of particular variations. Therefore, we have that

$$
\begin{align*}
D_{1} L_{h}^{e}\left(q_{0}, q_{1}, z_{0}\right) & =\left(\frac{\partial L}{\partial \dot{q}}\left(\gamma_{0,1}(h)\right) \frac{\partial q_{0,1}(h)}{\partial q_{0}^{i}}-\frac{\partial L}{\partial \dot{q}}\left(\gamma_{0,1}(0)\right) e^{b(h)} \frac{\partial q_{0,1}(0)}{\partial q_{0}^{i}}\right. \\
& \left.+\left(e^{b(h)}-1\right) \frac{\partial z_{0,1}(0)}{\partial q_{0}^{i}}\right) \mathrm{d} q_{0}^{i}  \tag{71}\\
& =-\frac{\partial L}{\partial \dot{q}^{i}}\left(\gamma_{0,1}(0)\right) e^{b(h)} \mathrm{d} q_{0}^{i}
\end{align*}
$$

since $q_{0,1}(h) \equiv q_{1}$ and so its derivative with respect to $q_{0}$ vanishes, $q_{0,1}(0) \equiv$ $q_{0}$ and so its derivative with respect to $q_{0}$ is the identity and, finally, $z_{0,1}(0) \equiv$ $z_{0}$ does not depend upon $q_{0}$. Likewise, the next derivative follows from applying similar arguments. Indeed, we have that

$$
\begin{align*}
D_{2} L_{h}^{e}\left(q_{0}, q_{1}, z_{0}\right) & =\left(\frac{\partial L}{\partial \dot{q}}\left(\gamma_{0,1}(h)\right) \frac{\partial q_{0,1}(h)}{\partial q_{1}^{i}}-\frac{\partial L}{\partial \dot{q}}\left(\gamma_{0,1}(0)\right) e^{b(h)} \frac{\partial q_{0,1}(0)}{\partial q_{1}^{i}}\right. \\
& \left.+\left(e^{b(h)}-1\right) \frac{\partial z_{0,1}(0)}{\partial q_{1}^{i}}\right) \mathrm{d} q_{1}^{i}  \tag{72}\\
& =\frac{\partial L}{\partial \dot{q}^{i}}\left(\gamma_{0,1}(h)\right) \mathrm{d} q_{1}^{i} .
\end{align*}
$$

Analogously, we also deduce

$$
\begin{align*}
D_{z} L_{h}^{e}\left(q_{0}, q_{1}, z_{0}\right) & =\left(\frac{\partial L}{\partial \dot{q}}\left(\gamma_{0,1}(h)\right) \frac{\partial q_{0,1}(h)}{\partial z_{0}}-\frac{\partial L}{\partial \dot{q}}\left(\gamma_{0,1}(0)\right) e^{b(h)} \frac{\partial q_{0,1}(0)}{\partial z_{0}}\right. \\
& \left.+\left(e^{b(h)}-1\right) \frac{\partial z_{0,1}(0)}{\partial z_{0}}\right) \mathrm{d} z_{0}  \tag{73}\\
& =\left(e^{b(h)}-1\right) \mathrm{d} z_{0}
\end{align*}
$$

Now, the result follows by the definition of the discrete Legendre transforms.

The commutativity of the following diagram summarizes the statement of the previous theorem

Now, we are going to relate the continuous contact Lagrangian flow with its discrete counterpart, when we take as discrete Lagrangian the corresponding exact discrete Lagrangian.
Theorem 4.4. Take a regular Lagrangian $L: T Q \rightarrow \mathbb{R}$ and fix a time step $h>0$. Then we have that:

1. $L_{h}^{e}$ is a regular discrete Lagrangian function;
2. If $H$ is the Hamiltonian function corresponding to $L$ introduced at the end of Section 2.2 and $\phi_{t}^{X_{H}}$ is its contact Hamiltonian flow, we have that

$$
\begin{equation*}
\mathbb{F}^{+} L_{h}^{e}=\phi_{h}^{X_{H}} \circ \mathbb{F}^{-} L_{h}^{e} . \tag{75}
\end{equation*}
$$

3. If $(q, z):[0, N h] \rightarrow Q \times \mathbb{R}$ is a solution of the Herglotz equations, then it is related to the solution of the discrete Herglotz equations $\left\{\left(q_{0}, z_{0}\right),\left(q_{1}, z_{1}\right), \ldots,\left(q_{N}, z_{N}\right)\right\}$ for the corresponding exact discrete Lagrangian with $(q(0), q(h), z(0))$ as initial conditions in the following way:

$$
\begin{equation*}
q_{k}=q(k h), \quad z_{k}=z(k h) \quad \text { for } k=0, \ldots, N . \tag{76}
\end{equation*}
$$

Proof. Item 1. is a consequence of of the theorem before, since $\mathbb{F}^{-} L_{h}^{e}$ is a composition of two local diffeomorphisms it is itself a local diffeomorphism. Item 2. comes from unyielding the definitions:

$$
\mathbb{F}^{+} L_{h}^{e}=\mathbb{F} L \circ R_{h}^{e+}=\mathbb{F} L \circ \phi_{h}^{\Gamma_{L}} \circ R_{h}^{e-}=\phi_{h}^{X_{H}} \circ \mathbb{F} L \circ R_{h}^{e-}=\phi_{h}^{X_{H}} \circ \mathbb{F}^{-} L_{h}^{e} .
$$

For item 3. observe that, by discrete Herglotz equations, for every $k=$ $1, \ldots, N-1$ we have that

$$
\mathbb{F}^{+} L_{h}^{e}(q(k-1), q(k), z(k-1))=\mathbb{F}^{-} L_{h}^{e}(q(k), q(k+1), z(k))
$$

so that $\left\{\left(q_{0}, z_{0}\right),\left(q_{1}, z_{1}\right), \ldots,\left(q_{N}, z_{N}\right)\right\}$ is indeed the solution of these equations.

## 5 Numerical examples

Given a mechanical contact Lagrangian with a euclidean metric and a potential function $V: Q \rightarrow \mathbb{R}$ of the type

$$
L(q, \dot{q}, z)=\frac{1}{2} \dot{q}^{2}-V(q)+\gamma z, \quad(q, \dot{q}, z) \in T Q \times \mathbb{R}, \quad \gamma<0 .
$$

one usually approximates the exact discrete Lagrangian associated to $L$ by means of a quadrature rule. Note that the restriction of $\gamma$ to negative values is necessary to model dissipative dynamics, though we could define the integrator for any value of $\gamma \in \mathbb{R}$. If we use the middle point rule to approximate the positions, i.e., $q \approx \frac{q_{1}+q_{0}}{2}$, one may define the discrete Lagrangian $L_{d}: Q \times Q \times \mathbb{R} \rightarrow \mathbb{R}$ in the following way

$$
L_{d}\left(q_{0}, q_{1}, z_{0}\right)=\frac{1}{2 h}\left(q_{1}-q_{0}\right)^{2}-h V\left(\frac{q_{1}+q_{0}}{2}\right)+h \gamma z_{0} .
$$

We remark that the value of $h$ should be chosen small enough so that the function $\sigma_{d}$ does not vanish anywhere. In this case, the discrete Herglotz
equations are of the type

$$
\begin{aligned}
& \frac{q_{1}-q_{0}}{h}-\frac{h}{2} \frac{\partial V}{\partial q}\left(\frac{q_{1}+q_{0}}{2}\right)=\frac{1}{(1+h \gamma)}\left(\frac{q_{2}-q_{1}}{h}+\frac{h}{2} \frac{\partial V}{\partial q}\left(\frac{q_{2}+q_{1}}{2}\right)\right) \\
& z_{1}=L_{d}\left(q_{0}, q_{1}, z_{0}\right)=\frac{1}{2 h}\left(q_{1}-q_{0}\right)^{2}-h V\left(\frac{q_{1}+q_{0}}{2}\right)+(h \gamma+1) z_{0}
\end{aligned}
$$

Example 1. The free single particle contact Lagrangian is

$$
L(q, \dot{q}, z)=\frac{1}{2} \dot{q}^{2}+\gamma z, \quad(q, \dot{q}, z) \in T Q \times \mathbb{R} .
$$

A simple discretization of this Lagrangian would be

$$
\begin{equation*}
L_{d}\left(q_{0}, q_{1}, z_{0}\right)=\frac{1}{2 h}\left(q_{1}-q_{0}\right)^{2}+h \gamma z_{0} \tag{77}
\end{equation*}
$$

Then, choosing $h$ small enough so that the function $\sigma_{d}$ is non-vanishing, the discrete Herglotz equations for $L_{d}$ are locally given by

$$
\begin{aligned}
& \frac{q_{1}-q_{0}}{h}=\frac{q_{2}-q_{1}}{h(1+h \gamma)} \Rightarrow q_{2}=(h \gamma+2) q_{1}-(h \gamma+1) q_{0} \\
& z_{1}=\frac{1}{2 h}\left(q_{1}-q_{0}\right)^{2}+(h \gamma+1) z_{0}
\end{aligned}
$$

The discrete flow obtained by solving these equations is plotted in Fig. (1)
In this case, one can also compute the exact discrete Lagrangian and solve the exact dynamics.

$$
\begin{equation*}
L_{h}^{e}\left(q_{0}, q_{1}, z_{0}\right)=\frac{\gamma\left(q_{1}-q_{0}\right)^{2} e^{\gamma h}}{2 e^{\gamma h}-2}-z_{0}\left(e^{\gamma h}-1\right) . \tag{78}
\end{equation*}
$$

Example 2. The damped harmonic oscillator is described by the Lagrangian

$$
L(q, \dot{q}, z)=\frac{1}{2} \dot{q}^{2}-\frac{1}{2} q^{2}+\gamma z, \quad(q, \dot{q}, z) \in T Q \times \mathbb{R} .
$$

Using a middle point discretization, i.e., $q \approx \frac{q_{1}+q_{0}}{2}$, one may define the discrete Lagrangian

$$
\begin{equation*}
L_{d}\left(q_{0}, q_{1}, z_{0}\right)=\frac{1}{2 h}\left(q_{1}-q_{0}\right)^{2}-\frac{h}{8}\left(q_{1}+q_{0}\right)^{2}+h \gamma z_{0} . \tag{79}
\end{equation*}
$$

In this case, after choosing $h$ small enough, the discrete Herglotz equations hold

$$
\begin{aligned}
& \frac{q_{1}-q_{0}}{h}-\frac{h}{4}\left(q_{1}+q_{0}\right)=\frac{1}{(1+h \gamma)}\left(\frac{q_{2}-q_{1}}{h}+\frac{h}{4}\left(q_{2}+q_{1}\right)\right) \\
& z_{1}=\frac{1}{2 h}\left(q_{1}-q_{0}\right)^{2}-\frac{h}{8}\left(q_{1}+q_{0}\right)^{2}+(h \gamma+1) z_{0}
\end{aligned}
$$



Figure 1: Position $q$ and $z$ and logarithm of the discrete Hamiltonian $H \circ \mathbb{F}^{-} L_{d}$ for a free particle, computed by solving the discrete Herglotz equations for the discrete Lagrangian (77) (continuous line) and the exact dynamics (dashed line), for $\gamma=-0.05$ and the time-step $h=0.5$. The initial conditions are $q_{0}=1, q_{1}=2$ and $z_{0}=0$.
which can be solved explicitly for $q_{2}$

$$
q_{2}=-\frac{\left(h^{3} \gamma+4 h \gamma+h^{2}+4\right) q_{0}+\left(h^{3} \gamma-4 h \gamma+2 h^{2}-8\right) q_{1}}{h^{2}+4}
$$

The discrete flow obtained by solving these equations is plotted in Fig. 2.
In this case, the exact discrete Lagrangian and the exact discrete dynamics can be computed with the aid of a Computer Algebra system, but the analytic expressions are complicated, so we only include their graph in Fig. 2.

## 6 Conclusions and future work

In this paper, we went deeper in the geometry of discrete contact mechanics following, as a starting point, the results by Vermeeren et al., 2019.


Figure 2: Position $q$ and $z$ and logarithm of the discrete Hamiltonian $H \circ$ $\mathbb{F}^{-} L_{d}$ for a harmonic oscillator, computed by solving the discrete Herglotz equations on the discrete Lagrangian (79) (continuous line) and the exact dynamics (dashed line), for $\gamma=-0.05$ and the time-step $h=0.5$. The initial conditions are $q_{0}=1, q_{1}=2$ and $z_{0}=0$.

We have done a detailed study of the discrete Herglotz principle and its geometric properties, including the discrete Legendre transforms and the associated discrete Lagrangian and Hamiltonian flows. Moreover, we have analyzed the existence of dissipated quantities related with symmetries of the system and the construction of the exact discrete Lagrangian function given the correspondence between the discrete and continuous system.

In future work, we will study the variational error analysis allowing us to estimate the error order of the proposed methods just from the error of approximation of the exact discrete Lagrangian function, that is, how well the discrete Lagrangian function matches the exact discrete Lagrangian function Marsden and West, 2001, Patrick and Cuell, 2009. Moreover, we will introduce higher-order methods for contact Lagrangian systems extending the theory of Morse functions to Legendrian submanifolds (see [Libermann

and Marle, 1987, Barbero Liñán et al., 2019, Ferraro et al., 2017). For instance, this theory will give a complete geometric explanation of other possible discretizations of the phase space, as for instance, the one used by Vermeeren et al which is $Q \times Q \times \mathbb{R}^{2}$ instead of $Q \times Q \times \mathbb{R}$.

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## References

[Barbero Liñán et al., 2019] Barbero Liñán, M., Cendra, H., García Toraño, E., and Martín de Diego, D. (2019). Morse families and Dirac systems. J. Geom. Mech., 11(4):487-510.
[Blanes and Casas, 2016] Blanes, S. and Casas, F. (2016). A concise introduction to geometric numerical integration. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL.
[Bloch, 2015] Bloch, A. (2015). Nonholonomic Mechanics and Control. Springer, Interdisciplinary Applied Mathematics 24.
[Bravetti, 2017] Bravetti, A. (2017). Contact Hamiltonian Dynamics: The Concept and Its Use. Entropy, 19(12):535.
[Bravetti, 2018] Bravetti, A. (2018). Contact geometry and thermodynamics. Int. J. Geom. Methods Mod. Phys., 16(supp01):1940003.
[Bravetti et al., 2020] Bravetti, A., Seri, M., Vermeeren, M., and Zadra, F. (2020). Numerical integration in Celestial Mechanics: a case for contact geometry. Celestial Mech. Dynam. Astronom., 132(1):Paper No. 7.
[de León and de Diego, 1996] de León, M. and de Diego, D. M. (1996). On the geometry of non-holonomic lagrangian systems. Journal of Mathematical Physics, 37:3389-3414.
[de León and Lainz Valcázar, 2019a] de León, M. and Lainz Valcázar, M. (2019a). Contact hamiltonian systems. Journal of Mathematical Physics, 60(10):102902.
[de León and Lainz Valcázar, 2019b] de León, M. and Lainz Valcázar, M. (2019b). Singular lagrangians and precontact hamiltonian systems. International Journal of Geometric Methods in Modern Physics, 16(10):1950158.
[de León and Valcázar, 2020] de León, M. and Valcázar, M. L. (2020). Infinitesimal symmetries in contact hamiltonian systems. Journal of Geometry and Physics(forthcoming).
[Ferraro et al., 2017] Ferraro, S., de León, M., Marrero, J. C., Martín de Diego, D., and Vaquero, M. (2017). On the geometry of the HamiltonJacobi equation and generating functions. Arch. Ration. Mech. Anal., 226(1):243-302.
[Gaset et al., 2019] Gaset, J., Gràcia, X., Muñoz-Lecanda, M. C., Rivas, X., and Román-Roy, N. (2019). New contributions to the Hamiltonian and Lagrangian contact formalisms for dissipative mechanical systems and their symmetries. arXiv.
[Hairer et al., 2010] Hairer, E., Lubich, C., and Wanner, G. (2010). Geometric numerical integration, volume 31 of Springer Series in Computational Mathematics. Springer, Heidelberg. Structure-preserving algorithms for ordinary differential equations, Reprint of the second (2006) edition.
[Herglotz, 1930] Herglotz, G. (1930). Beruhrungstransformationen. In Lectures at the University of Gottingen, Gottingen.
[Libermann and Marle, 1987] Libermann, P. and Marle, C.-M. (1987). Symplectic geometry and analytical mechanics, volume 35 of Mathematics and its Applications. D. Reidel Publishing Co., Dordrecht. Translated from the French by Bertram Eugene Schwarzbach.
[Marrero et al., 2016] Marrero, J., de Diego, D. M., and Martínez, E. (2016). On the exact discrete lagrangian function for variational integrators: theory and applications. arxiv:1608.01586v1 [math.dg].
[Marsden and West, 2001] Marsden, J. E. and West, M. (2001). Discrete mechanics and variational integrators. Acta Numer., 10:357-514.
[Patrick and Cuell, 2009] Patrick, G. W. and Cuell, C. (2009). Error analysis of variational integrators of unconstrained Lagrangian systems. Numer. Math., 113(2):243-264.
[Sanz-Serna and Calvo, 1994] Sanz-Serna, J. M. and Calvo, M. P. (1994). Numerical Hamiltonian problems, volume 7 of Applied Mathematics and Mathematical Computation. Chapman \& Hall, London.
[Simoes A. and de Diego D., 2020] Simoes A., M. J. and de Diego D., M. (2020). Exact discrete lagrangian mechanics for nonholonomic mechanics.
[Vermeeren et al., 2019] Vermeeren, M., Bravetti, A., and Seri, M. (2019). Contact variational integrators. J. Phys. A, 52(44):445206, 28.


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