# Near-pulse solutions of the FitzHugh-Nagumo equations on cylindrical surfaces 

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#### Abstract

We introduce a geometrical extension of the FitzHugh-Nagumo equations describing propagation of electrical impulses in nerve axons. In this extension, the axon is modelled as a warped cylinder, rather than a straight line, as is usually done, while pulses propagate on its surface, as is the case with real axons.

We prove the stability of electrical impulses for a standard cylinder and existence and stability of pulse-like solutions for warped cylinders whose radii are small and vary slowly along their lengths.


## 1 Introduction

The FitzHugh-Nagumo system ([11, 29]), modelling the propagation of electric impulses in nerve axons, is a simplified version of the Hodgkin-Huxley system [16] and is given as

$$
\begin{align*}
\partial_{t} u_{1} & =\partial_{x}^{2} u_{1}+f\left(u_{1}\right)-u_{2}, \\
\partial_{t} u_{2} & =\varepsilon\left(u_{1}-\gamma u_{2}\right), \tag{1.1}
\end{align*}
$$

where $u_{1}$ and $u_{2}$ are real functions of $x \in \mathbb{R}$ and $t \geq 0$, the parameters $\varepsilon$ and $\gamma$ are chosen to be positive and small and $f$ (the reaction term) is given by the cubic polynomial

$$
f\left(u_{1}\right):=-u_{1}\left(u_{1}-\alpha\right)\left(u_{1}-1\right),
$$

for $0<\alpha<\frac{1}{2}$. Here, an axon is modelled by a straight line without an internal geometric structure.

In our work, we make a first step in taking into account the geometry of the axon, namely, a cylindrical cable-like fiber, with electrical signals propagating on
its surface. Thus we consider an extension of the FitzHugh-Nagumo (FHN) system on a cylindrical surface, $\mathcal{S}$. The system has the form

$$
\begin{align*}
\partial_{t} u_{1} & =\Delta_{\mathcal{S}} u_{1}+f\left(u_{1}\right)-u_{2},  \tag{1.2}\\
\partial_{t} u_{2} & =\varepsilon\left(u_{1}-\gamma u_{2}\right),
\end{align*}
$$

where $\Delta_{\mathcal{S}}$ denotes the Laplace-Beltrami operator on $\mathcal{S}$ and $\varepsilon, \gamma$ and $f$ are the same as above. As in the original FitzHugh-Nagumo system, $u_{1}$ is the electrical potential across the axon membrane (fast variable), and $u_{2}$ is the $\mathrm{Na}^{+}$channel activation and inactivation parameters lumped into a single unknown (slow, recovery variable).

We call Eq. (1.2) the cylindrical FitzHugh-Nagumo system, or the FHNcyl system for short. Taking formally $\mathcal{S}=\mathbb{R}$ in Eq. (1.2) gives Eq. (1.1).

A solution to Eq. (1.1) which is a function of a single variable, $z=x-c t, c>0$, and vanishes at infinity is called a pulse. One of the first results on the existence of pulses is due to Hastings [13], who showed that when $\mathcal{S}$ is the real line, $0<\alpha<\frac{1}{2}$ and $\varepsilon, \gamma$ are positive and sufficiently small, Eq. (1.1) has a pulse solution, whose speed depends on $\alpha, \gamma$, and $\varepsilon$. The pulse is obtained as a homoclinic orbit in a related system of ordinary differential equations.

It turns out that when $\varepsilon>0$ is sufficiently small, Eq. (1.1) has at least two different pulse solutions, the fast pulse studied by Hastings, which travels with speed $c_{f}(\varepsilon)=\frac{\sqrt{2}}{2}(1-2 \alpha)+o(\varepsilon)$, and a slow pulse that travels with speed $c_{s}(\varepsilon)=O(\sqrt{\varepsilon})$ (Carpenter [3], Hastings [13, 14], Langer [26], Krupa, Sandstede and Szmolyan [25], Jones, Kopell and Langer [22], Arioli and Koch [1]). Langer [26] also proved uniqueness of the fast pulse. Jones [21], and independently, Yanagida [34], proved that the fast pulse is stable. In addition, fast pulses with oscillatory tails exist and are stable (Carter and Sandstede [5], Carter, de Rijk and Sandstede [4]). On the other hand, the slow pulse is always unstable (Flores [12], Evans [10], Ikeda, Mimura and Tsujikawa [19]).

Existence and stability for fast pulses have been studied for variants of Eq. (1.1), where the second equation also has a diffusion term (Cornwell and Jones [8], Chen and Choi [6], Chen and Hu [7]). Another system that admits stable fast pulses is the discrete analogue of Eq. (1.1) (Hupkes and Sandstede [18], Schouten-Straatman and Hupkes [32], Hupkes, Morelli, Schouten-Straatman and Van Vleck [17]).

There are a few results in higher dimensions. In $\mathbb{R}^{2}$, Mikhailov and Krinskii [28] and Keener [24] studied spiral solutions of Eq. (1.1). In $N$-dimensions, Tsujikawa, Nagai, Mimura, Kobayashi and Ikeda [33] proved that there exist fast pulse solutions propagating in a one-dimensional direction. Such solutions are stable.

In this paper we study solutions of the FHNcyl system, Eq. (1.2), on infinitely long, thin cylindrical surfaces. For $\mathcal{S}$ a standard cylinder, the pulse solutions to Eq. (1.1) are also (angle-independent) solutions to Eq. (1.2) and we continue to call them the pulses. We show that
(i) on a cylinder of small constant radius, the (fast) pulses are stable under general perturbations of the initial condition that depend on both spatial variables;
(ii) on a warped cylinder whose radius is small and varies slowly along its length, solutions that are initially close to a pulse stay near the family of pulses for all time.

Our extension, Eq. (1.2), of Eq. (1.1) is geometrical rather than biophysical. However, the techniques we use are fairly robust and could be easily modified for more realistic second order elliptic operators describing the surface evolution instead of the Laplacian $\Delta_{\mathcal{S}}$.

### 1.1 Main results

Consider Eq. (1.2) on the standard cylinder of constant radius $R$ centered about the $x$-axis in $\mathbb{R}^{3}$,

$$
\mathcal{S}_{R}:=\left\{(x, R \cos \theta, R \sin \theta) \in \mathbb{R}^{3} \mid x \in \mathbb{R}, \theta \in S^{1}\right\}
$$

where $S^{1}=\mathbb{R} /(2 \pi)$ is the unit circle. The Laplacian on this surface is defined by

$$
\Delta_{\mathcal{S}_{R}}=\partial_{x}^{2}+R^{-2} \partial_{\theta}^{2},
$$

and the Riemannian area element is $R d \theta d x$. Clearly, the cylindrical FitzHughNagumo system on $\mathcal{S}=\mathcal{S}_{R}$ is invariant under translations. If $u(x, \theta, t)$ is a solution, then so are its translates

$$
u_{h}(x, \theta, t):=u(x-h, \theta, t), \quad h \in \mathbb{R} .
$$

Each pulse $\Phi$ on $\mathcal{S}=\mathbb{R}$ defines a smooth axisymmetric traveling wave solution $u(x, \theta, t)=\Phi(x-c t)$ of Eq. (1.2) on $\mathcal{S}_{R}$. Its speed $c$ is determined by the parameters $\alpha, \gamma$, and $\varepsilon$. It is a consequence of translation invariance that all translates $\Phi_{h}$ of $\Phi$ are pulses of the same speed $c$.

Our first result concerns the stability of a particular fast pulse, $\Phi$, under perturbations of the initial values that need not be axisymmetric. We consider mild solutions, defined by an integral equation derived from Duhamel's formula, in the mixed Sobolev space

$$
H^{2,1}:=\left\{u=\left(u_{1}, u_{2}\right) \in L^{2}\left(\mathcal{S}_{R}\right) \times L^{2}\left(\mathcal{S}_{R}\right) \mid \Delta_{\mathcal{S}} u_{1} \in L^{2}\left(\mathcal{S}_{R}\right), \partial_{x} u_{2} \in L^{2}\left(\mathcal{S}_{R}\right)\right\} .
$$

The norm on this space is denoted by $\|\cdot\|_{2,1}$. With this notion of solution, the initial-value problem is locally well-posed, i.e., for each initial value $u_{0} \in H^{2,1}$, a
unique mild solution exists on some positive time interval, and this solution depends continuously on the initial data, see Proposition 1.3. The following theorem says that mild solutions which are initially close to $\Phi$ approach nearby translates of $\Phi(x-c t)$ as $t \rightarrow \infty$. In technical terms, the traveling pulse $\Phi(x-c t)$ is orbitally asymptotically stable.

Theorem 1.1 (Stability of pulses, standard cylinder). Consider Eq. (1.2) on the cylinder $\mathcal{S}_{R}$ of constant radius $R \leq 1$. Fix $\alpha \in\left(0, \frac{1}{2}\right), \varepsilon>0$, and $\gamma>0$ such that the equation has a fast pulse solution $\Phi(x-c t)$. If $\varepsilon$ is sufficiently small, then there is a neighborhood $\mathcal{U}$ of $\Phi$ in $H^{2,1}$ such that for every $u_{0} \in \mathcal{U}$, the mild solution $u(t)$ with initial value $\left.u\right|_{t=0}=u_{0}$ exists globally in time and satisfies

$$
\begin{equation*}
\left\|u(t)-\Phi_{c t+h_{*}}\right\|_{2,1} \leq C_{1} e^{-\xi t}\left\|u_{0}-\Phi\right\|_{2,1} \quad(t \geq 0) \tag{1.3}
\end{equation*}
$$

for some $\xi>0$ and $h_{*} \in \mathbb{R}$ (determined by $u_{0}$ ) with

$$
\left|h_{*}\right| \leq C_{2}\left\|u_{0}-\Phi\right\|_{2,1}
$$

Here, $C_{1}$ and $C_{2}$ are positive constants.
Theorem 1.1 is proved in Section 3. Though, our proof of Theorem 1.1 could be slightly shortened by appealing to the general Theorem 4.3.5 of [23], for the reader's convenience, we provide a self-contained proof which uses only a spectral result of [21] and [34], in addition to well-known results about semigroups. Under the assumptions of the theorem, Eq. (1.3) holds for any decay rate $\xi$ with $\xi<$ $\min \{\alpha, \beta, \varepsilon \gamma\}$, where $\beta$ is the exponent from Lemma 2.9. The neighborhood $\mathcal{U}$, as well as the constants $C_{1}, C_{2}$ and $\xi$ depend only on the parameters $\alpha, \gamma$, and $\varepsilon$.

The translates of $\Phi$ form a one-dimensional manifold of pulses

$$
\begin{equation*}
\mathcal{M}:=\left\{\Phi_{h} \mid h \in \mathbb{R}\right\} \tag{1.4}
\end{equation*}
$$

Denote by $\operatorname{dist}(v, \mathcal{M}):=\inf _{h}\left\|v-\Phi_{h}\right\|_{2,1}$ the distance of $v \in H^{2,1}$ from the manifold. By translation invariance, the conclusion of Theorem 1.1 yields a tubular neighborhood $\mathcal{W}=\left\{w \in H^{2,1} \mid \operatorname{dist}(w, \mathcal{M})<\eta\right\}$ such that

$$
\operatorname{dist}(u(t), \mathcal{M}) \leq C_{1} e^{-\xi t} \operatorname{dist}\left(u_{0}, \mathcal{M}\right)
$$

for all mild solutions with initial values in $\mathcal{M}$. As $t \rightarrow \infty$, each solution converges to a particular traveling pulse $\Phi\left(x-c t-h_{*}\right)$.

For our second result, we consider warped cylindrical surfaces, defined as graphs over the standard one, with a variable radius $\rho(x)$,

$$
\begin{equation*}
\mathcal{S}_{\rho}:=\left\{(x, \rho(x) \cos \theta, \rho(x) \sin \theta) \in \mathbb{R}^{3} \mid x \in \mathbb{R}, \theta \in S^{1}\right\} \tag{1.5}
\end{equation*}
$$

On $\mathcal{S}_{\rho}$, the Laplace-Beltrami operator is given by

$$
\begin{equation*}
\Delta_{\mathcal{S}_{\rho}}:=\frac{1}{\sqrt{g(x)}} \partial_{x}\left(\frac{\rho(x)}{\sqrt{1+\rho^{\prime}(x)^{2}}} \partial_{x}\right)+\rho^{-2}(x) \partial_{\theta}^{2} \tag{1.6}
\end{equation*}
$$

where $g(x):=\rho(x)^{2}\left(1+\rho^{\prime}(x)^{2}\right)$ is the squared Riemannian density.
We identify functions $u$ on $\mathcal{S}_{\rho}$ with functions on $\mathcal{S}_{R}$ via the coordinate map

$$
\begin{equation*}
\psi_{\rho}(x, R \cos \theta, R \sin \theta)=(x, \rho(x) \cos \theta, \rho(x) \sin \theta) \tag{1.7}
\end{equation*}
$$

from $\mathcal{S}_{R}$ to $\mathcal{S}_{\rho}$. Under the assumption that $\rho$ is twice continuously differentiable, positive, bounded, and bounded away from zero, $\psi_{\rho}$ is a diffeomorphism of class $C^{2}$. Via this identification, the norms $\|\cdot\|$ and $\|\cdot\|_{2,1}$ on the standard cylinder are pushed forward to apply to functions on $\mathcal{S}_{\rho}$.

When $\rho$ is non-constant, Eq. (1.2) cannot be expected to have pulse solutions. However, if $\rho$ is almost constant, then there are near-pulse solutions that remain in a neighborhood of $\mathcal{M}$ for all time. Generally, these solutions do not stay close to any particular pulse, but move slowly along the manifold:

Theorem 1.2 (Near-pulse solutions, warped cylinder). Consider the FHN system on a cylinder $\mathcal{S}_{\rho}$ of variable radius, as in Eq. (1.5), and let $\alpha, \varepsilon$ and $\gamma$ be as in Theorem 1.1. There are a constant $\delta_{*}>0$ and a tubular neighborhood $\mathcal{W}$ of $\mathcal{M}$ in $H^{2,1}$, with the following properties: If $R \leq 1$ and $\delta:=R^{-1}\|\rho-R\|_{C^{2}} \leq \delta_{*}$, then for every $u_{0} \in \mathcal{W}$, the unique mild solution $u(t)$ with initial value $\left.u\right|_{t=0}=u_{0}$ exists globally in time, and satisfies

$$
\begin{equation*}
\operatorname{dist}(u(t), \mathcal{M}) \leq C_{1} e^{-\xi t} \operatorname{dist}\left(u_{0}, \mathcal{M}\right)+C_{2} \delta, \quad(t \geq 0) \tag{1.8}
\end{equation*}
$$

for some positive constants $C_{1}, C_{2}$, and $\xi$.
Under the assumptions of the theorem, initial values $u_{0} \in \mathcal{M}$ give rise to nearpulse solutions that satisfy $\sup _{t>0} \operatorname{dist}(u(t), \mathcal{M}) \leq C_{2} \delta$. The Theorem 1.2 will be proved in Section 4. The tubular neighborhood $\mathcal{W}$, as well as the constants $\delta_{*}, C_{1}$, $C_{2}$, and $\xi$ depend only on the parameters $\alpha, \gamma$, and $\varepsilon$. In general the value of the exponential decay rate $\xi$ in Eq. (1.8) lies below that of Eq. (1.3).

### 1.2 Outline of the arguments

The key idea is to write a solution near a pulse $\Phi$ as the superposition of a modulated pulse with a transversal fluctuation. On the standard cylinder $\mathcal{S}_{R}$, Theorem 1.1 concludes that (for suitable values of the parameters) the fluctuation decreases exponentially over time, as the solution settles on a nearby translate of $\Phi$. In the proof, we first establish linearized stability (Section 2), and then apply a fixed point argument for the nonlinear evolution (Section 3).

On the warped cylinder $\mathcal{S}_{\rho}$, Theorem 1.2 provides bounds on the fluctuation of near-pulse solutions in terms of the distance of the variable radius $\rho$ from the constant $R$. In the proof, we use the standard cylinder for reference, and combine a perturbation result for the linearization with the exponential orbital stability proved in Theorem 1.1.

We briefly describe some key technical aspects of the proofs. Given a fast pulse $\Phi$, let $c$ be its speed, and consider the FHNcyl system on the standard cylinder of radius $R$. We employ a moving frame ( $z=x-c t$ ), where the pulse is stationary. In this frame the FHNcyl system becomes

$$
\begin{align*}
& \partial_{t} u_{1}=\Delta_{\mathcal{S}_{R}} u_{1}+c \partial_{z} u_{1}+f\left(u_{1}\right)-u_{2}  \tag{1.9}\\
& \partial_{t} u_{2}=c \partial_{z} u_{2}+\varepsilon\left(u_{1}-\gamma u_{2}\right) .
\end{align*}
$$

Denote by $L$ the linearization of Eq. (1.9) about the stationary solution $\Phi$. In Section 2, we prove that the semigroup generated by $L$ decays exponentially in directions transversal to the tangent space of $\mathcal{M}$ at $\Phi$, that is,

$$
\left\|e^{t L}(1-P)\right\|_{2,1} \lesssim e^{-\sigma t} \quad(t \geq 0)
$$

for some positive constant $\sigma$. Here, $P$ is a projection onto the tangent space that commutes with $L$, and $\|\cdot\|_{2,1}$ denotes the operator norm on $H^{2,1}$. The proof of this estimate is the most challenging part of the paper. Since $L$ is not self-adjoint, $P$ is not an orthogonal projection. Noting that the tangent space of $\mathcal{M}$ at $\Phi$ is spanned by the derivative $\tau:=-\partial_{z} \Phi$, and that $L \tau=0$, we construct $P$ as the spectral projection associated with the zero eigenvalue of $L$. The decay estimate follows from the fact that the remainder of the spectrum lies in the left half-plane.

In Section 3, we establish the nonlinear stability of the pulse (cf. [23]) and prove Theorem 1.1. We work again in the moving frame, and show that every mild solution of Eq. (1.9) that starts out sufficiently close to $\Phi$ converges exponentially to a translated pulse $\Phi_{h_{*}}$. As explained above, we decompose such a solution into a modulated pulse (moving on $\mathcal{M}$ ) and a fluctuation (transversal to $\mathcal{M}$ ). Concretely, we write $u=\Phi_{h}+v$, choosing $h$ in such a way that

$$
\Phi_{h}-\Phi \approx P(u-\Phi), \quad v \approx(1-P)(u-\Phi)
$$

with errors of order $\|u-\Phi\|_{2,1}^{2}$ in a neighborhood of $\Phi$. This transforms Eq. (1.9) into an equation for $v(t)$ in the canonical form

$$
\partial_{t} v=L v+N(v, h),
$$

where $N(v, h)$ is of order $\left(|h|+\|v\|_{2,1}\right)\|v\|_{2,1}$, coupled to an ordinary differential equation for the evolution of $h$. With the help of the linearized decay estimate from Section 2, we prove the bounds

$$
\|v(t)\|_{2,1} \lesssim e^{-\xi t}\left\|v_{0}\right\|, \quad|h(t)-h(0)| \lesssim e^{-\xi t}\left\|v_{0}\right\|^{2} \quad(t \geq 0)
$$

for any $\xi$ with $\xi<\sigma$, i.e., the fluctuation decays exponentially, while the modulation converges. Since $\left\|v_{0}\right\|_{2,1} \lesssim\left\|u_{0}-\Phi\right\|_{2,1}$ and $h_{0} \lesssim\left\|u_{0}-\Phi\right\|_{2,1}$, this yields the conclusion of Theorem 1.1.

Section 4 contains the proof of Theorem 1.2. We consider the variable radius $\rho(x)$ of a warped cylinder as a perturbation of $R$, and then appeal to Theorem 1.1. The basis for the argument is an estimate for the linearized evolution on $\mathcal{S}_{\rho}$. Note that in the moving frame which we used on the standard cylinder, the variation of the radius amounts to a time-dependent perturbation of the principal part, as $\Delta_{\mathcal{S}_{R}}$ becomes $\Delta_{\mathcal{S}_{\rho(z+c t}}$. To avoid this issue, we study the perturbation in the static frame, and linearize the FHNcyl system about the zero solution instead of the traveling pulse. It turns out that the linearized operator, $A_{\rho}$, is sectorial. Therefore, we can represent the semigroup $e^{t A_{\rho}}$ by an absolutely convergent contour integral, and control the perturbation via resolvent estimates. We use Grönwall's inequality to extend these perturbation estimates to the nonlinear evolution generated by the FHNcyl system on $\mathcal{S}_{\rho}$. In combination with the exponential decay of fluctuations for near-pulse solutions on $\mathcal{S}_{R}$ that was proved in Theorem 1.1, this yields Theorem 1.2.

### 1.3 Preliminaries and notation

We make the standing assumption that $\rho$ is of class $C^{2}$, positive, bounded, and bounded away from zero. Also, we assume that $\alpha \in\left(0, \frac{1}{2}\right)$, and that $\gamma>0, \varepsilon>0$ are small enough that the FHNcyl system (1.2) admits a fast pulse solution, $\Phi$. Furthermore, we assume that $\varepsilon$ is small enough so that the spectral results of [21] and [34] apply.

On a cylindrical surface $\mathcal{S}_{\rho}$, the FHNcyl system takes the form

$$
\begin{align*}
& \partial_{t} u_{1}=\Delta_{\mathcal{S}_{\rho}} u_{1}+f\left(u_{1}\right)-u_{2} \\
& \partial_{t} u_{2}=\varepsilon\left(u_{1}+\gamma u_{2}\right), \tag{1.10}
\end{align*}
$$

where the Laplace-Beltrami operator $\Delta_{\mathcal{S}_{\rho}}$ is given by Eq. (1.6). Denote the right hand side of Eq. (1.10) by $F_{\rho}(u)$.

We consider the initial-value problem

$$
\begin{equation*}
\partial_{t} u=F_{\rho}(u),\left.\quad u\right|_{t=0}=u_{0} \tag{1.11}
\end{equation*}
$$

on the space $H^{2,1}$. The principal part of $F_{\rho}$ is given by its Gâteaux derivative

$$
A_{\rho}:=d F_{\rho}(0)=\left(\begin{array}{cc}
\Delta_{\mathcal{S}_{\rho}}-\alpha & -1  \tag{1.12}\\
\varepsilon & -\varepsilon \gamma
\end{array}\right) .
$$

Since $A_{\rho}$ is a bounded perturbation of the diagonal operator that acts as $\Delta_{\mathcal{S}_{\rho}}$ on the first component, and vanishes on the second, it generates an analytic semigroup
$e^{t A_{\rho}}$ in $L^{2}$. We will show in Lemma 4.2 that the semigroup restricts to a uniformly bounded analytic semigroup on the dense invariant subspace $H^{2,1}$.

We start out by verifying that the initial-value problem for Eq. (1.10) is wellposed locally in time. Since $F(0)=0$, we can expand $F(u)=\partial_{t} u=A_{\rho} u+N(u)$, where $A_{\rho}=d F_{\rho}(0)$ was defined in Eq. (1.12), and

$$
\begin{equation*}
N(u):=F_{\rho}(u)-A_{\rho} u=\binom{-u_{1}^{3}+(\alpha+1) u_{1}^{2}}{0} . \tag{1.13}
\end{equation*}
$$

If $u(t)$ is a classical solution of Eq. (1.2), then by Duhamel's formula it also solves the integral equation

$$
\begin{equation*}
u(t)=e^{t A_{\rho}} u_{0}+\int_{0}^{t} e^{(t-s) A_{\rho}} N(u(s)) d s=: \mathcal{F}_{\rho}(u)(t) \tag{1.14}
\end{equation*}
$$

By definition, a mild solution of (1.11) is a strongly continuous function $u(t)$ taking values in $H^{2,1}$ that solves the fixed point problem $u=\mathcal{F}_{\rho}(u)$ in the space $C\left([0, T] ; H^{2,1}\right)$ for some $T>0$. Note that since $H^{2,1}$ is contained in the domain of $L$, the time derivative of a mild solution lies in $L^{2}$ and satisfies Eq. (1.2) in $L^{2}$. Moreover, smooth initial values rise to classical solutions.

Since the cylindrical surface has dimension 2 , the Sobolev space $H^{2}\left(\mathcal{S}_{R}\right)$ is a Banach algebra. It follows directly from the continuity of the multiplication that $N$, as a polynomial in the first component, is locally Lipschitz in $H^{2,1}$. Explicitly, for every $\eta>0$ there exists a constant $C_{\eta}>0$ (which depends on $\alpha, \gamma$, and $\varepsilon$, as well as $\eta$ ) such that

$$
\begin{equation*}
\|N(u)-N(w)\|_{2,1} \leq C_{\eta}\left\|u_{1}-w_{1}\right\|_{H^{2}} \tag{1.15}
\end{equation*}
$$

for all $u, w$ with $\left\|u_{1}\right\|_{H^{2}},\left\|w_{1}\right\|_{H^{2}} \leq \eta$. From here, local well-posedness of the initial-value problem follows by standard methods. For the sake of completeness, we construct the mild solution of Eq. (1.10), as follows.

Proposition 1.3 (Local well-posedness). Assume that $\rho$ is of class $C^{2}$, bounded, and bounded away from zero. Then for each $u_{0} \in H^{2,1}$, there exists $T>0$ (depending on $\left\|u_{0}\right\|_{2,1}$ ) such that Eq. (1.10) on $\mathcal{S}_{\rho}$ has a unique mild solution $u$ in $C\left([0, T], H^{2,1}\right)$ with initial condition $\left.u\right|_{t=0}=u_{0}$. The solution depends continuously on $u_{0}$.

Proof. We proceed by Picard iteration. Given $u_{0} \in H^{2,1}$, fix $\eta>0$ and $T>0$ (to be specified below) and consider

$$
\mathcal{B}:=\left\{u \in C\left([0, T], H^{2,1}\right) \mid\|u(t)\|_{2,1} \leq \eta \text { for all } 0 \leq t \leq T\right\}
$$

equipped with the norm $\|u\|_{T}:=\sup _{0 \leq t \leq T}\|u(t)\|_{2,1}$.

The map $\mathcal{F}_{\rho}$ defined by Eq. (1.14) is Lipschitz continuous on $\mathcal{B}$,

$$
\begin{aligned}
\left\|\mathcal{F}_{\rho}(u)-\mathcal{F}_{\rho}(w)\right\|_{T} & \leq \sup _{0 \leq t \leq T} \int_{0}^{t}\left\|e^{(t-s) A_{\rho}}\right\|_{2,1}\|N(u(s))-N(w(s))\|_{2,1} d s \\
& \leq C_{0} C_{\eta} T\|v-w\|_{T}
\end{aligned}
$$

where $C_{0}:=\sup _{t \geq 0}\left\|e^{t A_{\rho}}\right\|_{2,1}$, and $C_{\eta}$ is as in Eq. (1.15). Moreover,

$$
\left\|\mathcal{F}_{\rho}(0)\right\|_{T}=\sup _{0 \leq t \leq T}\left\|e^{t A_{\rho}} u_{0}\right\|_{2,1} \leq C_{0}\left\|u_{0}\right\|_{2,1}
$$

Choose $\eta=2 C_{0}\left\|u_{0}\right\|_{2,1}$, and $T=\left(2 C_{0} C_{\eta}\right)^{-1}$. Then $\mathcal{F}_{\rho}$ has Lipschitz constant $\frac{1}{2}$ and maps $\mathcal{B}$ into itself. By Banach's contraction mapping theorem, $\mathcal{F}_{\rho}$ has a unique fixed point in $\mathcal{B}$, which provides the desired mild solution of Eq. (1.10).

Let $w(t)$ be another mild solution, whose initial value $w_{0}:=\left.w\right|_{t=0}$ satisfies $\left\|w_{0}\right\|_{2,1}<\eta$. The difference between the solutions is bounded by

$$
\begin{aligned}
\|u(t)-w(t)\|_{2,1} & \leq\left\|e^{t A_{\rho}}\left(u_{0}-w_{0}\right)\right\|_{2,1}+\int_{0}^{t}\left\|e^{(t-s) A_{\rho}}(N(u(s))-N(w(s)))\right\|_{2,1} d s \\
& \leq C_{0}\left\|u_{0}-w_{0}\right\|_{2,1}+C_{0} C_{\eta} \int_{0}^{t}\|u(s)-w(s)\|_{2,1} d s
\end{aligned}
$$

so long as $\max \left\{\|u(s)\|_{2,1},\|w(s)\|_{2,1}\right\} \leq \eta$ for all $0 \leq s \leq t$. By Grönwall's inequality,

$$
\|u(t)-w(t)\|_{2,1} \leq C_{0} e^{C_{0} C_{\eta} t}\left\|u_{0}-w_{0}\right\|_{2,1}
$$

This proves continuous dependence on initial data.
Consider now the FHNcyl system on the standard cylinder $\mathcal{S}_{R}$. By translation invariance, the linearization $A_{R}$ commutes with translations in space and time. In the moving frame, as described by Eq. (1.9), the principal part of the system is given by $\bar{L}:=A_{R}+c \partial_{z}$. As a sum of commuting operators, $\bar{L}$ generates a semigroup $e^{t \bar{L}}$ on $L^{2}$ that acts as

$$
\left(e^{t \bar{L}} u\right)(z)=\left(e^{t A_{R}} u\right)(z+c t) .
$$

Like the group of translations, the semigroup $e^{t \bar{L}}$ is strongly continuous but not analytic. Since translations are isometries of $H^{2,1},\left\|e^{t \bar{L}}\right\|_{2,1}=\left\|e^{t A}\right\|_{2,1}$ for all $t>0$. In particular, mild solutions of Eq. (1.9) are equivalent to mild solutions of Eq. (1.2) on $\mathcal{S}_{R}$ via the transformation $z=x-c t$.

A general remark on the use of constants: In our estimates, it is understood that constants may vary from equation to equation, and depend on the fixed parameters $\alpha, \gamma, \varepsilon$ of Eq. (1.2). We frequently use the notation $\lesssim$ and $\gtrsim$ for the respective inequalities up to such constants. Dependence on other parameters, including $R$, $\rho$, and $u_{0}$ will be made explicit.

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## 2 Linearized stability on the standard cylinder

Fix a fast pulse $\Phi$ on $\mathcal{S}_{R}$, and let $\mathcal{M}$ be the manifold of its translates defined in Eq. (1.4). By definition, $\Phi$ is an axisymmetric traveling wave solution of Eq. (1.2). In this section we study the linear stability of $\Phi$.

In the moving frame where $\Phi$ is stationary, the system is given by Eq. (1.9). Let $G(u)=F_{R}(u)+\partial_{z} u$ be the right hand side of this equation. Since the pulse is a stationary solution, $G(\Phi)=0$. The linearization about $\Phi$ is given by the Gâteaux derivative

$$
L:=d G(\Phi)=\left(\begin{array}{cc}
\Delta_{\mathcal{S}_{R}}+c \partial_{z}+f^{\prime}\left(\phi_{1}\right) & -1  \tag{2.1}\\
\varepsilon & c \partial_{z}-\varepsilon \gamma
\end{array}\right) .
$$

The linearization defines a closed linear operator on the Hilbert space $L^{2}:=$ $L^{2}\left(\mathcal{S}_{R} ; \mathbb{C}^{2}\right)$ of two-component square integrable functions, with the inner product

$$
\begin{equation*}
\langle u, w\rangle:=\int_{\mathbb{R}} \int_{S^{1}}\left(u_{1} \bar{w}_{1}+\varepsilon^{-1} u_{2} \bar{w}_{2}\right) R d \theta d z \tag{2.2}
\end{equation*}
$$

and the corresponding norm $\|\cdot\|$. The domain of $L$ is the dense subspace

$$
\begin{equation*}
H^{2,1}:=\left\{u=\left(u_{1}, u_{2}\right) \in L^{2} \mid\|u\|_{2,1}:=\left\|\left(\Delta_{\mathcal{S}_{R}} u_{1}, \partial_{z} u_{2}\right)\right\|+\|u\|<\infty\right\} \tag{2.3}
\end{equation*}
$$

whose norm $\|u\|_{2,1}$ is equivalent to the graph norm of $L$, see Lemma 2.3. Note that $H^{2,1}$ properly contains $H^{2} \times H^{1}$, because Eq. (2.3) does not require $\partial_{\theta} u_{2}$ to lie in $L^{2}$. We will show that $L$ generates a strongly continuous semigroup $e^{t L}$.

Since $G(\Phi)=0$, by translation invariance we have that $G\left(\Phi_{h}\right)=0$ for any $h \in \mathbb{R}$. Differentiating this equation with respect to $h$, we find that $L \partial_{z} \Phi=0$. This means that 0 is an eigenvalue of $L$, and the tangent vector $\tau=-\partial_{z} \Phi$ is an eigenfunction. It turns out that the spectral projection $P$ associated with the zero eigenvalue of $L$ has rank one. The complementary projection $Q=1-P$ is the projection onto the range of $L$. Both $P$ and $Q$ commute with $L$ and with the semigroup $e^{t L}$.

The main result of this section is the following:

Proposition 2.1 (Linearized decay). Let $L$ be the operator defined by Eq. (2.1). If $\varepsilon>0$ is sufficiently small, and $0<R \leq 1$, then there exists $\sigma>0$ such that the semigroup $e^{t L}$ satisfies

$$
\begin{equation*}
\left\|e^{t L} Q\right\|_{2,1} \leq C e^{-\sigma t}, \quad(t \geq 0) \tag{2.4}
\end{equation*}
$$

for some constant $C$.
As an immediate consequence, the semigroup $e^{t L}$ is uniformly bounded. Indeed, by the triangle inequality,

$$
\left\|e^{t L}\right\|_{2,1} \leq\left\|e^{t L} P\right\|_{2,1}+\left\|e^{t L} Q\right\|_{2,1} \leq\|P\|_{2,1}+C\|Q\|_{2,1} \quad(t \geq 0)
$$

since $e^{t L}$ is constant on the range of $P$ and decays exponentialy on the range of $Q$.
The proof of Proposition 2.1 will be given in Subsection 2.4, and will specify the conditions on $\sigma$. An important tool is the following general result of Prüss [31, Corollary 4] that we state next:
Theorem 2.2 (Prüss). Suppose that $B$ generates a strongly continuous semigroup on a Hilbert space. If the resolvent $(\lambda-B)^{-1}$ is uniformly bounded on the halfplane $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq-\beta\}$, for some $\beta>0$, then there exists a constant $C>0$ such that $\left\|e^{t B}\right\| \leq C e^{-\beta t}$ for all $t \geq 0$.

### 2.1 The linear semigroup

We first justify the choice of the function space $H^{2,1}$.
Lemma 2.3 (Domain of $L$ ). The operator $L$ defined in Eq. (2.1) has domain $D(L)=H^{2,1}$, and its graph norm satisfies

$$
\|u\|_{2,1} \lesssim\|L u\|+\|u\| \lesssim\|u\|_{2,1}, \quad u \in H^{2,1}
$$

Proof. We use that $\left\|\partial_{z} u_{1}\right\| \leq \frac{1}{2 c}\left\|\Delta_{\mathcal{S}_{R}} u_{1}\right\|+\frac{c}{2}\left\|u_{1}\right\|$, and set

$$
\begin{equation*}
b:=\sup _{z \in \mathbb{R}}\left|f^{\prime}\left(\phi_{1}(z)\right)-f^{\prime}(0)\right|<\infty . \tag{2.5}
\end{equation*}
$$

By the reverse triangle inequality, this yields for the lower bound

$$
\|L u\| \geq \min \left\{\frac{1}{2}, c\right\}\left\|\left(\Delta_{\mathcal{S}_{R}} u_{1}, \partial_{z} u_{2}\right)\right\|-\left(\frac{c^{2}}{2}+b\right)\left\|u_{1}\right\|
$$

which implies that

$$
\begin{aligned}
\left(1+\frac{c^{2}}{2}+b\right)(\|L u\|+\|u\|) & \geq\|L u\|+\left(1+\frac{c^{2}}{2}+b\right)\|u\| \\
& \geq \min \left\{\frac{1}{2}, c\right\}\|u\|_{2,1} .
\end{aligned}
$$

For the upper bound, the triangle inequality yields

$$
\|L u\|+\|u\| \leq \max \left\{\frac{3}{2}, c, 1+\frac{c^{2}}{2}+b\right\}\|u\|_{2,1}
$$

An operator $B$ on a Hilbert space is called dissipative, if $\operatorname{Re}\langle B x, x\rangle \leq 0$ on its domain. We will frequently use the following convenient corollary of the Lumer-Phillips theorem (see [30, Theorem 1.4.3]).

Lemma 2.4 (Dissipative operators). Let $B$ be a closed, densely defined, dissipative operator on a Hilbert space $H$. Then $B$ generates a strongly continuous semigroup of contractions, $e^{t B}$. Its spectrum lies in the left half-plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq 0\}$, and its resolvent is bounded by

$$
\left\|(\lambda-B)^{-1}\right\| \leq \frac{1}{\operatorname{Re} \lambda}, \quad(\operatorname{Re} \lambda>0)
$$

Proof. Since $B$ is dissipative, the operator $1-B$ is injective,

$$
\|(1-B) v\| \geq\|v\|^{-1}|\operatorname{Re}\langle(1-B) v, v\rangle| \geq\|v\| .
$$

Since its adjoint is injective by the same argument, the range of $1-B$ is dense.
Let $w_{0} \in H$ be arbitrary. Choose a sequence $\left(w_{n}\right)$ in the range of $1-B$ with $\lim w_{n}=w_{0}$. For each $n$, let $v_{n}$ be in the domain of $B$ such that $(1-B) v_{n}=w_{n}$. Since

$$
\left\|v_{n}-v_{m}\right\| \leq\left\|(1-B)\left(v_{n}-v_{m}\right)\right\|=\left\|w_{n}-w_{m}\right\|
$$

by the Cauchy criterion the sequence $\left(v_{n}\right)$ converges to some limit, $v_{0}$. Since $B$ is a closed operator, $w_{0}=(1-B) v_{0}$. We conclude that $1-B$ is surjective.

By the Lumer-Phillips theorem, $B$ generates a strongly continuous semigroup of contractions on $H$. The resolvent bound follows from the Hille-Yosida theorem (see [30, Theorem 1.3.1]).

The next lemma will be used to construct the semigroup generated by $L$. It also plays a role in the spectral estimates.

Lemma 2.5 (Compact perturbation). The operator $L$ is a bounded, relatively compact perturbation of

$$
\bar{L}:=\left(\begin{array}{cc}
\Delta_{\mathcal{S}_{R}}+c \partial_{z}-\alpha & -1  \tag{2.6}\\
\varepsilon & c \partial_{z}-\varepsilon \gamma
\end{array}\right) .
$$

Proof. We decompose $L=\bar{L}+V$, where $V$ is the matrix multiplication operator

$$
V:=\left(\begin{array}{cc}
f^{\prime}\left(\phi_{1}\right)-f^{\prime}(0) & 0 \\
0 & 0
\end{array}\right) .
$$

Since $f$ is a polynomial and $\phi_{1}$ is a smooth, bounded function, $V$ is a bounded operator on $L^{2}$ and also on $H^{2,1}$. The term $f^{\prime}\left(\phi_{1}(z)\right)-f^{\prime}(0)$ is continuous and decays at infinity. By a standard result (see [23, Theorem, 3.1.11]), $V \bar{L}^{-1}$ is compact.

The operator $\bar{L}$ captures the behavior of $L$ as $z \rightarrow \pm \infty$. The next lemma implies that $\bar{L}$ generates a strongly continuous semigroup of contractions.
Lemma 2.6 (Principal part of $\bar{L}$ ). Let $\bar{L}$ be the operator defined by Eq. (2.6), and $\sigma=\min \{\alpha, \varepsilon \gamma\}$. Then $\operatorname{Re}\langle\bar{L} v, v\rangle \leq-\sigma\|v\|^{2}$.

Proof. By Lemmas 2.3 and 2.5, the domain of $\bar{L}$ is $H^{2,1}$. Since $\partial_{z}$ is skew-adjoint in $L^{2}\left(\mathcal{S}_{R}\right)$, and the off-diagonal terms in $\bar{L}$ are skew-adjoint with respect to the inner product from Eq. (2.2), we have

$$
\begin{aligned}
\operatorname{Re}\langle\bar{L} v, v\rangle & =\int_{\mathcal{S}_{R}}\left(\left(\Delta_{\mathcal{S}_{R}} v_{1}\right) \bar{v}_{1}-\alpha\left|v_{1}\right|^{2}-\gamma\left|v_{2}\right|^{2}\right) R d \theta d z \\
& =-\int_{\mathcal{S}_{R}}\left(\alpha\left|v_{1}\right|^{2}+\gamma\left|v_{2}\right|^{2}\right) R d \theta d z \\
& \leq-\min \{\alpha, \varepsilon \gamma\}\|v\|^{2}
\end{aligned}
$$

In the second step, we have used that $\Delta_{\mathcal{S}_{R}}$ is negative semi-definite, and in the third step we have applied the definition of the inner product in Eq. (2.2). It follows that $\operatorname{Re}\langle(\sigma+\bar{L}) v, v\rangle \leq 0$ for all $v \in H^{2,1}$.

In the Fourier representation, $\bar{L}$ is given by the matrix multiplication operator

$$
m(k, n):=\left(\begin{array}{cc}
-k^{2}-n^{2} R^{-2}+i c k-\alpha & -1  \tag{2.7}\\
\varepsilon & i c k-\varepsilon \gamma
\end{array}\right), \quad\left(k \in \mathbb{R}, n \in \mathbb{N}_{0}\right)
$$

Therefore $\bar{L}$ has only essential spectrum. Its resolvent set consists of those $\lambda \in \mathbb{C}$ for which $\lambda-m(k, n)$ is invertible (for all $k \in \mathbb{R}, n \in \mathbb{N}_{0}$ ) and

$$
\sup _{k \in \mathbb{R}, n \in \mathbb{N}_{0}}\left\|(\lambda-m(k, n))^{-1}\right\|<\infty .
$$

The spectrum contains the branch of eigenvalues of $m(k, 0)$ given by

$$
\begin{aligned}
\lambda_{+}(k, 0) & =i c k-\frac{1}{2}\left(k^{2}+\alpha+\varepsilon \gamma\right)+\frac{1}{2} \sqrt{\left(k^{2}+\alpha-\varepsilon \gamma\right)^{2}-4 \varepsilon} \\
& \sim i c k-\varepsilon \gamma \quad(|k| \rightarrow \infty) .
\end{aligned}
$$

Consequently, $\bar{L}$ is not sectorial, and $e^{t \bar{L}}$ is not an analytic semigroup.
Lemmas 2.5 and 2.6 imply, by Lemma 2.4 and a standard perturbation result (see [30, Theorem 3.1.1]), that $L=\bar{L}+V$ generates a strongly continuous semigroup on $L^{2}$, denoted by $e^{t L}$. Evidently, $e^{t L}$ also fails to be analytic. Since

$$
\begin{equation*}
f^{\prime}(y)-f^{\prime}(0)=-3 y^{2}+2(\alpha+1) y \leq 1, \quad\left(y \in \mathbb{R}, 0<\alpha<\frac{1}{2}\right) \tag{2.8}
\end{equation*}
$$

we have that $\langle V v, v\rangle \leq\|v\|^{2}$. By Lemma 2.6,

$$
\begin{equation*}
\operatorname{Re}\langle L v, v\rangle=\operatorname{Re}\langle(\bar{L}+V) v, v\rangle \leq\|v\|^{2} \tag{2.9}
\end{equation*}
$$

i.e., $-1+L$ is dissipative. By Lemma 2.4, the semigroup satisfies $\left\|e^{t L}\right\| \leq e^{t}$.

### 2.2 The spectral projection

In this subsection, we prove that
(i) $\operatorname{spec}(L) \subset\{0\} \cup\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda<-\sigma\}$ for some $\sigma>0$;
(ii) 0 is a simple eigenvalue of $L$ and $L^{*}$.

This will be used to construct the spectral projection $Q$ that appears in Proposition 2.1. We start with the essential spectrum of $L$.

Lemma 2.7 (Essential spectrum). The operator $L$ defined by Eq. (2.1) satisfies

$$
\operatorname{spec}_{\text {ess }}(L) \subset\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq-\sigma\}
$$

where $\sigma=\min \{\alpha, \varepsilon \gamma\}$.
Proof. By Lemma 2.5, $L$ is a relatively compact perturbation of the operator $\bar{L}$ from Eq. (2.6). It follows from Weyl's essential spectrum theorem (see [23, Chapter 2]) that

$$
\operatorname{spec}_{e s s}(L)=\operatorname{spec}_{e s s}(\bar{L})=\operatorname{spec}(\bar{L})
$$

Since $\sigma+\bar{L}$ is dissipative by Lemma 2.6, its spectrum lies in the left half-plane by Lemma 2.4.

To analyze the discrete spectrum of $L$, we expand functions on $\mathcal{S}_{R}$ as Fourier series in the angular variable, $\theta$,

$$
v(z, \theta)=\sum_{n \in \mathbb{Z}} v_{n}(z) e^{i n \theta}
$$

By Eq. (2.7), $L$ splits into a direct sum $L=\bigoplus_{n \geq 0} L_{n}$, where

$$
L_{n}:=\left(\begin{array}{cc}
\partial_{z}^{2}-n^{2} R^{-2}+c \partial_{z}+f^{\prime}\left(\phi_{1}(z)\right) & -1  \tag{2.10}\\
\varepsilon & c \partial_{z}-\varepsilon \gamma
\end{array}\right)
$$

is the restriction of $L$ to the invariant subspace corresponding to the Fourier modes $\pm n$. The next lemma concerns the positive modes.

Lemma 2.8 (Resolvent estimate, $n>0$ ). Let $L_{n}$ be given by Eq. (2.10). If $R \leq 1$, then

$$
\operatorname{spec}_{d i s c}\left(L_{n}\right) \subset\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq-\sigma\} \quad(n>0)
$$

and

$$
\left\|\left(\lambda-L_{n}\right)^{-1}\right\| \leq \frac{1}{\operatorname{Re} \lambda+\sigma}, \quad(\operatorname{Re} \lambda>-\sigma, n>0)
$$

Here, $\sigma=\min \{\alpha, \varepsilon \gamma\}$.

Proof. Using that $f^{\prime}\left(\phi_{1}(z)\right)-f^{\prime}(0) \leq 1$ by Eq. (2.8), we obtain

$$
\begin{aligned}
\operatorname{Re}\left\langle L_{n} v, v\right\rangle & =\int_{\mathbb{R}}\left(\left(\partial_{z}^{2} v_{1}\right) \bar{v}_{1}+\left(f^{\prime}\left(\phi_{1}(z)\right)-n^{2} R^{-2}\right)\left|v_{1}\right|^{2}-\gamma\left|v_{2}\right|^{2}\right) R d z \\
& \leq-\sigma\|v\|^{2}, \quad(n>0)
\end{aligned}
$$

provided that $R \leq 1$. (Recall that $f^{\prime}(0)=-\alpha$.) Since $\sigma+L_{n}$ is dissipative, the resolvent estimate follows from Lemma 2.4.

The heart of the matter is the discrete spectrum of the zero mode.
Lemma 2.9 (Spectrum of $L_{0}$ ). Let $L_{0}$ be given by Eq. (2.10) with $n=0$. If $\varepsilon>0$ is sufficiently small, then there exists $\beta>0$ such that

$$
\operatorname{spec}_{\text {disc }}\left(L_{0}\right) \subset\{0\} \cup\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq-\beta\}
$$

Moreover, 0 is a simple eigenvalue of both $L_{0}$ and its adjoint, $L_{0}^{*}$.
Proof. Let $\tau=-\partial_{z} \Phi$ be the tangent vector to $\mathcal{M}$ at $\Phi$. We have argued above that $L \tau=0$ by translation invariance, that is, 0 is an eigenvalue of $L$, with eigenfunction $\tau$. It remains to show that the eigenvalue 0 is simple, and that there are no other eigenvalues with nonnegative real part.

The operator $L_{0}$ agrees with the linearization of the FHN system in one spatial dimension. The spectrum of this operator in the space of bounded continuous functions was analyzed by Jones [21] and Yanagida [34]. Specifically, they proved that for $\varepsilon>0$ sufficiently small, 0 is a simple eigenvalue, and all other eigenvalues lie in some half-plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda<-\beta\}$.

We claim that the discrete spectrum of $L_{0}$ on $L^{2}(\mathbb{R})$ is contained in its discrete spectrum on the space of bounded continuous functions. Indeed, any generalized eigenfunction must lie in the domain of $L_{0}$, given by the mixed Sobolev space $H^{2,1}$, see Lemma 2.3. In particular, the generalized eigenfunctions are bounded and continuous. Therefore the results of Jones [21] and Yanagida [34] also apply to $L_{0}$.

Finally, by Lemma 2.5, $L_{0}$ is a Fredholm operator of the same index as $\bar{L}_{0}$. The Fredholm index of $\bar{L}_{0}$ is zero, because 0 lies in its resolvent set by Lemmas 2.6 and 2.4. Therefore, 0 is a simple eigenvalue also for $L_{0}^{*}$.

Combining Lemmas 2.7, 2.8, and 2.9, we conclude that

$$
\operatorname{spec}(L) \subset\{0\} \cup\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq-\sigma\}
$$

where $\sigma=\min \{\alpha, \beta, \varepsilon \gamma\}$, and $\beta$ is determined by Lemma 2.9. Since the eigenvalue at zero is isolated, the Riesz projection to the zero eigenspace is defined by the line integral

$$
P=\frac{1}{2 \pi i} \oint_{\Gamma_{0}}(\lambda-L)^{-1} d \lambda
$$

where $\Gamma_{0} \subset \mathbb{C}$ is any simple closed positively oriented curve that separates zero from the remainder of the spectrum of $L$ [23, Chapter 2]. By definition, $P$ is a bounded linear operator that commutes with $L$ and $e^{t L}$.

Lemma 2.10 (Spectral projection). Under the assumptions of Lemma 2.9, the Riesz projection is given by

$$
P u=\left\langle u, \tau^{*}\right\rangle \tau, \quad u \in L^{2},
$$

where $\tau=-\partial_{z} \phi_{1}$, and $\tau^{*}$ is the eigenfunction of the adjoint $L^{*}$ corresponding to the zero eigenvalue, normalized to $\left\langle\tau, \tau^{*}\right\rangle=1$.

Proof. The eigenfunction $\tau^{*}$ is well-defined because 0 is a simple eigenvalue for both $L$ and $L^{*}$ by Lemma 2.9. In particular, since $\tau$ does not lie in the range of $L, \tau^{*}$ is not orthogonal to $\tau$. The Riesz projection $P$ is uniquely determined by its action on the nullspace and range of $L$, i.e., the properties that $P \tau=\tau$ and $P L v=0$ for all $v \in H^{2,1}$ hold. We verify that $\left\langle\tau, \tau^{*}\right\rangle \tau=\tau$, and $\left\langle L v, \tau^{*}\right\rangle \tau=\left\langle v, L^{*} \tau^{*}\right\rangle \tau=0$, as required.

Lemma 2.11 (Resolvent estimate, $n=0$ ). Under the assumptions of Lemma 2.9, let $P$ be the projection to the nullspace of $L$ constructed in Lemma 2.10, let $Q=$ $1-P$ be the complementary projection, and let $Q_{0}$ be its restriction to the $n=0$ subspace of $L^{2}$. For every $\sigma>\min \{\alpha, \beta, \varepsilon \gamma\}$, there exists a positive constant $C$ such that

$$
\left\|\left(\lambda-L_{0}\right)^{-1} Q_{0}\right\| \leq C, \quad(\operatorname{Re} \lambda \geq-\sigma)
$$

Here, $\beta$ is as in Lemma 2.9.
The proof of this lemma is deferred to the next subsection.

### 2.3 Proof of Lemma 2.11

We estimate the resolvent of the operator $L_{0}$, given by Eq. (2.10) with $n=0$, separately in the three regions

$$
\begin{aligned}
& S_{1}:=\{\lambda \in \mathbb{C}|-\sigma \leq \operatorname{Re} \lambda \leq 2,|\operatorname{Im} \lambda| \leq N\} \\
& S_{2}:=\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq 2\} \\
& S_{3}:=\{\lambda \in \mathbb{C}|-\sigma \leq \operatorname{Re} \lambda \leq 2,|\operatorname{Im} \lambda| \geq N\}
\end{aligned}
$$

Here, $\sigma<\min \{\alpha, \beta, \varepsilon \gamma\}$, and $\beta$ is as in Lemma 2.9. The constant $N$ will be chosen in the proof of Lemma 2.14.

On $S_{1}$, we appeal to compactness.
Lemma 2.12 (Resolvent estimate on $S_{1}$ ). For any $N>0$,

$$
\sup _{\lambda \in S_{1}}\left\|\left(\lambda-L_{0}\right)^{-1} Q_{0}\right\|<\infty .
$$

Proof. By Lemma 2.9, $S_{1}$ intersects the spectrum of $L_{0}$ only at the simple eigenvalue 0 . Since the restriction of $L_{0}$ to the range of $Q_{0}$ has no spectrum in $S_{1}$, its resolvent is an analytic function of $\lambda$, and hence bounded on the compact set $S_{1}$.

The half-plane $S_{2}$ is treated by a dissipativity estimate.
Lemma 2.13 (Resolvent estimate on $S_{2}$ ). For any $N>0$,

$$
\begin{equation*}
\sup _{\lambda \in S_{2}}\left\|\left(\lambda-L_{0}\right)^{-1}\right\| \leq 1 \tag{2.11}
\end{equation*}
$$

Proof. Since $L_{0}$ is the restriction of $L$ to a subspace, Eq. (2.9) implies that

$$
\operatorname{Re}\left\langle L_{0} v, v\right\rangle \leq\|v\|^{2}
$$

By Lemma 2.4, $\lambda-L_{0}$ is invertible for $\operatorname{Re} \lambda>1$ and the inverse satisfies

$$
\left\|\left(\lambda-L_{0}\right)^{-1}\right\| \leq \frac{1}{R e \lambda-1} \quad(\operatorname{Re} \lambda>1)
$$

Since $\operatorname{Re} \lambda \geq 2$ on $S_{2}$, this implies Eq. (2.11).
For the region $S_{3}$, some explicit estimates are required.
Lemma 2.14 (Resolvent estimate on $S_{3}$ ). If $N$ is sufficiently large then

$$
\sup _{\lambda \in S_{3}}\left\|\left(\lambda-L_{0}\right)^{-1}\right\| \leq \frac{2}{\min \{\alpha, \varepsilon \gamma\}-\sigma}
$$

Here, $\sigma<\min \{\alpha, \beta, \varepsilon \gamma\}$, and $\beta$ is as in Lemma 2.9.
Proof. For $\lambda \in S_{3}$, we write $L_{0}=\bar{L}_{0}+V$, where $V$ is defined by this relation and

$$
\bar{L}_{0}:=\left(\begin{array}{cc}
\partial_{z}^{2}+c \partial_{z}+f^{\prime}(0) & -1 \\
\varepsilon & c \partial_{z}-\varepsilon \gamma
\end{array}\right)
$$

Since $\sigma+\bar{L}_{0}$ is dissipative, the resolvent set of $\bar{L}_{0}$ contains the half-plane $\{\lambda \in$ $\mathbb{C} \mid \operatorname{Re} \lambda \geq-\sigma\}$. We want to solve for $\left(\lambda-L_{0}\right)^{-1}$ in the resolvent identity

$$
\begin{equation*}
\left(\lambda-\bar{L}_{0}\right)^{-1}-\left(\lambda-L_{0}\right)^{-1}=-\left(\lambda-\bar{L}_{0}\right)^{-1} V\left(\lambda-L_{0}\right)^{-1} \tag{2.12}
\end{equation*}
$$

First, we prove that there exists an $N>0$ such that

$$
\begin{equation*}
\sup _{\lambda \in S_{3}}\left\|\left(\lambda-\bar{L}_{0}\right)^{-1} V\right\| \leq \frac{1}{2} \tag{2.13}
\end{equation*}
$$

Indeed, let $\lambda \in S_{3}$. Using that the (1,2)- and (2,2)-entries of the operator matrix $\left(\lambda-\bar{L}_{0}\right)^{-1} V$ vanish and that

$$
\left\|\left(\begin{array}{ll}
a_{11} & 0 \\
a_{21} & 0
\end{array}\right)\right\| \leq\left\|a_{11}\right\|+\left\|a_{21}\right\|,
$$

we find

$$
\left\|\left(\lambda-\bar{L}_{0}\right)^{-1} V\right\| \leq\left(\left\|\left(\left(\lambda-\bar{L}_{0}\right)^{-1}\right)_{11}\right\|+\left\|\left(\left(\lambda-\bar{L}_{0}\right)^{-1}\right)_{21}\right\|\right) \sup _{y \in \mathbb{R}}\left|f^{\prime}(y)-f^{\prime}(0)\right| .
$$

Since the last factor is bounded by Eq. (2.5), to verify Eq. (2.13) it suffices to show that

$$
\begin{equation*}
\lim _{N \rightarrow \infty} \sup _{\lambda \in S_{3}}\left\|\left(\left(\lambda-\bar{L}_{0}\right)^{-1}\right)_{i 1}\right\|=0, \quad i=1,2 \tag{2.14}
\end{equation*}
$$

Moreover, since the differential operator $\bar{L}_{0}$ has real coefficients, we may restrict the supremum to the intersection of $S_{3}$ with the upper half-plane.

In the Fourier representation, $\bar{L}_{0}$ becomes the matrix multiplication operator

$$
m(k, 0)=\left(\begin{array}{cc}
-k^{2}+i c k-\alpha & -1 \\
\varepsilon & i c k-\varepsilon \gamma
\end{array}\right), \quad(k \in \mathbb{R})
$$

see Eq. (2.7). In particular,

$$
\begin{equation*}
\left\|\left(\left(\lambda-\bar{L}_{0}\right)^{-1}\right)_{i j}\right\|=\sup _{k \in \mathbb{R}}\left|\left((\lambda-m(k, 0))^{-1}\right)_{i j}\right| \tag{2.15}
\end{equation*}
$$

By Cramer's rule (suppressing the dependence on $k$ in the notation),

$$
\left((\lambda-m)^{-1}\right)_{11}=\frac{\lambda-m_{22}}{\operatorname{det}(\lambda-m)}, \quad\left((\lambda-m)^{-1}\right)_{21}=\frac{m_{21}}{\operatorname{det}(\lambda-m)} .
$$

Passing to reciprocals, we compute for the first entry

$$
\frac{1}{\left((\lambda-m)^{-1}\right)_{11}}=\lambda-m_{11}-\frac{m_{12} m_{21}}{\lambda-m_{22}}
$$

We next separate the real and imaginary parts. Since $\operatorname{Re} \lambda \geq-\sigma$ on $S_{3}$, and $\sigma \leq \min \{\alpha, \varepsilon \gamma\}$, we have that

$$
\operatorname{Re}\left(\lambda-m_{11}\right) \geq k^{2}, \quad \operatorname{Re}\left(\lambda-m_{22}\right) \geq \varepsilon \gamma-\sigma>0
$$

Since $m_{12} m_{21}=-\varepsilon<0$, it follows that

$$
\operatorname{Re} \frac{1}{\left((\lambda-m)^{-1}\right)_{11}} \geq k^{2} .
$$

For the imaginary part, we have

$$
\operatorname{Im} \frac{1}{\left((\lambda-m)^{-1}\right)_{11}} \geq \operatorname{Im} \lambda-c k+\varepsilon|\operatorname{Im} \lambda-c k|^{-1}
$$

Combining the two estimates yields for $\operatorname{Im} \lambda \geq N$

$$
\begin{aligned}
\frac{1}{\left|\left((\lambda-m)^{-1}\right)_{11}\right|} & \geq \max \left\{k^{2},|\operatorname{Im} \lambda-c k|+\varepsilon|\operatorname{Im} \lambda-c k|^{-1}\right\} \\
& \geq \max \left\{\frac{N^{2}}{4 c^{2}}, \frac{N}{2}+\frac{2 \varepsilon}{N}\right\} \rightarrow \infty \quad(N \rightarrow \infty)
\end{aligned}
$$

The second inequality above holds since $k \geq \frac{N}{2 c}$ whenever $|\operatorname{Im} \lambda-c k| \leq \frac{N}{2}$. This implies Eq. (2.14) for $i=1$. Similarly,

$$
\frac{1}{\left((\lambda-m)^{-1}\right)_{21}}=\frac{\left(\lambda-m_{11}\right)\left(\lambda-m_{22}\right)}{m_{21}}-m_{12} .
$$

As before, we separately estimate the real and imaginary parts of each of the factors in the numerator

$$
\left|\lambda-m_{11}\right| \geq \max \left\{k^{2},|\operatorname{Im} \lambda-c k|\right\}, \quad\left|\lambda-m_{22}\right| \geq \varepsilon \gamma-\sigma .
$$

It follows that

$$
\begin{aligned}
\left|\frac{1}{\left((\lambda-m)^{-1}\right)_{12}}\right| \geq & \frac{\varepsilon \gamma-\sigma}{\varepsilon} \max \left\{k^{2},|\operatorname{Im} \lambda-c k|\right\}-1 \\
\geq & \frac{\varepsilon \gamma-\sigma}{\varepsilon} \max \left\{\frac{N^{2}}{4 c^{2}}, \frac{N}{2}\right\}-1 \\
& \rightarrow \infty \quad(N \rightarrow \infty) .
\end{aligned}
$$

This implies Eq. (2.14) for $i=2$.
Now choose $N$ so large that Eq. (2.13) holds. Since $1-\left(\lambda-\bar{L}_{0}\right)^{-1} V$ is invertible, we can solve for the resolvent of $L_{0}$ in Eq. (2.12) to obtain

$$
\left(\lambda-L_{0}\right)^{-1}=\left(1-\left(\lambda-\bar{L}_{0}\right)^{-1} V\right)^{-1}\left(\lambda-\bar{L}_{0}\right)^{-1}
$$

Using this relation and Eq (2.13), we estimate

$$
\begin{aligned}
\left\|\left(\lambda-L_{0}\right)^{-1}\right\| & \leq\left(\left\|1-\left(\lambda-\bar{L}_{0}\right)^{-1} V\right\|\right)^{-1}\left\|\left(\lambda-\bar{L}_{0}\right)^{-1}\right\| \\
& \leq 2\left\|\left(\lambda-\bar{L}_{0}\right)^{-1}\right\| \\
& \leq \frac{2}{\operatorname{Re} \lambda+\min \{\alpha, \varepsilon \gamma\}}
\end{aligned}
$$

where the last line follows from the dissipativity of $\min \{\alpha, \varepsilon \gamma\}+\bar{L}_{0}$ by Lemma 2.4. Since $\operatorname{Re} \lambda \geq-\sigma$ on $S_{3}$, this proves the claim.

Lemma 2.11 follows from Lemmas 2.12-2.14.

### 2.4 Proof of Proposition 2.1

Let $P$ be the projection constructed in Lemma 2.10, and $Q=1-P$ be the complementary projection to the range of $L$. Choose

$$
\sigma<\min \{\alpha, \beta, \varepsilon \gamma\}
$$

where $\beta$ is the constant from Lemma 2.9. We need to find a constant $C>0$ such that $\left\|e^{t L} Q\right\|_{2,1} \leq C e^{-\sigma t}$ for all $t>0$.

Splitting $L$ into the direct sum of its Fourier modes, we obtain

$$
\left\|(\lambda-L)^{-1} Q\right\|=\left\|\oplus_{n \geq 0}\left(\lambda-L_{n}\right)^{-1} Q_{n}\right\| \leq \sup _{n \geq 0}\left\{\left\|\left(\lambda-L_{n}\right)^{-1} Q_{n}\right\|\right\},
$$

see Eq. (2.10). Lemmas 2.8 and 2.11 imply that there exists a constant $C>0$ such that

$$
\left\|(\lambda-L)^{-1} Q\right\| \leq C, \quad(\operatorname{Re} \lambda \geq-\sigma)
$$

Since $Q$ commutes with $L$, this establishes the hypotheses of Prüss' theorem on the range of $Q$.

Applying Theorem 2.2 with $B$ equal to the restriction of $L$ to the range of $Q$, we obtain that

$$
\left\|e^{t L} Q\right\| \leq C e^{-\sigma t}
$$

for some constant $C>0$. Since $L$ commutes with $e^{t L}$, it follows that

$$
\left\|L e^{t L} Q u\right\|+\left\|e^{t L} Q u\right\| \leq C e^{-\sigma t}(\|L u\|+\|u\|) \quad\left(u \in H^{2,1}\right)
$$

Since the $H^{2,1}$-norm is equivalent to the graph norm of $L$ by Lemma 2.3, Eq. (2.4) follows. This completes the proof of Proposition 2.1.

## 3 Nonlinear stability on the standard cylinder

In this section, we return to the nonlinear system on $\mathcal{S}_{R}$ in the moving frame, and prove Theorem 1.1. Let $G(u)$ denote the right hand side of Eq. (1.9). By Proposition 1.3, the initial-value problem

$$
\begin{equation*}
\partial_{t} u=G(u),\left.\quad u\right|_{t=0}=u_{0} \tag{3.1}
\end{equation*}
$$

is locally well-posed in the class of mild solutions on $H^{2,1}$.

### 3.1 Decomposition of the solution near $\mathcal{M}$

Let $P$ be the projection to the tangent space of $\mathcal{M}$ at $\Phi$ from Lemma 2.10. By translation invariance,

$$
P_{h} v:=\left\langle v, \tau_{h}^{*}\right\rangle \tau_{h}, \quad v \in L^{2}
$$

defines the corresponding projection to the tangent space of $\mathcal{M}$ at the translated pulse $\Phi_{h}$. This is the spectral projection associated with the zero eigenspace of $L_{h}:=d G\left(\Phi_{h}\right)$.

Proposition 3.1. Under the assumptions of Proposition 2.1:
(i) (Projection onto $\mathcal{M}$.) There exists a tubular neighborhood $\mathcal{W}$ of $\mathcal{M}$ in $H^{2,1}$ such that every $u \in \mathcal{W}$ has a unique decomposition as

$$
\begin{equation*}
u=\Phi_{h}+v \quad \text { with } P_{h} v=0 . \tag{3.2}
\end{equation*}
$$

(ii) (Local projection near $\Phi$.) There exists a neighborhood $\mathcal{U}$ of $\Phi$ in $H^{2,1}$ such that every $u \in \mathcal{U}$ has a unique decomposition

$$
\begin{equation*}
u=\Phi_{h}+v \quad \text { with } P v=0 \tag{3.3}
\end{equation*}
$$

In both cases, $h$ and $v$ are smooth functions of $u$.
In the proof, we show that each of Eq. (3.2) and Eq. (3.3) defines a pair of complementary non-linear projections $\mathcal{P}: u \mapsto \Phi_{h(u)}$ (onto $\mathcal{M}$ ) and and $\mathcal{Q}: u \mapsto v$ (onto a transversal subspace), with

$$
\left.d \mathcal{P}\right|_{u=\Phi}=P,\left.\quad d \mathcal{Q}\right|_{u=\Phi}=Q
$$

In fact, Eq. (3.3) defines a diffeomorphism $u \mapsto(h, v)$ from $\mathcal{U}$ onto a neighborhood of the origin in $\mathbb{R} \times \operatorname{Ran}(Q)$. The proof relies on the Implicit Function Theorem. The following lemma provides the requisite smoothness.

Lemma 3.2 (Smooth dependence on $h$ ). The manifold of pulses $\mathcal{M}$ is a smooth simple curve in $H^{2,1}$. Moreover, the tangent vector $\tau_{h}$, the dual vector $\tau_{h}^{*}$, the projections $P_{h}, Q_{h}$, the linearization $L_{h}=d G\left(\Phi_{h}\right)$, and the nonlinearity $N_{h}(v):=$ $G\left(\Phi_{h}+v\right)-L_{h} v$ depend smoothly in $H^{2,1}$ on $h$, with bounded derivatives of all orders.

Proof. The smoothness of $\mathcal{M}$ follows from the smoothness and decay of $\Phi$ and its derivatives. This also proves the smoothness of $\tau_{h}, \tau_{h}^{*}$, and the projections. The linearization $L_{h}$ is a matrix-valued differential operator whose coefficients are smooth functions of $\Phi_{h}$; the linearity $N_{h}(v)=G\left(\Phi_{h}+v\right)-L_{h}\left(\Phi_{h}\right) v$ is a cubic polynomial in $v_{1}$ whose coefficients are smooth functions of $\Phi_{h}$. Since $H^{2}\left(\mathcal{S}_{R}\right)$ is a Banach algebra, $G\left(\Phi_{h}+v\right), L_{h} v$, and $N_{h}(v)$ all depend smoothly on $h$.

Proof of Proposition 3.1. (i). Given $u$ near $\mathcal{M}$, we need to find $h \in \mathbb{R}$ such that $P_{h}\left(u-\Phi_{h}\right)=0$. Choose $h_{0} \in \mathbb{R}$ with $\left\|u-\Phi_{h_{0}}\right\|=\operatorname{dist}(u, \mathcal{M})$. By applying the translation $\tau_{-h_{0}}$ to a neighborhod of $u$, we may assume that $h_{0}=0$. Thus we need to solve

$$
\mathcal{H}(u, h):=\left\langle u-\Phi_{h}, \tau_{h}^{*}\right\rangle=0
$$

near $(\Phi, 0)$. Clearly, $\mathcal{H}(\Phi, 0)=0$. Moreover, since $\Phi_{h}$ and $\tau_{h}^{*}$ are smooth in $h$, the map $\mathcal{H}$ is continuously differentiable in $h \in \mathbb{R}$ and $u \in L^{2}$. Since $\partial_{z} \Phi=-\tau$,

$$
\left.\partial_{h} \mathcal{H}(u, h)\right|_{u=\Phi}=\left\langle\tau, \tau^{*}\right\rangle=1 .
$$

By the Implicit Function Theorem, there is a unique solution $h=h(u)$ in a neighborhood $\mathcal{U}$ of $\Phi$, which is continuously differentiable in $u$ and satisfies $h(0)=0$. Since $\Phi$ is smooth, also $v(u)=u-\Phi_{h(u)}$ is smooth. The tubular neighborhood $\mathcal{W}$ is the union of all translates of $\mathcal{U}$.
(ii) Apply the Implicit Function Theorem to $\mathcal{H}(u, h):=\left\langle u-\Phi_{h}, \tau^{*}\right\rangle$.

Fix a pulse $\Phi \in \mathcal{M}$, and let $\mathcal{U}$ be the neighborhood constructed in the second part of Proposition 3.1. Consider a mild solution $u$ of Eq. (1.9) on $\mathcal{U}$. By Eq. (3.3), we can represent it uniquely as the superposition of a modulated pulse $\Phi_{h(t)}$ and a transversal fluctuation $v(t)$

$$
\begin{equation*}
u(t)=\Phi_{h(t)}+v(t) \quad \text { with } \quad P v(t)=0 . \tag{3.4}
\end{equation*}
$$

Since $\Phi_{h}$ is a stationary solution of Eq. (1.10), we have $G\left(\Phi_{h}\right)=0$. Its Taylor expansion about $\Phi_{h}$ is given by $G\left(\Phi_{h}+v\right)=L_{h} v+N_{h}(v)$, where $L_{h}=d G\left(\Phi_{h}\right)$ is as in Eq. (2.1) but with $\Phi_{h}$ in place of $\Phi$, and

$$
\begin{equation*}
N_{h}(v)=\binom{v_{1}^{2}\left(\alpha+1-3\left(\phi_{h}\right)_{1}-v_{1}\right)}{0} . \tag{3.5}
\end{equation*}
$$

Note that $N_{h}$ differs from the nonlinearity $N$ in Eq. (1.13) by a bounded multiplication operator that decays as $z \rightarrow \pm \infty$.

Assume for the moment that $u$ is a classical solution of Eq. (1.9). Substituting Eq. (3.4) into Eq. (3.1) and using that $\partial_{t} \Phi_{h}(z)=\dot{h} \tau_{h}$, we obtain

$$
\dot{h}(t) \tau_{h}+\partial_{t} v=G\left(\Phi_{h(t)}+v\right)=L_{h} v+N_{h}(v) .
$$

We next apply the spectral projections $P$ and $Q$. Since $P \partial_{t} v=0$, we have by the chain rule

$$
\begin{equation*}
\left\langle\tau_{h}, \tau^{*}\right\rangle \dot{h}=\left\langle L_{h} v+N_{h}(v), \tau^{*}\right\rangle . \tag{3.6}
\end{equation*}
$$

The complementary projection yields

$$
\partial_{t} v=Q\left(L_{h} v+N_{h}(v)-\dot{h} \tau_{h}\right) .
$$

In general, if $u$ is a mild solution of Eq. (1.10), we interpret $v$ as a mild solution of the equation

$$
\begin{equation*}
v(t)=e^{t L} v_{0}+\int_{0}^{t} e^{(t-s) L} Q\left(L_{h(s)} v(s)+N_{h(s)}(v(s))-\dot{h}(s) \tau_{h(s)}\right) d s \tag{3.7}
\end{equation*}
$$

By the same argument as in Eq. (1.15), the nonlinearity $N_{h}$ is locally Lipschitz on $H^{2,1}$. We will need the following refined estimate that takes advantage of the fact that $N_{h}$ vanishes quadratically at $v=0$.

Lemma 3.3 (Small Lipschitz estimate). For any $\eta>0$ there exists a constant $C_{\eta}>0$ such that the nonlinearity $N_{h}(v)$ defined in Eq. (3.5) satisfies

$$
\begin{equation*}
\left\|N_{h}(v)-N_{h}(w)\right\|_{2,1} \leq C_{\eta} \max \left\{\left\|v_{1}\right\|_{H^{2}},\left\|w_{1}\right\|_{H^{2}}\right\}\left\|v_{1}-w_{1}\right\|_{H^{2}} \tag{3.8}
\end{equation*}
$$

for all $v, w$ with $\left\|v_{1}\right\|_{H^{2}},\left\|w_{1}\right\|_{H^{2}} \leq \eta$ and all $h \in \mathbb{R}$.
Proof. We expand the first component of $N_{h}$ as
$\left(N_{h}(v)\right)_{1}-\left(N_{h}(w)\right)_{1}=\left(\left(\alpha+1-3\left(\phi_{h}\right)_{1}\right)\left(v_{1}+w_{1}\right)-\left(v_{1}^{2}+v_{1} w_{1}+w_{1}^{2}\right)\right)\left(v_{1}-w_{1}\right)$.
Eq. (3.8) follows directly from the continuity of the multiplication in $H^{2}$ and the fact that $\phi_{1} \in H^{2}$.

Lemma 3.4 (Evolution inequalities). With the notation and assumptions of Proposition 2.1, suppose $h(t)$ and $v(t)$ satisfy Eqs. (3.6)-(3.7) on some interval $[0, T]$, and that

$$
|h(t)| \leq \kappa, \quad\|v(t)\|_{2,1} \leq \eta
$$

for all $0 \leq t \leq T$, where $\kappa>0$ is sufficiently small, and $\eta>0$. Then there exists a constant $C>0$ (depending on $\kappa$ and $\eta$ ) such that

$$
\begin{equation*}
|\dot{h}| \leq C\left(|h|+\|v\|_{2,1}\right)\|v\|_{2,1}, \quad(0 \leq t \leq T) \tag{3.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\|v(t)\|_{2,1} \leq C_{0} e^{-\sigma t}\left\|v_{0}\right\|_{2,1}+C \int_{0}^{t} e^{-\sigma(t-s)}\left(|h(t)|+\|v(t)\|_{2,1}\right)\|v(t)\|_{2,1} d s \tag{3.10}
\end{equation*}
$$

Proof. Choose $\kappa>0$ such that $\left\langle\tau_{h}, \tau^{*}\right\rangle \geq \frac{1}{2}$ for all $h$ with $|h| \leq \kappa$. This is possible because $\left\langle\tau, \tau^{*}\right\rangle=1$, and $\left\langle\tau_{h}, \tau^{*}\right\rangle$ depends smoothly on $h$ by Lemma 3.2. For $|h| \leq \kappa$, Eq. (3.6) yields

$$
|\dot{h}| \leq 2\left(\left|\left\langle L_{h} v, \tau^{*}\right\rangle\right|+\left|\left\langle N_{h}(v), \tau^{*}\right\rangle\right|\right) .
$$

Since $L^{*} v=0$, the first summand is bounded by

$$
\left|\left\langle L_{h} v, \tau^{*}\right\rangle\right|=\left|\left\langle v,\left(L_{h}^{*}-L^{*}\right) \tau^{*}\right\rangle\right| \leq C|h|\|v\|_{2,1}
$$

for some constant $C$. By Lemma 3.3, the second summand satisfies $\left\|N_{h}(v)\right\|_{2,1} \leq$ $C_{\eta}\|v\|_{2,1}^{2}$. Combining these two inequalities yields the bound on $|\dot{h}|$.

For $v(t)$, we separately estimate each term on the right hand side of Eq. (3.7). In the integrand, we use that $L v=0$ and $\left\|L_{h}-L\right\|_{2,1}$ is of order $|h|$ by Lemma 3.2. For the nonlinearity, we use Lemma 3.3, and for the last term, we use Eq. (3.9). The result is

$$
\|v(t)\|_{2,1} \leq\left\|e^{t L} Q\right\|_{2,1}\left\|v_{0}\right\|_{2,1}+C \int_{0}^{t}\left\|e^{-(t-s) L} Q\right\|_{2,1}\left(|h(s)|+\|v(s)\|_{2,1}\right)\|v(s)\|_{2,1} d s
$$

The proof is completed with Proposition 2.1.
We end this subsection with a differential inequality that will be used below.
Lemma 3.5 (Estimates on $h$ ). Let $y$ be a nonnegative, nondecreasing function on $[0, t]$, and let $C, \xi, \eta$ be nonnegative constants. If $h$ satisfies the differential inequality

$$
|\dot{h}| \leq C e^{-\xi t} y(|h|+y) \quad(0 \leq t \leq T)
$$

with $h(0)=0$, and $y(T) \leq \eta$, then

$$
|h(t)| \leq C_{1} y(t), \quad|\dot{h}(t)| \leq C_{2} e^{-\xi t} y(t)^{2} \quad(0 \leq t \leq T)
$$

where $C_{1}=\left(e^{\frac{C \eta}{\xi}}-1\right), C_{2}=C e^{\frac{C \eta}{\xi}}$.
Proof. Fix $t_{0} \in(0, T]$. Since $y$ is non-decreasing, $|\dot{h}(s)| \leq C y(t)(|h(t)|+y(t))$ for all $0 \leq s \leq t$. We separate variables and integrate from $h(0)=0$ to obtain

$$
\log \left(\frac{|h(s)|+y(t)}{y(t)}\right) \leq \frac{C y(t)}{\xi}\left(1-e^{-\xi t}\right) \leq \frac{C \eta}{\xi} \quad(0 \leq s \leq t)
$$

which yields the first claim after solving for $h(s)$ and setting $s=t$. The second claim follows by substituting this bound back into the differential inequality.

### 3.2 Proof of Theorem 1.1

Let $L$ be the linearization about $\Phi$ in the moving frame, defined in Eq. (2.1), and $\xi \in(0, \sigma)$. We will construct a neighborhood $\mathcal{U}$ of $\Phi$ in $H^{2,1}$ such that for every solution $u(t)$ with initial value $u_{0} \in \mathcal{U}$ there is a real-valued function $h(t)$ such that

$$
\left\|u(t)-\Phi_{h(t)}\right\| \lesssim e^{-\xi t}\left\|u_{0}-\Phi\right\|_{2,1} \quad(t \geq 0)
$$

where $|h(0)| \lesssim\left\|u_{0}-\Phi\right\|_{2,1}$, and there exists $h_{*} \in \mathbb{R}$ such that

$$
\left|h(t)-h_{*}\right| \lesssim e^{-\xi t}\left\|u_{0}-\Phi\right\|_{2,1}^{2} \quad(t \geq 0)
$$

Transforming back to the static frame, this will prove the theorem.
By Eq. (3.2) of Proposition 3.1, there is a neighborhood $\mathcal{U}$ of $\Phi$ such that each $u_{0} \in \mathcal{U}$ can be written uniquely as $u_{0}=\Phi_{h_{0}}+v_{0}$, where $P_{h_{0}} v=0$. Since $h_{0}$ depends smoothly on $u$,

$$
\left|h_{0}\right| \lesssim\left\|u_{0}-\Phi\right\|_{2,1}, \quad\left\|v_{0}\right\|_{2,1}=\left\|u_{0}-\Phi_{h_{0}}\right\|_{2,1} \lesssim\left\|u_{0}-\Phi\right\|_{2,1}
$$

Replacing $u_{0}$ with its translate $\left(u_{0}\right)_{-h_{0}}$, we may assume that $h_{0}=0$, that is,

$$
u_{0}=\Phi+v_{0}, \quad P v_{0}=0 .
$$

By the second part of Proposition 3.1, the solution of Eq. (1.9) with initial value $u_{0}$ can be written uniquely as

$$
u(t)=\Phi_{h(t)}+v(t), \quad P v(t)=0
$$

so long as $u(t) \in \mathcal{U}$. The functions $h(t)$ and $v(t)$ satisfy inequalities (3.6) and (3.7) with initial values $h(0)=0$ and $v(0)=v_{0}$.

Since the map $u \mapsto(h, v)$ is a diffeomorphism from $\mathcal{U}$ to a neighborhood of the origin in $\mathbb{R} \times \operatorname{Ran}(Q)$, by replacing $\mathcal{U}$ with a smaller neighborhood we may assume that it has the form $\mathcal{U}=\left\{\Phi_{h}+v\left|(h, v) \in \mathbb{R} \times \operatorname{Ran}(Q),|h|<\kappa,\|v\|_{2,1}<\eta\right\}\right.$, where $\kappa$ is so small that $\left\langle\tau_{h}, \tau^{*}\right\rangle \geq \frac{1}{2}$ whenever $|h| \leq \kappa$. The value of $\eta>0$ will be further specified below.

Let $\sigma$ be the exponent from Proposition 2.1, and let $C_{0}$ be the multiplicative constant. Choose $\xi \in(0, \sigma)$, define the monotonically increasing function

$$
y(t):=\sup _{0 \leq s \leq t} e^{-\xi s}\|v(s)\|_{2,1},
$$

and let

$$
T:=\inf \left\{t \geq 0| | h(s) \mid \geq \kappa \text { or }\|y(s)\|_{2,1} \geq \eta\right\}
$$

Assume that $\left\|v_{0}\right\|_{2,1}<\frac{\eta}{2 C}$, and apply Lemma 3.4. By Eq. (3.9),

$$
|\dot{h}(t)| \leq C e^{-\xi t}(|h(t)|+y(t)) y(t), \quad(0 \leq t \leq T)
$$

where we have used that $\|v(t)\|_{2,1} \leq e^{-\xi t} y(t)$ by definition of $y$. It follows by Lemma 3.5 that $h(t) \leq C_{\eta} y(t)$ for $0 \leq t \leq T$ for some constant $C_{\eta}$. Since $y(t)<\eta$ for $t<T$, by reducing the value of $\eta$ we can achieve that $|h(t)|<\kappa$ for all $t \in[0, T)$. Inserting this estimate into Eq. (3.10) yields

$$
y(t) \leq C_{0} e^{-(\sigma-\xi) t}\left\|v_{0}\right\|_{2,1}+C \int_{0}^{t} e^{-(\sigma-\xi)(t-s)} y^{2}(s) d s
$$

where $C$ is the product of $C_{0}, C_{\eta}$, and the constant from Lemma 3.3. Since $y$ is nondecreasing, taking it out of the integral yields the upper bound

$$
\begin{equation*}
y(t) \leq C_{0}\left\|v_{0}\right\|_{2,1}+C y^{2}(t) \quad(0 \leq t \leq T) \tag{3.11}
\end{equation*}
$$

with a suitably adjusted constant $C$.
Consider the quadratic polynomial $p(y):=C_{0}\left\|v_{0}\right\|_{2,1}-y+C y^{2}$. If $d:=$ $4 C_{0} C\left\|v_{0}\right\|_{2,1}<1$, then $P$ has two positive real roots, and is positive on the interval between them. The smaller root satisfies

$$
y_{*}=\frac{1}{2 C}\left(1-\sqrt{1-4 C_{0} C\left\|v_{0}\right\|_{2,1}}\right) \leq 2 C\left\|v_{0}\right\|_{2,1}<\eta .
$$

Since $C_{0} \geq 1$, we have that $\left\|v_{0}\right\|_{2,1}<y_{*}$. Eq. (3.11) implies, by continuity, that $y(t) \leq y_{*}$ for all $0 \leq t \leq T$. If $T<\infty$, then by continuity also $y(T) \leq y_{*}<\eta$, contradicting the definition of $T$. Hence $T=+\infty$, and

$$
\|v(t)\|_{2,1} \leq e^{-\xi t} y(t) \leq 2 C_{0} e^{-\xi t}\left\|v_{0}\right\|_{2,1} \quad(t \geq 0)
$$

Since $|h(t)| \leq \kappa$ and $\|v(t)\|<\eta$, we conclude that $\Phi_{h(t)}+v(t) \in \mathcal{U}$ for all $t \geq 0$, and $\|v(t)\|_{2,1}$ converges exponentially to zero.

To show that $h(t)$ converges as well, we use again Lemma 3.5 to see that

$$
|\dot{h}(t)| \leq C e^{-\xi t}\left\|v_{0}\right\|_{2,1}^{2}
$$

It follows that $h(t)$ converges exponentially to a limit, $h_{*}$, with $\left|h_{*}\right| \lesssim\left\|v_{0}\right\|_{2,1}^{2}$. Since $\left\|v_{0}\right\|_{2,1} \lesssim\left\|u_{0}-\Phi\right\|_{2,1}$, this proves the estimate for $h$. The proof of the theorem is completed by shrinking the neighborhood once more, to

$$
\mathcal{U}=\left\{\Phi_{h}+v\left|(h, v) \in \mathbb{R} \times \operatorname{Ran}(Q),|h|<\kappa,\|v\|_{2,1}<\frac{\eta}{2 C}\right\}\right.
$$

## 4 Near-pulse solutions on warped cylinders

This section is dedicated to the proof of Theorem 1.2. Consider the FHNcyl system (1.2) on a warped cylinder $\mathcal{S}_{\rho}$, given by Eq. (1.10).

In the special case where $\rho \equiv R$, Eq. (1.10) equivalent to Eq. (1.9), expressed in the static frame. The pulse defines a traveling wave solution $\Phi(x-c t)$ on $\mathcal{S}_{R}$. As discussed in the introduction, the proof of Theorem 1.2 relies on a perturbation estimate that controls the dependence of solutions on $\rho$. The size of the perturbation is measured in terms of the essential parameter $\delta:=R^{-1}\|\rho-R\|_{C^{2}}$.

Proposition 4.1 (Perturbation of the radius). Suppose that $u \in C\left([0, T], H^{2,1}\right)$ is a mild solution of Eq. (1.10) on a standard cylinder $\mathcal{S}_{R}$, with initial value $u_{0}:=\left.u\right|_{t=0}$. There are positive constants $\delta_{*}$ and $C$ (which depend on $T$ and on $\left.\sup _{0 \leq t \leq T}\|u(t)\|_{2,1}\right)$ such that if $0<R \leq 1$ and $\delta:=R^{-1}\|\rho-R\|_{C^{2}} \leq \delta_{*}$, then the unique mild solution of Eq. (1.10) on $\mathcal{S}_{\rho}$ with initial values $\left.u_{\rho}\right|_{t=0}=u_{0}$ satisfies

$$
\sup _{0 \leq t \leq T}\left\|u_{\rho}(t)-u(t)\right\|_{2,1} \leq C \delta
$$

The Riemannian structure on $\mathcal{S}_{\rho}$ induces an alternative inner product on $L^{2}\left(\mathcal{S}_{\rho}\right)$,

$$
\begin{equation*}
\langle u, w\rangle_{\rho}:=\int_{\mathcal{S}_{\rho}}\left(u_{1} \bar{w}_{1}+\varepsilon^{-1} u_{2} \bar{w}_{2}\right) d \mu_{\rho} \tag{4.1}
\end{equation*}
$$

where $d \mu_{\rho}=\sqrt{g} d \theta d x$ is the Riemannian area element whose density is determined by $g=\rho^{2}\left(1+\rho^{\prime 2}\right)$. The corresponding norm will be denoted by $\|\cdot\|_{\rho}$. We also define the mixed Sobolev spaces

$$
\begin{equation*}
H^{2 k, \ell}\left(\mathcal{S}_{\rho}\right):=\left\{u \in L^{2} \mid\left(\Delta_{\mathcal{S}_{\rho}}\right)^{k} u_{1} \in L^{2}\left(\mathcal{S}_{\rho}\right),\left(\partial_{x}\right)^{\ell} u_{2} \in L^{2}\left(\mathcal{S}_{\rho}\right)\right\} \tag{4.2}
\end{equation*}
$$

for $k, \ell=0,1$, with norms

$$
\|u\|_{2 k, \ell ; \rho}:=\sum_{0 \leq i \leq k}\left\|\left(\Delta_{\mathcal{S}_{\rho}}\right)^{i} u_{1}\right\|_{\rho}+\varepsilon^{-1} \sum_{0 \leq j \leq \ell}\left\|\partial_{x}^{j} u_{2}\right\|_{\rho}
$$

For $k=1, \ell=0$, the space $H^{2,0}$ agrees with the corresponding Sobolev space $H^{2} \times L^{2}$, and $H^{0,0}=L^{2}$. However, for $\ell=1$, since $H^{2 k, 1}$ places no condition on $\partial_{\theta} u_{2}$, the space $H^{0,1}$ properly contains $L^{2} \times H^{1}$, and $H^{2,1}$ properly contains $H^{2} \times H^{1}$. On the standard cylinder $\mathcal{S}_{R}$, Eq. (4.2) with $k=\ell=1$ coincides with the definition of $H^{2,1}$ in Eq. (2.3).

We will show in Lemma 4.4 that the norms $\|\cdot\|_{2 k, \ell ; \rho}$ are equivalent to $\|\cdot\|_{2 k, \ell}$. Hence the cylindrical surface $\mathcal{S}_{\rho}$ will be omitted from the notation whenever this is possible without causing confusion.

### 4.1 The linear semigroup

The linearization of Eq. (1.10) about zero is given by the Gâteaux derivative $A_{\rho}=$ $d F_{\rho}(0)$. We start with some basic properties of $A_{\rho}$. Throughout this subsection, $\rho$ is fixed subject to the standing assumption. The domain of $A_{\rho}$ (as an operator on $\left.L^{2}\left(\mathcal{S}_{\rho}\right)\right)$ is $H^{2,0}\left(\mathcal{S}_{\rho}\right)$, and its graph norm is equivalent to $\|\cdot\|_{2,0 ; \rho}$. Since

$$
\operatorname{Re}\left\langle A_{\rho} u, u\right\rangle_{\rho} \leq-\sigma\|u\|_{\rho}^{2}, \quad u \in H^{2,1}\left(\mathcal{S}_{\rho}\right)
$$

where $\sigma=\{\alpha, \varepsilon \gamma\}$, the graph norm of $A_{\rho}$ is equivalent to $\left\|A_{\rho} u\right\|_{\rho}$,

$$
\begin{equation*}
\|u\|_{2,0 ; \rho} \lesssim\left\|A_{\rho} u\right\|_{\rho} \lesssim\|u\|_{2,0 ; \rho} . \tag{4.3}
\end{equation*}
$$

In the same way as for the operator $\bar{L}$ in Lemma 2.6, it follows with Lemma 2.4 that $A_{\rho}$ generates a strongly continuous, exponentially decaying semigroup on $L^{2}\left(\mathcal{S}_{\rho}\right)$ that has $H^{2,0}\left(\mathcal{S}_{\rho}\right)$ as an invariant subspace.

We want to work in the subspace $H^{2,1}$ that was used for Theorem 1.1. To this end, we first restrict $A_{\rho}$ to the intermediate subspace $H^{0,1}\left(\mathcal{S}_{\rho}\right)$.
Lemma 4.2 (Domain of $A_{\rho}$ in $H^{0,1}$ ). Let $\alpha, \gamma, \varepsilon$ be fixed positive constants, and let $\rho$ be a positive function of class $C^{2}$ on the real line. Then the operator $A_{\rho}$ maps $H^{2,1}\left(\mathcal{S}_{\rho}\right)$ bijectively onto $H^{0,1}\left(\mathcal{S}_{\rho}\right)$, and

$$
\begin{equation*}
\|u\|_{2,1 ; \rho} \lesssim\left\|A_{\rho} u\right\|_{0,1 ; \rho} \lesssim\|u\|_{2,1 ; \rho} \quad\left(u \in H^{2,1}\left(\mathcal{S}_{\rho}\right)\right) \tag{4.4}
\end{equation*}
$$

Proof. Fix $\rho$ as in the assumptions, and let $u \in H^{2,1}\left(\mathcal{S}_{\rho}\right)$. To simplify notation, we momentarily suppress the dependence of the spaces and norms on $\rho$ in the notation.

Write $\left\|A_{\rho} u\right\|_{0,1}=\left\|A_{\rho} u\right\|+\left\|\partial_{x}\left(\varepsilon u_{1}-\varepsilon \gamma u_{2}\right)\right\|$, and combine the upper bound in Eq. (4.3) with the estimates $\left\|\partial_{x} u_{1}\right\| \leq\|u\|_{2,1}$ and $\left\|\partial_{x} u_{2}\right\| \leq\|u\|_{0,1}$. It follows that $A_{\rho} u \in H^{0,1}$, and the upper bound in Eq. (4.4) holds. In particular, $A_{\rho}$ maps $H^{2,1}$ to $H^{0,1}$. By Eq. (4.3), this map is injective.

To show that this map is also surjective, let $w \in H^{0,1}$. Since $w \in L^{2}$, the equation $A_{\rho} u=w$ has a unique solution $u \in H^{2,0}$. The lower bound in Eq. (4.3) yields $u_{1} \in H^{2}$, and $\left\|u_{1}\right\|_{H^{2}} \lesssim\left\|A_{\rho} u\right\| \leq\left\|A_{\rho} u\right\|_{0,1}$. For the second component, we use that $\varepsilon u_{1}-\varepsilon \gamma u_{2}=w_{2}$, and estimate

$$
\left\|\partial_{x} u_{2}\right\| \leq \gamma^{-1}\left\|\partial_{x} u_{1}\right\|+(\varepsilon \gamma)^{-1}\left\|\partial_{x} w_{2}\right\| \lesssim\|u\|_{2,0}+\|w\|_{0,1} \lesssim\left\|A_{\rho} u\right\|_{0,1} .
$$

This proves surjectivity, and the lower bound.
A useful consequence of Lemma 4.2 is that

$$
\left\|B_{0,1 ; \rho}\right\| \lesssim\|B\|_{2,1 ; \rho} \lesssim\|B\|_{0,1 ; \rho}
$$

for every bounded linear operator $B$ on $H^{0,1}\left(\mathcal{S}_{\rho}\right)$ that commutes with $A_{\rho}$. The next lemma provides spectral estimates on $A_{\rho}$ that are needed to construct the semigroup $e^{t A_{\rho}}$ on $H^{0,1}\left(\mathcal{S}_{\rho}\right)$.
Lemma 4.3 ( $A_{\rho}$ is sectorial). Let $\alpha, \gamma, \varepsilon$ be positive constants, and let $\rho$ be a real-valued function on $\mathbb{R}$ that is bounded and bounded away from zero. Then $A_{\rho}$ generates an analytic semigroup $e^{t A_{\rho}}$ on $H^{2,1}$. The spectrum of $A_{\rho}$ on $H^{0,1}\left(\mathcal{S}_{\rho}\right)$ is contained in the truncated sector

$$
\Sigma:=\left\{\lambda \in \mathbb{C} \left\lvert\, \operatorname{Re} \lambda \leq-\sigma \min \left\{1, \varepsilon^{-\frac{1}{2}}|\operatorname{Im} \lambda|\right\}\right.\right\}
$$

where $\sigma:=\min \{\alpha, \varepsilon \gamma\}$. Moreover, we have the resolvent estimate

$$
\begin{equation*}
\left\|(\lambda-A)^{-1}\right\|_{2 k, 1 ; \rho} \leq C\left(1+\sup \left|\rho^{\prime}\right|\right) \min \left\{1,|\lambda|^{-1}\right\}, \quad(k=0,1) \tag{4.5}
\end{equation*}
$$

for all $\lambda$ with $\operatorname{Re} \lambda \geq-\frac{1}{2} \sigma \min \left\{1, \varepsilon^{-\frac{1}{2}}|\operatorname{Im} \lambda|\right\}$, for some constant $C$. The semigroup satisfies

$$
\sup _{t>0}\left\|e^{t A_{\rho}}\right\|_{2,1} \leq C\left(1+\sup \left|\rho^{\prime}\right|\right) e^{-\sigma t}
$$

Proof. We will bound the numerical range of $A_{\rho}$ with respect to a certain weighted inner product, and then apply [30, Theorem 1.3.9]. As in the proof of Lemma 4.2, $\rho$ is fixed and will be suppressed in the notation. For $s>0$, define

$$
\langle u, v\rangle_{s}:=\left\langle u_{1}, v_{1}\right\rangle+\varepsilon^{-1}\left\langle u_{2}, v_{2}\right\rangle+s \varepsilon^{-1}\left\langle\partial_{x} u_{2}, \partial_{x} v_{2}\right\rangle .
$$

The corresponding norm $\|\cdot\|_{s}$ is equivalent to the norm $\|\cdot\|_{0,1}$ from Eq. (4.2),

$$
\min \left\{1, s^{\frac{1}{2}}\right\}\|u\|_{0,1} \leq\|u\|_{s} \leq \max \left\{1, s^{\frac{1}{2}}\right\}\|u\|_{0,1}, \quad(s>0)
$$

We compute

$$
\begin{align*}
& \operatorname{Re}\left\langle A_{\rho} u, u\right\rangle_{s}=-\alpha\left\|u_{1}\right\|^{2}-\gamma\left\|u_{2}\right\|^{2}+\left\langle\Delta_{\mathcal{S}_{\rho}} u_{1}, u_{1}\right\rangle+s \operatorname{Re}\left\langle\partial_{x} u_{1}-\gamma \partial_{x} u_{2}, \partial_{x} u_{2}\right\rangle \\
& \quad \leq-\sigma\|u\|^{2}-\frac{1}{1+\sup \left|\rho^{\prime}\right|^{2}}\left\|\partial_{x} u_{1}\right\|^{2}+s\left\|\partial_{x} u_{1}\right\|\left\|\partial_{x} u_{2}\right\|-s \gamma\left\|\partial_{x} u_{2}\right\|^{2} \\
& \quad \leq-\sigma\|u\|_{s}^{2}+\frac{s^{2}}{4}\left(1+\sup \left|\rho^{\prime}\right|^{2}\right)\left\|\partial_{x} u_{2}\right\|^{2} . \tag{4.6}
\end{align*}
$$

Note that the inner products and norms on the right hand side of the first line are the standard Riemannian ones for scalar functions in $L^{2}\left(\mathcal{S}_{\rho}\right)$. In the second line, we have integrated the Laplacian term by parts. The last step follows by completing the square. For any $q \in(0,1)$, we can achieve $\operatorname{Re}\left\langle A_{\rho} u, u\right\rangle_{s} \leq-q \sigma\|u\|_{s}^{2}$ by choosing $s$ sufficiently small. By Lemma $2.4, A_{\rho}$ generates a semigroup of contractions with respect to the norm $\|\cdot\|_{s}$. Moreover, the spectrum of $A_{\rho}$ is contained in each of the half-planes $\{\operatorname{Re} \lambda \leq-q \sigma\}$, and hence in their intersection.

Likewise,

$$
\begin{aligned}
\operatorname{Im}\left\langle A_{\rho} u, u\right\rangle_{s} & =2 \operatorname{Im}\left\langle u_{1}, u_{2}\right\rangle+s \operatorname{Im}\left\langle\partial_{x} u_{1}, \partial_{x} u_{2}\right\rangle \\
& \leq \sqrt{\varepsilon}\|u\|^{2}+s\left\|\partial_{x} u_{1}\right\|\left\|\partial_{x} u_{2}\right\| .
\end{aligned}
$$

Comparing with the second line of Eq. (4.6), we see that for $s>0$ sufficiently small

$$
\operatorname{Re}\left\langle A_{\rho} u, u\right\rangle_{s} \leq-\sigma \varepsilon^{-\frac{1}{2}}\left|\operatorname{Im}\left\langle A_{\rho} u, u\right\rangle_{s}\right| .
$$

Since the resolvent set of $A_{\rho}$ contains 0, by [30, Theorem 1.3.9], it contains the entire complement of the sector $\left\{\operatorname{Re} \lambda \leq-\sigma \varepsilon^{-\frac{1}{2}}|\operatorname{Im} \lambda|\right\}$. In summary, for $s$ sufficiently small, the numerical range of $A_{\rho}$ with respect to $\langle\cdot, \cdot\rangle_{s}$ lies in

$$
\Sigma_{q}:=\left\{\lambda \in \mathbb{C} \left\lvert\, \operatorname{Re} \lambda \leq-\sigma \min \left\{q, \varepsilon^{-\frac{1}{2}}|\operatorname{Im} \lambda|\right\}\right.\right\}
$$

Moreover, the resolvent satisfies

$$
\left\|(\lambda-A)^{-1}\right\|_{s} \leq\left(\inf _{z \in \Sigma_{q}}\|\lambda-z\|_{s}\right)^{-1}, \quad\left(\lambda \notin \Sigma_{q}\right)
$$

To obtain Eq. (4.5), we choose $q=\frac{3}{4}, s=\gamma\left(1+\sup \left|\rho^{\prime}\right|^{2}\right)^{-1}$, and compare $\|\cdot\|_{s}$ with $\|\cdot\|_{0,1}$. Since the resolvent commutes with $A_{\rho}$, by Lemma 4.2, the estimate holds, with a suitably adjusted constant, also for the norm $\|\cdot\|_{2,1}$. By Lemma 2.4, the bound on the semigroup follows from the dissipativity of $A_{\rho}$.

### 4.2 Comparison with the standard cylinder

In this subsection we compare solutions of the FHNcyl system on $\mathcal{S}_{\rho}$ with solutions on the standard cylinder $\mathcal{S}_{R}$. For this purpose, we consider functions $u(x, \theta)$ on $\mathcal{S}_{\rho}$ as functions on $\mathcal{S}_{R}$, via the coordinate diffeomorphism $\psi_{\rho}: \mathcal{S}_{R} \rightarrow \mathcal{S}_{\rho}$ defined in Eq. (1.7). We next show that the composition $u \mapsto u \circ \psi_{\rho}$ induces a bounded linear transformation from $H^{2,1}\left(\mathcal{S}_{\rho}\right)$ to $H^{2,1}\left(\mathcal{S}_{R}\right)$. With a slight abuse of notation, we identify the composition $u \circ \psi_{\rho}$ (viewed as a function on the standard cylinder $\mathcal{S}_{R}$ ) with $u$ itself (on $\mathcal{S}_{\rho}$ ), and the norm $\|\cdot\|_{2,1 ; \rho}$ (originally defined on $H^{2,1}\left(\mathcal{S}_{\rho}\right)$ ) with its pull-back to $H^{2,1}\left(\mathcal{S}_{R}\right)$.

Lemma 4.4 (Equivalence of Sobolev spaces). Let $\alpha, \gamma, \varepsilon$ be fixed positive constants, and let $\rho$ be a positive function of class $C^{2}$. Under the assumptions of Proposition 4.6, if $\delta:=R^{-1}\|\rho-R\|_{C^{2}} \leq \frac{1}{16}$, then

$$
\frac{1}{2}\|u\|_{2 k, \ell} \leq\|u\|_{2 k, \ell ; \rho} \leq 2\|u\|_{2 k, \ell} \quad(k, \ell \in\{0,1\})
$$

Proof. We will show that for every scalar-valued function $w$ on $\mathcal{S}_{\rho}$,

$$
\left\{\begin{array}{l}
\left|\|w\|_{L^{2}\left(\mathcal{S}_{\rho}\right)}-\|w\|_{L^{2}\left(\mathcal{S}_{R}\right)}\right| \leq 2 \delta\|w\|_{L^{2}\left(\mathcal{S}_{R}\right)}  \tag{4.7}\\
\left|\left\|\partial_{x} w\right\|_{L^{2}\left(\mathcal{S}_{\rho}\right)}-\left\|\partial_{x} w\right\|_{L^{2}\left(\mathcal{S}_{R}\right)}\right| \leq 2 \delta\left\|\partial_{x} w\right\|_{L^{2}\left(\mathcal{S}_{R}\right)} \\
\left|\left\|\Delta_{\mathcal{S}_{\rho}} w\right\|_{L^{2}\left(\mathcal{S}_{\rho}\right)}-\left\|\Delta_{\mathcal{S}_{R}} w\right\|_{L^{2}\left(\mathcal{S}_{R}\right)}\right| \leq 8 \delta\left(\left\|\Delta_{\mathcal{S}_{R}} w\right\|_{L^{2}\left(\mathcal{S}_{R}\right)}+\|w\|_{L^{2}\left(\mathcal{S}_{R}\right)}\right)
\end{array}\right.
$$

and then apply the triangle inequality.
For the first line, we have by Eq. (4.1)

$$
\|w\|_{\rho}^{2}=\int_{\mathbb{R}} \int_{S^{1}}|w|^{2} \sqrt{g(x)} d \theta d x
$$

where $g=\rho^{2}\left(1+\left(\rho^{\prime}\right)^{2}\right)$. The pointwise bound $|\sqrt{g(x)}-R| \leq 2 \delta$ yields the first line of Eq. (4.7). The second line follows by applying the first one to $\partial_{x} w$.

For the third line, we write the difference between the Laplacians as

$$
\Delta_{\mathcal{S}_{\rho}}-\Delta_{\mathcal{S}_{R}}=a(x) \partial_{x}^{2}+b(x) \partial_{x}+c(x)|\rho(x)-R|\left(R^{-2} \partial_{\theta}^{2}\right),
$$

where the coefficients are pointwise bounded by $0<a(x) \leq \frac{1}{2} R \delta,|b(x)| \leq \delta+R^{2} \delta^{2}$, and $c(x) \leq 6 R \delta$, see Eq. (1.6). Since $\delta \leq 1, R \leq 1$, and $\left\|\partial_{x} w\right\| \leq \frac{1}{2}\left(\left\|\Delta_{\mathcal{S}_{R}} w\right\|+\|w\|\right)$, we obtain with the triangle inequality that

$$
\begin{equation*}
\left\|\left(\Delta_{\mathcal{S}_{\rho}}-\Delta_{\mathcal{S}_{R}}\right) w\right\| \leq 6 \delta\left(\left\|\Delta_{\mathcal{S}_{R}} w\right\|+\|w\|\right) \tag{4.8}
\end{equation*}
$$

Using once more the triangle inequality, as well as the first line of Eq. (4.7), we arrive at

$$
\begin{aligned}
\left|\left\|\Delta_{\mathcal{S}_{\rho}} w\right\|_{\rho}-\left\|\Delta_{\mathcal{S}_{R}} w\right\|\right| & \leq\left\|\left(\Delta_{\mathcal{S}_{\rho}}-\Delta_{\mathcal{S}_{R}}\right) w\right\|+\left|\left\|\Delta_{\mathcal{S}_{\rho}} w\right\|_{\rho}-\left\|\Delta_{\mathcal{S}_{\rho}} w\right\|\right| \\
& \leq 8 \delta\left(\left\|\Delta_{\mathcal{S}_{R}} w\right\|+\|w\|\right) .
\end{aligned}
$$

When $\delta \leq \frac{1}{16}$, we can solve Eq. (4.7) for the norm on $L^{2}\left(\mathcal{S}_{\rho}\right)$ to obtain

$$
\frac{1}{2}\|w\|_{L^{2}\left(\mathcal{S}_{R}\right)} \leq\|w\|_{L^{2}\left(\mathcal{S}_{\rho}\right)} \leq 2\|w\|_{L^{2}\left(\mathcal{S}_{R}\right)}
$$

and likewise for $\left\|\partial_{x} w\right\|_{\rho}$ and $\left\|\Delta_{\mathcal{S}_{\rho}} w\right\|_{\rho}$. By the definition of the norms in Eqs. (2.3) and (4.2), this proves the claim.

From now on, we identify the spaces $H^{2 k, \ell}\left(\mathcal{S}_{\rho}\right)$ with $H^{2 k, \ell}\left(\mathcal{S}_{R}\right)$, and use the standard norms $\|\cdot\|_{2 k, \ell}$. Under the assumptions of Lemma 4.4, the inequalities in Lemmas 4.2 and 4.3 hold also for these norms.

The next lemma bounds the difference between the resolvents of $A_{\rho}$ and $A_{R}$.
Lemma 4.5 (Perturbation estimate for the resolvent). Let $\alpha, \gamma$ and $\varepsilon$ be positive constants, and $0<R \leq 1$. There exists a constant $C$ such that if $\delta:=R^{-1} \| \rho-$ $R \|_{C^{2}} \leq \frac{1}{16}$, then

$$
\left\|\left(\lambda-A_{\rho}\right)^{-1}-\left(\lambda-A_{R}\right)^{-1}\right\|_{2,1} \leq C \delta \min \left\{1,|\lambda|^{-1}\right\}
$$

for all $\lambda$ with $\operatorname{Re} \lambda \geq-\frac{1}{2} \sigma \min \left\{1, \varepsilon^{-\frac{1}{2}}|\operatorname{Im} \lambda|\right\}$.
Proof. If $\lambda$ is as in the statement of the lemma, then by Lemma 4.3 it lies in the resolvent set of both $A_{\rho}$ and $A_{R}$. To estimate the difference, we write

$$
A_{\rho}-A_{R}=\left(\begin{array}{cc}
\Delta_{\mathcal{S}_{\rho}}-\Delta_{\mathcal{S}_{R}} & 0 \\
0 & 0
\end{array}\right)=: W
$$

and apply the resolvent identity

$$
\left(\lambda-A_{\rho}\right)^{-1}-\left(\lambda-A_{R}\right)^{-1}=\left(\lambda-A_{R}\right)^{-1} W\left(\lambda-A_{\rho}\right)^{-1}
$$

For the factor on the right, we use Lemmas 4.4 and 4.2 to see that

$$
\left\|\left(\lambda-A_{\rho}\right)^{-1}\right\|_{2,1} \lesssim\left\|\left(\lambda-A_{\rho}\right)^{-1}\right\|_{2,1 ; \rho} \lesssim\left\|\left(\lambda-A_{\rho}\right)^{-1}\right\|_{0,1 ; \rho} \lesssim \min \left\{1,|\lambda|^{-1}\right\}
$$

The second inequality holds because the resolvent commutes with $A_{\rho}$. By Eq. (4.8), the middle factor maps $H^{2,1}$ into $H^{0,1}$ and satisfies

$$
\|W u\|_{0,1}=\left\|\left(\Delta_{\mathcal{S}_{\rho}}-\Delta_{\mathcal{S}_{R}}\right) u_{1}\right\| \lesssim \delta\|u\|_{2,1}
$$

for all $u \in H^{2,1}$. By Lemma 4.2, the factor on the left maps $H^{0,1}$ back into $H^{2,1}$, and

$$
\begin{aligned}
\left\|\left(\lambda-A_{R}\right)^{-1} u\right\|_{2,1} & \lesssim\left\|A_{R}\left(\lambda-A_{R}\right)^{-1} u\right\|_{0,1} \\
& \leq\|u\|_{0,1}+|\lambda|\left\|\left(\lambda-A_{R}\right)^{-1} u\right\|_{0,1} \\
& \lesssim\|u\|_{0,1}
\end{aligned}
$$

for all $u \in H^{0,1}$. In the second line we have written $A_{R}=-\left(\lambda-A_{R}\right)+\lambda$ and applied the triangle inequality, and in the last line we have used that $\left\|\left(\lambda-A_{R}\right)^{-1}\right\|_{0,1} \lesssim$ $\min \left\{1,|\lambda|^{-1}\right\}$ by Lemma 4.3.

Combining the inequalities for the three factors, we conclude that

$$
\left\|\left(\left(\lambda-A_{\rho}\right)^{-1}-\left(\lambda-A_{R}\right)^{-1}\right) u\right\|_{2,1} \lesssim \delta \min \left\{1,|\lambda|^{-1} \mid\right\}\|u\|_{2,1}
$$

for all $u \in H^{2,1}$, proving the claim.
Proposition 4.6 (Perturbation estimate for the semigroup). Let $\alpha, \gamma$ and $\varepsilon$ be positive constants. There exists a constant $C$ such that, if $0<R \leq 1$ and $\delta:=$ $R^{-1}\|\rho-R\|_{C^{2}} \leq \frac{1}{16}$, then the semigroup generated by $A_{\rho}$ on $H^{2,1}$ satisfies

$$
\left\|e^{t A_{R}}-e^{t A_{\rho}}\right\|_{2,1} \leq C \delta\left(1+\log t^{-1}\right)
$$

for all $t \geq 0$.
Proof. Let $\Gamma$ be the contour consisting of the two half-lines $\operatorname{Re} \lambda=-\frac{1}{2} \sigma \varepsilon^{-\frac{1}{2}}|\operatorname{Im} \lambda|$, traversed counterclockwise. By Lemma 4.3, $\Gamma$ encloses the spectrum of $A_{\rho}$ and $A_{R}$. Since $A_{\rho}$ is sectorial, the semigroup $e^{t A_{\rho}}$ is represented by the contour integral

$$
e^{t A_{\rho}}=\frac{1}{2 \pi i} \oint_{\Gamma} e^{\lambda t}\left(\lambda-A_{\rho}\right)^{-1} d \lambda,
$$

and correspondingly for $A_{R}$. Parametrizing $\Gamma$ by $\lambda(s)=-|s|+2 i \sigma^{-1} \sqrt{\varepsilon}$, we see that for each $t>0$, the integral converges absolutely with respect to the operator norm on $H^{2,1}$.

We estimate the difference from $e^{t A_{R}}$ by

$$
\begin{aligned}
\left\|e^{t A_{\rho}}-e^{t A_{R}}\right\|_{2,1} & =\left\|\frac{1}{2 \pi i} \oint_{\Gamma} e^{\lambda t}\left(\left(\lambda-A_{\rho}\right)^{-1}-\left(\lambda-A_{R}\right)^{-1}\right) d \lambda\right\|_{2,1} \\
& \leq \frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{t \operatorname{Re} \lambda(s)}\left\|\left(\lambda-A_{\rho}\right)^{-1}-\left(\lambda-A_{R}\right)^{-1}\right\|_{2,1}\left|\lambda^{\prime}(s)\right| d s \\
& \lesssim \delta \int_{0}^{\infty} e^{-t s} \min \left\{1, s^{-1}\right\} d s
\end{aligned}
$$

where we have applied Lemma 4.5 to the integrand in the last step. For $t \geq 1$, the integral is uniformly bounded. For $t<1$, we have

$$
\int_{0}^{\infty} e^{-t s} \min \left\{1, s^{-1}\right\} d s \leq 1+\int_{1}^{t^{-1}} s^{-1} d s+\int_{t^{-1}}^{\infty} t e^{-t s} d s \leq 2+\log t^{-1}
$$

proving the claim.
Next, we address the dependence of the nonlinear evolution on $\rho$.

### 4.3 Proof of Proposition 4.1

Given a solution $u(t)$ of the FHN system on $\mathcal{S}_{R}$, set $\eta=2 \sup _{0 \leq t \leq T}\|u(t)\|_{2,1}$. Let $u_{\rho}(t)$ be the solution on $\mathcal{S}_{\rho}$ with the same initial value, $u_{0}$. By definition of the mild solutions,

$$
u_{\rho}(t)-u(t)=\left(e^{t A_{\rho}}-e^{t A_{R}}\right) u_{0}+\int_{0}^{t}\left(e^{(t-s) A_{\rho}} N\left(u_{\rho}(s)\right)-e^{(t-s) A_{R}} N(u(s))\right) d s
$$

so long as both solutions exist. By Proposition 4.6, for $\delta \leq \frac{1}{16}$ the difference of the semigroups is bounded by

$$
\left\|e^{t A_{\rho}}-e^{t A_{R}}\right\| \leq C_{0} \delta\left(1+\log t^{-1}\right), \quad(t>0)
$$

with some constant $C_{0}$. We use the triangle inequality on the integrand, and then apply the semigroup estimate and Eq. (1.15),

$$
\begin{array}{rl}
\| e^{(t-s) A_{\rho}} N\left(u_{\rho}\right)-e^{(t-s) A_{R}} & N(u) \|_{2,1} \\
& \leq\left\|\left(e^{(t-s) A_{\rho}}-e^{(t-s) A_{R}}\right) N(u)\right\|_{2,1}+\left\|e^{(t-s) A_{\rho}}\left(N\left(u_{\rho}\right)-N(u)\right)\right\|_{2,1} \\
& \leq C_{0} \delta\left(1+\log t^{-1}\right)\|N(u)\|_{2,1}+C_{1}\left\|N\left(u_{\rho}\right)-N(u)\right\|_{2,1} \\
& \leq C_{0} C_{\eta} \delta\left(1+\log t^{-1}\right)\|u\|_{2,1}+C_{1} C_{\eta}\left\|u_{\rho}-u\right\|_{2,1},
\end{array}
$$

so long as $\left\|u_{\rho}(s)\right\| \leq \eta$. Here, $C_{1}=\sup _{t} e^{t A_{\rho}}$, see Lemma 4.3, and $C_{\eta}$ is the Lipschitz constant Eq. (1.15). For the integral, it follows that

$$
\left\|u_{\rho}(t)-u(t)\right\|_{2,1} \leq C_{2}(T) \delta\left\|u_{0}\right\|_{2,1}+C_{3} \int_{0}^{t}\left\|u_{\rho}(s)-u(s)\right\|_{2,1} d s
$$

where $C_{2}(t)=C_{1}\left(1+2 C_{\eta} t\right)\left(1+\log t^{-1}\right)$, and $C_{3}=C_{2} C_{\eta}$. In the bound on the nonlinearity, we have used that $\left\|u_{\rho}(t)\right\|_{2,1} \leq \frac{1}{2} \eta$ for $0 \leq t \leq T$. Set $C:=C_{2}(T) e^{C_{3} T}$. By Grönwall's inequality,

$$
\left\|u_{\rho}(t)-u(t)\right\|_{2,1} \leq C \delta\left\|u_{0}\right\|_{2,1}, \quad(0 \leq t \leq T)
$$

provided that $\sup _{0 \leq t \leq T}\left\|u_{\rho}(t)\right\|_{2,1} \leq \eta$. Since $\|u(t)\|_{2,1} \leq \frac{1}{2} \eta$ for $0 \leq t \leq T$, by the triangle inequality this is guaranteed by setting $\delta_{*}=\min \left\{\frac{1}{16}, \frac{\eta}{2 C}\right\}$.

We are now ready to construct the near-pulse solutions on warped cylinders.

### 4.4 Proof of Theorem 1.2

For reference, consider the FHNcyl system on a standard cylinder $\mathcal{S}_{R}$ in a neighborhood of $\mathcal{M}$. Fix a pulse $\Phi \in \mathcal{M}$. Under the assumptions of Theorem 1.1 there are constants $C_{0} \geq 1$ and $\xi_{0}>0$ and a neighborhood $\mathcal{U}$ of $\Phi$ in $H^{2,1}$ such that $\operatorname{dist}(u(t), \mathcal{M}) \leq C_{0} e^{-\xi_{0} t}\left\|u_{0}-\Phi\right\|_{2,1}$ for every solution with initial values in $\mathcal{U}$. By translation invariance, it follows that

$$
\begin{equation*}
\operatorname{dist}(u(t), \mathcal{M}) \leq C_{0} e^{-\xi_{0} t} \operatorname{dist}\left(u_{0}, \mathcal{M}\right) \tag{4.9}
\end{equation*}
$$

for all solutions with initial values in some tubular neighborhood $\mathcal{W}$ of $\mathcal{M}$. We take $\mathcal{W}$ to have the form

$$
\mathcal{W}=\left\{w \in H^{2,1} \mid \operatorname{dist}(w, \mathcal{M})<\eta\right\}
$$

for some $\eta>0$ with $C_{0} \eta \leq\|\Phi\|_{2,1}$. Set $T:=\frac{1}{\xi_{0}} \log \left(2 C_{0}\right)$, so that $C_{0} e^{-\xi_{0} T}=\frac{1}{2}$. By the triangle inequality,

$$
\sup _{0 \leq t \leq T}\|u(t)\| \leq \sup _{0 \leq t \leq T} \operatorname{dist}(u(t), \mathcal{M})+\sup _{\Phi_{h} \in \mathcal{M}}\left\|\Phi_{h}\right\|_{2,1} \leq 2\|\Phi\|_{2,1}
$$

for all solutions on $\mathcal{S}_{R}$ with initial value $\left.u\right|_{t=0} \in \mathcal{W}$.
Consider now the FHNcyl system on a warped cylinder $\mathcal{S}_{\rho}$ with $\delta:=R^{-1} \| \rho-$ $R \|_{C^{2}} \leq \delta_{*}$ (to be determined below). Given an initial condition $u_{0} \in \mathcal{W}$, let $u(t)$ be the mild solution of the reference system on $\mathcal{S}_{R}$ with $\left.u\right|_{t=0}=u_{0}$. Since $\|u(t)\|_{2,1} \leq 2\|\Phi\|_{2,1}$ for all $t \geq 0$, by Proposition 4.1 there is a value $\delta_{0}>0$ (determined by $T$ ) and $C>0$ such that

$$
\begin{equation*}
\sup _{0 \leq t \leq T}\left\|u_{\rho}(t)-u(t)\right\|_{2,1} \leq C \delta \tag{4.10}
\end{equation*}
$$

provided that $\delta \leq \delta_{0}$. Choose $\delta_{*}:=\min \left\{\delta_{0}, \frac{\eta}{2 C}\right\}$. Assuming that $\delta \leq \delta_{*}$, we combine Eq. (4.9) with Eq. (4.10) to see that

$$
\begin{align*}
\operatorname{dist}\left(u_{\rho}(t), \mathcal{M}\right) & \leq \operatorname{dist}(u(t), \mathcal{M})+\left\|u_{\rho}(t)-u(t)\right\|_{2,1} \\
& \leq C_{0} e^{-\xi_{0} t} \operatorname{dist}\left(u_{0}, \mathcal{M}\right)+C \delta  \tag{4.11}\\
& <2 C_{0} \eta
\end{align*}
$$

for all $t \in[0, T]$. It follows that $\left\|u_{\rho}(t)\right\|_{2,1} \leq 2\|\Phi\|_{2,1}$. Furthermore, by the choice of $\delta_{*}$ and $T$,

$$
\operatorname{dist}\left(u_{\rho}(T), \mathcal{M}\right) \leq \frac{1}{2} \operatorname{dist}\left(u_{0}, \mathcal{M}\right)+C \delta<\eta
$$

Therefore $u_{\rho}(T) \in \mathcal{W}$ whenever $u_{\rho}(0) \in \mathcal{W}$. We repeat the estimate to obtain inductively

$$
\operatorname{dist}\left(u_{\rho}((k+1) T), \mathcal{M}\right) \leq \frac{1}{2} \operatorname{dist}\left(u_{\rho}(k T), \mathcal{M}\right)+C \delta, \quad\left(k \in \mathbb{N}_{0}\right)
$$

Solving the recursion, we conclude that dist $\left(u_{\rho}(k T), \mathcal{M}\right) \leq 2^{-k} \operatorname{dist}\left(u_{0}, \mathcal{M}\right)+2 C \delta$, and further, by Eq. (4.11)

$$
\operatorname{dist}\left(u_{\rho}(t), \mathcal{M}\right) \leq 2^{-k} C_{0} \operatorname{dist}\left(u_{0}, \mathcal{M}\right)+\left(2+C_{0}\right) C \delta
$$

for all $t$ with $k T \leq t \leq(k+1) T$ and all $k \in \mathbb{N}_{0}$. By the choice of $T$, this implies Eq. (1.8) with $\xi=\frac{\log 2}{\log 2+\log C_{0}} \xi_{0}, C_{1}=2 C_{0}$, and $C_{2}=\left(2+C_{0}\right) C$.

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