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ON THE EXISTENCE OF SYMMETRIC BICIRCULAR CENTRAL CONFIGURATIONS OF THE $3n$–BODY PROBLEM

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Abstract. In this paper we consider central configurations of the planar $3n$–body problem consisting of n masses at the vertices of a regular n–gon inscribed in a circle of radius r and $2n$ masses at the vertices of a second (not necessarily regular) concentric $2n$–gon inscribed in a circle of radius ar which are symmetric in the sense that the set of positions of the $3n$ masses and the set of the corresponding masses are invariant under the action of a finite subgroup of $O(2)$. There are two different types of such configurations. In the first type, called *regular bicircular central configurations of the* $3n$*–body problem*, the second $2n$–gon is regular, n of the vertices of the second n–gon are aligned with the vertices of the first regular n–gon and the masses at the vertices of this $2n$–gon alternate values. In the second type, called *semiregular bicircular central configurations of the* $3n$*–body problem*, the second $2n$–gon is semiregular and the masses at its vertices are all of them equal. A semiregular $2n$–gon has n pair of vertices symmetric by a reflection of an angle β with respect to the axis of symmetry of the first regular n–gon. Our aim is to analyze the set of values of the parameter a for the regular $2n$–gon and of the parameters (a, β) for the semiregular $2n$–gon providing symmetric bicircular central configurations. In particular, for all $n \geq 2$ we prove analytically the existence of symmetric bicircular central configurations with a (respectively (a, β)) satisfying some particular conditions. Using either computer assisted results or numerical results we also describe the complete set of values of a (respectively, (a, β)) providing symmetric bicircular central configurations for $n = 2, 3, 4, 5$ and we give numerical evidences that the pattern for $n > 5$ is the same as the one for $n = 5$.

1. Introduction

We consider the planar Newtonian N–body problem

$$m_k\,\ddot{\mathbf{q}}_k = -\sum_{j=1,\,j\neq k}^{N} G\,m_k\,m_j\,\frac{\mathbf{q}_k-\mathbf{q}_j}{|\mathbf{q}_k-\mathbf{q}_j|^3}\,, \qquad k=1,\dots,N,$$

where $\mathbf{q}_k \in \mathbb{R}^2$ is the position vector of the point mass m_k in an inertial coordinate system and G is the gravitational constant, which can be taken equal to one by choosing conveniently the unit of time. A configuration of the N bodies is called *central* if the acceleration vector of each body is proportional to its position vector with respect to the center of mass with the same constant of proportionality. In other words, given $m_1, \dots, m_N$ a configuration $(\mathbf{q}_1, \dots, \mathbf{q}_N)$ with $\mathbf{q}_i \neq \mathbf{q}_j$ for all $i \neq j$ is central if there exists a constant λ such that

$$\sum_{j=1,\,j\neq k}^{N} m_j\,\frac{\mathbf{q}_k-\mathbf{q}_j}{|\mathbf{q}_k-\mathbf{q}_j|^3} = \lambda\,(\mathbf{q}_k-\mathbf{c}), \qquad k=1,\dots,N, \tag{1}$$

where $\mathbf{c} = \sum_{k=1}^{N} m_k\,\mathbf{q}_k / \sum_{k=1}^{N} m_k$ is the center of mass. For a classical background on the study of central configurations, see for instance [16] and [5]. In this paper we deal with symmetric central configurations having the masses at the vertices of two concentric polygons.

A (regular) polygon of n–vertices is usually called a (regular) n–gon. Concentric n–gons are called *nested* when the vertices of all n–gons are aligned and they are called *twisted* when the vertices of at least one of the n–gons is rotated by an angle of π/n with respect to the other ones.

The simplest central configuration of the planar Newtonian N–body problem consists of N equal masses at the vertices of a regular N–gon. At our knowledge the first author studying central configurations having the masses at the vertices of two concentric n–gons was Hoppe [7] in 1879. In his work Hope showed that if n equal masses are placed at the vertices of a regular n–gon, then n

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other equal masses may be placed at the vertices of a second nested/twisted regular n–gon in such a way that the resulting configuration is central (see for more details [9]). Since then several authors have studied central configurations having the masses at the vertices of concentric regular n–gons. Assuming that the masses on each n–gon are equal and masses on different n–gons could be different, Klemplerer [8] in 1967, gave the relation between the ratio of the masses and the ratio of the radii for two twisted regular n–gons central configurations when $n = 2, 3, 4$. More recently Moeckel and Simó in [11] proved that for any n and for every ratio of the masses there are exactly two nested regular n–gons central configurations one with the ratios of the sizes of the n–gons less than one and the other with the ratio of the sizes greater than one.

There are also several works studying central configurations with kn masses at the vertices of k concentric n–gons such that all the masses on the same n–gon are equal and masses on different n–gons could be different. These central configurations are called a crown of k rings of n bodies or simply a (k, n)-crown in [2], but since only few authors use this nomenclature we will not use it here. Llibre and Melo in [3] proved the existence of central configurations of three twisted regular n–gons with $n = 2, 3$ and of four twisted regular 2–gons. Corbera and Llibre in [4] proved the existence of central configurations of p nested regular n–gons for all $p \geqslant 2$ and $n \geqslant 2$ (see also [6]). Zhao and Chen in [19] proved the existence of central configurations of the $(pn + gn)$–body problem with pn masses at the vertices of p nested regular n–gons and gn masses at the vertices of g twisted regular n–gons. Barrabés and Cors in [2] derived the generic equations of central configurations having kn masses at the vertices of k concentric regular n–gons. They also prove the existence of at least a two twisted regular n–gons central configuration for any mass ratio and for any $n \geqslant 1$ and they give the exact number and location of two twisted n–gons central configurations for $n = 3, 4$.

Up to here it is assumed that the masses on each n–gon are equal and masses of different n–gons could be different and that the n–gons are either nested or twisted by an angle of π/n. Without imposing the condition that the masses on each n–gon are equal Zhang and Zhou in [20] proved that if the configuration with $2n$ masses at the vertices of two nested/twisted n–gons is central then the masses on each n–gon must be equal. Assuming that the n–gons can be twisted by an angle $\theta \in [0, \pi/n]$ Yu and Zhang in [17] proved that if a configuration having n equal masses at the vertices a regular n–gon and n additional masses at the vertices of a second concentric n–gon rotated by an angle of θ with respect to the first one is central, then the angle θ must be either 0 or π/n.

The configurations considered in all the previous papers are highly symmetric. Montaldi in [12], using variational arguments, proved that for every possible symmetry type given by a finite subgroup of $O(2)$ and for any symmetric choice of the masses there is at least a central configuration. In particular he gives an alternative proof for the existence of nested/twisted regular n–gons central configurations when all the masses on each n–gon are equal and masses of different n–gons could be different. He also proved the existence of symmetric central configurations of concentric polygons that are combinations of nested/twisted regular n–gons and semiregular $2n$–gons. Notice that in this last cast the number of masses on each concentric polygon is not necessarily equal. See below for a more precise explanation of Montaldi's work.

There are some other works studying central configurations with the masses at the vertices of concentric polygons having different number of vertices. Yu and Zhang in [18] proved that if a configuration with n-equal masses at the vertices of a regular n–gon and ℓ equal masses at the vertices of a regular ℓ–gon with a common center is central, then $n = \ell$. Siluszyk in [13, 14] studied central configurations consisting of n equal masses at the vertices of a regular n–gon and $2n$ masses at the vertices of a second concentric $2n$–gon with a common center. In particular in [13] the author found the expressions of the masses in function of the sizes of the n–gons and in [14], by using computer assisted methods, she studied the existence of central configurations of this type for $n = 2$. Marchesin in [10] proved the existence of central configurations consisting of 4 equal masses at the vertices of an square, 4 equal masses at the vertices of a second concentric square twisted by an angle of $\pi/4$ and 8 equal masses on a third concentric polygon having the vertices at the bisectors of the angles formed by each pair of position vectors of two consecutive masses of the previous two squares.

From now on a central configuration is *symmetric* if the set of positions and the set of masses are invariant under the action of a finite subgroup of $O(2)$. Inspired by the works of Siluszyk, Montaldi and Marchesin, in this work we consider symmetric central configurations consisting of n equal masses

at the vertices of a regular n–gon and $2n$ masses (not all them necessarily equal) at the vertices of a second (not necessarily regular) concentric $2n$–gon. We call this kind of configurations *symmetric bicircular central configurations of the* $3n$*–body problem.* Notice that by [18] if the $2n$–gon is regular, then the masses at this $2n$–gon cannot be all of them equal. In fact, when the $2n$–gon is regular the symmetric bicircular central configurations of the $3n$–body problem can be thought as the limit case when $a_2 \to a_3$ of the central configurations of the $3n$–body problem consisting of n masses equal to m_1 at the vertices of a regular n–gon inscribed in a circle of radius a_1, n masses equal to m_2 at the vertices of a nested regular n–gon inscribed in a circle of radius a_2 and n additional masses equal to m_3 (with $m_2 \neq m_3$ by [18]), at the vertices of a regular n–gon inscribed in a circle of radius a_3 twisted by an angle of π/n with respect the previous two. Thus when the $2n$–gon is regular the masses at the vertices of this $2n$–gon alternate the values m_2 and m_3. Using the nomenclature in [2], the symmetric bicircular central configurations of the $3n$–body problem are a $(3, n)$–crown.

2. Statement of the problem and of the main results

According to Montaldi [12] we have two different types of symmetric bicircular central configurations of the $3n$–body problem. Indeed, in Montaldi [12] it is proved that a generic symmetric planar central configuration consists of k_O (with $k_O = 0$ or $k_O = 1$) masses at the origin, $n \cdot k_N$ masses at the vertices of k_N regular nested n–gons, $n \cdot k_T$ masses at the vertices of k_T regular n–gons twisted by an angle of π/n, and $2n \cdot k_S$ masses at the vertices of k_S nested semiregular $2n$–gons, all centered at the origin. Moreover the masses on each polygon must be equal but masses on different polygons can be different. A *semiregular* $2n$*–gon* is a symmetric polygon of $2n$ vertices which is invariant by reflections with respect to all symmetry axis of the regular n–gon.

The symmetric bicircular central configurations of the $3n$–body problem does not contain any mass at the origin, so $k_O = 0$. Moreover since in these configurations we have $3n$–bodies, $k_N + k_T + 2\,k_S = 3$. Thus we have only three possibilities: either $k_N = 1$, $k_T = 2$ and $k_S = 0$, or $k_N = 2$, $k_T = 1$ and $k_S = 0$, or $k_N = 1$, $k_T = 0$ and $k_S = 1$. Since central configurations are invariant under rotations the case $k_N = 2$, $k_T = 1$ and $k_S = 0$ and the case $k_N = 1$, $k_T = 2$ and $k_S = 0$ provide the same configuration up to a rotation. So we have only two different types of symmetric bicircular central configurations of the $3n$–body problem, the configurations with $k_N = 1$, $k_T = 2$ and $k_S = 0$ which are called *regular bicircular central configurations of the* $3n$*–body problem* and the configurations with $k_N = 1$, $k_T = 0$ and $k_S = 1$ which are called *semiregular bicircular central configurations of the* $3n$*–body problem.*

In short, a regular bicircular central configuration of the $3n$–body problem consists of n bodies with masses equal to m_1 at the vertices of a regular n–gon inscribed in a circle of radius $a_1 = r$ and $2n$ bodies with alternating masses m_2 and m_3 with $m_2 \neq m_3$ at the vertices of a second regular $2n$–gon inscribed in a circle of radius $a_2 = ar$ with a common center having n vertices aligned with the vertices of the first regular n–gon, see Fig. 1(a). Moreover, a semiregular bicircular central configuration of the $3n$–body problem consists of n bodies with masses equal to m_1 at the vertices of a regular n–gon inscribed in a circle of radius $a_1 = r$ and $2n$ bodies with masses equal to m_2 at the vertices of a second concentric semiregular $2n$–gon inscribed in a circle of radius $a_2 = ar$ having n pair of vertices symmetric by a reflection of angle β with respect to the axis of symmetry of the regular n–gon, see Fig. 1(b). Without loss of generality we choose the unit of mass and the unit of length so that $m_1 = 1$ and $r = 1$.

Notice that the regular bicircular central configurations of the $3n$–body problem are the ones studied in [13, 14], but the author in these two works does not analyze whether the expressions of the masees are positive or negative and in case of being positive for which values of the ratios of the radius they are positive and consequently they provide central configurations.

In the first part of the paper (Section 3) we go a step further in the study of the regular bicircular central configurations of the $3n$–body problem. In particular we prove analytically for all $n \geq 3$ the existence of central configurations for either sufficiently small values, or sufficiently large values of the ratio of the radius of the circles a and for $n = 2$ we prove the existence of central configurations for sufficiently large values of a. Then using computer assisted methods we find the complete set of values of a for which there exist regular bicircular central configurations of the $3n$–body problem for

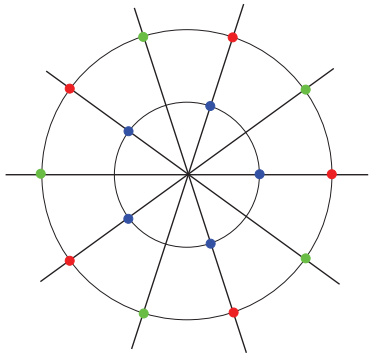

Fig.1 a Regular bicircular configurations

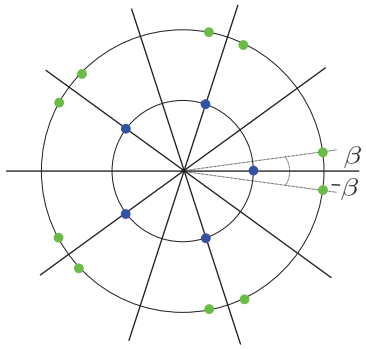

Fig.1 b Semiregular bicircular configurations

FIGURE 1. Examples of bicircular configurations

$n = 2, 3, 4, 5$ and we analyze these families of central configurations. Finally we make a numerical exploration of the families of central configurations of the regular bicircular central configurations of the $3n$–body problem for $n = 6, ..., 500$. We remark that the result obtained here for $n = 2$ does not coincide with the one given in [14].

In the second part of the paper (Section 4) we study the semiregular bicircular central configurations of the $3n$–body problem. In particular, we show that the only values of β that can provide central configurations are $\beta \in (\pi/2n, \pi/n)$ and we prove analytically the existence of semiregular bicircular central configurations of the $3n$–body problem for all values of β in a sufficiently small neighborhood of $\pi/2n$ and of π/n for all $n \geq 2$. Then we make a numerical exploration of the families of semiregular bicircular central configurations of the $3n$–body problem for $n = 2, \ldots, 100$.

Next we give a complete summary of the obtained results.

2.1. **Summary of the results concerning regular bicircular central configurations of the $3n$–body problem.** The analytical results that we have obtained for the regular bicircular central configurations of the $3n$–body problem are summarized in the following theorem.

Theorem 1. *For each $n \geqslant 2$ there exist a nonemptyset D and functions $m_2 = m_2(a)$ and $m_3 = m_3(a)$ that provide regular bicircular central configurations of the $3n$–body problem for all $a \in D$. In particular,*

(a) *If $n = 2$, then we can assure the existence of central configurations at least for all $a \in (a^*, \infty)$, where $a^* \in (1, \infty)$ is the largest zero of m_3. In this configurations $m_3 \to 0$ and $m_2 \to m_2(a^*)$ when $a \to a^{*+}$ and $m_2, m_3 \to \infty$ when $a \to \infty$.*

(b) *If $n \geqslant 3$, then we can assure the existence of central configurations at least for all $a \in (0, a_1^*) \cup (a_2^*, \infty)$, where a_1^* is the minimum between the first positive zero and the pole of m_2 and a_2^* is the largest zero of m_3. In these configurations $m_2, m_3 \to 0$ when $a \to 0^+$, $m_3 \to 0$ and $m_2 \to m_2(a_2^*)$ when $a \to a_2^{*+}$, and $m_2, m_3 \to \infty$ when $a \to \infty$.*

The computed assisted results that we have obtained for the regular bicircular central configurations of the $3n$–body problem with $n = 2, 3, 4, 5$ are summarized in the following results.

Theorem 2. *For $n = 2$ we can find functions $m_2 = m_2(a)$ and $m_3 = m_3(a)$ that provide regular bicircular central configurations of the 6–body problem for all $a \in D = (a_1, a_2) \cup (a^*, \infty)$ where a_1 is the pole of m_2, a_2 is the first zero of m_2 and a^* is the largest zero of m_3. The approximate values of a_1, a_2 and a^* are given in Table 1. Moreover the functions m_2 and m_3 satisfy the following properties (see Fig. 2):*

(a) *$m_2, m_3 \to \infty$ with $m_2/m_3 \to 1^-$ when $a \to a_1^+$;*

(b) *in the interval (a_1, a_2) the function m_2 is decreasing and m_3 has a unique critical point at $a = a_0 = 0.5670013389..$ which is a minimum with $m_3(a_0) = 4.7014182338...$ Moreover $m_2 < m_3$;*

n	2	3	4	5
D	$(a_1, a_2) \cup (a^*, \infty)$	$(0, a_1^*) \cup (a_1, a_2) \cup (a_2^*, \infty)$	$(0, a_1^*) \cup (a_1, a_2) \cup (a_2^*, \infty)$	$(0, a_1^*) \cup (a_2^*, \infty)$
		$a_1^* = 0.13753297..$	$a_1^* = 0.35631015..$	$a_1^* = 0.63131956..$
	$a_1 = 0.53178607..$	$a_1 = 0.58231561..$	$a_1 = 0.61035220..$	
	$a_2 = 0.61614478..$	$a_2 = 0.63460382..$	$a_2 = 0.61188342...$	
	$a^* = 3.52823222..$	$a_2^* = 2.63616425..$	$a_2^* = 2.26854434..$	$a_2^* = 2.06282906..$

TABLE 1. The set D providing regular bicircular central configurations of the $3n$–body problem. The values of a in the cells colored in light gray correspond to zeros of the function m_2, the ones in the cells colored in dark gray correspond to poles of the function m_2 and the ones in the non colored cells correspond to the largest zero of the function m_3.

(c) $m_2 \to 0$ *and* $m_3 \to 5.950134407..$ *when* $a \to a_2^-$;
(d) $m_2 \to 1.6921282709..$ *and* $m_3 \to 0$ *when* $a \to a^{*+}$;
(e) *in the interval* (a^*, ∞) *both functions* m_2 *and* m_3 *are increasing and* $m_2 > m_3$;
(f) $m_2, m_3 \to \infty$ *with* $m_2/m_3 \to 1^+$ *when* $a \to \infty$.

Theorem 3. *For* $n = 3$ *we can find functions* $m_2 = m_2(a)$ *and* $m_3 = m_3(a)$ *that provide regular bicircular central configurations of the* 9*–body problem for all* $a \in D = (0, a_1^*) \cup (a_1, a_2) \cup (a_2^*, \infty)$ *where* a_1^* *is the first zero of* m_2, a_1 *is the pole of* m_2, a_2 *is the second zero of* $m_2(a)$ *and* a_2^* *is the largest zero of* m_3. *The approximate values of* a_1^*, a_1, a_2 *and* a_2^* *are given in Table 1. Moreover the functions* m_2 *and* m_3 *satisfy the following properties (see Fig. 3):*

(a) $m_2, m_3 \to 0$ *with* $m_2/m_3 \to 1^-$ *when* $a \to 0^+$;
(b) *in the interval* $(0, a_1^*)$ *the function* m_2 *has a unique critical point at* $a = a_0 = 0.1030896914..$ *which is a maximum with* $m_2(a_0) = 0.0003095830...$ *Moreover* m_3 *is increasing and* $m_2 < m_3$;
(c) $m_2 \to 0$ *and* $m_3 \to 0.0060680996..$ *when* $a \to a_1^{*-}$;
(d) $m_2, m_3 \to \infty$ *when* $a \to a_1^+$ *with* $\lim_{a \to a_1^+} m_2/m_3 = 1^-$;
(e) *in the interval* (a_1, a_2) *the function* m_2 *is decreasing and* m_3 *has a unique critical point at* $a = a_3 = a_3 = 0.6096781095..$ *which is a minimum with* $m_3(a_3) = 4.7014182338...$ *Moreover* $m_2 < m_3$;
(f) $m_2 \to 0$ *and* $m_3 \to 4.4805332525..$ *when* $a \to a_2^-$;
(g) $m_2 \to 1.3553872894..$ *and* $m_3 \to 0$ *when* $a \to a_2^*$;
(h) *in the interval* (a_2^*, ∞) *both functions* m_2 *and* m_3 *are increasing and* $m_2 > m_3$;
(i) $m_2, m_3 \to \infty$ *with* $m_2/m_3 \to 1^+$ *when* $a \to \infty$.

Theorem 4. *For* $n = 4$ *we can find functions* $m_2 = m_2(a)$ *and* $m_3 = m_3(a)$ *that provide regular bicircular central configurations of the* 12*–body problem for all* $a \in D = (0, a_1^*) \cup (a_1, a_2) \cup (a_2^*, \infty)$ *where* a_1^* *is the first zero of* m_2, a_1 *is the second zero of* m_2, a_2 *is the pole of* m_2 *and* a_2^* *is the largest zero fo* m_3. *The approximate values of* a_1^*, a_1, a_2 *and* a_2^* *are given in Table 1. Moreover the functions* m_2 *and* m_3 *satisfy the following properties (see Fig. 4):*

(a) $m_2, m_3 \to 0$ *with* $m_2/m_3 \to 1^-$ *when* $a \to 0^+$;
(b) *in the interval* $(0, a_1^*)$ *the function* m_2 *has a unique critical point at* $a = a_0 = 0.2746698699..$ *which is a maximum with* $m_2(a_0) = 0.0085881109..$ *and the function* m_3 *is increasing. Moreover* $m_2 < m_3$;
(c) $m_2 \to 0$ *and* $m_3 \to 0.1238514421..$ *when* $a \to a_1^{*-}$;
(d) $m_2 \to 0$ *and* $m_3 \to 2.3831374646..$ *when* $a \to a_1^+$;
(e) *in the interval* (a_1, a_2) *both functions* m_2 *and* m_3 *are increasing and* $m_2 < m_3$;
(f) $m_2, m_3 \to \infty$ *with* $m_2/m_3 \to 1^-$ *when* $a \to a_2^-$;
(g) $m_2 \to 1.0670996767..$ *and* $m_3 \to 0$ *when* $a \to a_2^{*+}$;
(h) *in the interval* (a_2^*, ∞) *both functions* m_2 *and* m_3 *are increasing and* $m_2 > m_3$;
(i) $m_2, m_3 \to \infty$ *with* $m_2/m_3 \to 1^+$ *when* $a \to \infty$.

Theorem 5. *For $n = 5$ we can find functions $m_2 = m_2(a)$ and $m_3 = m_3(a)$ that provide regular bicircular central configurations of the 15–body problem for all $a \in D = (0, a_1^*) \cup (a_2^*, \infty)$ where a_1^* is the pole of m_2 and a_2^* is the largest zero of m_3. The approximate values of a_1^*, a_2 and a_2^* are given in Table 1. Moreover the functions m_2 and m_3 satisfy the following properties (see Fig. 5):*

(a) *$m_2, m_3 \to 0$ with $m_2/m_3 \to 1^-$ when $a \to 0^+$;*
(b) *in the interval $(0, a_1^*)$ both functions m_2 and m_3 are increasing and $m_2 < m_3$;*
(c) *$m_2, m_3 \to \infty$ with $m_2/m_3 \to 1^-$ when $a \to a_1^{*-}$;*
(d) *$m_2 \to 0.8426164718..$ and $m_3 \to 0$ when $a \to a_2^{*+}$;*
(e) *in the interval (a_2^*, ∞) both functions m_2 and m_3 are increasing and $m_2 > m_3$;*
(f) *$m_2, m_3 \to \infty$ with $m_2/m_3 \to 1^+$ when $a \to \infty$.*

The functions $m_2 = m_2(a)$ and $m_3 = m_3(a)$ in Theorems 1-5 are expressed by the formulas in (4).

After a numerical exploration of the cases $n = 6, \dots, 500$ we see that the pattern observed for $n = 6, ..., 500$ is the same as the one observed for $n = 5$. So we make the following conjecture.

Conjecture 6. *For all $n \geq 5$ there exist functions $m_2 = m_2(a)$ and $m_3 = m_3(a)$ that provide regular bicircular central configurations of the $3n$-body problem for all a in the set $D \subset (0, a_1^*) \cup (a_2^*, \infty)$ where a_1^* is the pole of m_2 and a_2^* is the largest zero of m_3. Moreover the functions m_2 and m_3 are increasing in D and they satisfy that $m_2 < m_3$ when $a \subset (0, a_1^*)$, $m_2 > m_3$ when $a \subset (a_2^*, \infty)$, $m_2, m_3 \to 0$ with $m_2/m_3 \to 1$ when $a \to 0^+$; $m_2, m_3 \to \infty$ with $m_2/m_3 \to 1$ when $a \to a_1^{*-}$; $m_3 \to 0$ when $a \to a_2^{*+}$; and $m_2, m_3 \to \infty$ with $m_2/m_3 \to 1$ when $a \to \infty$.*

In short, when $n = 2$ we have proved analytically (see Theorem 1(a)) the existence of central configurations for all a in the interval (a^*, ∞) where $a^* > 1$ is the largest zero of m_3. Using computer assisted results we prove the existence of an additional interval (a_1, a_2) providing central configurations, where $a_1 < 1$ is the pole of m_2 and $a_2 < 1$ is a zero of m_2 (see Theorem 2).

When $n \geq 3$ we have proved analytically (see Theorem 1(b)) the existence of central configurations for $a \in (0, a_1^*) \cup (a_2^*, \infty)$ where $0 < a_1^* < 1$ is the minimum between the first positive zero and the pole of m_2 and $a_2^* > 1$ is the largest zero of m_3. For $n = 3, 4$ using computer assisted results we have proved the existence of an additional interval (a_1, a_2) with $a_1, a_2 < 1$ providing central configurations. Moreover for $n = 3$ we have proved that a_1^* is the first positive zero of m_2, a_1 is the pole of m_2 and a_2 is the second zero of m_2 (see Theorem 3). When $n = 4$ we have proved that a_1^* is the first zero of m_2, a_1 is the second zero of m_2 and a_2 is the pole of m_2 (see Theorem 4). For $n = 5$ we have proved using computer assisted results that the only set of values of a providing central configurations is $(0, a_1^*) \cup (a_2^*, \infty)$; that is, the set given in Theorem 1(b) (see Theorem 5). Finally after a numerical explorations of the cases $n = 6, \dots, 500$ we conclude that for all $n = 6, \dots, 500$ that the only set of values of a providing central configurations is the one given in Theorem 1(b).

We must remark that fixed a value of n there are no central configurations of the regular bicircular $3n$–body problem with $a \to 1$. We have observed that as n increases the value a_1^* increases and the value of a_2^* decreases, but we do not know whether $a_1^* \to 1^-$ and $a_2^* \to 1^+$ when $n \to \infty$ or on the contrary they could tend to a value different from one. This question remains as an open problem.

On the other hand, as n increases the values of m_2 and m_3 are getting closer. We know from results in [18] that $m_2 \neq m_3$ because we have two concentric regular n–gons with different number of vertices but it seems that if $n \to \infty$ then $m_2 \to m_3$.

2.2. **Summary of the results concerning semiregular bicircular central configurations of the $3n$–body problem.** The main analytic results for the semiregular bicircular central configurations of the $3n$–body problem are summarized in the following theorem.

Theorem 7. *There exist a function $m = m(a, \beta)$ that provides semiregular bicircular central configurations of the $3n$–body problem for some values of $a > 0$ and $\beta \in (\pi/2n, \pi/n)$. In particular,*

(a) *If $n \geq 3$, then for each β in a sufficiently small neighborhood of π/n there exist at least two central configurations, one with $0 < a < 1$ and one with $a > 1$. Moreover, if $n = 2$, then for each β in a sufficiently small neighborhood of π/n there exists at least one central configuration with $a > 1$. The values of m at these central configurations satisfy that $m \to 0$ as $\beta \to \pi/n$.*

(b) *If $n \geq 3$, then for each β in a sufficiently small neighborhood of $\beta = \pi/2n$ there exist at least four central configurations, one with a sufficiently close to the origin, one with $0 < a < 1$ not necessarily close to the origin, one with a sufficiently large and one with $a > 1$ not necessarily large. Moreover, if $n = 2$, then for each β in a sufficiently small neighborhood of $\beta = \pi/2n$ there exists at least three central configurations, one with a sufficiently large, one with $a > 1$ not necessarily large and one with $0 < a < 1$. The values of m at the central configurations with $0 < a < 1$ not close to the origin and with $a > 1$ not being large satisfy that $m \to \infty$ as $\beta \to \pi/2n$.*

The function $m(a, \beta)$ is given in (16).

After a numerical study of the behavior of the families of semiregular bicircular central configurations of the $3n$–body problem for $n = 2, 3, 4$ we get the following numerical results.

Result 1 *When $n = 2$, we can find continuous functions $\alpha_1(\beta)$ defined for $\beta \in (\pi/4, \pi/2)$, and $\alpha_2(\beta)$ and $\alpha_3(\beta)$ defined for $\beta \in (\pi/4, b^*]$ with $b^* = 0.9195936184..$ and a function $m = m(a, \beta)$ such that the following statements hold for the semiregular bicircular central configurations of the 6–body problem (see Fig. 9a).*

(a) *If $\beta \in (\pi/4, b^*)$, then $m = m(\beta, a)$ provides three families of central configurations, one with $a = \alpha_1(\beta)$, one with $a = \alpha_2(\beta)$ and one with $a = \alpha_3(\beta)$.*
(b) *If $\beta = b^*$, then $m = m(\beta, a)$ provides two central configurations, one with $a = \alpha_1(b^*)$ and one with $a = \alpha_2(b^*) = \alpha_3(b^*)$ (the families with $a = \alpha_2(\beta)$ and $a = \alpha_3(\beta)$ coincide at this point).*
(c) *If $\beta \in (b^*, \pi/2)$, then $m = m(\beta, a)$ provides a unique family of central configurations with $a = \alpha_1(\beta)$.*

Moreover $m(\alpha_1(\beta), \beta) \to 0$ when $\beta \to \pi/2^-$; and $m(\alpha_1(\beta), \beta), m(\alpha_2(\beta), \beta), m(\alpha_3(\beta), \beta) \to \infty$ when $\beta \to \pi/4^+$.

Result 2 *When $n = 3$, we can find continuous functions $\alpha_1(\beta)$ and $\alpha_2(\beta)$ defined for $\beta \in (\pi/6, b^*]$, functions $\alpha_1(\beta)$ and $\alpha_2(\beta)$ defined for $\beta \in (\pi/6, \pi/3)$, with $b^* = 0.7119233940..$ and a function $m = m(a, \beta)$ such that the following statements hold for the semiregular bicircular central configurations of the 9–body problem (see Fig. 9b).*

(a) *If $\beta \in (\pi/6, b^*)$, then $m = m(\beta, a)$ provides four families of central configurations, one with $a = \alpha_1(\beta)$, one with $a = \alpha_2(\beta)$, one with $a = \alpha_3(\beta)$, and one with $a = \alpha_4(\beta)$.*
(b) *If $\beta = b^*$, then $m = m(\beta, a)$ provides three central configurations, one with $a = \alpha_1(b^*) = \alpha_2(b^*)$ (the families with $a = \alpha_1(\beta)$ and $a = \alpha_2(\beta)$ coincide at this point), one with $\alpha_3(b^*)$ and one with $a = \alpha_4(b^*)$.*
(c) *If $\beta \in (b^*, \pi/3)$, then $m = m(\beta, a)$ provides two family of central configurations, one with $a = \alpha_3(\beta)$ and one with $a = \alpha_4(\beta)$.*

Moreover $m(\alpha_3(\beta), \beta), m(\alpha_4(\beta), \beta) \to 0$ when $\beta \to \pi/3^-$; $m(\alpha_1(\beta), \beta), m(\alpha_2(\beta), \beta), m(\alpha_3(\beta), \beta) \to \infty$ and $m(\alpha_4(\beta), \beta) \to 0$ when $\beta \to \pi/6^+$.

Result 3 *When $n = 4$, we can find continuous functions $\alpha_1(\beta)$ and $\alpha_4(\beta)$ defined for $\beta \in (\pi/8, b^*]$, functions $\alpha_2(\beta)$ and $\alpha_3(\beta)$ defined for $\beta \in (\pi/8, \pi/4)$, with $b^* = 0.4665964724..$ and a function $m = m(a, \beta)$ such that the following statements hold for the semiregular bicircular central configurations of the 12–body problem (see Fig. 9c).*

(a) *If $\beta \in (\pi/8, b^*)$, then $m = m(\beta, a)$ provides four families of central configurations, one with $a = \alpha_1(\beta)$, one with $a = \alpha_2(\beta)$, one with $a = \alpha_3(\beta)$, and one with $a = \alpha_4(\beta)$.*
(b) *If $\beta = b^*$, then $m = m(\beta, a)$ provides three central configurations, one with $a = \alpha_1(b^*) = \alpha_4(b^*)$ (the families with $a = \alpha_1(\beta)$ and $a = \alpha_4(\beta)$ coincide at this point), one with $\alpha_2(b^*)$ and one with $a = \alpha_3(b^*)$.*
(c) *If $\beta \in (b^*, \pi/4)$, then $m = m(\beta, a)$ provides two family of central configurations, one with $a = \alpha_2(\beta)$ and one with $a = \alpha_3(\beta)$.*

Moreover $m(\alpha_3(\beta), \beta), m(\alpha_2(\beta), \beta) \to 0$ when $\beta \to \pi/4^-$; and $m(\alpha_1(\beta), \beta), m(\alpha_2(\beta), \beta), m(\alpha_3(\beta), \beta) \to \infty$ and $m(\alpha_4(\beta), \beta) \to 0$ when $\beta \to \pi/8^+$.

We have also studied numerically the families of semiregular bicircular central configurations of the $3n$–body for $n = 5, \dots, 100$ and we have observed that the behavior is the same as the one for $n = 4$. So we state the following conjecture.

Conjecture 8. *For all $n \geq 4$ we can find a value $\beta = b^*$ and functions $\alpha_i(\beta)$ for $i = 1, 2, 3, 4$ such that*

(a) *for $\beta \in (\pi/2n, b^*)$ there exist exactly four different families of semiregular bicircular central configurations, one emanating from a point $(a_1, \pi/2n)$ with $0 < a_1 < 1$ (the family with $a = \alpha_1(\beta)$), one emanating from a point $(a_2, \pi/2n)$ with $a_2 > 1$ (the family with $a = \alpha_1(\beta)$), one emanating from the point $(\infty, 0)$ (the family with $a = \alpha_3(\beta)$), and one emanating from the point $(0, \pi/2n)$ ($\alpha_4(\beta)$);*

(b) *for $\beta = b^*$ there exist exactly three different central configurations, the one with $a = a_1(b^*) = a_4(b^*)$, the one with $a = a_2(b^*)$ and the one with $a = a_3(b^*)$;*

(c) *for $\beta \in (b^*, \pi/n)$ there exist exactly two different central configurations, one tending to a point $(a_1^*, \pi/n)$ with $0 < a_1^* < 1$ when $\beta \to \pi/n^-$ (the family with $a = a_2(\beta)$) and one tending to a point $(a_2^*, \pi/n)$ with $a_2^* > 1$ when $\beta \to \pi/n^-$ (the family with $a = a_3(\beta)$).*

Moreover the masses associated to these families of central configurations $m = m(a, \beta)$ satisfy that $m(\alpha_3(\beta), \beta), m(\alpha_2(\beta), \beta) \to 0$ when $\beta \to \pi/2n^-$; and $m(\alpha_1(\beta), \beta), m(\alpha_2(\beta), \beta), m(\alpha_3(\beta), \beta) \to \infty$ and $m(\alpha_3(\beta), \beta) \to 0$ when $\beta \to \pi/n^+$.

We note that the relative length of the interval of values of the parameter β providing four different families of central configurations gets smaller as n increases. With relative lenght we mean realitve lengt with respect to the total length of the interval $(\pi/2n, \pi/n)$.

3. Regular bicircular central configurations of the $3n$–body problem

3.1. **The equations.** As we have seen in the introduction the regular bicircular central configurations of the $3n$–body problem can be thought as the limit case of a $(3, n)$–crown (see [2]) where two twisted n-gons are inscribed in a circle of the same radius.

From equation (9) in [2] the equations for central configurations of any $(3, n)$–crown with n masses equal to $m_1 = 1$ at the vertices of a first n-gon inscribed in a circle of radius $a_1 = 1$ and twisted an angle $\varpi_1 = 0$, n masses equal to m_2 at the vertices of a second n-gon inscribed in a circle of radius a_2 and twisted an angle $\varpi_2 = 0$, and n masses equal to m_3 at the vertices of a third n-gon inscribed in a circle of radius a_3 and twisted an angle $\varpi_3 = \pi/n$ are

$$\begin{aligned} C_{21} - S_n\, a_2 + \left(\frac{S_n}{a_2^2} - a_2\, C_{12}\right) m_2 + \Big(C_{23} - a_2\, C_{13}\Big) m_3 = 0, \\ C_{31} - S_n\, a_3 + \Big(C_{32} - a_3\, C_{12}\Big) m_2 + \left(\frac{S_n}{a_3^2} - a_3\, C_{13}\right) m_3 = 0, \end{aligned} \tag{2}$$

where

$$S_n = \sum_{j=1}^{n-1} \frac{1}{4\sin\left(\frac{\pi j}{n}\right)} = \sum_{j=1}^{n-1} \frac{1 - \cos\left(\frac{2\pi j}{n}\right)}{\left(2 - 2\cos\left(\frac{2\pi j}{n}\right)\right)^{3/2}} = \sum_{j=1}^{n-1} \frac{1}{2\Big(2 - 2\cos\left(\frac{2\pi j}{n}\right)\Big)^{1/2}},$$

$$C_{k\ell} = C_{k\ell}(a_k, a_\ell) = \sum_{j=1}^{n} \frac{a_k - a_\ell \cos\left(\varpi_k - \varpi_\ell + \frac{2\pi j}{n}\right)}{\left(a_k^2 + a_\ell^2 - 2a_k a_\ell \cos\left(\varpi_k - \varpi_\ell + \frac{2\pi j}{n}\right)\right)^{3/2}},$$

for $k \neq \ell$.

Thus the equations of the regular bicircular central configurations of the $3n$–body problem are given by (2) with $a_2 = a_3 = a$ and they can be written as the linear system in the variables m_2, m_3

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{3}$$

where

$$a_{11} = \frac{K_1}{a^3} - K_2, \qquad a_{12} = \frac{K_5}{a^3} - K_3, \qquad b_1 = K_1 - K_4,$$
$$a_{21} = \frac{K_5}{a^3} - K_2, \qquad a_{22} = \frac{K_1}{a^3} - K_3, \qquad b_2 = K_1 - K_6,$$

and

$$\begin{aligned}
K_1 &= S_n, \\
K_2 &= K_2(a) = C_{12}(1,a) = \sum_{j=1}^{n} \frac{1 - a\cos\left(\frac{2\pi j}{n}\right)}{\left(1 + a^2 - 2a\cos\left(\frac{2\pi j}{n}\right)\right)^{3/2}}, \\
K_3 &= K_3(a) = C_{13}(1,a) = \sum_{j=1}^{n} \frac{1 - a\cos\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right)}{\left(1 + a^2 - 2a\cos\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right)\right)^{3/2}}, \\
K_4 &= K_4(a) = \frac{C_{21}(a,1)}{a} = \sum_{j=1}^{n} \frac{1 - 1/a\cos\left(\frac{2\pi j}{n}\right)}{\left(1 + a^2 - 2a\cos\left(\frac{2\pi j}{n}\right)\right)^{3/2}}, \\
K_5 &= a^2 C_{23}(a,a) = a^2 C_{32}(a,a) \\
&= \sum_{j=1}^{n} \frac{1 - \cos\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right)}{\left(2 - 2\cos\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right)\right)^{3/2}} = \sum_{j=1}^{n} \frac{1}{2\left(2 - 2\cos\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right)\right)^{1/2}}, \\
K_6 &= K_6(a) = \frac{C_{31}(a,1)}{a} = \sum_{j=1}^{n} \frac{1 - 1/a\cos\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right)}{\left(1 + a^2 - 2a\cos\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right)\right)^{3/2}}.
\end{aligned}$$

Solving system (3) we get the solution

$$\begin{aligned}
m_2 = m_2(a) &= \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} = \frac{a^3\, m_{N,2}}{m_D}, \\
m_3 = m_3(a) &= \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}} = \frac{a^3\, m_{N,3}}{m_D}
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
m_{N,2} &= K_1^2 - K_1K_4(a) - K_1K_5 + K_5K_6(a) - a^3K_3(a)(K_6(a) - K_4(a)), \\
m_{N,3} &= K_1^2 - K_1K_5 + K_4(a)K_5 - K_1K_6(a) + a^3K_2(a)(K_6(a) - K_4(a)), \\
m_D &= (K_5 - K_1)\Delta(a),
\end{aligned}$$

and

$$\Delta(a) = -K_1 - K_5 + a^3(K_2(a) + K_3(a)).$$

The solution (4) is defined when the denominator m_D is different from zero. By Lemma 3 (c) in Appendix 1 $K_5 - K_1 \neq 0$ and by Lemma 4 in Appendix 1 the function $\Delta(a)$ has a unique zero and it belongs to the interval $(0,1)$. So the set where m_D is different from zero is not empty. So in what follows we will assume that $m_D \neq 0$. In fact, this not seems to be restrictive because, as we will see in Section 3.3, we have numerical evidences that there are no solutions of (3) with $m_D = 0$ (m_D, $m_{N,2}$, and $m_{N,3}$ are not simultaneously 0).

These expressions have been obtained in [13] by using a different approach.

Note that

$$m_3 = m_2 + \frac{a^3(K_6(a) - K_4(a))}{K_5 - K_1}. \tag{5}$$

In short, a configuration of the regular bicircular $3n$–body problem is central if m_2 and m_3 are given by (4) and a is so that $m_2, m_3 > 0$.

3.2. **Proof of Theorem 1.** The central configurations of the regular bicircular $3n$–body problem are given by the solutions of (3) with $m_2, m_3 > 0$. Let $m_2 = m_2(a)$ and $m_3 = m_3(a)$ be the solutions of (3) given in (4). We need the following auxiliary result that give some properties of m_2 and m_3.

Proposition 9. *Let $m_2 = m_2(a)$ and $m_3 = m_3(a)$ be the functions defined in* (4)*. Then the following statements hold for $n \geq 2$.*

(a) $m_2 \to 0^+$ *when* $a \to 0^+$ *for* $n \geq 3$ *and* $m_2 \to 0^-$ *when* $a \to 0^+$ *for* $n = 2$*;*
(b) $m_3 \to 0^+$ *when* $a \to 0^+$*;*
(c) $m_3 \to -\infty$ *when* $a \to 1^+$*;*
(d) $m_2 \to \infty$ *and* $m_3 \to \infty$ *when* $a \to \infty$*;*
(e) $m_2 > m_3$ *when* $a > 1$ *and* $m_2 < m_3$ *when* $a \in (0, 1)$*.*

The proof of Proposition 9 is given in Appendix 1.

From Proposition 9(a) and (b), for each $n \geq 3$ there exist a sufficiently small interval $I_1 = (0, a_1^*)$ such that $m_2, m_3 > 0$. Since $m_2 < m_3$ when $0 < a < 1$ (see Proposition 9(e)) this value a_1^* corresponds either to a zero of m_2 or to a point where the denominator m_D vanishes. In Section 3.3 we prove, by using a computer assisted proof, that a_1^* is a zero of m_2 when $n = 3, 4$ and it is the zero of m_D when $n = 5$. Moreover we give strong numerical evidences that a_1^* is the zero of m_D for $n > 5$.

We continue with the proof of Theorem 1. From Proposition 9(d) for each $n \geq 2$ there exists a sufficiently large value $a_2^* > 1$ such that $m_2, m_3 > 0$ for all $a \in I_2 = (a_2^*, \infty)$. Using Proposition 9(c) and (d) we get that m_3 has at least a zero for $a > 1$ and from Lemma 4 in Appendix 1 we obtain that m_D has no zeros for $a > 1$. Since $m_2 > m_3$ when $a > 1$ (see again Proposition 9), and connecting all above comments together we can assure that a_2^* is the largest zero of m_3. In Section 3.3 using a computer assisted proof we prove, for $n = 2, 3, 4, 5$, that m_3 has a unique zero, a_2^*, for $a > 1$ and we give strong numerical evidences that this also happens for $n > 5$.

These arguments, together with Proposition 9, complete the proof of Theorem 1.

3.3. **Particular cases of regular bicircular central configurations of the $3n$–body problem.**

3.3.1. *Case $n = 2$.* Computing the values of K_i with $i = 1, \ldots, 6$ for $n = 2$ we get

$$\begin{aligned} &K_1 = \frac{1}{4}, \qquad K_3(a) = K_6(a) = \frac{2}{h_{22}}, \qquad K_5 = \frac{1}{\sqrt{2}}, \\ &K_2(a) = \frac{1-a}{h_{12}} + \frac{1}{(a+1)^2}, \qquad K_4(a) = -\frac{1-a}{a\,h_{12}} + \frac{1}{a(a+1)^2}, \end{aligned} \tag{6}$$

where $h_{12} = ((1-a)^2)^{3/2}$ and $h_{22} = (1+a^2)^{3/2}$. Thus when $n = 2$, the solutions $m_2 = m_2(a)$ and $m_3 = m_3(a)$ of system (3) with $m_D \neq 0$ are given by (4) with K_i given by (6). Note that in order that these solutions provide central configurations both m_2 and m_3 have to be positive. Next, we find the set of values of a satisfying these conditions.

Since $a > 0$, the possible changes of sign of m_2 and m_3 are given by the zeroes of $m_{N,2}$ and m_D, and $m_{N,3}$ and m_D respectively, see (4). We start computing the zeroes of m_D.

First we transform equation $m_D = 0$ into a polynomial equation having all the solutions of equation $m_D = 0$ and probably new ones in the following way. Dropping the denominators we get the equation

$$\begin{aligned} g = (a+1)^2 \left(4a^3((a-1)h_{22} - 2h_{12}) + \left(1 + 2\sqrt{2}\right) h_{12}h_{22}\right) \\ - 4a^3 h_{12}h_{22} = 0. \end{aligned}$$

In order to drop the square roots we consider equation $g = 0$ as a polynomial equation in the variables a, h_{12} and h_{22}. Then the zeroes of m_D can be thought as solutions of the polynomial system $g = 0$, $e_1 = 0$ and $e_2 = 0$ with

$$e_1 = h_{12}^2 - ((1-a)^2)^3, \qquad e_2 = h_{22}^2 - (1+a^2)^3.$$

We eliminate the variable h_{12} by means of the resultant $R_1 = \mathrm{Res}[g, e_1, h_{12}]$ and the variable h_{22} by means of the resultant $R_2 = \mathrm{Res}[R_1, e_2, h_{22}]$. The resulting polynomial is a polynomial in the variable a with irrational coefficients (the coefficients depend on $\sqrt{2}$) that has all the zeroes of m_D and

probably new ones. To avoid irrational coefficients, we introduce a new variable $h_{32} = \sqrt{2}$ and a new equation $e_3 = h_{32}^2 - 2$. We eliminate the variable h_{32} by means of the resultant $R_3 = \mathrm{Res}[R_2, e_3, h_{32}]$ obtaining in this way a polynomial with integer coefficients having all the zeroes of m_D and probably new ones. The obtained polynomial is $(a-1)^8 P_{60}(a)$ where $P_{60}(a)$ is given by

$$\begin{aligned}
&(45041a^{28} + 471582a^{26} + 1263627a^{24} + 900044a^{22} + 1046137a^{20} + 172866a^{18} + 233227a^{16}\\
&+47784a^{14} + 27723a^{12} - 18750a^{10} + 6329a^8 - 564a^6 - 693a^4 - 98a^2 + 49)(4096a^{32}\\
&-2048a^{31} + 32128a^{30} - 12064a^{29} + 176561a^{28} - 42784a^{27} + 495902a^{26} - 50496a^{25}\\
&+1006987a^{24} + 27328a^{23} + 759372a^{22} + 93472a^{21} + 1118393a^{20} + 64800a^{19} + 259394a^{18}\\
&+11904a^{17} + 310283a^{16} - 31104a^{15} + 38184a^{14} - 41696a^{13} + 1675a^{12} - 15072a^{11}\\
&-10814a^{10} - 1344a^9 + 3641a^8 - 1344a^7 - 1204a^6 + 224a^5 - 245a^4 + 224a^3 - 98a^2 + 49)
\end{aligned}$$

From now on we will denote by $P_n(a)$ a polynomial of degree n in the variable a. Applying Sturm's Theorem we get that the polynomial equation $P_{60}(a) = 0$ has exactly four real solutions with $a > 0$. We solve numerically the equation $P_{60}(a) = 0$ and we found the solutions $a = a_1 = 0.4656636054..$, $a = a_2 = d_2 = 0.5317860740..$, $a = a_3 = 0.5390030006..$, $a = a_4 = 0.5824356327...$ By substituting these solutions into the initial equation $m_D = 0$ we see that only the solution $a = d_2$ can be a solution of the initial equation. This can be proved in a more rigorous way by using intervalar aritmethics (see [15]). We have used Mathematica's capability of operating on interval objects to get an interval enclosure of the function m_D in a sufficiently small interval containing the possible solutions of the equation $m_D = 0$. We start proving that $a_1 \in \mathbf{a_1} = [4656636054/10^{10}, 4656636055/10^{10}]$ cannot be a solution of the initial equation. Notice that since $a_1 < 1$ we have that $((1-a_1)^2)^{3/2} = (1-a_1)^3$. Using intervalar arithmetics we get

$$\begin{aligned}
&1 + a_1^2 \in [h_{21}, h_{22}] = [\tfrac{30421064834853172729}{25\cdot 10^{18}}, \tfrac{4867370373949038521}{4\cdot 10^{18}}],\\
&(1-a_1)^3 \in \mathbf{h_1} = [\tfrac{12204900804202460241055050 69}{8\cdot 10^{27}}, \tfrac{19070157517273170971924514317}{125\cdot 10^{2}7}].
\end{aligned}$$

Moreover we can see that

$$h_{21}^{3/2} \in [\tfrac{13423062258}{10^{10}}, \tfrac{13423062259}{10^{10}}], \quad h_{22}^{3/2} \in [\tfrac{13423062259}{10^{10}}, \tfrac{13423062260}{10^{10}}],$$

so $(1+a_1^2)^{3/2} \in \mathbf{h_2} = [13423062258/10^{10}, 13423062260/10^{10}]$, moreover $\sqrt{2} \in \mathbf{h_3} = [14142135623/10^{10}, 14142135624/10^{10}]$. By substituting into the expression of m_D the values of a, $((1-a)^2)^{3/2}$, $(1+a^2)^{3/2}$ and $\sqrt{2}$ by $\mathbf{a_1}$, $\mathbf{h_1}$, $\mathbf{h_2}$ and $\mathbf{h_3}$ respectively and doing intervalar arithmetics again we get that $m_D \in [-0.1855803641, -0.1855803639]$, so a_1 does not satisfy equation $m_D = 0$. Repeating this procedure for the remaining solutions we get that $m_D \in [-0.0000000001, 0.0000000004]$ for $a_2 \in [5317860740/10^{10}, 5317860741/10^{10}]$, $m_D \in [0.0271889606, 0.0271889613]$ for $a_3 \in [5390030006/10^{10}, 5390030007/10^{10}]$ and $m_D \in [0.2330977572, 0.2330977582]$ for $a_4 \in [5824356327/10^{10}, 5824356328/10^{10}]$. This proves that the unique solution of $P_{60}(a) = 0$ providing a solution of the initial equations is the solution $a = d_2$.

Using the same procedure the equation $m_{N,2} = 0$ can be transformed into a polynomial equation of the form $(a-1)^8 a^4 P_{84}(a) = 0$ where the polynomial $P_{84}(a)$ has exactly 8 real roots with $a > 0$. Among these solutions only $a = a_{12} = 0.6161447847..$ and $a = a_{22} = 2.8235222602..$ are solutions of the initial equation $m_{N,2} = 0$. Doing it again we get that $m_{N,3} = 0$ can be transformed into a polynomial equation of the form $(a-1)^{16} a^4 P_{92}(a) = 0$ where the polynomial $P_{92}(a)$ has exactly 8 real roots with $a > 0$. Among these solutions only $a = b_{12} = 0.5161941182..$ and $a = b_{22} = 3.5282322274..$ are solutions of the initial equation $m_{N,3} = 0$.

Note that $m_{N,2}$, $m_{N,3}$ and m_D are not simultaneously zero. This means that when $n = 2$ there could not be solutions of system (3) with $m_D = 0$.

Finally we analyze the signs of m_2 and m_3. By substituting $a = 4/10$, $(1-(6/10)^2)^{3/2} \in [12493582352/10^{10}, 12493582353/10^{10}]$ and $\sqrt{2} \in [14142135623/10^{10}, 14142135624/10^{10}]$ into m_2 and m_3 and doing intervalar arithmetics again we get that $m_2 \in [-0.3673154049, -0.3673154050]$ and $m_3 \in [0.6505307765, 0.6505307766]$, so m_2 is negative and m_3 is positive in $(0, b_{12})$. Doing the same for $a = 6/10$, $a = 2$, $a = 3$ and $a = 4$ we conclude that $m_2 > 0$ for $a \in (d_2, a_{12}) \cup (a_{22}, \infty)$ and $m_3 > 0$ for $a \in (0, b_{12}) \cup (d_2, 1) \cup (b_{22}, \infty)$. So the region where the masses m_2 and m_3 given in (4) provide central configurations is $a \in (d_2, a_{12}) \cup (b_{22}, \infty)$, see Fig. 2 for the plots of m_2 and m_3.

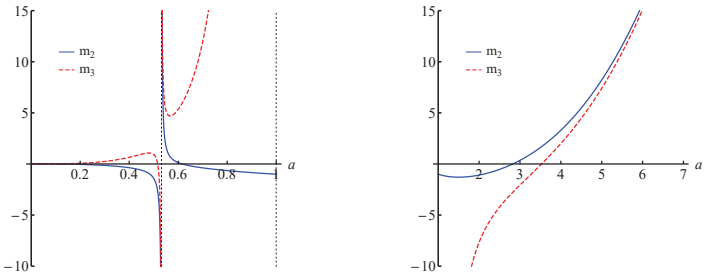

Fig.2a $(0 < a < 1)$ Fig.2b $(a > 1)$

FIGURE 2. Plot of the masses m_2 (continuous line) and m_3 (dashed line) for $n = 2$.

Examining the properties of the functions m_2 and m_3 we get that in the interval (d_2, a_{12}) the functions $m_2, m_3 \to \infty$ when $a \to d_2^+$. Since m_2 and m_3 satisfy the relation (5) and $(K_6(a)-K_4(a))/(K_5-K_1)$ is a finite number for all $a \neq 1$ we can easily see that $\lim_{a\to d_2^+} m_2/m_3 = 1$. By computing the derivatives of the functions m_2 and m_3 and analyzing its zeros as we have done with the zeroes of m_2 and m_3, we see that the function m_2 is decreasing for $a \in (0, d_2) \cup (d_2, 1) \cup (1, 1.5015204804..)$ and increasing for $a \in (1.5015204804.., \infty)$; and the function m_3 is increasing for $a \in (0, 0.4812067311..) \cup (a_{c2}, 1) \cup (1, +\infty)$ with $a_{c2} = 0.5670013389..$ and decreasing for $a \in (0.4812067311.., d_2) \cup (d_2, a_{c2})$ (see Fig. 2). Moreover $m_2 \to 0$ when $a \to a_{12}^-$, $m_3(a_{c2}) = 4.7014182338..$, $m_3 \to 5.950134407..$ when $a \to a_{12}^-$, $m_2 \to 1.6921282709..$ and $m_3 \to 0$ when $a \to b_{22}^+$. Finally we get that $m_2, m_3 \to \infty$ when $a \to \infty$ with $\lim_{a\to\infty} m_2/m_3 = 1$ (see Fig. 2 again).

3.3.2. *Case* $n = 3$. When $n = 3$ the values of K_i for $i = 1, \dots 6$ are

$$\begin{aligned} K_1 &= \frac{1}{\sqrt{3}}, & K_5 &= \frac{5}{4}, \\ K_2(a) &= \frac{1-a}{h_{13}} + \frac{2+a}{h_{33}}, & K_3(a) &= \frac{1}{(1+a)^2} + \frac{2-a}{h_{23}}, \\ K_4(a) &= -\frac{1-a}{a\,h_{13}} + \frac{1+2a}{a\,h_{33}}, & K_6(a) &= \frac{1}{a\,(1+a)^2} - \frac{1-2a}{a\,h_{23}}, \end{aligned} \tag{7}$$

where $h_{13} = ((1-a)^2)^{3/2}$, $h_{23} = (1-a+a^2)^{3/2}$, and $h_{33} = (1+a+a^2)^{3/2}$. Thus when $n = 3$, the solutions $m_2 = m_2(a)$ and $m_3 = m_3(a)$ of system (3) with $m_D \neq 0$ are given by (4) with K_i given by (7). Next we find the values of a for which $m_2 = m_2(a) > 0$ and $m_3 = m_3(a) > 0$.

First we find all the real solutions of equations $m_D = 0$, $m_{N,2} = 0$ and $m_{N,3} = 0$ with $a > 0$. Following step by step the procedure explained in Section 3.3.1 by introducing the new variable $h_{43} = \sqrt{3}$ we transform equations $m_D = 0$, $m_{N,2} = 0$ and $m_{N,3} = 0$ into polynomial equations with integer coefficients having the same solutions as the initial equations and probably new ones. Then we solve numerically the obtained polynomial equations and we check which of their solutions with $a > 0$ provide solutions of the corresponding initial equations. The results that we have obtained are summarized in Table 2. Note that $m_{N,2}$, $m_{N,3}$ and m_D are not simultaneously zero. This means that when $n = 3$ there could not be solutions of system (3) with $m_D = 0$.

Proceeding again as in Section 3.3.1 we analyze the signs of m_2 and m_3 and we get that $m_2 > 0$ for $a \in (0, a_{13}) \cup (d_3, a_{23}) \cup (a_{33}, \infty)$ and $m_3 > 0$ for $a \in (0, b_{13}) \cup (d_3, 1) \cup (b_{23}, \infty)$. In short, the masses m_2 and m_3 given in (4) can provide central configurations when $a \in (0, a_{13}) \cup (d_3, a_{23}) \cup (b_{23}, \infty)$, see Fig. 3 for the plots of m_2 and m_3.

Examining the properties of the functions m_2 and m_3 we get that in the interval $(0, a_{13})$ the functions $m_2, m_3 \to 0$ when $a \to 0^+$, moreover $\lim_{a\to 0^+} m_2/m_3 = 1$; the function m_2 is increasing for $a \in (0, a_{c_13})$, decreasing for $a \in (a_{c_13}, a_{13})$, it has a maximum at $a = a_{c_13} = 0.1030896914..$ with $m_2(a_{c_13}) = 0.0003095830..$, and $m_2 \to 0$ when $a \to a_{13}^-$; and the function m_3 is increasing in $a \in (0, a_{13})$ and $m_3 \to 0.0060680996..$ when $a \to a_{13}^-$ (see Fig. 3a). In the interval (d_3, a_{23}) the functions $m_2, m_3 \to \infty$ when $a \to d_3^+$ with $\lim_{a\to d_3^+} m_2/m_3 = 1$; the function m_2 is decreasing and $m_2 \to 0$ when $a \to a_{23}^-$; the function m_3 is decreasing for $a \in (d_3, a_{c_23})$, is increasing for

Equation	Polynomial equation	$\sharp$	Solutions
$m_D = 0$	$6561(-1+a)^{16}P_{170}(a) = 0$	12	$a = d_3 = 0.5823156170..$
$m_{N,2} = 0$	$(-1+a)^{16}a^4P_{252}(a) = 0$	27	$a = a_{13} = 0.1375329706..$ $a = a_{23} = 0.6346038252..$ $a = a_{33} = 1.9309056653..$
$m_{N,3} = 0$	$(-1+a)^{32}a^4P_{252}(a) = 0$	18	$a = b_{13} = 0.5726308779..$ $a = b_{23} = 2.6361642533..$

TABLE 2. Real positive solutions of $m_D = 0$, $m_{N,2} = 0$, and $m_{N,3} = 0$ for $n = 3$. Here, $\sharp$ means the number of real roots of the polynomial equation $P_n(a) = 0$ for $a > 0$.

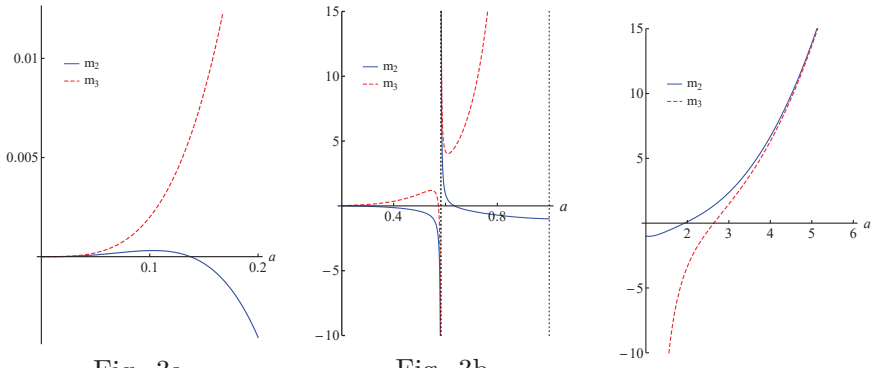

Fig. 3a ($0 < a < 0.2$) Fig. 3b ($0.2 < a < 1$) Fig. 3c ($a > 1$)

FIGURE 3. Plot of the masses m_2 (continuous line) and m_3 (dashed line) for $n = 3$.

$a \in (a_{c_{23}}, a_{23})$, it has a minimum at $a = a_{c_{23}} = 0.6096781095..$ with $m_3(a_{c_{23}}) = 4.7014182338..$, and $m_3 \to 4.4805332525..$ when $a \to a_{23}^-$ (see Fig. 3b). In the interval (b_{23}, ∞) both functions m_2 and m_3 are increasing, $m_2 \to 1.3553872894..$ and $m_3 \to 0$ when $a \to b_{23}^+$ and $m_2, m_3 \to \infty$ when $a \to \infty$ with $\lim_{a\to\infty} m_2/m_3 = 1$ (see Fig. 3c).

3.3.3. *Case* $n = 4$. When $n = 4$ the values of K_i for $i = 1, \dots 6$ are

$$\begin{aligned} K_1 &= \frac{1}{4} + \frac{1}{\sqrt{2}}, & K_5 &= \sqrt{2+\sqrt{2}}, \\ K_2(a) &= \frac{1-a}{h_{14}} + \frac{1}{(1+a)^2} + \frac{2}{h_{24}}, & K_3(a) &= \frac{2-\sqrt{2}a}{h_{34}} + \frac{2+\sqrt{2}a}{h_{44}}, \\ K_4(a) &= -\frac{1-a}{a\,h_{14}} + \frac{1}{a(1+a)^2} + \frac{2}{h_{24}}, & K_6(a) &= \frac{-\sqrt{2}+2a}{ah_{34}} + \frac{\sqrt{2}+2a}{ah_{44}}, \end{aligned} \tag{8}$$

where $h_{14} = ((1-a)^2)^{3/2}$, $h_{24} = \left(a^2+1\right)^{3/2}$, $h_{34} = \left(a^2-\sqrt{2}a+1\right)^{3/2}$ and $h_{44} = \left(a^2+\sqrt{2}a+1\right)^{3/2}$. Thus when $n = 4$, the solutions $m_2 = m_2(a)$ and $m_3 = m_3(a)$ of system (3) with $m_D \neq 0$ are given by (4) with K_i given by (8). Next we find the values of a for which $m_2 = m_2(a) > 0$ and $m_3 = m_3(a) > 0$.

First, proceeding as in Sections 3.3.1 and 3.3.2 by introducing the new variables $h_{54} = \sqrt{2}$ and $h_{64} = \sqrt{2+\sqrt{2}}$, we transform equations $m_D = 0$, $m_{N,2} = 0$ and $m_{N,3} = 0$ into polynomial equations with integer coefficients. We find all the real solutions with $a > 0$ of these equations and the results that we have obtained are summarized in Table 3. Note that $m_{N,2}$, $m_{N,3}$ and m_D are not simultaneously zero. This means that when $n = 4$ there could not be solutions of system (3) with $m_D = 0$.

Equation	Polynomial equation	$\sharp$	Solutions
$m_D = 0$	$(-1+a)^{64}P_{880}(a)$	50	$a = d_4 = 0.6118834277..$
$m_{N,2} = 0$	$(-1+a)^{64}a^{16}P_{1264}(a)$	107	$a = a_{14} = 0.3563101506..$ $a = a_{24} = 0.6103522047..$ $a = a_{34} = 1.4630863479..$
$m_{N,3} = 0$	$(-1+a)^{128}a^{16}P_{1328}(a)$	76	$a = b_{14} = 0.6121322975..$ $a = b_{24} = 2.2685443458..$

TABLE 3. Real positive solutions of $m_D = 0$, $m_{N,2} = 0$, and $m_{N,3} = 0$ for $n = 4$. Here, $\sharp$ means the number of real roots of the polynomial equation $P_n(a) = 0$ for $a > 0$.

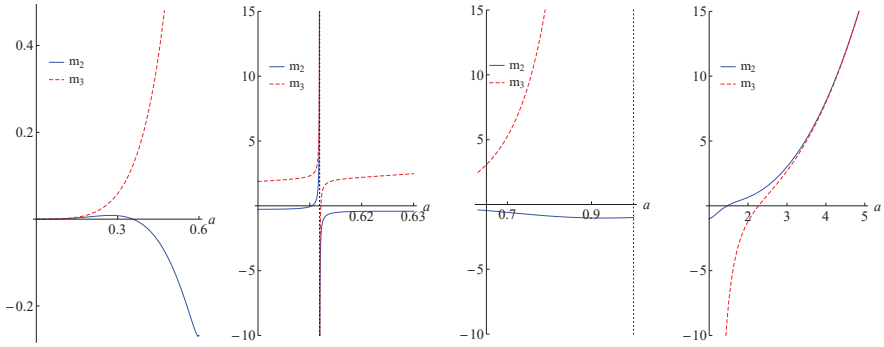

Fig. 4a ($0 < a < 0.6$) Fig. 4b ($0.6 < a < 0.63$) Fig. 4c ($0.63 < a < 1$) Fig. 4d ($a > 1$)

FIGURE 4. Plot of the masses m_2 (continuous line) and m_3 (dashed line) for $n = 4$.

Next we analyze the signs of m_2 and m_3 and we see the region where the masses m_2 and m_3 given in (4) can provide central configurations is $a \in (0, a_{14}) \cup (a_{24}, d_4) \cup (b_{24}, \infty)$. See Fig. 4 for the plots of m_2 and m_3.

Finally, we examine the properties of the functions m_2 and m_3 we get that in the interval $(0, a_{14})$ both functions $m_2, m_3 \to 0$ when $a \to 0^+$, moreover $\lim_{a\to 0^+} m_2/m_3 = 1$; the function m_2 is increasing for $a \in (0, a_{c4})$, decreasing for $a \in (a_{c4}, a_{14})$, it has a maximum at $a = a_{c4} = 0.2746698699..$ with $m_2(a_{c4}) = 0.0085881109..$, and $m_2 \to 0$ when $a \to a_{14}^-$; and the function m_3 is increasing in $a \in (0, a_{14})$ and $m_3 \to 0.1238514421..$ when $a \to a_{14}^-$ (see Fig. 4a). In the interval (a_{24}, d_4), (see Fig. 4b) both functions m_2 and m_3 are increasing, $m_2, m_3 \to \infty$ when $a \to d_4^-$ with $\lim_{a\to d_4^-} m_2/m_3 = 1$, and $m_2 \to 0$ and $m_3 \to 2.3831374646..$ when $a \to a_{24}^+$. Finally in the interval (b_{24}, ∞) both functions m_2 and m_3 are increasing (see Fig. 4d), $m_2 \to 1.0670996767..$ and $m_3 \to 0$ when $a \to b_{24}^+$ and $m_2, m_3 \to \infty$ when $a \to \infty$ with $\lim_{a\to\infty} m_2/m_3 = 1$.

Case $0 < a < 1$

Equation	Polynomial equation	$\sharp_1$	Solutions
$m_D = 0$	$\kappa\,(-1+a)^{64}\,P_{1084}(a)$	50	$a = d_5 = 0.6313195684..$
$m_{N,2} = 0$	$\kappa\,(-1+a)^{64}\,a^4\,P_{1596}(a)$	131	—
$m_{N,3} = 0$	$\kappa\,(-1+a)^{128}\,a^4\,P_{1596}(a)$	85	$a = b_{15} = 0.6425878402..$

Case $a > 1$

Equation	Polynomial equation	$\sharp_2$	Solutions
$m_D = 0$	$\kappa\,(-1+a)^{64}\,P_{1072}(a)$	59	—
$m_{N,2} = 0$	$\kappa\,(-1+a)^{64}\,a^4\,P_{1596}(a)$	108	$a = a_{15} = 1.2290630401..$
$m_{N,3} = 0$	$\kappa\,(-1+a)^{128}\,a^4\,P_{1596}(a)$	97	$a = b_{25} = 2.0628290636..$

TABLE 4. Real positive solutions of $m_D = 0$, $m_{N,2} = 0$, and $m_{N,3} = 0$ for $n = 5$. Here $\sharp_1$ (respectively, $\sharp_2$) means the number of real roots of the polynomial equation $P_n(a) = 0$ for $0 < a < 1$ (respectively, $a > 1$), and κ is a constant.

3.3.4. *Case* $n = 5$. When $n = 5$ the values of K_i for $i = 1, \dots 6$ are

$$\begin{aligned} K_1 &= \sqrt{1+\frac{2}{\sqrt{5}}}, \qquad K_5 = \frac{1}{4}+\sqrt{5}, \\ K_2(a) &= \frac{1-a}{h_{15}} + \frac{\sqrt{2}(4+\eta a)}{h_{45}} + \frac{\sqrt{2}(4+\xi a)}{h_{55}}, \\ K_3(a) &= \frac{1}{(1+a)^2} + \frac{\sqrt{2}(4-\xi a)}{h_{25}} + \frac{\sqrt{2}(4-\eta a)}{h_{35}}, \\ K_4(a) &= -\frac{1-a}{a h_{15}} + \frac{\sqrt{2}(\eta+4a)}{a h_{45}} + \frac{\sqrt{2}(\xi+4a)}{a h_{55}}, \\ K_6(a) &= \frac{1}{a(1+a)^2} + \frac{\sqrt{2}(4a-\xi)}{a h_{25}} + \frac{\sqrt{2}(4a-\eta)}{a h_{35}}, \end{aligned} \tag{9}$$

where $h_{15} = ((1-a)^2)^{3/2}$, $h_{25} = \left(2a^2 - a\xi + 2\right)^{3/2}$, $h_{35} = \left(2a^2 - a\eta + 2\right)^{3/2}$, $h_{45} = \left(2a^2 + a\eta + 2\right)^{3/2}$, $h_{55} = \left(2a^2 + a\xi + 2\right)^{3/2}$, $\xi = 1+\sqrt{5}$, and $\eta = 1-\sqrt{5}$. So when $n = 5$, the solutions $m_2 = m_2(a)$ and $m_3 = m_3(a)$ of system (3) with $m_D \neq 0$ are given by (4) with K_i for $i = 1, \dots, 6$ given by (9).

We proceed in a similar way than in the cases $n = 2, 3, 4$ to find all the real solutions of equations $m_D = 0$, $m_{N,2} = 0$ and $m_{N,3} = 0$ with $a > 0$, but in this case to shorten the computations we consider separately the cases $a > 1$ and $0 < a < 1$ and we eliminate the square root corresponding to h_{15} directly by simplification instead of eliminating it by means of a resultant with respect h_{15}. To transform equations $m_D = 0$, $m_{N,2} = 0$ and $m_{N,3} = 0$ into polynomial equations with integer coefficients, we introduce the new variables $h_{65} = \sqrt{2}$, $h_{75} = \sqrt{5}$ and $h_{85} = \sqrt{1+2/h_{75}}$. The results that we have obtained are summarized in Table 4. Note that $m_{N,2}$, $m_{N,3}$ and m_D are not simultaneously zero. This means that when $n = 5$ there could not be solutions of system (3) with $m_D = 0$.

Analyzing the signs of m_2 and m_3 we see that the region where the masses m_2 and m_3 given in (4) can provide central configurations is $a \in (0, d_5) \cup (b_{25}, \infty)$, see Fig. 5 for the plots of m_2 and m_3.

Examining the properties of the functions m_2 and m_3 we get that in the interval $(0, d_5)$, m_2 and m_3 are increasing (see Fig. 5a), $m_2, m_3 \to 0$ when $a \to 0^+$ with $\lim_{a\to 0^+} m_2/m_3 = 1$, and $m_2, m_3 \to \infty$ when $a \to d_5^-$ with $\lim_{a\to d_5^-} m_2/m_3 = 1$. In the interval (b_{25}, ∞), m_2 and m_3 are increasing (see Fig. 5b), $m_2 \to 0.8426164718..$ and $m_3 \to 0$ when $a \to b_{25}^+$, and $m_2, m_3 \to \infty$ when $a \to \infty$ with $\lim_{a\to\infty} m_2/m_3 = 1$.

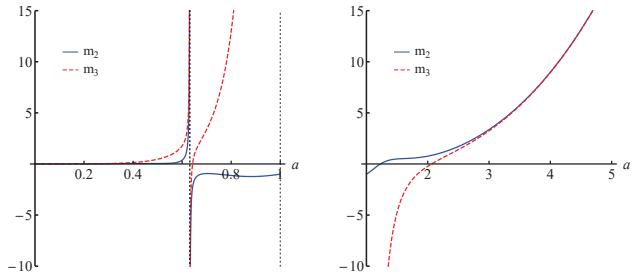

Fig. 5a ($0 < a < 1$) Fig. 5b ($a > 1$)

FIGURE 5. Plot of the masses m_2 (continuous line) and m_3 (dashed line) for $n = 5$.

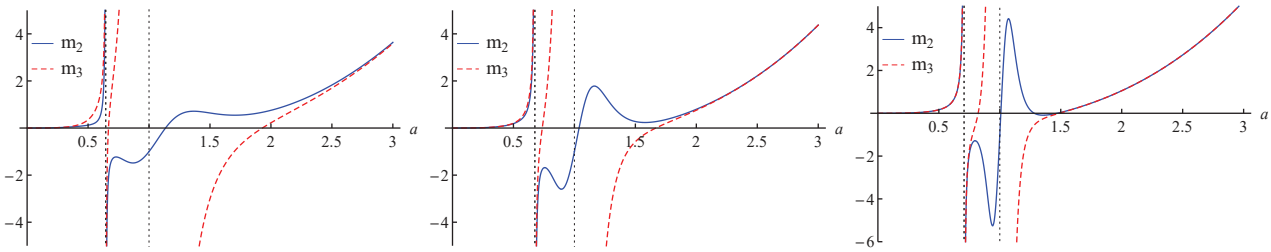

Fig. 6a ($n = 6$) Fig. 6b ($n = 10$) Fig. 6c ($n = 20$)

FIGURE 6. Plot of the masses m_2 (continuous line) and m_3 (dashed lien) when $n = 6$ in Fig. 6a, $n = 10$ in Fig. 6b and $n = 20$ in Fig. 6c.

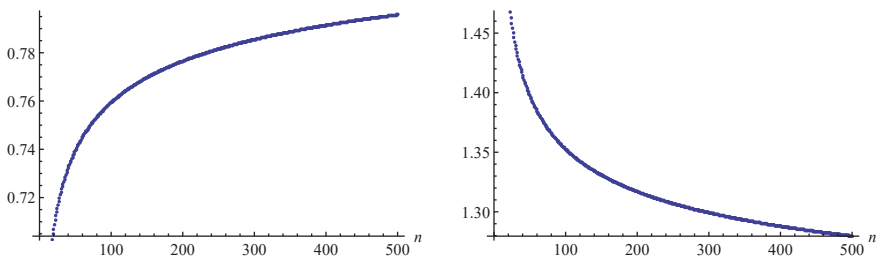

Fig. 7a Fig. 7b

FIGURE 7. Plot of d_n (the zero of m_D) in Fig. 7a and plot of b_n (the zero of $m_{N,3}$ with $a > 1$) in Fig. 7b.

3.3.5. *Numerical study for $n > 5$.* We have analyzed the behavior of m_2 and m_3 as a function of a for $n = 6, 7, \ldots, 500$ and we have seen that it is essentially the same as the one for $n = 5$ (see Fig. 6 for $n = 6, 10, 20$). More precisely, for all $n = 5, \ldots, 500$, the denominator m_D has a unique zero $d_n < 1$ (the existence of such zero has been proved analytically in Lemma 4(d) in Appendix 1). We have computed numerically the value of d_n for $n = 6, \ldots, 500$ and we have plotted it in Fig. 7(a). Note that $d_n < d_{n+1} < 1$ for all $n = 6, \ldots, 499$. We observe that for all $n = 6, \ldots, 500$ the functions m_2 and m_3 are increasing in the interval $(0, d_n)$. Moreover $m_2 < m_3$ in this interval (this has already been proved analytically in Proposition 9(e)), $m_2, m_3 \to 0^+$ when $a \to 0^+$ (this has already been proved analytically in Proposition 9(a) and (b)) and $m_2, m_3 \to \infty$ when $a \to d_n^-$. So $0 < m_2 < m_3$ for all $a \in (0, d_n)$ and therefore the interval $(0, d_n)$ provides central configurations. We also observe that for all $n = 6, \ldots, 500$ the function m_2 is negative in the interval $(d_n, 1)$, thus this interval does not provide central configurations. Finally, we observe that for all $n = 6, \ldots, 500$, $m_3 \to -\infty$ when $a \to 1^+$ and $m_3 \to \infty$ when $a \to \infty$ (this has already been proved analytically in Proposition 9(c) and (d))). Moreover we observe that m_3 is increasing in the interval $(1, \infty)$ for all $n = 6, \ldots, 500$ and therefore m_3 has a unique zero b_n with $a > 1$ (from Proposition 9(c) and (d)

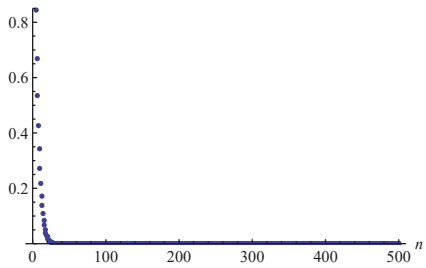

FIGURE 8. Plot of $m_2(b_n)$.

again we can prove analytically the existence of at least one zero of m_3 with $a > 1$, numerically we see that this zero is unique). We have computed numerically the value of b_n for $n = 6, \ldots, 500$ and we have plotted it in Fig. 7b. We see that $b_n > b_{n+1} > 1$ for all $n = 6, \ldots, 499$. Since $m_2 > m_3$ when $a > 1$ (this is proved analytically in Proposition 9(e)), the interval (b_n, ∞) provides central configurations.

In short, the set of values of a where the solutions m_2 and m_3 given by (4) are positive is $a \in (0, d_n) \cup (b_n, \infty)$, where d_n is the zero of m_D and b_n is the zero of the numerator $m_{N,3}$ with $a > 1$. In the region $(0, d_n)$ both masses go from zero (when a tends to 0) to infinity (when a tends to d_n), whereas in the region (b_n, ∞) the mass m_3 goes from 0 (when $a = b_n$) to infinity (when $a \to \infty$) and the mass m_2 goes from the positive value $m_2(b_n)$ to infinity (when $a \to \infty$). We have computed numerically the value $m_2(b_n)$ and we have plotted it in Fig. 8. We observe that the value of $m_2(b_n)$ tends rapidly to 0 as n increases. For instance, when $n = 5$, $m_2(b_n) = 0.8426164718..$; when $n = 20$, $m_2(b_n) = 0.0236101462..$; when $n = 40$, $m_2(b_n) = 0.0000894392..$; when $n = 100$, $m_2(b_n) = 1.1846352161.. \times 10^{-11}...$ All numerical computations have been done with a minimum of 100 digit precision and we have ensured that all the digits given here are exact.

We observe that as n increases the difference between m_2 and m_3 decreases rapidly, see again Fig. 6. Thus as n increases, the masses m_2 and m_3 in a regular bicircular central configuration of the $3n$–body problem tend to be equal.

4. SEMIREGULAR BICIRCULAR CENTRAL CONFIGURATIONS OF THE $3n$–BODY PROBLEM

4.1. **The equations.** Now we consider semiregular bicircular central configurations of the $3n$–body problem which consists of n bodies with masses $m_1 = \cdots = m_n = 1$ at the vertices of a regular n–gon inscribed in a circle of radius 1 and $2n$ bodies with masses equal $m_{n+1} = \cdots = m_{3n} = m$ at the vertices of a semiregular $2n$–gon inscribed in a circle of radius a. By using complex coordinates the positions of the vertices of the initial n-gon can be written as $\mathbf{q}_j = e^{i\beta_j}$ with $\beta_j = 2\pi j/n$ for $j = 1, \ldots, n$ and the vertices of the semiregular $2n$–gon situated on the circle of radius a can be written as $\mathbf{q}_{j+n} = ae^{i(\beta_j - \beta)}$ and $\mathbf{q}_{j+2n} = ae^{i(\beta_j + \beta)}$ with $\beta \in (0, \pi/n)$ and $j = 1, \ldots, n$, see Fig. 1(b).

It is easy to check that the center of mass of the system is at the origin. Under these hypothesis, the first n equations of (1) become

$$\sum_{j=1, j\neq k}^{n} \frac{\mathbf{q}_k - \mathbf{q}_j}{|\mathbf{q}_k - \mathbf{q}_j|^3} + m \sum_{j=1}^{n} \frac{\mathbf{q}_k - \mathbf{q}_{j+n}}{|\mathbf{q}_k - \mathbf{q}_{j+n}|^3} + m \sum_{j=1}^{n} \frac{\mathbf{q}_k - \mathbf{q}_{j+2n}}{|\mathbf{q}_k - \mathbf{q}_{j+2n}|^3} = \lambda \mathbf{q}_k, \tag{10}$$

for $k = 1, \ldots, n$, the following n equations become

$$\begin{aligned} \sum_{j=1}^{n} \frac{\mathbf{q}_{k+n} - \mathbf{q}_j}{|\mathbf{q}_{k+n} - \mathbf{q}_j|^3} + m \sum_{j=1, j\neq k}^{n} \frac{\mathbf{q}_{k+n} - \mathbf{q}_{j+n}}{|\mathbf{q}_{k+n} - \mathbf{q}_{j+n}|^3} \\ + m \sum_{j=1}^{n} \frac{\mathbf{q}_{k+n} - \mathbf{q}_{j+2n}}{|\mathbf{q}_{k+n} - \mathbf{q}_{j+2n}|^3} = \lambda \mathbf{q}_{k+n}, \end{aligned} \tag{11}$$

for $k=1,\dots,n$, and the last n equations of (1) are

$$\begin{aligned}\sum_{j=1}^{n}\frac{\mathbf{q}_{k+2n}-\mathbf{q}_j}{|\mathbf{q}_{k+2n}-\mathbf{q}_j|^3}+m\sum_{j=1}^{n}\frac{\mathbf{q}_{k+2n}-\mathbf{q}_{j+n}}{|\mathbf{q}_{k+2n}-\mathbf{q}_{j+n}|^3}\\ +m\sum_{j=1,j\neq k}^{n}\frac{\mathbf{q}_{k+2n}-\mathbf{q}_{j+2n}}{|\mathbf{q}_{k+2n}-\mathbf{q}_{j+2n}|^3}=\lambda\mathbf{q}_{k+2n},\end{aligned}\tag{12}$$

for $k=1,\dots,n$.

Proceeding in a similar way than in [4], we divide the k-th equation of (10) by $\mathbf{q}_k$, k-th equation of (11) by $\mathbf{q}_{k+n}$ and the k-th equation of (12) by $\mathbf{q}_{k+2n}$ and we get system

$$\begin{aligned}&\sum_{j=1,j\neq k}^{n}\frac{1-e^{i(\beta_j-\beta_k)}}{|e^{i\beta_k}-e^{i\beta_j}|^3}+m\sum_{j=1}^{n}\frac{1-a\,e^{i(\beta_j-\beta_k-\beta)}}{|e^{i\beta_k}-a\,e^{i(\beta_j-\beta)}|^3}\\ &\quad+m\sum_{j=1}^{n}\frac{1-a\,e^{i(\beta_j-\beta_k+\beta)}}{|e^{i\beta_k}-a\,e^{i(\beta_j+\beta)}|^3}=\lambda,\\ &\sum_{j=1}^{n}\frac{1-1/a\,e^{i(\beta_j-\beta_k+\beta)}}{|a\,e^{i(\beta_k-\beta)}-e^{i\beta_j}|^3}+\frac{m}{a^3}\sum_{j=1,j\neq k}^{n}\frac{1-e^{i(\beta_j-\beta_k)}}{|e^{i(\beta_k-\beta)}-e^{i(\beta_j-\beta)}|^3}\\ &\quad+\frac{m}{a^3}\sum_{j=1}^{n}\frac{1-e^{i(\beta_j-\beta_k+2\beta)}}{|e^{i(\beta_k-\beta)}-e^{i(\beta_j+\beta)}|^3}=\lambda,\\ &\sum_{j=1}^{n}\frac{1-1/a\,e^{i(\beta_j-\beta_k-\beta)}}{|a\,e^{i(\beta_k+\beta)}-e^{i\beta_j}|^3}+\frac{m}{a^3}\sum_{j=1}^{n}\frac{1-e^{i(\beta_j-\beta_k-2\beta)}}{|e^{i(\beta_k+\beta)}-e^{i(\beta_j-\beta)}|^3}\\ &\quad+\frac{m}{a^3}\sum_{j=1,j\neq k}^{n}\frac{1-e^{i(\beta_j-\beta_k)}}{|e^{i(\beta_k+\beta)}-e^{i(\beta_j+\beta)}|^3}=\lambda,\end{aligned}\tag{13}$$

for $k=1,\dots,n$. Here

$$\begin{aligned}\left|e^{i\beta_k}-e^{i\beta_j}\right|&=\left|e^{i(\beta_k\pm\beta)}-e^{i(\beta_j\pm\beta)}\right|=(2-2\cos(\beta_j-\beta_k))^{1/2},\\ \left|e^{i\beta_k}-a\,e^{i(\beta_j\pm\beta)}\right|&=\left(1+a^2-2a\cos(\beta_j-\beta_k\pm\beta)\right)^{1/2},\\ \left|a\,e^{i(\beta_k\pm\beta)}-e^{i\beta_j}\right|&=\left(1+a^2-2a\cos(\beta_j-\beta_k\mp\beta)\right)^{1/2},\\ \left|e^{i(\beta_k\mp\beta)}-e^{i(\beta_j\pm\beta)}\right|&=(2-2\cos(\beta_j-\beta_k\pm2\beta))^{1/2}.\end{aligned}$$

Since for all $k=1,\dots,n$ the set $\{\beta_j-\beta_k+\varphi\}_{j=1,\dots,n}$ modulus 2π is equal to the set $\{2\pi j/n+\varphi\}_{j=1,\dots,n}$ for all $\varphi\in\mathbb{R}$, the equations of system (13) are independent of k. So it is not restrictive to take $k=n$. After straightforward computations we can see that system (13) is equivalent to the system

$$\begin{aligned}&K_1+m\big(L_2+N_2i\big)+m\big(L_3+N_3i\big)=\lambda,\\ &L_4+N_4i+\frac{m}{a^3}K_1+\frac{m}{a^3}\big(L_5+N_5i\big)=\lambda,\\ &L_6+N_6i+\frac{m}{a^3}\big(L_7+N_7i\big)+\frac{m}{a^3}K_1=\lambda,\end{aligned}\tag{14}$$

where K_1 is defined as in Section 3 (see (3)) and

$$\begin{aligned}L_2&=L_2(a,\beta)=\sum_{j=1}^{n}\frac{1-a\cos\left(\frac{2\pi j}{n}-\beta\right)}{\left(1+a^2-2a\cos\left(\frac{2\pi j}{n}-\beta\right)\right)^{3/2}},\\ N_2&=N_2(a,\beta)=\sum_{j=1}^{n}\frac{-a\sin\left(\frac{2\pi j}{n}-\beta\right)}{\left(1+a^2-2a\cos\left(\frac{2\pi j}{n}-\beta\right)\right)^{3/2}},\end{aligned}$$

$$
\begin{aligned}
L_4 &= L_4(a,\beta) = \sum_{j=1}^{n} \frac{1 - 1/a\cos\left(\frac{2\pi j}{n}+\beta\right)}{\left(1+a^2-2a\cos\left(\frac{2\pi j}{n}+\beta\right)\right)^{3/2}},\\
N_4 &= N_4(a,\beta) = \sum_{j=1}^{n} \frac{-1/a\sin\left(\frac{2\pi j}{n}+\beta\right)}{\left(1+a^2-2a\cos\left(\frac{2\pi j}{n}+\beta\right)\right)^{3/2}},\\
L_5 &= L_5(\beta) = \sum_{j=1}^{n} \frac{1 - \cos\left(\frac{2\pi j}{n}+2\beta\right)}{\left(2-2\cos\left(\frac{2\pi j}{n}+2\beta\right)\right)^{3/2}},\\
N_5 &= N_5(\beta) = \sum_{j=1}^{n} \frac{-\sin\left(\frac{2\pi j}{n}+2\beta\right)}{\left(2-2\cos\left(\frac{2\pi j}{n}+2\beta\right)\right)^{3/2}},\\
L_3 &= L_3(a,\beta) = L_2(a,-\beta), \qquad N_3 = N_3(a,\beta) = N_2(a,-\beta),\\
L_6 &= L_6(a,\beta) = L_4(a,-\beta), \qquad N_6 = N_6(a,\beta) = N_4(a,-\beta),\\
L_7 &= L_7(\beta) = L_5(-\beta), \qquad N_7 = N_7(\beta) = N_5(-\beta).
\end{aligned}
$$

Note that

$$
\sum_{j=1}^{n-1} \frac{-\sin\left(\frac{2\pi j}{n}\right)}{\left(2-2\cos\left(\frac{2\pi j}{n}\right)\right)^{3/2}} = 0.
$$

Since for all $\varphi \in \mathbb{R}$

$$
\cos\left(\tfrac{2\pi j}{n}-\varphi\right) = \cos\left(\tfrac{2\pi (n-j)}{n}+\varphi\right), \qquad \sin\left(\tfrac{2\pi j}{n}-\varphi\right) = -\sin\left(\tfrac{2\pi (n-j)}{n}+\varphi\right),
$$

we see that

$$
\begin{aligned}
&L_2(a,\beta) = L_3(a,\beta), \quad N_2(a,\beta) = -N_3(a,\beta), \quad L_4(a,\beta) = L_6,\\
&N_4(a,\beta) = -N_6(a,\beta), \quad L_5(\beta) = L_7(\beta), \quad N_5(\beta) = -N_7(\beta).
\end{aligned}
$$

So system (14) is equivalent to system

$$
K_1 + 2mL_2(a,\beta) = \lambda, \quad L_4(a,\beta) + \frac{m}{a^3}(K_1 + L_5(\beta)) = \lambda, \quad N_4(a,\beta) + \frac{m}{a^3}N_5(\beta) = 0. \tag{15}
$$

Solving the third identity in (15) we get

$$
m = -a^3 \frac{N_4(a,\beta)}{N_5(\beta)} \tag{16}
$$

and from the first and second identities in (15) we obtain

$$
m = a^3 \frac{(K_1 - L_4(a,\beta))}{K_1 - 2a^3 L_2(a,\beta) + L_5(\beta)}. \tag{17}
$$

Therefore, from (16) and (17) we have

$$
F(a,\beta) := N_5(\beta)(K_1 - L_4(a,\beta)) + N_4(a,\beta)(K_1 - 2a^3 L_2(a,\beta) + L_5(\beta)) = 0.
$$

In short, a semiregular bicircular configuration of the $3n$–body problem is central if $m = m(a,\beta)$ is given by (16) and a, β are such that $F(a,\beta) = 0$ and $m(a,\beta) > 0$.

4.2. **Admissible values of β.** The following proposition provides the range of values of β that can provide semiregular bicircular central configurations of the $3n$–body problem.

Proposition 10. *A necessary condition to have a semiregular bicircular central configuration of the $3n$–body problem is that $\beta \in (\pi/2n, \pi/n)$.*

Proof. We will see that $m = -a^3 N_4(a,\beta)/N_5(\beta) > 0$ if and only if $\beta \in (\pi/2n, \pi/n)$. We recall that

$$
N_4(a,\beta) = \frac{1}{a^2}\frac{d}{d\beta}\sum_{j=1}^{n} \frac{1}{\left(1+a^2-2a\cos\left(\frac{2\pi j}{n}+\beta\right)\right)^{1/2}}
$$

and that

$$N_5(\beta) = \frac{1}{2}\lim_{a\to 1}\frac{d}{d\beta}\sum_{j=1}^{n}\frac{1}{\left(1+a^2-2a\cos\left(\frac{2\pi j}{n}+2\beta\right)\right)^{1/2}}.$$

Consider first the case in which $a \in [0,1)$. In this case using Proposition 14 in Appendix 1 with $\alpha = 1/2$ and $u = \beta$ and taking derivatives with respect to β we get

$$\begin{aligned} N_4(a,\beta) &= \frac{n}{a^2\pi}\int_0^1 \frac{d}{d\beta}\left(\frac{t^{-1/2}}{(1-t)^{1/2}}\,\frac{1}{(1-a^2t)^{1/2}}\,\frac{1-(at)^{2n}}{B_1}\right)dt \\ &= \frac{n\sin(n\beta)}{a^2\pi}\int_0^1 \frac{-2n(at)^n(1-(at)^{2n})}{t^{1/2}(1-t)^{1/2}(1-a^2t)^{1/2}B_1^2}\,dt, \end{aligned}$$

with $B_1 = 1+(at)^{2n}-2(at)^n\cos(n\beta)$, which is negative for $\beta \in (0,\pi/n)$ because the integrand in the integral is negative. Moreover, using Proposition 14 again with $\alpha = 1/2$ and $u = 2\beta$ and taking derivatives with respect to β we get

$$\begin{aligned} N_5(\beta) &= \frac{1}{2}\lim_{a\to 1^-}\frac{n}{\pi}\int_0^1 \frac{d}{d\beta}\left(\frac{t^{-1/2}}{(1-t)^{1/2}}\,\frac{1}{(1-a^2t)^{1/2}}\,\frac{1-(at)^{2n}}{B_2}\right)dt \\ &= \frac{n\sin(2n\beta)}{\pi}\lim_{a\to 1^-}\int_0^1 \frac{-2n(at)^n(1-(at)^{2n})}{t^{1/2}(1-t)^{1/2}(1-a^2t)^{1/2}B_2^2}\,dt, \end{aligned}$$

with $B_2 = 1+(at)^{2n}-2(at)^n\cos(2n\beta)$, which is negative if $\beta \in (0,\pi/2n)$ and positive if $\beta \in (\pi/2n,\pi/n)$ because again the integrand is negative.

Consider now the case $a > 1$. In this case using Proposition 15 in Appendix 1 with $\alpha = 1/2$ and $u = \beta$ and taking derivatives with respect to β we get that

$$\begin{aligned} N_4(a,\beta) &= \frac{n}{a^2\pi}\int_0^1 \frac{d}{d\beta}\left(\frac{a^{2n}-t^{2n}}{t^{1/2}(1-t)^{1/2}(a^2-t)^{1/2}B_3}\right)dt \\ &= \frac{n\sin(n\beta)}{a^2\pi}\int_0^1 \frac{-2n(at)^n(a^{2n}-t^{2n})}{t^{1/2}(1-t)^{1/2}(a^2-t)^{1/2}B_3^2}\,dt, \end{aligned}$$

with $B_3 = a^{2n}+t^{2n}-2a^nt^n\cos(n\beta)$, which is negative for $\beta \in (0,\pi/n)$. Moreover, using Proposition 15 again with $\alpha = 1/2$ and $u = 2\beta$ and taking derivatives with respect to β we get that

$$\begin{aligned} N_5(\beta) &= \frac{1}{2}\lim_{a\to 1^+}\frac{n}{\pi}\int_0^1 \frac{d}{d\beta}\left(\frac{a^{2n}-t^{2n}}{t^{1/2}(1-t)^{1/2}(a^2-t)^{1/2}B_4}\right)dt \\ &= \frac{n\sin(2n\beta)}{\pi}\lim_{a\to 1^+}\int_0^1 \frac{-2n(at)^n(a^{2n}-t^{2n})}{t^{1/2}(1-t)^{1/2}(1-a^2t)^{1/2}B_4^2}\,dt, \end{aligned}$$

with $B_4 = a^{2n}+t^{2n}-2a^nt^n\cos(2n\beta)$, which is negative if $\beta \in (0,\pi/2n)$ because the integrand is negative and positive if $\beta \in (\pi/2n,\pi/n)$, again because the integrand is negative. Therefore, $N_4(a,\beta)$ is negative for $\beta \in (0,\pi/n)$ and $N_5(\beta)$ is negative for $\beta \in (0,\pi/2n)$ and positive for $\beta \in (\pi/2n,\pi/n)$.

In short, $m = -a^3N_4(a,\beta)/N_5(\beta)$ is positive if and only if $\beta \in (\pi/2n,\pi/n)$ and the proposition is proved. □

4.3. **Proof of Theorem 7(a).** We study the existence of central configurations of the semiregular bicircular $3n$–body problem around $\beta = \pi/n$. For proving Theorem 7(a) we need the following auxiliary proposition concerning the sign of the function $F(a,\beta)$ as $\beta \to \pi/n$.

Proposition 11. *The following holds for* $\bar{F}(a) = \lim_{\beta\to\pi/n} F(a,\beta)$.

(a) $\bar{F}(a) > 0$ *when* $a \to \infty$;
(b) $\bar{F}(a) < 0$ *when* $a \to 1$;
(c) $\bar{F}(a) > 0$ *when* $a \to 0$ *and* $n \geq 3$ *and* $\bar{F}(a) < 0$ *when* $a \to 0$ *and* $n = 2$.

We note that in Proposition 11 we have that $\bar{F}(a)$ could be $\pm\infty$.

Proposition 11 is proved in Appendix 2.

Proof of Theorem7(a). It follows from Proposition 10 that all solutions of $F(a,\beta) = 0$ for $\beta \in (\pi/2n, \pi/n)$ satisfy $m > 0$, therefore all solutions of $F(a,\beta) = 0$ provide central configurations of the semiregular bicircular $3n$–body problem. Notice that the function F is continuous for $a \in (0,\infty)$ and $\beta \in (\pi/2n, \pi/n)$, therefore the points where the sign of F changes provides always solutions of $F(a,\beta) = 0$.

In view of Proposition 11 we have that when $n \geq 3$, fixed a value of β in a sufficiently small neighborhood of π/n, the function F has at least one change of sign with $a > 1$ and one change of sign with $0 < a < 1$. So for each β in a sufficiently small neighborhood of π/n there are at least two values of a for which $F(a,\beta) = 0$, one with $a > 1$ and one with $0 < a < 1$. When $n = 2$ the function F has at least one change of sign with $a > 1$. So for each β in a sufficiently small neighborhood of π/n there is at least one value of a with $a > 1$ for which $F(a,\beta) = 0$.

On the other hand, from the proofs of Lemma 5 and Proposition 11 (see Appendix 2) we have that $\lim_{\beta\to\pi/n} N_4(a,\beta) = 0$ and $\lim_{\beta\to\pi/n} N_5(\beta) = \infty$, respectively. Therefore from (16), $m \to 0$ when $\beta \to \pi/n$.

□

4.4. **Proof of Theorem 7(b).** Now we study the existence of central configurations of the semiregular bicircular $3n$–body problem around $\beta = \pi/2n$. For proving Theorem 7(b) we need the following two auxiliary propositions concerning the study of the sign of the function $F(a,\beta)$.

Proposition 12. *The following holds for* $\bar{F}(a) = \lim_{\beta\to\pi/2n} F(a,\beta)$.

(a) $\bar{F}(a) < 0$ *when* $a \to 0$*;*
(b) $\bar{F}(a) < 0$ *when* $a \to \infty$*;*
(c) $\bar{F}(a) > 0$ *when* $a \to 1$.

Proposition 13. *The following statements hold for* $\bar{F}(\beta) = \lim_{a\to 0} F(a,\beta)$ *and* $\tilde{F}(\beta) = \lim_{a\to\infty} F(a,\beta)$.

(a) *For all* $\beta \in (\pi/2n, \pi/n)$ *we have* $\bar{F}(\beta) > 0$ *for* $n \geq 3$ *and* $\bar{F}(\beta) < 0$ *for* $n = 2$.
(b) *For all* $\beta \in (\pi/2n, \pi/n)$ *and* $n \geq 2$ *we have* $\tilde{F}(\beta) > 0$.

We note that in Proposition 13, both $\bar{F}(\beta)$ and $\tilde{F}(\beta)$ could be $\pm\infty$.

The proof of Propositions 12 and 13 can be found in Appendix 3.

Proof of Theorem 7(b). As in Theorem 7(a) recall that all solutions of $F(a,\beta) = 0$ with $\beta \in (\pi/2n, \pi/n)$ satisfy $m > 0$ and therefore they provide central configurations of the semiregular bicircular $3n$–body problem. Moreover the points where the sign of F changes provide solutions of $F(a,\beta) = 0$.

In view of Proposition 12 we have that in a sufficiently small neighborhood of $\beta = \pi/2n$ the function F has at least one change of sign with $a > 1$ and one change of sign with $0 < a < 1$. Therefore fixed a value of β in a sufficiently small neighborhood of $\beta = \pi/2n$, there exist at least two zeros of F, one with $a < 1$ and one with $0 < a < 1$.

In view of Propositions 12 and 13 we have that when $n \geq 3$

$$\lim_{a\to 0}\lim_{\beta\to\pi/2n} F(a,\beta) < 0, \qquad \lim_{\beta\to\pi/2n}\lim_{a\to 0} F(a,\beta) > 0.$$

Therefore fixed a value of β in a sufficiently small neighborhood of $\pi/2n$, there exist at least one zero of F with a sufficiently close to the origin.

Again, in view of Propositions 12 and 13 we have that for all $n \geq 2$

$$\lim_{a\to\infty}\lim_{\beta\to\pi/2n} F(a,\beta) < 0, \qquad \lim_{\beta\to\pi/2n}\lim_{a\to\infty} F(a,\beta) > 0$$

Therefore fixed a value of β in a sufficiently small neighborhood of $\pi/2n$, there exists at least one zero of F with a sufficiently large.

In short, fixed a value of β in a sufficiently small neighborhood of $\pi/2n$ there exist at least four zeroes of F when $n \geq 3$, one near $a = 0$, one with $0 < a < 1$ not necessarily small, one with a sufficiently large and one with $a > 1$ not necessarily large.

When $n = 2$ there exist at least three solutions of F one with a sufficiently large, one with $a > 1$ not necessarily large, and one with $0 < a < 1$.

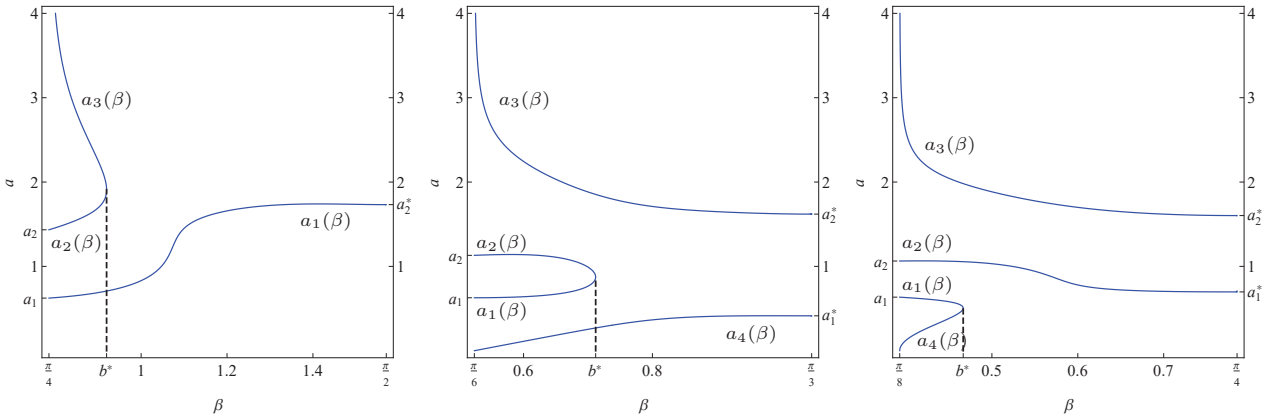

Fig. 9a $n = 2$ Fig. 9b $n = 3$ Fig. 9c $n = 4$

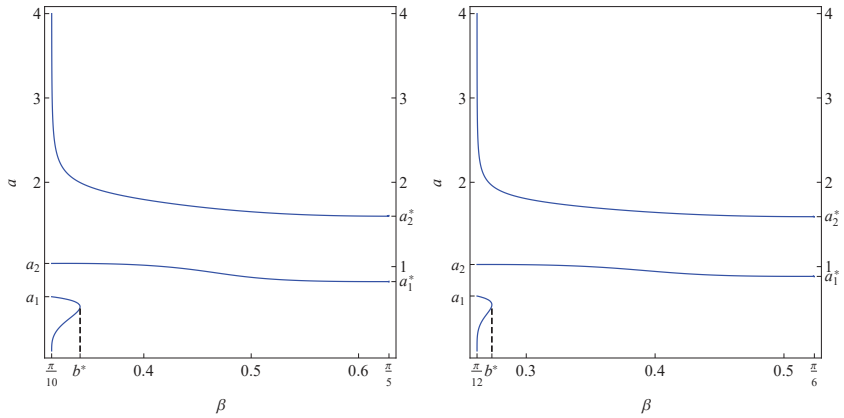

Fig. 9d $n = 5$ Fig. 9e $n = 6$

FIGURE 9. Solutions of $F(a, \beta) = 0$.

On the other hand, in the proof of Lemma 6 (see Appendix 3) we have seen that $\lim_{\beta\to\pi/2n} N_5(\beta) = 0$. Moreover from (4.2) we have that $a \notin \{0; \infty\}$ then $\lim_{\beta\to\pi/2n} N_4(a, \beta) \neq 0$. Therefore from (16) we can guarantee that $m \to \infty$ at the central configurations coming from the zeroes of F with $0 < a < 1$ not small and $a > 1$ not large.

This completes the proof of Theorem 7(b). □

5. Particular cases of semiregular bicircular central configurations of the $3n$–body problem

When $n = 2$ from Theorem 7(b) we have at least three families (depending on β) of central configurations in a neighborhood of $\beta = \pi/4$, one emanating from a point $(a_1, \pi/4)$, one emanating from $(a_2, \pi/4)$ with $a_2 > 1$, and one emanating from $(\infty, \pi/4)$. From Theorem 7(a) we have at least one family of central configurations in a neighborhood of $\beta = \pi/2$ that emanates from a point $(a_2^*, \pi/2)$ with $a_2^* > 1$. They are given by the families of zeroes of F for $n = 2$.

We have studied numerically these families of zeroes and we have obtained the following (see Fig. 9a) the family of central configurations emanating from the point $(a_1, \pi/4) = (0.6240605991.., \pi/4)$ joins the family emanating from $(a_2^*, \pi/2) = (\sqrt{3}, \pi/2)$ and the family emanating from $(a_2, \pi/4) = (1.4339374069.., \pi/4)$ joins the family emanating from $(\infty, \pi/4)$. Moreover, these are the only families of central configurations. In particular if $\beta \in (\pi/4, b^*)$ with $b^* = 0.9195936184..$ the semiregular bicircular 6–body problem has three different central configurations, if $b = b^*$ it has two central configurations and if $\beta > b^*$ it has only one central configuration.

When $n \geq 3$ from Theorem 7(a) we have at least four families of central configurations of the semiregular bicircular $3n$–body problem in a neighborhood of $\beta = \pi/2n$, one emanating from $(0, 2\pi/n)$, one emanating from a point $(a_1, \pi/2n)$ with $a_1 \in (0, 1)$, one emanating from a point

$(a_2, \pi/2n)$ with $a_2 > 1$ and one emanating from $(\infty, \pi/2n)$. Moreover, from Theorem 7(b) we have at least two families of central configurations in a neighborhood of $\beta = \pi/n$, one emanating from a point $(a_1^*, \pi/n)$ with $a_1^* \in (0,1)$ and one emanating from a point $(a_2^*, \pi/n)$ with $a_2^* > 1$. As above they are given by the families of zeroes of F.

We have studied numerically these families of zeroes for $n = 3, 4, 5, 6$ and we have obtained the following (see again Fig. 9).

When $n = 3$ the family of central configurations emanating from the point $(0, \pi/6)$ joins the family emanating from $(a_1^*, \pi/3) = (0.4138879324.., \pi/3)$, the family emanating from $(a_1, \pi/6) = (0.6280478552.., \pi/6)$ joins the family emanating from $(a_2, \pi/6) = (1.1308109202.., \pi/6)$ and the family emanating from $(\infty, \pi/6)$ joins the family emanating from $(a_2^*, \pi/3) = (1.6197896088.., \pi/3)$. Moreover, these are the only families of central configurations when $n = 3$. In particular if $\beta \in (\pi/6, b^*)$ with $b^* = 0.7119233840..$ the semiregular bicircular 9–body problem has four different central configurations, if $b = b^*$ it has three central configurations and if $\beta > b^*$ it has two central configurations.

When $n = 4$ the family of central configurations emanating from $(0, \pi/8)$ joins the family emanating from $(a_1, \pi/8) = (0.6351161391.., \pi/8)$, the family emanating from $(a_2, \pi/8) = (1.0636734282.., \pi/8)$ joins the family emanating from $(a_1^*, \pi/4) = (0.697380509.., \pi/4)$ and the family emanating from $(\infty, \pi/8)$ joins the family emanating from $(a_2^*, \pi/4) = (1.6024084862.., \pi/4)$. Moreover, these are the only families of central configurations when $n = 4$. In particular if $\beta \in (\pi/2n, b^*)$ with $b^* = 0.4665964724..$ the semiregular bicircular 12–body problem has four different central configurations, if $b = b^*$ it has three central configurations and if $\beta > b^*$ it has two central configurations. The same behavior occurs for $n = 5, \dots, 100$ (see Fig. 9 for $n = 5, 6$), so we conjecture that this happens for all $n > 6$. We note that when $n = 5$, $a_1 = 0.6434495204..$, $a_2 = 1.0379259369..$, $a_1^* = 0.822828699..$, $a_2^* = 1.5979217289..$ and $b^* = 0.3406546931..$; and when $n = 6$, $a_1 = 0.6515248377..$, $a_2 = 1.0252694202..$, $a_1^* = 0.8843211381..$, $a_2^* = 1.5922353553..$ and $b^* = 0.2733239284...$

Finally have compute the values of the masses for the families of central configurations with $n = 2, 3, 4$ and we have plotted them in Fig. 10.

Appendix 1. Proof of Proposition 9

We state and prove some auxiliary results that will be used in the proof of Proposition 9. We need the following two propositions taken from [1].

Proposition 14 ([1, Proposition 7]). *For $0 \leqslant a < 1$, $\alpha \in (0,1)$ and $u \in [0, 2\pi)$ we have*

$$\sum_{j=1}^{n} \frac{1}{\left(1 + a^2 - 2a\cos\left(\frac{2\pi j}{n} + u\right)\right)^{\alpha}} = \frac{n\,\sin(\pi\alpha)}{\pi} \int_0^1 \frac{t^{\alpha-1}}{(1-t)^{\alpha}} \frac{1}{(1-a^2t)^{\alpha}} \frac{1-(at)^{2n}}{1+(at)^{2n} - 2(at)^n\cos(nu)} dt.$$

Proposition 15 ([1, Proposition 8]). *For $a > 1$, $\alpha \in (0,1)$ and $u \in [0, 2\pi)$ we have*

$$\sum_{j=1}^{n} \frac{1}{\left(1 + a^2 - 2a\cos\left(\frac{2\pi j}{n} + u\right)\right)^{\alpha}} = \frac{n\,\sin(\pi\alpha)}{\pi} \int_0^1 \frac{t^{\alpha-1}}{(1-t)^{\alpha}} \frac{1}{(a^2-t)^{\alpha}} \frac{a^{2n}-t^{2n}}{a^{2n}+t^{2n} - 2(at)^n\cos(nu)} dt.$$

We need the following auxiliary result.

Lemma 1. *Let $u \in \mathbb{R}$ and let $\gamma = 2\pi\, j/n + u$.*

(a) *The following identites hold for all $\ell \in \mathbb{N}$*

$$\sum_{j=1}^{n} \cos\left(\frac{2\ell\pi\, j}{n} + u\right) = 0, \qquad \sum_{j=1}^{n} \sin\left(\frac{2\ell\pi\, j}{n} + u\right) = 0, \tag{18}$$

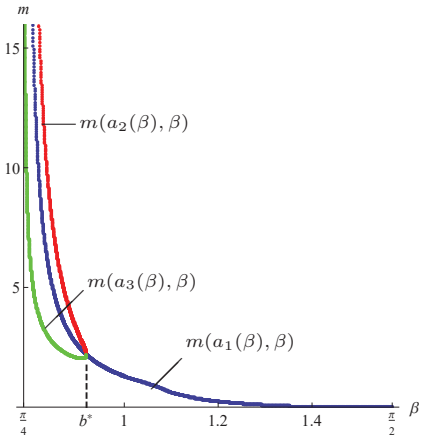

Fig. 10a $n = 2$

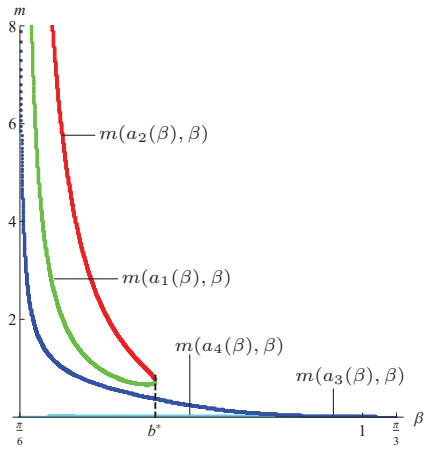

Fig. 10b $n = 3$

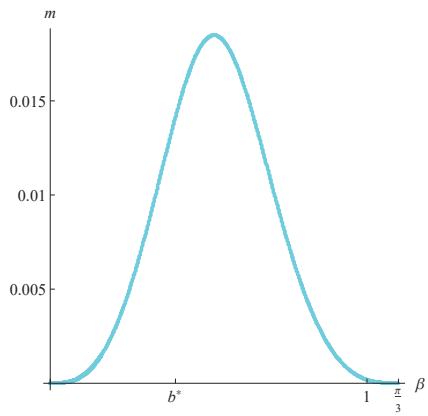

Fig. 10b The plot of $m(a_4(\beta),\beta)$ for $n = 3$

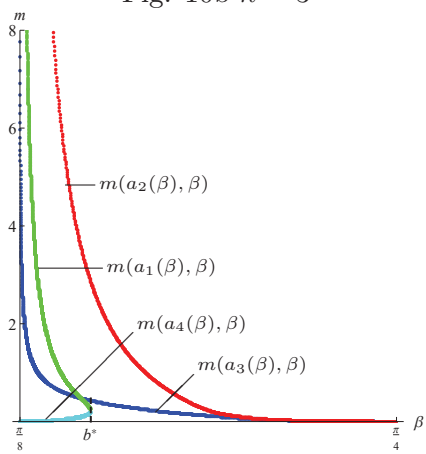

Fig. 9c $n = 4$

FIGURE 10. Values of m at the families of semiregular bicircular central configurations of the $3n$–body problem.

when $n \geq 2$ *and* $n \neq \ell$.

(b) *For* $n \geq 3$ *we have*

$$\sum_{j=1}^{n} \cos^2 \gamma = \frac{n}{2}, \tag{19}$$

and for $n = 2$ *we have*

$$\sum_{j=1}^{2} \cos^2 \left(\pi\, j + u\right) = 2\cos^2 u. \tag{20}$$

(c) *For all* $n \geq 1$ *we get*

$$\sum_{j=1}^{n} \frac{a - \cos\gamma}{\left(1 + a^2 - 2a\cos\gamma\right)^{3/2}} = -\frac{d}{da}\sum_{j=1}^{n} \frac{1}{\left(1 + a^2 - 2a\cos\gamma\right)^{1/2}}.$$

(d) *Let*

$$L(a,u) = \sum_{j=1}^{n} \frac{1 - 1/a\cos\gamma}{\left(1 + a^2 - 2a\cos\gamma\right)^{3/2}}.$$

We have $L(a,u) = -n/2 + O(a)$ *when* $n \geq 3$ *and* $L(a,u) = 2 - 6\cos^2 u + O(a)$ *when* $n = 2$.

Proof. Using the sum of the first n terms of a geometric series we get

$$\sum_{j=1}^{n} e^{i(2\ell\pi j/n+u)} = 0,$$

for all $\ell \in \mathbb{Z}$ with $\ell \neq n$ (here $i = \sqrt{-1}$). This proves statement (a). From the formula of the cosinus of twice an angle an applying statement (a) with $\ell = 2$ and $2u$ instead of u we get

$$\sum_{j=1}^{n} \cos^2\left(\frac{2\pi\, j}{n} + u\right) = \frac{1}{2}\sum_{j=1}^{n}\left(\cos\left(2\Big(\frac{2\pi\, j}{n} + u\Big)\right) + 1\right) = \frac{n}{2}.$$

This proves statement (b) for $n > 2$. Statement (b) for $n = 2$ and statement (c) follows from direct computations.

Expanding the function L in Laurent series around $a = 0$ we have

$$L(a,u) = -\frac{1}{a}\sum_{j=1}^{n}\cos\big(\frac{2\pi\, j}{n} + u\big) + \sum_{j=1}^{n} 1 - 3\sum_{j=1}^{n}\cos^2\big(\frac{2\pi\, j}{n} + u\big) + O(a).$$

Then, when $n \geq 3$ in view of (18) with $\ell = 1$ together with (19) we obtain

$$L(a,u) = n - \frac{3n}{2} + O(a) = -\frac{n}{2} + O(a)$$

and when $n = 2$ in view of (18) with $\ell = 1$ and (20) we obtain

$$L(a,u) = 2 - 6\cos^2 u + O(a).$$

This completes the proof of statement (d).

□

We need the following technical lemma.

Lemma 2. *We have* $f(v) = (1+v)(1-v^{2n}) - 4nv^n(1-v) > 0$ *for* $n \geq 2$ *and* $v \in (0,1)$.

Proof. Note that

$$\begin{aligned} f(v) =& (1+v)(1-v^n)(1+v^n) - 4nv^n(1-v) \\ =& (1-v)\big((1+v)(1+v+v^2+\ldots+v^{n-1})(1+v^n) - 4nv^n\big) \\ :=& (1-v)g(v) \end{aligned}$$

where

$$\begin{aligned} g(v) &= (1+v)(1+v+v^2+\ldots+v^{n-1})(1+v^n) - 4nv^n \\ &= 1 + 2v + 2v^2 + 2v^3 + \ldots + 2v^{n-1} + 2v^n + 2v^n + 2v^{n+1} + \ldots \\ &\quad + 2v^{2n-1} + v^{2n} - 4nv^n \\ &= 1 + 2v + \ldots + 2v^{n-1} + (2-4n)v^n + 2v^{n+1} + \ldots + 2v^{2n-1} + v^{2n}. \end{aligned} \tag{21}$$

We see that

$$g(1) = 0 \quad \text{and} \quad g'(1) = 0.$$

Indeed,

$$g(1) = 1 + 2(n-1) + 2 - 4n + 2(n-1) + 1 = 0,$$

and since

$$\begin{aligned} g'(v) = 2 + 4v + 6v^2 + \ldots + 2(n-1)v^{n-2} + (2-4n)nv^{n-1} \\ + 2(n+1)v^n + \ldots + 2(2n-1)v^{2n-2} + 2nv^{2n-1}, \end{aligned}$$

then

$$\begin{aligned} g'(1) &= 2(1+2+3+\ldots+(n-1)) + n(2-4n) \\ &\quad + 2(n+1+n+2+\ldots+2n-1) + 2n \\ &= 2\Big(\sum_{j=1}^{2n-1} j - n + (1-2n)n + n\Big) = 2\Big(\sum_{j=1}^{2n-1} j + n(1-2n)\Big) = 0. \end{aligned}$$

Therefore,

$$g(v) = (1-v)^2 h(v), \quad h(v) = \sum_{j=0}^{2n-2} c_j v^j$$

for some coefficients c_j.

We will show by induction that

$$c_j = \begin{cases} (j+1)^2 & \text{for } j = 0, \dots, n-1, \\ (2n-j-1)^2 & \text{for } j = n, \dots, 2n-2. \end{cases}$$

In view of (21) we have that the coefficients c_j satisfy $c_0 = 1$, $c_1 = 2 + 2c_0 = 4$,

$$\begin{aligned} &c_k + c_{k-2} - 2c_{k-1} = 2 \quad \text{for } k = 2, \dots, n-1, n+1, \dots, 2n-2\ , \\ &c_n + c_{n-2} - 2c_{n-1} = 2 - 4n, \end{aligned}$$

$c_{2n-3} - 2c_{2n-2} = 2$ and $c_{2n-2} = 1$ (and so $c_{2n-3} = 4$).

We prove by induction the cases for $j = 0, \dots, n-1$. It is clear for $j = 0, 1$ and we will prove it for some $2 \le j < n-1$. Note that by the induction hypotheses

$$c_{j+1} = -c_{j-1} + 2c_j + 2 = -j^2 + 2j^2 + 4j + 2 + 2 = j^2 + 4j + 4 = (j+2)^2$$

and so the induction is satisfied for $j = 0, \dots, n-1$. For $j = n$ we have

$$c_n = 2 - 4n - (n-1)^2 + 2n^2 = 2 - 4n - n^2 + 2n - 1 + 2n^2 = n^2 - 2n + 1 = (n-1)^2,$$

and for $j = n+1$ we have

$$c_{n+1} = -c_{n-1} + 2c_n + 2 = -(n-1+1)^2 + 2(n-1)^2 + 2 = (2n - (n+1) - 1)^2.$$

For $j = n, \dots, 2n-2$ the induction hypotheses yields $c_j = (2n-j-1)^2$. Note that it is clear for $j = n$ and $j = n+1$ and we will show it for $n+1 < j \le 2n-2$. In particular the cases $j = 2n-3$ and $j = 2n-2$ are also trivially satisfied. So, we only need to show it for $n+1 < j < 2n-3$. By the induction hypotheses for any $n+1 < j < 2n-3$ we have

$$\begin{aligned} c_j &= 2 - c_{j-2} + 2c_{j-1} = 2 - (2n - (j-2) - 1)^2 + 2(2n - (j-1) - 1)^2 \\ &= 2 - (2n-j+1)^2 + 2(2n-j)^2 = (2n-j-1)^2 \end{aligned}$$

and the induction hypotheses holds. In short the lemma is proved. □

The next result concern properties of the functions $K_i(a)$ introduced in Section 3.

Lemma 3. *The following statements hold for all $n \ge 2$:*

(a) $K_6(a) > K_4(a)$ *for* $a \in (0,1)$*;*
(b) $K_6(a) < K_4(a)$ *for* $a > 1$*;*
(c) $K_5 > K_1 > 0$.
(d) $a^3 K_4(a) = K_2(1/a)$ *and* $a^3 K_6(a) = K_3(1/a)$.

Proof. Note that

$$\begin{aligned} aK_6(a) &= \sum_{j=1}^{n} \frac{a - \cos\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right)}{\left(1 + a^2 - 2a\cos\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right)\right)^{3/2}}, \\ aK_4(a) &= \sum_{j=1}^{n} \frac{a - \cos\left(\frac{2\pi j}{n}\right)}{\left(1 + a^2 - 2a\cos\left(\frac{2\pi j}{n}\right)\right)^{3/2}}. \end{aligned}$$

When $a \in (0,1)$, using Lemma 1(c) and Proposition 14 with $\alpha = 1/2$ and $u = 0$ we get

$$aK_4(a) = -\frac{n}{\pi} \int_0^1 \frac{d}{da} \left(\frac{t^{-1/2}}{(1-t)^{1/2}} \frac{1}{(1-a^2t)^{1/2}} \frac{1-(at)^{2n}}{1+(at)^{2n} - 2(at)^n} \right) dt.$$

and using Lemma 1(c) and Proposition 14 with $\alpha = 1/2$ and $u = \pi/n$ we get

$$aK_6(a) = -\frac{n}{\pi} \int_0^1 \frac{d}{da} \left(\frac{t^{-1/2}}{(1-t)^{1/2}} \frac{1}{(1-a^2t)^{1/2}} \frac{1-(at)^{2n}}{1+(at)^{2n} + 2(at)^n} \right) dt.$$

Hence, for $a \in (0,1)$,

$$K_6(a) - K_4(a) = -\frac{n}{\pi a}\int_0^1 \frac{d}{da}\left(\frac{t^{-1/2}}{(1-t)^{1/2}}\frac{1}{(1-a^2t)^{1/2}}\frac{4(at)^n}{((at)^{2n}-1)}\right)dt$$
$$= -\frac{n}{\pi a}\int_0^1 \frac{4(ta)^n(ta^2(-1+(ta)^{2n})+n(ta^2-1)(1+(ta)^{2n}))}{a(1-t)^{1/2}t^{1/2}(1-a^2t)^{3/2}((ta)^{2n}-1)^2}\,dt > 0,$$

because the integrand is negative for $a \in (0,1)$. Therefore, $K_6(a) > K_4(a)$ for $0 < a < 1$ and so statement (a) is proved.

For $a > 1$ using Lemma 1(c) and Proposition 15 with $\alpha = 1/2$ and $u = 0$ we get

$$aK_4(a) = -\frac{n}{\pi}\int_0^1 \frac{d}{da}\left(\frac{t^{-1/2}}{(1-t)^{1/2}}\frac{1}{(a^2-t)^{1/2}}\frac{a^{2n}-t^{2n}}{a^{2n}+t^{2n}-2a^nt^n}\right)dt.$$

and using Lemma 1(c) and Proposition 15 with $\alpha = 1/2$ and $u = \pi/n$ and taking derivatives with respect to a we get

$$aK_6(a) = -\frac{n}{\pi}\int_0^1 \frac{d}{da}\left(\frac{t^{-1/2}}{(1-t)^{1/2}}\frac{1}{(a^2-t)^{1/2}}\frac{a^{2n}-t^{2n}}{a^{2n}+t^{2n}+2a^nt^n}\right)dt.$$

Hence, for $a > 1$,

$$K_6(a) - K_4(a) = -\frac{n}{\pi a}\int_0^1 \frac{d}{da}\left(\frac{t^{-1/2}}{(1-t)^{1/2}}\frac{1}{(a^2-t)^{1/2}}\frac{4(at)^n}{t^{2n}-a^{2n}}\right)dt$$
$$= -\frac{n}{\pi a}\int_0^1 \frac{4(ta)^n(a^2(a^{2n}-t^{2n})+n(a^2-t)(a^{2n}+t^{2n}))}{a(1-t)^{1/2}t^{1/2}(a^2-t)^{3/2}(t^{2n}-a^{2n})^2}\,dt < 0,$$

because the integrand is positive for $a > 1$. Therefore, $K_6(a) < K_4(a)$ for $a > 1$ and statement (b) is proved.

To prove statement (c) we proceed as follows. Note that

$$K_5 = \lim_{a\to 1}\frac{1}{2}\sum_{j=1}^{n}\frac{1}{\left(1+a^2-2a\cos\left(\frac{2\pi j}{n}+\frac{\pi}{n}\right)\right)^{1/2}} := \lim_{a\to 1}\frac{1}{2}A_0,$$
$$K_1 = \lim_{a\to 1}\frac{1}{2}\sum_{j=1}^{n-1}\frac{1}{\left(1+a^2-2a\cos\left(\frac{2\pi j}{n}\right)\right)^{1/2}}$$
$$= \lim_{a\to 1}\frac{1}{2}\left(\sum_{j=1}^{n}\frac{1}{\left(1+a^2-2a\cos\left(\frac{2\pi j}{n}\right)\right)^{1/2}} - \sum_{j=1}^{1}\frac{1}{(1+a^2-2a\cos(2\pi j))^{1/2}}\right)$$
$$:= \lim_{a\to 1}\frac{1}{2}(A_1 - A_2),$$

Thus

$$K_5 - K_1 = \lim_{a\to 1}\frac{1}{2}(A_0 - A_1 + A_2) \tag{22}$$

where A_0, A_1, A_2 are the summations defined above. Applying Proposition 14 we have that $A_0 - A_1 + A_2$ when $a \in (0,1)$ is given by

$$A = \frac{n}{\pi}\int_0^1 \frac{1-(ta)^n}{(1-t)^{1/2}t^{1/2}(1-a^2t)^{1/2}(1+(ta)^n)}\,dt$$
$$-\frac{n}{\pi}\int_0^1 \frac{1+(ta)^n}{(1-t)^{1/2}t^{1/2}(1-a^2t)^{1/2}(1-(ta)^n)}\,dt$$
$$+\frac{1}{\pi}\int_0^1 \frac{1+ta}{(1-t)^{1/2}t^{1/2}(1-a^2t)^{1/2}(1-ta)}\,dt$$
$$= \frac{n}{\pi}\int_0^1 \left(\frac{4(ta)^n}{-1+(ta)^{2n}} + \frac{1+ta}{n(1-ta)}\right)\frac{1}{(1-t)^{1/2}t^{1/2}(1-a^2t)^{1/2}}\,dt.$$

On the other hand, applying Proposition 15 we have that $A_0 - A_1 + A_2$ when $a > 1$ is given by

$$\overline{A} = \frac{n}{\pi}\int_0^1 \left(-\frac{4(ta)^n}{a^{2n}-t^{2n}} + \frac{a+t}{n(a-t)}\right)\frac{1}{(1-t)^{1/2}t^{1/2}(a^2-t)^{1/2}}\,dt.$$

After doing the substitution $a \to 1/a$, the expression $\overline{A}$ can be written as $a\,A$. Thus

$$\lim_{a\to 1^-} A = \lim_{a\to 1^+} \overline{A} = \lim_{a\to 1} A_0 - A_1 + A_2. \tag{23}$$

Now we show that $A > 0$. Note that taking $v = ta$ we get

$$\frac{4(ta)^n}{-1+(ta)^{2n}} + \frac{1+ta}{n(1-ta)} = \frac{4v^n}{-1+v^{2n}} + \frac{1}{n}\frac{1+v}{1-v}.$$

Using Lemma 2 we get that

$$\frac{4v^n}{-1+v^{2n}} + \frac{1}{n}\frac{1+v}{1-v} = \frac{(1+v)(1-v^{2n}) - 4nv^n(1-v)}{n(1-v)(1-v^{2n})} > 0,$$

for $v < 1$. So $A > 0$ and taking the limit when $a \to 1^-$ together with (23) and (22) we get $\lim_{a\to 1^-} A = \lim_{a\to 1^+} \overline{A} = 2(K_5 - K_1) > 0$ for all $n \geq 2$. Moreover K_1 is positive by definition. So, we have proved statement (c).

Statement (d) follows from direct computations. □

In the following lemma we provide properties of the function $\Delta(a) = -K_1 - K_5 + a^3(K_2(a) + K_3(a))$ that appears in the denominator m_D in (4).

Lemma 4. *For all $n \geq 2$ the following statements hold.*

(a) $\Delta(a)$ *is increasing for all* $a \in (0,1)$*;*
(b) $\Delta(0) < 0$, $\Delta(a) \to \infty$ *when* $a \to 1^-$ *and* $\Delta(a) \to -\infty$ *when* $a \to 1^+$*;*
(c) $\Delta(a) < 0$ *for all* $a > 1$*;*
(d) $\Delta(a)$ *has a unique zero and it belongs to the interval* $(0,1)$*.*

Proof. We first note that setting $b = 1/a$ we have

$$\begin{aligned} a^3K_2(a) &= \frac{1}{b}\sum_{j=1}^n \frac{b - \cos\left(\frac{2\pi j}{n}\right)}{\left(1+b^2-2b\cos\left(\frac{2\pi j}{n}\right)\right)^{3/2}} := \frac{1}{b}\bar{K}_2(b),\\ a^3K_3(a) &= \frac{1}{b}\sum_{j=1}^n \frac{b - \cos\left(\frac{2\pi j}{n}+\frac{\pi}{n}\right)}{\left(1+b^2-2b\cos\left(\frac{2\pi j}{n}+\frac{\pi}{n}\right)\right)^{3/2}} := \frac{1}{b}\bar{K}_3(b). \end{aligned} \tag{24}$$

Let $\overline{\Delta}(b) = (1/b\bar{K}_2(b) + 1/b\bar{K}_3(b))$. We want to show that for $a \in (0,1)$

$$\Delta'(a) = \left.\frac{d\overline{\Delta}(b)}{db}\right|_{b=1/a} \cdot \frac{db}{da} = \left.(\overline{\Delta}(b))'\right|_{b=1/a} \cdot \left(-\frac{1}{a^2}\right) > 0.$$

So, it is sufficient to show that $\Delta(b)' < 0$ for $b > 1$.

Let

$$T = \sum_{j=1}^n \frac{1}{\left(1+b^2-2b\cos\left(\frac{2\pi j}{n}\right)\right)^{1/2}} + \sum_{j=1}^n \frac{1}{\left(1+b^2-2b\cos\left(\frac{2\pi j}{n}+\frac{\pi}{n}\right)\right)^{1/2}}.$$

Using Lemma 1(c) we get that

$$\bar{K}_2(b) + \bar{K}_3(b) = -dT/db \tag{25}$$

and so $(\bar{\Delta}(b))' = \frac{1}{b^2}T_1 - \frac{1}{b}T_2$ where

$$T_1 = \frac{dT}{db}, \quad T_2 = \frac{d^2T}{db^2}.$$

Since from Proposition 15 we have

$$\begin{aligned} T_1 &= -\frac{2n}{\pi}\int_0^1 \frac{d}{db}\left(\frac{1}{t^{1/2}(1-t)^{1/2}(b^2-t)^{1/2}}\,\frac{t^{2n}+b^{2n}}{t^{2n}-b^{2n}}\right)dt \\ &= \frac{n}{\pi}\int_0^1 \frac{1}{t^{1/2}(1-t)^{1/2}}\left(\frac{2b(t^{2n}+b^{2n})}{(b^2-t)^{3/2}(t^{2n}-b^{2n})}-\frac{8nt^{2n}b^{2n-1}}{(b^2-t)^{1/2}(t^{2n}-b^{2n})^2}\right)dt, \end{aligned}$$

taking another derivative with respect to b we get

$$\begin{aligned} T_2 = \frac{n}{\pi}\int_0^1 \frac{1}{t^{1/2}(1-t)^{1/2}}\Bigg(&\frac{-8nt^{2n}b^{2n-2}((2n-1)t^{2n}+(2n+1)b^{2n})}{(b^2-t)^{1/2}(t^{2n}-b^{2n})^3} \\ &+\frac{2\left(8nb^{2n}\left(b^2-t\right)t^{2n}+\left(2b^2+t\right)\left(b^{4n}-t^{4n}\right)\right)}{\left(b^2-t\right)^{5/2}\left(t^{2n}-b^{2n}\right)^2}\Bigg)dt. \end{aligned}$$

Since $T_1<0$ (the integrand is negative) and $T_2>0$ (the integrand is positive) we readily obtain that $(\bar{\Delta}(b))'<0$ and so $\Delta'(a)>0$ for $a\in(0,1)$. In short statement (a) is proved.

It is clear that $\Delta(0)=-K_1-K_5<0$, see Lemma 3(c). Moreover

$$\lim_{a\to 1}K_2(a)=\sum_{j=1}^{n-1}\frac{1-\cos\left(\frac{2\pi j}{n}\right)}{\left(2-2\cos\left(\frac{2\pi j}{n}\right)\right)^{3/2}}+\lim_{a\to 1}\frac{1-a}{\left(1+a^2-2a\right)^{3/2}},$$

so

$$\lim_{a\to 1^-}K_2(a)=\infty \quad\text{and}\quad \lim_{a\to 1^+}K_2(a)=-\infty. \tag{26}$$

Furthermore $\lim_{a\to 1}K_3(a)=K_5$ which is different from zero and from infinity. Hence, using (26) we get

$$\lim_{a\to 1^-}\Delta(a)=\infty \quad\text{and}\quad \lim_{a\to 1^+}\Delta(a)=-\infty.$$

This completes the proof of statement (b).

Now we show that $\Delta(a)<0$ for $a>1$. To do so, we will show that $a^3K_2(a)+a^3K_3(a)<0$. Note that this is sufficient because $-K_1-K_5<0$. Clearly in view of (24) and (25) we have $a^3K_2(a)+a^3K_3(a)=\frac{1}{b}(\bar{K}_2(b)+\bar{K}_3(b))=-\frac{1}{b}\frac{dT}{db}$ with $b<1$. So, applying Proposition 14 we get

$$\begin{aligned} -\frac{1}{b}\frac{dT}{db} &= -\frac{2n}{\pi b}\int_0^1 \frac{d}{db}\left(\frac{1}{t^{1/2}(1-t)^{1/2}(1-b^2t)^{1/2}}\,\frac{1+(bt)^{2n}}{1-(bt)^{2n}}\right)dt \\ &= -\frac{2n}{\pi b}\int_0^1\Bigg(\frac{4n(tb)^{2n}}{b\,t^{1/2}(1-t)^{1/2}(1-b^2t)^{1/2}(1-(bt)^{2n})^2} \\ &\qquad +\frac{bt(1+(bt)^{2n})}{t^{1/2}(1-t)^{1/2}(1-b^2t)^{3/2}(1-(bt)^{2n})}\Bigg)dt \end{aligned}$$

and since the integrand is positive we readily have that $a^3K_2(a)+a^3K_3(a)<0$ for $a>1$ and so $\Delta(a)<0$ for $a>1$. In short, statement (c) is proved.

The proof of statement (d) is a direct consequence of the statements (a)–(c) together with the Bolzano-Cauchy theorem. □

Proof of Proposition 9. We expand $m_2=a^3m_{N,2}/m_D$ in Laurent series around $a=0$, see (4). The expansion of m_D around $a=0$ is given by

$$m_D=(K_5-K_1)(-K_1-K_5+O(a^3)).$$

Using Lemma 1(d) with $u=0$ (respectively, $u=\pi/n$) to expand K_4 (respectively, K_6) in Laurent series around $a=0$ we get

$$K_4(a)=-\frac{n}{2}+O(a) \qquad\text{and}\qquad K_6(a)=-\frac{n}{2}+O(a) \tag{27}$$

when $n\geq 3$ and

$$K_4(a)=-4+O(a) \qquad\text{and}\qquad K_6(a)=2+O(a) \tag{28}$$

when $n=2$. Therefore, when $n\geq 3$ we have

$$m_2 = a^3\frac{K_1^2+\frac{n}{2}K_1-K_1K_5-\frac{n}{2}K_5}{(K_5-K_1)(-K_1-K_5)}+O(a^4) = a^3\frac{K_1+\frac{n}{2}}{K_5+K_1}+O(a^4)$$

and since $K_1, K_5>0$ we obtain $\lim_{a\to 0^+} m_2 = 0^+$.

When $n=2$ we compute directly the quantities K_1 and K_5 and we have $K_1=1/4$, $K_5=1/\sqrt{2}$. So expanding m_3 in Laurent series around $a=0$ we get

$$m_2 = -a^3\Big(\frac{17}{7}+2\sqrt{2}\Big)+O(a^4).$$

Hence, $\lim_{a\to 0^+} m_2 = 0^-$. This completes the proof of statement (a).

For statement (b) we note that by (5)

$$m_3 = m_2 + \frac{a^3(K_6(a)-K_4(a))}{K_5-K_1}.$$

Using the Laurent series of K_4 and K_6 around $a=0$ given in (27) we get

$$\frac{a^3(K_6-K_4(a))}{K_5-K_1} = O(a^4)$$

when $n\geq 3$ and so

$$m_3 = a^3\frac{K_1+\frac{n}{2}}{K_5+K_1}+O(a^4),$$

which yields $\lim_{a\to 0^+} m_3 = 0^+$ when $n\geq 3$.

On the other hand, for $n=2$, using the Laurent series of K_4 and K_6 around $a=0$ given in (28) and the values of K_1 and K_5 for $n=2$ computed above we get

$$\frac{a^3(K_6-K_4(a))}{K_5-K_1} = \frac{a^3}{K_5-K_1}(6+O(a))$$

which yields

$$m_3 = \frac{a^3}{7}(7+34\sqrt{2})+O(a^4),$$

and so $\lim_{a\to 0^+} m_3 = 0^+$. This completes the proof of statement (b).

We expand $m_3 = a^3 m_{N,3}/m_D$ in Laurent series around $a=1$ (with $a>1$), see (4). Note that

$$K_2(a) = K_1 - \frac{1}{(a-1)^2}+O(a-1), \quad K_3(a) = K_5+O(a-1),$$
$$K_4(a) = K_1 + \frac{1}{(a-1)^2} - \frac{1}{a-1}+1+O(a-1), \quad K_6 = K_5+O(a-1),$$

and $a^3 = 1+O(a-1)$. Hence,

$$m_{N,3} = \frac{1}{(a-1)^4}+O((a-1)^{-3}) \quad \text{and} \quad m_D = -\frac{(K_5-K_1)}{(a-1)^2}+O((a-1)^{-1}).$$

Therefore,

$$m_3 = -\frac{1}{(a-1)^2(K_5-K_1)}+O((a-1)^{-1}).$$

Since in view of Lemma 3(c) we have $K_5>K_1>0$, then $\lim_{a\to 1^+} m_3 = -\infty$. which completes the proof of statement (c).

For statement (d), note that taking $b=1/a$ and using Lemma 3(d) we have $K_2(a)=K_2(1/b)=b^3K_4(b)$, $K_3(a)=K_3(1/b)=b^3K_6(b)$, $K_4(a)=K_2(1/a)/a^3=b^3K_2(b)$ and $K_6(a)=K_3(1/a)/a^3=b^3K_3(b)$. Thus, using Lemma 1(d) for $n\geq 3$ we have

$$K_4(b) = -\frac{n}{2}+O(b), \quad K_6(b) = -\frac{n}{2}+O(b).$$

Moreover expanding in power series around $b=0$ we get

$$K_2(b) = n+O(b), \quad K_3(b) = n+O(b).$$

Hence for $n \geq 3$

$$\begin{aligned} K_2(b) &= -\frac{n}{2}b^3 + O(b^4), \qquad K_3(b) = -\frac{n}{2}b^3 + O(b^4), \\ K_4(b) &= nb^3 + O(b^4), \qquad K_6(b) = nb^3 + O(b^4), \end{aligned} \tag{29}$$

Proceeding in the same way for $n = 2$ we get

$$\begin{aligned} K_2(b) &= -4b^3 + O(b^4), \qquad K_3(b) = 2b^3 + O(b^4), \\ K_4(b) &= 2b^3 + O(b^4), \qquad K_6(b) = 2b^3 + O(b^4). \end{aligned} \tag{30}$$

Using (29) and (30) the numerators of m_2 and m_3 (see (4)) can be written as

$$\frac{1}{b^3}(K_1^2 - K_1K_5) + O(1),$$

and the denominators of m_2 and m_3 (see again (4)) can be written as

$$(K_5 - K_1)(-K_1 - K_5 - n) + O(b)$$

for all $n \geq 2$. Thus the Laurent expansion of m_2 and m_3 around $b = 0$ becomes (after simplifying $K_5 - K_1$)

$$\frac{1}{b^3}\frac{K_1}{K_1 + K_5 + n} + O(b^{-2}).$$

Then taking into account that in view of Lemma 3(c) we have $K_5 > K_1 > 0$, we conclude that

$$\lim_{a\to\infty} m_2 = \lim_{b\to 0^+} m_2 = \infty, \qquad \lim_{a\to\infty} m_3 = \lim_{b\to 0^+} m_3 = \infty.$$

This proves statement (d).

From Lemma 3(a) we get $K_6 - K_4(a) > 0$ for $a \in (0,1)$. From Lemma 3(b) we get $K_6 - K_4(a) < 0$ for $a > 1$ and from Lemma 3(c) we get $K_5 - K_1 > 0$. Thus from (5) we get $m_2 < m_3$ when $a \in (0,1)$ and $m_2 > m_3$ when $a > 1$, which proves statement (e). □

Appendix 2. Proof of Proposition 11

We need the following auxiliary lemma.

Lemma 5. *Let*

$$E(a) := \sum_{j=0}^{[n/2]-1} \frac{-2(1+a^2)\cos\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right) + 5a - a\cos\left(\frac{4\pi j}{n} + \frac{2\pi}{n}\right)}{a\Big(1 + a^2 - 2a\cos\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right)\Big)^{5/2}}.$$

Then

$$\lim_{\beta\to\pi/n} N_4(a,\beta)(K_1 - 2a^3L_2(a,\beta) + L_5(\beta)) = \frac{1}{4}E(a)$$

if n is even and

$$\lim_{\beta\to\pi/n} N_4(a,\beta)(K_1 - 2a^3L_2(a,\beta) + L_5(\beta)) = \frac{1}{4}\left(E(a) + \frac{1}{a(1+a)^3}\right)$$

if n is odd.

Proof. First note that

$$\lim_{\beta\to\pi/n} N_4(a,\beta)(K_1 - 2a^3L_2(a,\beta)) = 0$$

because

$$\lim_{\beta\to\pi/n} N_4(a,\beta) = \sum_{j=1}^{n} \frac{-1/a\sin\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right)}{\Big(1 + a^2 - 2a\cos\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right)\Big)^{3/2}} = 0$$

and

$$\lim_{\beta\to\pi/n} L_2(a,\beta) = \sum_{j=1}^{n} \frac{1 - a\cos\left(\frac{2\pi j}{n} - \frac{\pi}{n}\right)}{\Big(1 + a^2 - 2a\cos\left(\frac{2\pi j}{n} - \frac{\pi}{n}\right)\Big)^{3/2}}$$

is finite for all $a>0$. On the other hand,

$$L_5(\beta)=\left(\sum_{j=0}^{n-2}\frac{1-\cos\left(\frac{2\pi\, j}{n}+2\beta\right)}{\left(2-2\cos\left(\frac{2\pi\, j}{n}+2\beta\right)\right)^{3/2}}\right)+\frac{1-\cos\left(\frac{2\pi\,(n-1)}{n}+2\beta\right)}{\left(2-2\cos\left(\frac{2\pi\,(n-1)}{n}+2\beta\right)\right)^{3/2}}$$
$$:=L_{5,1}(\beta)+L_{5,2}(\beta).$$

Clearly $\lim_{\beta\to\pi/n}N_4(a,\beta)L_{5,1}(\beta)=0$. Therefore, we need to study $\lim_{\beta\to\pi/n}N_4(a,\beta)L_{5,2}(\beta)$. Note that expanding $L_{5,2}$ around $\beta=\pi/n$ we have

$$L_{5,2}(\beta)=\frac{1-\cos\left(\frac{2\pi(n-1)}{n}+2\beta\right)}{\left(2-2\cos\left(\frac{2\pi(n-1)}{n}+2\beta\right)\right)^{3/2}}=\frac{1}{4\left(\beta-\frac{\pi}{n}\right)}+\frac{1}{24}\left(\beta-\frac{\pi}{n}\right)+O\Big(\left(\beta-\frac{\pi}{n}\right)^2\Big).$$

On the other hand, we can rewrite $N_4(a,\beta)$ as

$$\bar{N}_4(a,\beta)=\sum_{j=0}^{[n/2]-1}\left(\frac{-1/a\sin\left(\frac{2\pi\, j}{n}+\beta\right)}{\left(1+a^2-2a\cos\left(\frac{2\pi\, j}{n}+\beta\right)\right)^{3/2}}-\frac{1/a\sin\left(\frac{2\pi\,(-j-1)}{n}+\beta\right)}{\left(1+a^2-2a\cos\left(\frac{2\pi\,(-j-1)}{n}+\beta\right)\right)^{3/2}}\right)$$

if n is even and as $\bar{N}_4(a,\beta)+\bar{N}_4^*(a,\beta)$ with

$$\bar{N}_4^*(a,\beta)=\frac{-1/a\sin\left(\frac{2\pi\,[n/2]}{n}+\beta\right)}{\left(1+a^2-2a\cos\left(\frac{2\pi\,[n/2]}{n}+\beta\right)\right)^{3/2}}$$

if n is odd. Expanding $\bar{N}_4$ around $\beta=\pi/n$ we get

$$\bar{N}_4(a,\beta)=E(a)(\beta-\pi/n)+O((\beta-\pi/n)^2),$$

and expanding $\bar{N}_4^*$ also around $\beta=\pi/n$ we get

$$\bar{N}_4^*(a,\beta)=\frac{1}{a(1+a)^3}(\beta-\pi/n)+O((\beta-\pi/n)^3).$$

Thus, expanding $N_4(a,\beta)L_{5,2}(\beta)$ around $\beta=\pi/n$ we obtain

$$N_4(a,\beta)L_{5,2}(\beta)=\frac{1}{4}E(a)+O((\beta-\pi/n))$$

if n is even and

$$N_4(a,\beta)L_{5,2}(\beta)=\frac{1}{4}\left(E(a)+\frac{1}{a(1+a)^3}\right)+O((\beta-\pi/n))$$

if n is odd. Taking the limit as $\beta\to\pi/n$ we obtain the result that we wanted to prove. □

Proof of Proposition 11. Note that

$$\lim_{\beta\to\pi/n}N_5(\beta)=\sum_{j=1,j\neq n-1}^{n}\frac{-\sin\left(\frac{2\pi\,(j+1)}{n}\right)}{\left(2-2\cos\left(\frac{2\pi\,(j+1)}{n}\right)\right)^{3/2}}$$
$$+\lim_{\beta\to\pi/n}\frac{-\sin\left(\frac{2\pi\,(n-1)}{n}+2\beta\right)}{\left(2-2\cos\left(\frac{2\pi\,(n-1)}{n}+2\beta\right)\right)^{3/2}}=\infty.$$

In view of Lemma 5, it is clear that the sign of $F(a,\beta)$ as $\beta\to\pi/n$ is determined by the sign of the difference G given by

$$G=\lim_{\beta\to\pi/n}(K_1-L_4(a,\beta))=K_1-\bar{L}_4(a),$$

where

$$\bar{L}_4(a)=\lim_{\beta\to\pi/n}L_4(a,\beta)=\sum_{j=1}^{n}\frac{1-1/a\cos\left(\frac{2\pi\, j}{n}+\frac{\pi}{n}\right)}{\left(1+a^2-2a\cos\left(\frac{2\pi\, j}{n}+\frac{\pi}{n}\right)\right)^{3/2}}.$$

When $a \to \infty$ we have

$$\lim_{a\to\infty} G = K_1 - \lim_{a\to\infty} \bar{L}_4(a) = K_1 > 0.$$

This proves statement (a) of the proposition.

Furthermore, when $a \to 1$, using Lemma 3(c), we obtain

$$\lim_{a\to 1} G = K_1 - \sum_{j=1}^{n} \frac{1 - \cos\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right)}{\left(2 - 2\cos\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right)\right)^{3/2}} = K_1 - K_5 < 0.$$

So, statement (b) is proved.

Using Lemma 1(d) with $u = \pi/n$, the expansion of $\bar{L}_4$ around $a = 0$ is given by $\bar{L}_4(a) = -n/2 + O(a)$, when $n \geq 3$ and $\bar{L}_4(a) = 2 + O(a)$, when $n = 2$. Hence, when $n \geq 3$ the expansion of G around $a = 0$ is $G = K_1 + n/2 + O(a)$ and so $\lim_{a\to 0} G > 0$. For $n = 2$, computing the value of K_1, the expansion of G around $a = 0$ is $G = 1/4 - 2 + O(a)$ and so $\lim_{a\to 0} G = -7/4 < 0$. This yields statement (c) of the proposition. □

Appendix 3. Proof of Propositions 12 and 13

We need the following auxiliary lemma.

Lemma 6. *We have*

$$\lim_{\beta\to\pi/2n} N_5(\beta)(K_1 - L_4(a,\beta)) = 0.$$

Proof. First note that

$$\lim_{\beta\to\pi/2n} L_4(a,\beta) = \sum_{j=1}^{n} \frac{1 - 1/a\cos\left(\frac{2\pi j}{n} + \frac{\pi}{2n}\right)}{\left(1 + a^2 - 2a\cos\left(\frac{2\pi j}{n} + \frac{\pi}{2n}\right)\right)^{3/2}},$$

which is finite for all $a > 0$. Moreover, it is easy to see that

$$\lim_{\beta\to\pi/2n} N_5(\beta) = \sum_{j=1}^{n} \frac{-\sin\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right)}{\left(2 - 2\cos\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right)\right)^{3/2}} = 0$$

and so $\lim_{\beta\to\pi/2n} N_5(\beta)(K_1 - L_4(a,\beta)) = 0$, as we wanted to prove. □

Proof of Proposition 12. In view of Lemma 6 the sign of $F(a,\beta)$ around $\beta = \pi/2n$ is determined by the sign of $N_4(a,\beta)(K_1 - 2a^3L_2(a,\beta) + L_5(\beta))$ unless $\lim_{\beta\to\pi/2n} N_4(a,\beta)(K_1 - 2a^3L_2(a,\beta) + L_5(\beta)) = 0$. From the analysis of the sign of $N_4(a,\beta)(K_1 - 2a^3L_2(a,\beta) + L_5(\beta))$ around $\beta \to \pi/2n$ we will see that $\lim_{\beta\to\pi/2n} N_4(a,\beta)(K_1 - 2a^3L_2(a,\beta) + L_5(\beta)) \neq 0$.

In view of Proposition 10 we have that $N_4(a,\beta) < 0$ for $\alpha \in (0, \pi/n)$ and so $N_4(a,\beta) < 0$. So, we need to study the sign of $K_1 - 2a^3L_2(a,\beta) + L_5(\beta)$ as $\beta \to \pi/2n$.

Hence,

$$\lim_{\beta\to\pi/2n} K_1 - 2a^3L_2(a,\beta) + L_5(\beta) = K_1 - 2a^3\bar{L}_2(a) + K_5 := H,$$

where

$$\bar{L}_2(a) = \lim_{\beta\to\pi/2n} L_2(a,\beta) = \sum_{j=1}^{n} \frac{1 - a\cos\left(\frac{2\pi j}{n} - \frac{\pi}{2n}\right)}{\left(1 + a^2 - 2a\cos\left(\frac{2\pi j}{n} - \frac{\pi}{2n}\right)\right)^{3/2}}.$$

When $a \to 0$ we have that $\bar{L}_2(a) \to n$. So $\lim_{a\to 0} H = K_1 + K_5 > 0$ in view of Lemma 3(c). This proves statement (a) of the proposition.

To study the behavior when $a \to \infty$ we first observe that making the change $b = 1/a$ we get $a^3\bar{L}_2(a) = L(b, \pi/2n)$ where L is the function defined in Lemma 1(d). Thus applying Lemma 1(d) with $u = \pi/2n$ and using Lemma 3(c) we have

$$\lim_{a\to\infty} H = K_1 + n + K_5 > 0,$$

when $n \geq 3$ and

$$\lim_{a\to\infty} H = K_1 - 2\big(2 - 6\cos^2\left(\tfrac{\pi}{4}\right)\big) + K_5 = K_1 + 2 + K_5 > 0,$$

when $n=2$. This proves statement (b).

Finally, to prove statement (c) we need to study the sing of H when $a\to 1$. Note that

$$\lim_{a\to 1} H = \frac{1}{2}\sum_{j=1}^{n-1}\frac{1}{\left(2-2\cos\left(\frac{2\pi j}{n}\right)\right)^{1/2}} - \sum_{j=1}^{n}\frac{1}{\left(2-2\cos\left(\frac{2\pi j}{n}-\frac{\pi}{2n}\right)\right)^{1/2}} + \frac{1}{2}\sum_{j=1}^{n}\frac{1}{\left(2-2\cos\left(\frac{2\pi j}{n}+\frac{\pi}{n}\right)\right)^{1/2}}.$$

In order to study the sign of $\lim_{a\to 1} H$ we rewrite $\lim_{a\to 1} H$ as $\lim_{a\to 1} H_1$ where

$$H_1 = \frac{1}{2}\sum_{j=1}^{n}\frac{1}{\left(1+a^2-2a\cos\left(\frac{2\pi j}{n}\right)\right)^{1/2}} - \frac{1}{2}\sum_{j=1}^{1}\frac{1}{(1+a^2-2a\cos(2\pi j))^{1/2}} - \sum_{j=1}^{n}\frac{1}{\left(1+a^2-2a\cos\left(\frac{2\pi j}{n}-\frac{\pi}{2n}\right)\right)^{1/2}} + \frac{1}{2}\sum_{j=1}^{n}\frac{1}{\left(1+a^2-2a\cos\left(\frac{2\pi j}{n}+\frac{\pi}{n}\right)\right)^{1/2}}.$$

Now applying Proposition 14 with $\alpha=1/2$ and $u=0$; $n=1$, $\alpha=1/2$ and $u=0$; $\alpha=1/2$ and $u=-\pi/2n$; and $\alpha=1/2$ and $u=\pi/n$ respectively we get

$$H_1 = \frac{1}{\pi}\int_0^1 \frac{1}{t^{1/2}(1-t)^{1/2}(1-a^2t)^{1/2}}\left(\frac{4(at)^{2n}n}{1-(at)^{4n}} - \frac{1+at}{2(1-at)}\right)dt$$

Setting $v=at$ and $N=2n$ and using Lemma 2 we get

$$\frac{4(at)^{2n}n}{1-(at)^{4n}} - \frac{1+at}{2(1-at)} = \frac{4v^N\frac{N}{2}}{1-v^{2N}} - \frac{1+v}{2(1-v)} = \frac{4v^N N(1-v)-(1+v)(1-v^{2N})}{(1-v^{2N})2(1-v)} < 0.$$

Therefore $\lim_{a\to 1^-} H_1 < 0$.

Applying Proposition 15 to H_1 with $\alpha=1/2$ and $u=0$; $n=1$, $\alpha=1/2$ and $u=0$; $\alpha=1/2$ and $u=-\pi/2n$; and $\alpha=1/2$ and $u=\pi/n$ respectively we get

$$\bar{H}_1 = \frac{1}{\pi}\int_0^1 \frac{1}{t^{1/2}(1-t)^{1/2}(a^2-t)^{1/2}}\left(\frac{4(at)^{2n}n}{a^{4n}-t^{4n}} - \frac{a+t}{2(a-t)}\right)dt.$$

After doing the substitution $a\to 1/a$ we get that $\bar{H}_1$ can be written as $a\,H_1$. Thus $\lim_{a\to 1^+} H_1 < 0$. In short we have that $H<0$ when $a\to 1$ and statement (c) is proved. □

Proof of Proposition 13. We start proving statemt (a) for $n=2$. Computing directly the quantities K_1, $L_2(a,\beta)$, $L_4(a,\beta)$, $N_4(a,\beta)$, $L_5(\beta)$, and $N_5(\beta)$ for a fixed β and expanding F around $a=0$ we get

$$F(a,\beta) = -\frac{1}{16}\tan\beta\sec\beta\left(18\cos\beta+6\cos(3\beta)+17\cot^3\beta+7\right)+O(a^2).$$

So $\bar{F}(\beta)<0$ for all $\beta\in(\pi/2n,\pi/n)$ and $n=2$.

Now we will show that for any $\beta\in(\pi/2n,\pi/n)$ and $n\geq 3$ we have $\bar{F}(\beta)>0$. We fix β and we expand F around $a=0$. First, expanding N_4 around $a=0$ we get

$$N_4(a,\beta) = -\frac{1}{a}\sum_{j=1}^{n}\sin\left(\frac{2\pi j}{n}+\beta\right) - \frac{3}{2}\sum_{j=1}^{n}\sin\left(\frac{4\pi j}{n}+2\beta\right)+O(a)$$

and in view of (18) we have $N_4(a,\beta)=O(a)$. So expanding $N_4(a,\beta)(K_1-2a^3L_2(a,\beta)+L_5(\beta))$ around $a=0$ we obtain

$$N_4(a,\beta)(K_1-2a^3L_2(a,\beta)+L_5(\beta)) = O(a).$$

On the other hand, using Lemma 1(d) with $u = \beta$ and $n \geq 3$, the expansion of L_4 around $a = 0$ is given by $L_4(a, \beta) = -n/2 + O(a)$. Therefore, around $a = 0$ we get

$$F(a, \beta) = (K_1 + \frac{n}{2})N_5(\beta) + O(a). \tag{31}$$

The sign of $N_5(\beta)$ for any $\beta \in (0, \pi/n)$ was studied in Proposition 10 and we obtained that $N_5(\beta)$ is negative if $\beta \in (0, \pi/2n)$ and positive if $\beta \in (\pi/2n, \pi/n)$. Therefore, from (31) we have that for any $\beta \in (\pi/2n, \pi/n)$ and $n \geq 3$, $\bar{F}(\beta) > 0$ which completes the proof of statement (a).

Now we consider the case in which $a \to \infty$. Fixed $\beta \in (\pi/2n, \pi/n)$ we have that $N_5(\beta)$ is positive. Moreover,

$$\lim_{a\to\infty} L_4(a, \beta) = \lim_{a\to\infty} N_4(a, \beta) = 0.$$

Making the change $b = 1/a$ we get $a^3 L_2(a, \beta) = L(b, \beta)$, where L is the function defined in Lemma 1(d). Thus, applying Lemma 1(d) with $u = \beta$ we get that $\lim_{a\to\infty} a^3 L_2(a, \beta) = \lim_{b\to 0} -n/2 + O(b)$ when $n \geq 3$ and $\lim_{a\to\infty} a^3 L_2(a, \beta) = \lim_{b\to 0} 2 - 6\cos^2(\beta) + O(b)$ when $n = 2$. So,

$$\lim_{a\to\infty} (K_1 - 2a^3 L_2(a, \beta) + L_5(\beta))N_4(a, \beta) = 0,$$
$$\lim_{a\to\infty} N_5(\beta)(K_1 - L_4(a, \beta)) = N_5(\beta)K_1 > 0,$$

for all $n \geq 2$. In short, for any $\beta \in (\pi/2n, \pi)$ and $n \geq 2$, $\tilde{F}(\beta) > 0$. This concludes the proof of the proposition. □

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