# SEMIGLOBAL OBLIQUE PROJECTION EXPONENTIAL DYNAMICAL OBSERVERS FOR NONAUTONOMOUS SEMILINEAR PARABOLIC-LIKE EQUATIONS 

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#### Abstract

The estimation of the full state of a nonautonomous semilinear parabolic equation is achieved by a Luenberger type dynamical observer. The estimation is derived from an output given by a finite number of average measurements of the state on small regions. The state estimate given by the observer converges exponentially to the real state, as time increases. The result is semiglobal in the sense that the error dynamics can be made stable for an arbitrary given initial condition, provided a large enough number of measurements, depending on the norm of the initial condition, is taken. The output injection operator is explicit and involves a suitable oblique projection. The results of numerical simulations are presented showing the exponential stability of the error dynamics.


## 1. Introduction

We consider evolutionary nonlinear parabolic-like equations, for time $t \geq 0$, as

$$
\begin{equation*}
\dot{y}+A y+A_{\mathrm{rc}} y+\mathcal{N}(y)=f, \quad w=\mathcal{Z}_{S} y \tag{1.1}
\end{equation*}
$$

evolving in a Hilbert space $V . A$ and $A_{\mathrm{rc}}=A_{\mathrm{rc}}(t)$ are, respectively, a time-independent symmetric linear diffusion-like operator and a time-dependent linear reaction-convectionlike operator. Further $\mathcal{N}(y)=\mathcal{N}(t, y)$ is a time-dependent nonlinear operator and $f=$ $f(t)$ is a time-dependent external forcing. The triple $\left(A, A_{\mathrm{rc}}, \mathcal{N}, f\right)$, defining the dynamics, is assumed to be known.

The unknown state of the equation is the variable $y=y(t) \in V$, where $V$ is a suitable Hilbert space. The vector output $w=w(t)=\mathcal{Z}_{S} y(t) \in \mathbb{R}^{S_{\sigma} \times 1}$ consists of a finite number of measurements, where $S_{\sigma}$ is a positive integer. The output operator $\mathcal{Z}_{S}: V \rightarrow \mathbb{R}^{S_{\sigma} \times 1}$ is linear.

The initial state $y(0) \in V$, at time $t=0$, is assumed to be unknown. Our task is to estimate the state $y$ from the output $w$, which is assumed to be given in the form of "averages" as

$$
w(t)=\left[\begin{array}{c}
w_{1}(t)  \tag{1.2a}\\
w_{2}(t) \\
\vdots \\
w_{S}(t)
\end{array}\right], \quad w_{i}(t):=\left(\mathfrak{w}_{i}, y(t)\right)_{H}, \quad 1 \leq i \leq S_{\sigma},
$$

where $(\cdot, \cdot)_{H}$ is the scalar product in a pivot Hilbert space $H \supset V$. Each $\mathfrak{w}_{i} \in H$ will be referred to as a sensor, and we assume that
the family of sensors, $W_{S}:=\left\{\mathfrak{w}_{i} \mid 1 \leq i \leq S_{\sigma}\right\} \subset H$, is linearly independent.

[^0]We consider the case where we can place the sensors, depending on their number $S_{\sigma}$, so that we will actually have

$$
\begin{equation*}
w_{i}=w_{i, S}, \quad \mathfrak{w}_{i}(t)=\mathfrak{w}_{i, S}(t) \subset H, \quad 1 \leq i \leq S_{\sigma} \tag{1.2c}
\end{equation*}
$$

Remark 1.1. For simplicity, we may think of $S_{\sigma}=S$. In the application of the result to concrete examples, it is convenient to have a particular subsequence $\left(S_{\sigma}\right)_{S \in \mathbb{N}_{0}}$ of positive integer numbers, as we shall see in Section 4 , where we shall take $\sigma(S)=(2 S)^{d}$, for scalar parabolic equations evolving in rectangular spatial domains $\Omega \subset \mathbb{R}^{d}$.

In real applications, for a fixed instant of time $t$, it is not possible to recover $y(t)$ from $w(t)$, in general. However, from the knowledge of the dynamics of (1.1), it may be possible to construct a Luenberger type dynamical observer, giving us an estimate $\widehat{y}(t)$ of $y(t)$, so that $\widehat{y}(t)$ converges exponentially to $y(t)$ as time increases.

Together with the family of sensors we will need also a family of auxiliary functions

$$
\widetilde{W}_{S}:=\left\{\widetilde{\mathfrak{w}}_{i}=\widetilde{\mathfrak{w}}_{i, S} \mid 1 \leq i \leq S_{\sigma}\right\} \subset \mathrm{D}(A), \text { which is linearly independent, }
$$

where $\mathrm{D}(A) \subset V$ is another Hilbert space, to be precised later on, namely, as the domain of the diffusion operator $A$,. We will also consider the corresponding linear spans

$$
\mathcal{W}_{S}:=\operatorname{span} W_{S} \subset H \quad \text { and } \quad \widetilde{\mathcal{W}}_{S}:=\operatorname{span} \widetilde{W}_{S} \subset \mathrm{D}(A) .
$$

Remark 1.2. Sometimes, in [3, 14 this problem of constructing a dynamic state estimate is referred to as "continuous data assimilation".
1.1. The main result. We shall show that Luenberger observers as

$$
\begin{equation*}
\dot{\hat{y}}+A \widehat{y}+A_{\mathrm{rc}} \widehat{y}+\mathcal{N}(\widehat{y})=f+\mathfrak{I}_{S}^{[\lambda, \ell]}(\mathcal{Z} \widehat{y}-w), \quad \widehat{y}(0)=\widehat{y}_{0} \in V \tag{1.3a}
\end{equation*}
$$

with the output injection operator given by

$$
\begin{equation*}
\mathfrak{I}_{S}^{[\lambda, \ell]}:=-\lambda A^{-1} P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{S}^{\perp}} A^{\ell} P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}} \mathbf{Z}^{W_{s}}, \quad \lambda>0, \quad 0 \leq \ell \leq 2 \tag{1.3b}
\end{equation*}
$$

are able to estimate the state $y$ of system (1.1), for any given $\ell \in[0,2]$ and suitable tuples $\left(\lambda, \mathcal{W}_{S}, \widetilde{\mathcal{W}}_{S}\right)$. Here $\mathbf{Z}^{W_{s}}: \mathbb{R}^{S_{\sigma}} \rightarrow \mathcal{W}_{S}$ is the linear operator defined by

$$
\begin{equation*}
\mathbf{Z}^{W_{S}} \mathbf{z}:=\sum_{i=1}^{S_{\sigma}}\left(\left[\mathcal{V}_{S}\right]^{-1} \mathbf{z}\right)_{i} \mathfrak{m}_{i, S}, \quad \mathbf{z} \in \mathbb{R}^{S_{\sigma}} \tag{1.3c}
\end{equation*}
$$

where $\left[\mathcal{V}_{S}\right] \in \mathbb{R}^{S_{\sigma} \times S_{\sigma}}$ is the generalized Vandermonde matrix, whose entries in the $i$ th row and $j$ th column are

$$
\begin{equation*}
\left[\mathcal{V}_{S}\right]_{(i, j)}=\left(\mathfrak{m}_{i, S}, \mathfrak{m}_{j, S}\right)_{H} \tag{1.3d}
\end{equation*}
$$

and $P_{F}^{G}$ denotes the oblique projection in $H$ onto $F$ along $G$.
For suitable $\varrho \geq 1$ and $\mu>0$, we will have the inequality

$$
\begin{equation*}
|\widehat{y}(t)-y(t)|_{V} \leq \varrho \mathrm{e}^{-\mu(t-s)}|\widehat{y}(s)-y(s)|_{V}, \quad \text { for all } \quad t \geq s \geq 0 \tag{1.4}
\end{equation*}
$$

Note that, from (1.1) and (1.3), the error $z=\widehat{y}-y$ satisfies

$$
\begin{equation*}
\dot{z}+A z+A_{\mathrm{rc}} z+\mathfrak{N}_{y}(z)=\mathfrak{I}_{S}^{[\lambda, \ell]} \mathcal{Z} z, \quad z(0)=z_{0} \in V \tag{1.5a}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathfrak{N}_{y}(z):=\mathcal{N}(y+z)-\mathcal{N}(y)=\mathcal{N}(\widehat{y})-\mathcal{N}(y) \tag{1.5b}
\end{equation*}
$$

and $z_{0}=\widehat{y}_{0}-y(0)$. Our goal (1.4), reads now

$$
\begin{equation*}
|z(t)|_{V} \leq \varrho \mathrm{e}^{-\mu(t-s)}|z(s)|_{V}, \quad \text { for all } \quad t \geq s \geq 0 \tag{1.6}
\end{equation*}
$$

Remark 1.3. Observe that $z_{0}$ in 1.5) is unknown for us, because so is $y(0)$. On the other hand, the choice of $\widehat{y}_{0}=\widehat{y}(0)$ is at our disposal, for example, we can choose $\widehat{y}_{0}$ as an initial guess we might have for $y(0)$.

Remark 1.4. We can see that in 1.3b we have that $\mathfrak{I}_{S}^{[\lambda, \ell]} \mathcal{Z} z=-\lambda A^{-1} P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{s}^{\perp}} A^{\ell} q$ with $q \in \widetilde{\mathcal{W}}_{S} \subset \mathrm{D}(A)$. Hence, if $\ell>1$ we may have that $p:=A^{\ell} q \in \mathrm{D}\left(A^{1-\ell}\right) \backslash \mathrm{D}\left(A^{0}\right)$. Therefore, it is clear that in the case $p \notin \mathrm{D}\left(A^{0}\right)=H$ we cannot see $P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}^{\perp}} p$ as an oblique projection in $H$, but rather as an extension of such oblique projection to $\mathrm{D}\left(A^{1-\ell}\right)$. Such extensions are well defined, as we shall see later in Proposition 3.7.

Omitting the details at this point, the main result of this paper is as follows.
Main Result. Under general conditions on the tuple ( $\widehat{y}, A, A_{\mathrm{rc}}, \mathcal{N}, y$ ) and under particular conditions on the tuple $\left(\widetilde{W}_{S}, W_{S}\right)$ it holds the following. For any given $\ell \in[0,2]$, $R>0, \varrho>1$, and $\mu>0$, there are large enough $S \in \mathbb{N}_{0}$ and $\lambda>0$ such that: for all initial error $z_{0}$ satisfying $\left|z_{0}\right|_{V} \leq R$, it follows that the corresponding solution of (1.5), with the output injection operator as in 1.3b, satisfies (1.6).

Definition 1.5. If 1.6 holds true, we say that the error dynamics is exponential stable with rate $-\mu<0$ and transient bound $\varrho \geq 1$.

Note that Main Result says that we can stabilize the nonlinear error dynamics for arbitrary large initial errors $\left|z_{0}\right|_{V} \leq R$, with an arbitrary small exponential rate $-\mu<0$, and arbitrary small transient bound $\varrho>1$. For that, we simply have to take a large enough number of suitable sensors $S_{\sigma}$ and a large enough $\lambda$. In general, the "optimal" transient bound $\varrho=1$ cannot be taken in Main Result. However, later on, in Section 6 we shall give classes of systems where we can indeed take $\varrho=1$. Such classes include linear and suitable semilinear systems. The case $\varrho=1$ is interesting simply because it means that the error norm is strictly decreasing. Observe also that $\varrho=1$ is the smallest value possible for $\varrho$ in 1.6) (e.g., by taking $t=s$ ).

In the particular case our system is linear, $\mathcal{N}=0$, then it is not difficult to show that the observer proposed here is a global observer. By a global observer we understand that the output injection operator $\mathfrak{I}_{S}^{[\lambda, \ell]}$ can be taken independent of the norm of the initial error. The observer proposed here is different from the one proposed in 18, 19, hence this manuscript also contributes with a new result to the linear case.
1.2. Motivation. Observers are demanded in applications, for example, in the implementation of output based stabilizing controls. Suppose we have a feedback operator $K(t)$ such that the associated feedback control $f(t)=K(t) y(t)$ stabilizes system (1.1). See [20] for such stabilizing feedback control. In the case where the state $y$ is modeled by partial differential equations, the state is infinite-dimensional and it is not realistic to expect that we will be able to know/measure the entire state $y(t)$ at each instant of time $t$. However, we can expect that, with a good enough estimate $\widehat{y}(t)$ for $y(t)$, the approximated control $f(t)=K(t) \widehat{y}(t)$ will be able to stabilize (1.1).

We cannot expect that an infinite-dimensional state $y(t)$ can be reconstructed from the finite set $w(t)=\mathcal{Z}_{S} y(t)$ at a fixed time $t$, hence we look for a dynamical observer in order to construct an estimate $\widehat{y}(t)$ for $y(t)$, which will be improving as time increases.
1.3. On previous related works in literature. For partial differential equations, the results in the literature on state estimation concern mainly the autonomous case. For example, we refer to [1, 5] 8, 10, 15, 17, 23]. Exceptions are [9, 11, 12] for one-dimensional parabolic equations, $d=1$, by using the nontrivial backstepping and Cole-Hopf transformations. In [11 reaction type Lipschitz nonlinearities are considered, while in 9 ] convection nonlinearities are also included, where some details are omitted concerning the stability of the semilinear error dynamics, as also referred by the authors in 9 , Sect. C]. See also the auxiliary nonautonomous heat equation in [12, see Eq. (17)].

In the investigation of the autonomous case, as in [17, the spectral properties of the time-independent operator dynamics play a crucial role in the derivation of the results. The (un)stability results in [22] suggest that such spectral properties in the
nonautonomous case (at each fixed time $t>0$ ) are not an appropriate tool to deal with the nonautonomous case. The recent work [2] also shows that, in general, the state estimation problem in the nonautonomous case is not an easy task even for the case of finite-dimensional systems.

The approach in [3] is applicable to state estimation of parabolic-like systems for which we can derive the existence of a finite set of so-called determining parameters. This includes the 2D Navier-Stokes equations, whose weak solutions are well posed and are exponentially stable under the absence of external forces, i.e., when $f=0$. The method in [3] is quite interesting because, depending on the nature of the "chosen" determining parameters, it can be applied to several types of measurements, including average-like measurements as we are particularly interested in. However, in this manuscript we consider a class of nonlinear equations whose free dynamics evolution is not well posed in the sense of weak solutions (for initial states given in the pivot Hilbert space $H$ ). We will need strong solutions (for "more regular" initial states given in the Hilbert space $V \subset H$ ), but even for such solutions the free dynamics evolution will be well posed only for short time, that is, in general the free dynamics has strong solutions which blow up in finite time. Hence, to deal with state estimation for such class of systems, the method in [3] is (or, seems to be) not appropriate.

In [18] a global observer was presented to estimate the state of linear parabolic equations, where the placement of the actuators play an important role. The results in this manuscript are also derived under the assumption that we are allowed to suitably place the sensors. Such assumption seems to be natural and to reflect common sense: it matters (or, may matter) where we take our measurements in. Again, the observer in [18] provides an estimate for the weak solution and the exponential convergence is derived in the pivot $H$ norm. As we said above, weak solutions do not necessarily exist for the class of nonlinear systems we consider, this is one reason we will (need to) use a different output injection operator in this manuscript, to deal with strong solutions and derive the exponential convergence in the stronger $V$ norm.

Finally, we must say that some of the above mentioned works, as [5, 10, do not consider the observer design problem alone, but (already) coupled with a stabilization problem (output based feedback control). Also, some of the above works deal with boundary measurements, while here we deal with internal measurements.
1.4. Illustrating example. Scalar parabolic equations. The results will follow under general assumptions on the plant dynamics operators, on the external force, and on the targeted real state. We shall need also a particular assumption involving the set of sensors. Such assumptions will be presented later on and will be satisfied, in particular, for a general class of semilinear parabolic equations, under either Dirichlet or Neumann boundary conditions, including

$$
\begin{align*}
& \frac{\partial}{\partial t} y+(-\Delta+\mathbf{1}) y+a y+b \cdot \nabla y+\widetilde{a}|y|_{\mathbb{R}}^{r-1} y+(\widetilde{b} \cdot \nabla y)|y|_{\mathbb{R}}^{s-1} y=f,  \tag{1.7a}\\
& \left.\mathcal{G} y\right|_{\Gamma}=g, \quad w=\mathcal{Z}_{S} y, \tag{1.7b}
\end{align*}
$$

with $r \in(1,5)$ and $s \in[1,2)$, defined in a bounded connected open spatial subset $\Omega \in \mathbb{R}^{d}$, $d \in\{1,2,3\}$, with boundary $\Gamma=\partial \Omega . \Omega$ is assumed to be either smooth or a convex polygon. The state is a function $y=y(x, t)$, defined for $(x, t) \in \Omega \times(0,+\infty)$. The operator $\mathcal{G}$ imposes the boundary conditions,

$$
\begin{aligned}
& \mathcal{G}=\mathbf{1}, \quad \text { for Dirichlet boundary conditions, } \\
& \mathcal{G}=\mathbf{n} \cdot \nabla=\frac{\partial}{\partial \mathbf{n}}, \quad \text { for Neumann boundary conditions, }
\end{aligned}
$$

where $\mathbf{n}=\mathbf{n}(\bar{x})$ stands for the outward unit normal vector to $\Gamma$, at $\bar{x} \in \Gamma$.
The functions $a=a(x, t), b=b(x, t), \widetilde{a}=\widetilde{a}(x, t), \widetilde{b}=\widetilde{b}(x, t)$, and $f=f(x, t)$ are defined in $\Omega \times(0,+\infty)$, and the function $g=g(\bar{x}, t)$ is defined for $(\bar{x}, t) \in \Gamma \times(0,+\infty)$.

Thus the data tuple $(a, b, \widetilde{a}, \widetilde{b}, f, g)$ is allowed to depend on both space and time variables. We assume that,
$a$ and $\widetilde{a}$ are in $L^{\infty}(\Omega \times(0,+\infty))$,
$b$ and $\widetilde{b}$ are in $L^{\infty}(\Omega \times(0,+\infty))^{d}$,
There exists $\tau_{y}>0$ such that $\sup _{t \geq 0}\left(|y(t)|_{H^{1}(\Omega)}+|y|_{L^{2}\left(\left(t, t+\tau_{y}\right), H^{2}(\Omega)\right)}\right)<+\infty$.
Remark 1.6. In 1.8 c we assume, in particular, that the real state $y$ must be a globally defined strong solution $y \in \mathcal{Y}:=L_{\mathrm{loc}}^{\infty}\left((0,+\infty), H^{1}(\Omega)\right) \bigcap L_{\mathrm{loc}}^{2}\left((0,+\infty), H^{2}(\Omega)\right)$. In general, for regular enough external force $f$ (e.g., for $f=0$ ) we will only have the local existence in time: for a suitable $\tau_{*}>0, y \in L^{\infty}\left((0, \tau), H^{1}(\Omega)\right) \bigcap L^{2}\left((0, \tau), H^{2}(\Omega)\right)$, for $\tau<\tau_{*}$. There are, however, cases where 1.8 c will hold true, for example, for the case where $g=0$ and $f=K y$ is a stabilizing feedback control. See Section 1.2 Another example is the case of time-periodic systems having time-periodic solutions. A third example are Lyapunov stable (not necessarily asymptotic stable) systems.

As output we take the averages of the solution in subdomains $\omega_{i}=\omega_{i, S} \subset \Omega$, as

$$
\begin{equation*}
w_{i}(t)=\left(1_{\omega_{i}}, y(\cdot, t)\right)_{L^{2}(\Omega)}=\int_{\omega_{i}} y(x, t) \mathrm{d} x, \quad 1 \leq i \leq S_{\sigma} . \tag{1.9}
\end{equation*}
$$

We will be interested in the case the regions $\omega_{i}$, where we take the measurements in, are constrained to cover an a priori fixed volume, namely, $\operatorname{vol}\left(\bigcup_{i=1}^{S_{\sigma}} \omega_{i, S}\right) \leq r \operatorname{vol}(\Omega)$ with $0<r<1$ independent of $S$. In other words, we allow ourselves to take/place as many sensors as we want/need, but we are allowed to perform measurements only in (at most) a fixed percentage of the spatial domain $\Omega$, namely, $100 r \%$.
Remark 1.7. The usual average over $\omega_{i}$ is $\breve{w}_{i}:=\frac{\int_{\omega_{i}} y(x, t) \mathrm{d} x}{\int_{\omega_{i}} \mathrm{~d} x}$. However, we assume that we know our sensors, that is, we know the regions $\omega_{i}$ where we take the measurements in. In this case, knowing/measuring $\breve{w}_{i}$ is equivalent to knowing/measuring $w_{i}$.

In order to apply our results to system (1.7), we have just to rewrite (1.7) as an evolutionary equation (1.1). To this purpose, we define for both Dirichlet, respectively Neumann, boundary conditions the spaces

$$
\mathrm{D}(A)=H_{\mathcal{G}}^{2}(\Omega):=\left\{h \in H^{2}(\Omega)|\mathcal{G} h|_{\Gamma}=0\right\}, \quad \text { for } \quad \mathcal{G} \in\left\{\mathbf{1}, \frac{\partial}{\partial \mathbf{n}}\right\}
$$

and

$$
V=H_{\mathbf{1}}^{1}(\Omega):=H_{0}^{1}(\Omega)=\left\{h \in H^{1}(\Omega)|h|_{\Gamma}=0\right\}, \quad H_{\frac{\partial}{\partial \mathbf{n}}}^{1}(\Omega):=H^{1}(\Omega),
$$

with the operators
$A:=-\nu \Delta+\mathbf{1}, \quad A_{\mathrm{rc}}:=a \mathbf{1}+b \cdot \nabla, \quad$ and $\quad \mathcal{N}(t, y):=\widetilde{a}(t)|y|_{\mathbb{R}}^{r-1} y+(\widetilde{b}(t) \cdot \nabla y)|y|_{\mathbb{R}}^{s-1} y$. Then, we just construct the Luenberger observer as in 1.3 and apply the Main Result.
1.5. Contents and notation. In Section 2 we present the assumptions we require for the dynamics plant operators and for all the "parameters" involved in the output injection operator. In Section 3 we prove that under such assumptions the error of the observer estimate decreases exponentially to zero. In Section 4 we show that the required assumptions are satisfiable for standard parabolic equations evolving in rectangular domains. In Section 5 we present the results of numerical simulations showing the exponential stability of the error dynamics, for a rectangular domain, namely the unit square. In Section 6 we comment on the derived results. Finally, the Appendix gathers the proofs of auxiliary results needed to derive the main result.

Concerning the notation, we write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and nonnegative integers, respectively, and we set $\mathbb{R}_{r}:=(r,+\infty), r \in \mathbb{R}$, and $\mathbb{N}_{0}:=\mathbb{N} \backslash\{0\}$.

Given two Banach spaces $X$ and $Y$, if the inclusion $X \subseteq Y$ is continuous, we write $X \hookrightarrow Y$. We write $X \xrightarrow{\text { d }} Y$, respectively $X \xrightarrow{\text { c }} Y$, if the inclusion is also dense, respectively compact.

Let $X \subseteq Z$ and $Y \subseteq Z$ be continuous inclusions, where $Z$ is a Hausdorff topological space. Then we can define the Banach spaces $X \times Y, X \cap Y$, and $X+Y$, endowed with the norms $|(h, g)|_{X \times Y}:=\left(|h|_{X}^{2}+|g|_{Y}^{2}\right)^{\frac{1}{2}},|\hat{h}|_{X \cap Y}:=|(\hat{h}, \hat{h})|_{X \times Y}$, and $|\tilde{h}|_{X+Y}:=$ $\inf _{(h, g) \in X \times Y}\left\{|(h, g)|_{X \times Y} \mid \tilde{h}=h+g\right\}$, respectively. In case we know that $X \cap Y=\{0\}$, we say that $X+Y$ is a direct sum and we write $X \oplus Y$ instead.

The space of continuous linear mappings from $X$ into $Y$ is denoted by $\mathcal{L}(X, Y)$. In case $X=Y$ we write $\mathcal{L}(X):=\mathcal{L}(X, X)$. The continuous dual of $X$ is denoted $X^{\prime}:=$ $\mathcal{L}(X, \mathbb{R})$. The adjoint of an operator $L \in \mathcal{L}(X, Y)$ will be denoted $L^{*} \in \mathcal{L}\left(Y^{\prime}, X^{\prime}\right)$.

The space of continuous functions from $X$ into $Y$ is denoted by $\mathcal{C}(X, Y)$. The space of real valued increasing functions, defined in $\overline{\mathbb{R}_{0}}$ and vanishing at 0 is denoted by:

$$
\mathcal{C}_{0, t}\left(\overline{\mathbb{R}_{0}}, \mathbb{R}\right):=\left\{\mathfrak{i} \in \mathcal{C}\left(\overline{\mathbb{R}_{0}}, \mathbb{R}\right) \mid \mathfrak{i}(0)=0, \quad \text { and } \mathfrak{i}\left(\varkappa_{2}\right) \geq \mathfrak{i}\left(\varkappa_{1}\right) \text { if } \varkappa_{2} \geq \varkappa_{1} \geq 0\right\}
$$

We also denote the vector subspace $\mathcal{C}_{\mathrm{b}, \iota}(X, Y) \subset \mathcal{C}(X, Y)$ by

$$
\mathcal{C}_{\mathrm{b}, \iota}(X, Y):=\left\{f \in \mathcal{C}(X, Y)\left|\exists \mathfrak{i} \in \mathcal{C}_{0, \iota}\left(\overline{\mathbb{R}_{0}}, \mathbb{R}\right) \forall x \in X:|f(x)|_{Y} \leq \mathfrak{i}\left(|x|_{X}\right)\right\} .\right.
$$

The orthogonal complement to a given subset $B \subset H$ of a Hilbert space $H$, with scalar product $(\cdot, \cdot)_{H}$, is denoted $B^{\perp}:=\left\{h \in H \mid(h, s)_{H}=0\right.$ for all $\left.s \in B\right\}$.

Given two closed subspaces $F \subseteq H$ and $G \subseteq H$ of the Hilbert space $H=F \oplus G$, we denote by $P_{F}^{G} \in \mathcal{L}(H, F)$ the oblique projection in $H$ onto $F$ along $G$. That is, writing $h \in H$ as $h=h_{F}+h_{G}$ with $\left(h_{F}, h_{G}\right) \in F \times G$, we have $P_{F}^{G} h:=h_{F}$. The orthogonal projection in $H$ onto $F$ is denoted by $P_{F} \in \mathcal{L}(H, F)$. Notice that $P_{F}=P_{F}^{F^{\perp}}$.

Given a sequence $\left(a_{j}\right)_{j \in\{1,2, \ldots, n\}}$ of real nonnegative constants, $n \in \mathbb{N}_{0}, a_{i} \geq 0$, we denote $\|a\|:=\max _{1 \leq j \leq n} a_{j}$.

By $\bar{C}_{\left[a_{1}, \ldots, a_{n}\right]}$ we denote a nonnegative function that increases in each of its nonnegative arguments $a_{i}, 1 \leq i \leq n$.

Finally, $C, C_{i}, i=0,1, \ldots$, stand for unessential positive constants.

## 2. Assumptions

The results will follow under general assumptions on the plant dynamics operators $A$, $A_{\mathrm{rc}}, \mathcal{N}$, and on our targeted real state $y$. We will also need a particular assumption on the triple $\left(\mathcal{W}_{S}, \widetilde{\mathcal{W}}_{S}, \lambda\right)$.

The Hilbert space $H$, in which system (1.5) is evolving in, will be set as a pivot space, that is, we identify, $H^{\prime}=H$. Let $V$ be another Hilbert space with $V \subset H$.

Assumption 2.1. $A \in \mathcal{L}\left(V, V^{\prime}\right)$ is symmetric and $(y, z) \mapsto\langle A y, z\rangle_{V^{\prime}, V}$ is a complete scalar product in $V$.

From now on, we suppose that $V$ is endowed with the scalar product $(y, z)_{V}:=$ $\langle A y, z\rangle_{V^{\prime}, V}$, which still makes $V$ a Hilbert space. Necessarily, $A: V \rightarrow V^{\prime}$ is an isometry.

Assumption 2.2. The inclusion $V \subseteq H$ is dense, continuous, and compact.
Necessarily, we have that

$$
\langle y, z\rangle_{V^{\prime}, V}=(y, z)_{H}, \quad \text { for all }(y, z) \in H \times V,
$$

and also that the operator $A$ is densely defined in $H$, with domain $\mathrm{D}(A)$ satisfying

$$
\mathrm{D}(A) \stackrel{\mathrm{d}, \mathrm{c}}{\longrightarrow} V \stackrel{\mathrm{~d}, \mathrm{c}}{\longrightarrow} H \stackrel{\mathrm{~d}, \mathrm{c}}{\longrightarrow} V^{\prime} \stackrel{\mathrm{d}, \mathrm{c}}{\longrightarrow} \mathrm{D}(A)^{\prime} .
$$

Further, $A$ has a compact inverse $A^{-1}: H \rightarrow \mathrm{D}(A)$, and we can find a nondecreasing system of (repeated accordingly to their multiplicity) eigenvalues $\left(\alpha_{n}\right)_{n \in \mathbb{N}_{0}}$ and a corresponding complete basis of eigenfunctions $\left(e_{n}\right)_{n \in \mathbb{N}_{0}}$ :

$$
0<\alpha_{1} \leq \alpha_{2} \leq \cdots \leq \alpha_{n} \leq \alpha_{n+1} \rightarrow+\infty \quad \text { and } \quad A e_{n}=\alpha_{n} e_{n}
$$

We can define, for every $\xi \in \mathbb{R}$, the fractional powers $A^{\xi}$, of $A$, by

$$
y=\sum_{n=1}^{+\infty} y_{n} e_{n}, \quad A^{\xi} y=A^{\xi} \sum_{n=1}^{+\infty} y_{n} e_{n}:=\sum_{n=1}^{+\infty} \alpha_{n}^{\xi} y_{n} e_{n}
$$

and the corresponding domains $\mathrm{D}\left(A^{|\xi|}\right):=\left\{y \in H \mid A^{|\xi|} y \in H\right\}$, and $\mathrm{D}\left(A^{-|\xi|}\right):=$ $\mathrm{D}\left(A^{|\xi|}\right)^{\prime}$. We have that $\mathrm{D}\left(A^{\xi}\right) \stackrel{\mathrm{d}, \mathrm{c}}{\longrightarrow} \mathrm{D}\left(A^{\zeta_{1}}\right)$, for all $\xi>\xi_{1}$, and we can see that $\mathrm{D}\left(A^{0}\right)=$ $H, \mathrm{D}\left(A^{1}\right)=\mathrm{D}(A), \mathrm{D}\left(A^{\frac{1}{2}}\right)=V$.

For the time-dependent operator and external forcing we assume the following:
Assumption 2.3. For almost every $t>0$ we have $A_{\mathrm{rc}}(t) \in \mathcal{L}(V, H)$, and we have a uniform bound as $\left|A_{\mathrm{rc}}\right|_{L^{\infty}\left(\mathbb{R}_{0}, \mathcal{L}(V, H)\right)}=C_{\mathrm{rc}}<+\infty$.

Assumption 2.4. We have $\mathcal{N}(t, \cdot) \in \mathcal{C}_{\mathrm{b}, t}(\mathrm{D}(A), H)$ and there exist constants $C_{\mathcal{N}} \geq 0$, $n \in \mathbb{N}_{0}, \zeta_{1 j} \geq 0, \zeta_{2 j} \geq 0, \delta_{1 j} \geq 0, \delta_{2 j} \geq 0$, with $j \in\{1,2, \ldots, n\}$, such that for all $t>0$ and all $\left(y_{1}, y_{2}\right) \in \mathrm{D}(A) \times \mathrm{D}(A)$, we have

$$
\left|\mathcal{N}\left(t, y_{1}\right)-\mathcal{N}\left(t, y_{2}\right)\right|_{H} \leq C_{\mathcal{N}} \sum_{j=1}^{n}\left(\left|y_{1}\right|_{V}^{\zeta_{1 j}}\left|y_{1}\right|_{\mathrm{D}(A)}^{\zeta_{2 j}}+\left|y_{2}\right|_{V}^{\zeta_{1 j}}\left|y_{2}\right|_{\mathrm{D}(A)}^{\zeta_{2 j}}\right)|d|_{V}^{\delta_{1 j}}|d|_{\mathrm{D}(A)}^{\delta_{2 j}},
$$

with $d:=y_{1}-y_{2}, \zeta_{2 j}+\delta_{2 j}<1$ and $\delta_{1 j}+\delta_{2 j} \geq 1$.
Assumption 2.5. The targeted real state y, satisfying (1.1), satisfies the uniform persistent boundedness estimate as follows. There are constants $C_{y} \geq 0$ and $\tau_{y}>0$ such that

$$
\sup _{s \geq 0}|y(s)|_{V} \leq C_{y} \quad \text { and } \quad \sup _{s \geq 0}|y|_{L^{2}\left(\left(s, s+\tau_{y}\right), \mathrm{D}(A)\right)}<C_{y} .
$$

Assumption 2.6. The pair $\left(\sigma,\left(\widetilde{W}_{S}, W_{S}\right)_{S \in \mathbb{N}_{0}}\right)$ satisfies:

$$
\sigma: \mathbb{N}_{0} \rightarrow \mathbb{N}_{0} \text { is strictly increasing }
$$

and, with $S_{\sigma}:=\sigma(S), \widetilde{\mathcal{W}}_{S}=\operatorname{span} \widetilde{W}_{S}$, and $\mathcal{W}_{S}=\operatorname{span} W_{S}$,

$$
\begin{aligned}
& W_{S}:=\left\{\mathfrak{w}_{j} \mid 1 \leq j \leq S_{\sigma}\right\} \subset H \\
& \widetilde{W}_{S}:=\left\{\widetilde{\mathfrak{w}}_{j} \mid 1 \leq j \leq S_{\sigma}\right\} \subset \mathrm{D}(A) \subset H \\
& \operatorname{dim} \mathcal{W}_{S}=S_{\sigma}=\operatorname{dim} \widetilde{\mathcal{W}}_{S} \text { and } H=\mathcal{W}_{S} \oplus \widetilde{\mathcal{W}}_{S}^{\perp}
\end{aligned}
$$

The key assumption concerns the following Poincaré-like constant

$$
\begin{equation*}
\beta_{S_{\sigma+}}:=\inf _{Q \in\left(\mathrm{D}(A) \cap \mathcal{W}_{S}^{\perp}\right) \backslash\{0\}} \frac{|Q|_{\mathrm{D}(A)}^{2}}{|Q|_{V}^{2}} . \tag{2.1}
\end{equation*}
$$

Assumption 2.7. The sequence $\left(\beta_{S_{\sigma+}}\right)_{S \in \mathbb{N}_{0}}$ in 2.1) is divergent, $\lim _{S \rightarrow+\infty} \beta_{S_{\sigma+}}=+\infty$.
The last assumption concerns the type of outputs.
Assumption 2.8. The output $w=\mathcal{Z} y \in \mathbb{R}^{S_{\sigma}}$ is of the form $w_{i}(t)=\left(\mathfrak{w}_{i}, y(t)\right)_{H}$, with $\mathfrak{w}_{i} \in W_{S}$.

Assumptions 2.1 2.6 are satisfiable for parabolic systems as (1.7). Assumptions 2.1 2.3 are usually not hard to check for such systems. Assumption 2.4 is satisfied by a general class of polynomial nonlinearities as in 1.7. Assumption 2.5 is a requirement on our targeted state, which simply says that the real state to be estimated is a strong
solution which is bounded in a general appropriate way. It is also not difficult to construct spaces satisfying Assumption 2.6, and then in Assumption 2.8 we are simply requiring the form of the output.

The satisfiability of Assumption 2.7 is nontrivial. We shall prove in Section 4 that it is satisfied for scalar parabolic equations evolving in rectangular spatial domains $\Omega \subset \mathbb{R}^{d}$, for suitable placement of the sensors (as indicator functions). The proof can be adapted to general convex polynomial domains. The satisfiability of the Assumption 2.7 for general smooth domains is an open question. See the discussion in [18, Sect. 7.3].
Remark 2.9. Note that Assumption 2.3 is stronger than the one taken in [18, Assum. 2.3] in the linear setting. We need extra regularity for $A_{\mathrm{rc}}$ because weak solutions, as considered in [18], living in $W_{\text {loc }}\left(\mathbb{R}_{0}, V, V^{\prime}\right)$, are not regular enough to deal with the entire class of nonlinear systems we shall consider here. We need strong solutions, living in $W_{\text {loc }}\left(\mathbb{R}_{0}, \mathrm{D}(A), H\right)$, to guarantee the existence and uniqueness of solutions for all systems involved in our analysis.

## 3. Exponential stability of the error dynamics

For given $S \in \mathbb{N}_{0}$ and $\ell \in \mathbb{R}$, we define another Poicaré-like constant as follows

$$
\begin{equation*}
0<\underline{\alpha}_{S, \ell}:=\inf _{q \in \widetilde{\mathcal{W}}_{S} \backslash\{0\}} \frac{|q|_{\mathrm{D}}^{2}\left(A_{A} \frac{\ell}{2}\right)}{|q|_{\mathrm{D}(A)}^{2}} . \tag{3.1}
\end{equation*}
$$

We prove the following more general abstract version of the main Main Result.
Theorem 3.1. Let Assumptions 2.1 2.6 hold true and let us be given $\ell \in[0,2], R>0$, $\varrho>1$, and $\mu>0$. Then there exists a pair $\left(S^{*}, \lambda^{*}\right) \in \mathbb{N}_{0} \times \mathbb{R}_{0}$ such that: for all pairs $(S, \lambda)$ satisfying $S \geq S^{*}$ and $\lambda \geq \lambda^{*}(S)$, the error dynamical system

$$
\begin{equation*}
\dot{z}+A z+A_{\mathrm{rc}}(t) z+\mathfrak{N}_{y}(t, z)=\widetilde{\mathfrak{I}}_{S}^{[\lambda, \ell]} z, \quad z(0)=z_{0} \in V, \quad t \geq 0 \tag{3.2a}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathfrak{N}_{y}(t, z):=\mathcal{N}(t, z+y)-\mathcal{N}(t, y) \quad \text { and } \quad \widetilde{\mathfrak{I}}_{S}^{[\lambda, \ell]} z=-\lambda A^{-1} P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{S}^{\perp}} A^{\ell} P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}} z \tag{3.2b}
\end{equation*}
$$

is exponentially stable with rate $-\mu$ and transient bound $\varrho$. For all $z_{0} \in V$ the solution of (3.2) satisfies

$$
\begin{equation*}
|z(t)|_{V} \leq \varrho \mathrm{e}^{-\mu(t-s)}|z(s)|_{V}, \quad \text { for all } \quad t \geq s \geq 0 \tag{3.2c}
\end{equation*}
$$

Furthermore, the constants $S^{*}$ and $\lambda^{*}(S)$ can be taken of the form
and

$$
\begin{equation*}
\lambda^{*}(S)=\bar{C}_{\left.\left[\frac{1}{Q_{S, \ell}}, R, \mu, \varrho, \frac{1}{e^{\frac{1}{2}-1}}, \frac{1}{\tau_{y}}, \tau_{y}, \frac{2\left\|\zeta_{1}\right\|}{1-\left\|\delta_{2}+\zeta_{2}\right\|}, \frac{2+\| \frac{2 \zeta_{2}}{1-\delta_{2} \|}}{2-\left\|\frac{2 \zeta_{2}}{1-\delta_{2}}\right\|} \|, \frac{2\left\|\delta_{1}+\delta_{2}+\zeta_{1}+\zeta_{2}\right\|-2}{1-\left\|\delta_{2}+\zeta_{2}\right\|}, C_{\mathrm{rc}}, C_{y}\right], .\right] ., ~} \tag{3.2e}
\end{equation*}
$$

where $\left(C_{\mathrm{rc}}, C_{y}, \tau_{y}, \delta, \zeta\right)$ is the data in Assumptions 2.32 .5 .
Remark 3.2. Recall that $\|a\|:=\max _{1 \leq j \leq n} a_{j}$, for example, $\left\|\frac{2 \zeta_{2}}{1-\delta_{2}}\right\|=\max _{1 \leq j \leq n} \frac{2 \zeta_{2 j}}{1-\delta_{2 j}}$.
Remark 3.3. Observe that from 3.2 e , if we can show that for a given $\ell \in[0,2]$ we have that $\underline{\alpha}_{S, \ell} \geq \underline{\alpha}>0$ with $\underline{\alpha}$ independent of $S$, then we can conclude that the lower bound $\lambda^{*}(S)$ can be taken independent of $S$. This is always the case for $\ell=2$ because $\underline{\alpha}_{S, 2}=1$. For $0 \leq \ell<2$ the existence of such $\underline{\alpha}>0$ is not clear and will/may depend on $\widetilde{\mathcal{W}}_{S}$. We will come back to this point in Section 4 see Proposition 4.7, where we give an example where such strictly positive lower bound $\underline{\alpha}$ does not exist for $\ell \in\{0,1\}$.

Note that 1.5 a is equivalent to (3.2a). Indeed, denoting by $P_{\mathcal{W}_{S}}=P_{\mathcal{W}_{S}}^{\mathcal{W}}$ the orthogonal projection in $H$ onto $\mathcal{W}_{S}$, from [18, sect. 2], we know that

$$
\begin{equation*}
\mathbf{Z}^{W_{s}} \mathcal{Z}=P_{\mathcal{W}_{S}} \tag{3.3}
\end{equation*}
$$

which gives us $\widetilde{\mathfrak{I}}_{S}^{[\lambda, \ell]} z=\widetilde{\mathfrak{I}}_{S}^{[\lambda, \ell]} P_{\mathcal{W}_{S}} z=\mathfrak{I}_{S}^{[\lambda, \ell]} \mathcal{Z} z=\mathfrak{I}_{S}^{[\lambda, \ell]}(\mathcal{Z} \widehat{y}-w)$.
3.1. Auxiliary results. In the proof of Theorem 3.1, given in Section 3.2, we will use some auxiliary results, which are gathered in this section.

We start with results on appropriate estimates for the nonlinear term.
Lemma 3.4. Let Assumptions 2.1, 2.2, and 2.4 hold true, and let $P \in \mathcal{L}(H)$. Then there is a constant $\bar{C}_{\mathcal{N} 1}>0$ such that: for all $\widehat{\gamma}_{0}>0$, all $t>0$, and all $\left(y_{1}, y_{2}\right) \in$ $\mathrm{D}(A) \times \mathrm{D}(A)$, we have

$$
\begin{aligned}
& 2\left(P\left(\mathcal{N}\left(t, y_{1}\right)-\mathcal{N}\left(t, y_{2}\right)\right), A\left(y_{1}-y_{2}\right)\right)_{H} \\
& \leq \widehat{\gamma}_{0}\left|y_{1}-y_{2}\right|_{\mathrm{D}(A)}^{2}+\left(1+\widehat{\gamma}_{0}^{-\frac{1+\left\|\delta_{2}\right\|}{1-\left\|\delta_{2}\right\|}}\right) \bar{C}_{\mathcal{N} 1} \sum_{j=1}^{n}\left|y_{1}-y_{2}\right|_{V}^{\frac{2 \delta_{1 j}}{1-\delta_{2 j}}} \sum_{k=1}^{2}\left|y_{k}\right|_{V}^{\frac{2 \zeta_{1 j}}{1-\delta_{2 j}}}\left|y_{k}\right|_{\mathrm{D}(A)}^{\frac{2 \zeta_{2 j}}{1-\delta_{j}}} .
\end{aligned}
$$

Further, the constant $\bar{C}_{\mathcal{N} 1}$ is of the form $\bar{C}_{\mathcal{N} 1}=\bar{C}_{\left[n, \frac{1}{1-\left\|\delta_{2}\right\|}, C_{\mathcal{N}},|P|_{\mathcal{L}(H)}\right]}$.
The proof of the lemma is given in [20, Sect. A.1] for operators as $P=P \frac{\mathcal{W}_{S}}{\mathcal{W}_{S}^{1}}$, however the steps of such proof can be repeated for a general operator $P \in \mathcal{L}(H)$. See 20, Proposition 3.5].

Now, we present a sequence of auxiliary results as the following propositions. The corresponding proofs are presented later in the Appendix.

An estimate for $\mathfrak{N}_{y}(t, \widehat{y}-y)=\mathcal{N}(t, \widehat{y})-\mathcal{N}(t, y)$ is as follows.
Proposition 3.5. Let Assumptions 2.1, 2.2, 2.4, and 2.5 hold true. Then there are constants $\widetilde{C}_{\mathfrak{N} 1}>0$, and $\widetilde{C}_{\mathfrak{N} 2}>0$ such that: for all $\widehat{\gamma}_{0}>0$, all $t>0$, all $\left(z_{1}, z_{2}\right) \in$ $\mathrm{D}(A) \times \mathrm{D}(A)$, we have

$$
\begin{align*}
& 2\left(\mathfrak{N}_{y}\left(t, z_{1}\right)-\mathfrak{N}_{y}\left(t, z_{2}\right), A\left(z_{1}-z_{2}\right)\right)_{H} \leq \widehat{\gamma}_{0}\left|z_{1}-z_{2}\right|_{\mathrm{D}(A)}^{2}  \tag{3.4}\\
& \quad+\left(1+\widehat{\gamma}_{0}^{-\frac{1+\left\|\delta_{2}\right\|}{1-\mid \delta_{2} \|}}\right) \widetilde{C}_{\mathfrak{N} 1} \sum_{j=1}^{n}\left|z_{1}-z_{2}\right|_{V}^{\frac{2 \delta_{1 j}}{1-\delta_{2 j}}} \sum_{k=1}^{2}\left|y+z_{k}\right|_{V}^{\frac{2 \zeta_{1 j}}{1-\delta_{2 j}}}\left|y+z_{k}\right|_{\mathrm{D}(A)}^{\frac{2 \zeta_{2 j}}{1-\delta_{j}}} \\
& 2\left(\mathfrak{N}_{y}\left(t, z_{1}\right), A z_{1}\right)_{H} \leq \widehat{\gamma}_{0}\left|z_{1}\right|_{\mathrm{D}(A)}^{2}  \tag{3.5}\\
& +\widetilde{C}_{\mathfrak{N} 2}\left(1+\widehat{\gamma}_{0}^{-\chi_{5}}\right)\left(1+\widehat{\gamma}_{0}^{-\frac{\left(\chi_{5}+1\right) \chi_{2} \chi_{4}}{2}}\right)\left(1+|y|_{V}^{\chi_{1}}\right)\left(1+|y|_{\mathrm{D}(A)}^{\chi_{2}}\right)\left(1+\left|z_{1}\right|_{V}^{\chi_{3}}\right)\left|z_{1}\right|_{V}^{2}
\end{align*}
$$

with $\widetilde{C}_{\mathfrak{N} 2}=\bar{C}_{\left[n, \widetilde{C}_{\mathcal{N} 1},\left\|\zeta_{1}\right\|,\left\|\zeta_{2}\right\|, \frac{1}{1-\left\|\delta_{2}\right\|}, \frac{1}{1-\| \zeta_{2}+\delta_{2} \pi}\right]}$ and

$$
\begin{array}{ll}
\chi_{1}:=\frac{2\left\|\zeta_{1}\right\|}{1-\left\|\delta_{2}+\zeta_{2}\right\|} \geq 0, & \chi_{2}:=\left\|\frac{2 \zeta_{2}}{1-\delta_{2}}\right\| \in[0,2), \\
\chi_{3}:=\frac{2\left\|\delta_{1}+\delta_{2}+\zeta_{1}+\zeta_{2}\right\|-2}{1-\left\|\delta_{2}+\zeta_{2}\right\|} \geq 0, & \chi_{4}:=\frac{1}{1-\left\|\delta_{2}+\zeta_{2}\right\|}>1, \quad \chi_{5}:=\frac{1+\left\|\delta_{2}\right\|}{1-\left\|\delta_{2}\right\|}>1 . \tag{3.7}
\end{array}
$$

The next auxiliary results concern properties of oblique projections. Recall that $\mathcal{W}_{S} \subset$ $H=\mathrm{D}\left(A^{0}\right)$ and $\widetilde{\mathcal{W}}_{S} \subset \mathrm{D}(A)=\mathrm{D}\left(A^{1}\right)$, due to Assumption 2.6

Proposition 3.6. Let $\xi \in[0,1]$. The restriction of the oblique projection $P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}} \in \mathcal{L}(H)$ to $\mathrm{D}\left(A^{\xi}\right) \subseteq H$ is the oblique projection in $\mathrm{D}\left(A^{\xi}\right)$ onto $\widetilde{\mathcal{W}}_{S}$ along $\mathcal{W}_{S}^{\perp} \bigcap \mathrm{D}\left(A^{\xi}\right)$. That is, $\left.P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}}\right|_{\mathrm{D}\left(A^{\xi}\right)}=P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp} \cap \mathrm{D}\left(A^{\xi}\right)} \in \mathcal{L}\left(\mathrm{D}\left(A^{\xi}\right)\right)$.

For given $\xi \in[0,1]$, let us define the mapping $\left.P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{S}^{\perp}}\right|^{\mathrm{D}\left(A^{-\xi}\right)}: \mathrm{D}\left(A^{-\xi}\right) \rightarrow \mathrm{D}\left(A^{-\xi}\right)$ by

$$
\begin{equation*}
\left\langle\left. P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}^{\frac{1}{S}}}\right|^{\mathrm{D}\left(A^{-\xi}\right)} z, w\right\rangle_{\mathrm{D}\left(A^{-\xi}\right), \mathrm{D}\left(A^{\xi}\right)}:=\left\langle z, \mathcal{P}_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}} w\right\rangle_{\mathrm{D}\left(A^{-\xi}\right), \mathrm{D}\left(A^{\xi}\right)} \tag{3.8}
\end{equation*}
$$

for all $(z, w) \in \mathrm{D}\left(A^{-\xi}\right) \times \mathrm{D}\left(A^{\xi}\right)$.
Proposition 3.7. Let $\xi \in[0,1]$. The mapping $\left.P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{S}^{\perp}}\right|^{\mathrm{D}\left(A^{-\xi}\right)}$ is an extension of the oblique projection $P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{s}^{\perp}} \in \mathcal{L}(H)$ to $\mathrm{D}\left(A^{-\xi}\right) \supseteq H$, and we have the adjoint and norm identities as
$\left.P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{S}^{\perp}}\right|^{\mathrm{D}\left(A^{-\xi}\right)}=\left(\left.P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}}\right|_{\mathrm{D}\left(A^{\xi}\right)}\right)^{*}$ and $\left.\quad\left|P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{S}^{\perp}}\right|^{\mathrm{D}\left(A^{-\xi}\right)}\right|_{\mathcal{L}\left(\mathrm{D}\left(A^{-\xi}\right)\right)}=\left.\left|P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}}\right|_{\mathrm{D}\left(A^{\xi}\right)}\right|_{\mathcal{L}\left(\mathrm{D}\left(A^{\xi}\right)\right)}$, where $\left.P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}}\right|_{\mathrm{D}\left(A^{\xi}\right)}$ is the restriction in Proposition 3.6 .

Finally, we present auxiliary results that we use to analyze the stability of the nonlinear error dynamics.

Proposition 3.8. Let $\eta_{1}>0, \eta_{2}>0$ and $\mathfrak{s} \in(0,1)$. Then

$$
\max _{\tau \geq 0}\left\{-\eta_{1} \tau+\eta_{2} \tau^{\mathfrak{s}}\right\}=(1-\mathfrak{s}) \mathfrak{s}^{\frac{s}{1-s}} \eta_{2}^{\frac{1}{1-\mathfrak{s}}} \eta_{1}^{\frac{\mathfrak{s}}{\mathfrak{s}-1}}
$$

Proposition 3.9. Let $T>0, C_{h}>0, \mathfrak{r}>1$, and $h \in L_{\mathrm{loc}}^{\mathfrak{r}}\left(\mathbb{R}_{0}, \mathbb{R}\right)$ satisfying

$$
\begin{equation*}
\sup _{s \geq 0}|h|_{L^{\mathrm{r}}((s, s+T), \mathbb{R})}=C_{h} \leq+\infty . \tag{3.9}
\end{equation*}
$$

Let also, $\mu>0$, and $\varrho>1$. Then for every scalar $\bar{\mu}>0$ satisfying

$$
\begin{equation*}
\bar{\mu} \geq \max \left\{2 \frac{\mathfrak{r}-1}{\mathfrak{r}}\left(\frac{C_{h}^{\mathfrak{r}}}{\mathfrak{r} \log (\varrho)}\right)^{\frac{1}{\mathfrak{r}-1}}, 2 \mu\right\}+T^{\frac{-1}{\mathfrak{r}}} C_{h} \tag{3.10}
\end{equation*}
$$

we have that the scalar ODE system

$$
\begin{equation*}
\dot{v}=-\left(\bar{\mu}-|h|_{\mathbb{R}}\right) v, \quad v(0)=v_{0} \tag{3.11}
\end{equation*}
$$

is exponentially stable with rate $-\mu$ and transient bound $\varrho$. For every $v_{0} \in \mathbb{R}$,

$$
|v(t)|=\varrho \mathrm{e}^{-\mu(t-s)}|v(s)|, \quad t \geq s \geq 0, \quad v(0)=v_{0}
$$

Proposition 3.10. Let $T>0, C_{h}>0, \mathfrak{r}>1$, and $h \in L_{\mathrm{loc}}^{\mathfrak{r}}\left(\mathbb{R}_{0}, \mathbb{R}\right)$ satisfy (3.9). Let also $R>0, p>0, \mu>0, \varrho>1$, and $c>1$. Then the scalar ODE

$$
\begin{equation*}
\dot{\varpi}=-\left(\bar{\mu}-|h|_{\mathbb{R}}\left(1+|\varpi|_{\mathbb{R}}^{p}\right)\right) \varpi, \quad \varpi(0)=\varpi_{0}, \tag{3.12}
\end{equation*}
$$

is exponentially stable with transient bound $\varrho$ and rate $-\mu_{0}<-\mu$ as

$$
\begin{equation*}
\mu_{0}:=\max \left\{\mu, \frac{\log (2)}{p T},\left(\frac{\varrho^{2 p+1} R^{p} C_{h}}{\varrho^{\frac{1}{2}}-1}\right)^{\frac{\mathfrak{r}}{\mathfrak{r}-1}}\left(\frac{\mathfrak{r}-1}{\mathfrak{r}}\right) 2^{\frac{1}{\mathfrak{r}-1}}, 2^{\frac{\mathfrak{r}+1}{\mathfrak{r}-1}}\left(\varrho^{2 p+\frac{1}{2}} C_{h} \frac{p+1}{p} R^{p} c\right)^{\frac{\mathfrak{r}}{\mathfrak{r}-1}} p^{\frac{1}{\mathrm{r}-1}}\right\} \tag{3.13}
\end{equation*}
$$

if

$$
\begin{equation*}
\left|\varpi_{0}\right| \leq R \quad \text { and } \quad \bar{\mu} \geq \bar{\mu}_{*}:=\max \left\{2 \frac{\mathfrak{r}-1}{\mathfrak{r}}\left(\frac{2 C_{h}^{\mathfrak{r}}}{\mathfrak{r} \log (\varrho)}\right)^{\frac{1}{\mathfrak{r}-1}}, 4 \mu_{0}\right\}+T^{\frac{-1}{\mathrm{r}}} C_{h} . \tag{3.14}
\end{equation*}
$$

That is, the solution satisfies

$$
\begin{equation*}
|\varpi(t)|_{\mathbb{R}} \leq \varrho \mathrm{e}^{-\mu_{0}(t-s)}|\varpi(s)|_{\mathbb{R}}, \quad \text { for all } \quad t \geq s \geq 0, \quad \text { if } \quad\left|\varpi_{0}\right|<R \tag{3.15}
\end{equation*}
$$

3.2. Proof of the main Theorem 3.1. We split the error into oblique components as

$$
z=\theta+\Theta, \quad \text { with } \quad \theta:=P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}} z \quad \text { and } \quad \Theta:=P_{\mathcal{W}_{S}^{\perp}}^{\widetilde{\mathcal{W}}_{S}} z
$$

and observe that

$$
\begin{aligned}
\dot{z} & =-A z-A_{\mathrm{rc}} z-\mathcal{N}(\widehat{y})+\mathcal{N}(y)-\lambda A^{-1} P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{S}}_{S}^{\perp}} A^{\ell} P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}} z \\
& =-A z-A_{\mathrm{rc}} z-\mathfrak{N}_{y}(z)-\lambda A^{-1} P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{S}^{\perp}} A^{\ell} \theta
\end{aligned}
$$

from which we obtain

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t}|z|_{V}^{2}=2\left(-A z-A_{\mathrm{rc}} z-\mathfrak{N}_{y}(z)-\lambda A^{-1} P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{S}^{\perp}} A^{\ell} \theta, A z\right)_{H} . \tag{3.16}
\end{equation*}
$$

Observe that, by direct computations, using Assumptions 2.1/2.3 and the Young inequality, we find for all $\left(\gamma_{1}, \gamma_{2}\right) \in(0,2) \times \mathbb{R}_{0}$,

$$
\begin{align*}
2(-A z & \left.-A_{\mathrm{rc}} z, A z\right)_{H} \leq-\left(2-\gamma_{1}\right)|z|_{\mathrm{D}(A)}^{2}+\gamma_{1}^{-1} C_{\mathrm{rc}}^{2}|z|_{V}^{2} \\
\leq & -\left(2-\gamma_{1}\right)\left(1-\gamma_{2}\right)|\Theta|_{\mathrm{D}(A)}^{2}-\left(2-\gamma_{1}\right)\left(1-\gamma_{2}^{-1}\right)|\theta|_{\mathrm{D}(A)}^{2}+\gamma_{1}^{-1} C_{\mathrm{rc}}^{2}|z|_{V}^{2} \\
\leq & -\left(2-\gamma_{1}\right)\left(1-\gamma_{2}\right)|\Theta|_{\mathrm{D}(A)}^{2}+2 \gamma_{1}^{-1} C_{\mathrm{rc}}^{2}|\Theta|_{V}^{2} \\
& -\left(2-\gamma_{1}\right)\left(1-\gamma_{2}^{-1}\right)|\theta|_{\mathrm{D}(A)}^{2}+2 \gamma_{1}^{-1} C_{\mathrm{rc}}^{2}|\theta|_{V}^{2} \tag{3.17}
\end{align*}
$$

Direct computations also give us

$$
\begin{align*}
2\left(-\lambda A^{-1} P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{S}^{\perp}} A^{\ell} \theta, A z\right)_{H} & =-2 \lambda\left(P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{S}^{\perp}} A^{\ell} \theta, z\right)_{H}=-2 \lambda\left(A^{\ell} \theta, P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}} z\right)_{H} \\
& \left.=-2 \lambda|\theta|_{\mathrm{D}\left(A^{\frac{\ell}{2}}\right.}^{2}\right) \tag{3.18}
\end{align*}
$$

For the nonlinear term, using (3.5) and the Young inequality, we find for all $\gamma_{3} \in \mathbb{R}_{0}$,

$$
\begin{aligned}
& 2\left(\mathfrak{N}_{y}(t, z), A z\right)_{H} \leq \gamma_{3}|z|_{\mathrm{D}(A)}^{2} \\
& +\widetilde{C}_{\mathfrak{N} 2}\left(1+\widehat{\gamma}_{0}^{-\chi_{5}}\right)\left(1+\widehat{\gamma}_{0}^{-\frac{\left(x_{5}+1\right) \chi_{2} \chi_{4}}{2}}\right)\left(1+|y|_{V}^{\chi_{1}}\right)\left(1+|y|_{\mathrm{D}(A)}^{\chi_{2}}\right)\left(1+|z|_{V}^{\chi_{3}}\right)|z|_{V}^{2}
\end{aligned}
$$

which implies

$$
\begin{equation*}
2\left(\mathfrak{N}_{y}(t, z), A z\right)_{H} \leq \gamma_{3}|z|_{\mathrm{D}(A)}^{2}+\widehat{C} \Psi(y)\left(1+|z|_{V}^{\chi_{3}}\right)|z|_{V}^{2}, \tag{3.19a}
\end{equation*}
$$

with

$$
\begin{align*}
\widehat{C} & =\bar{C}_{\left[n, \widetilde{C}_{\mathcal{N} 1},\left\|\zeta_{1}\right\|,\left\|\zeta_{2}\right\|, \frac{1}{1-\left\|\delta_{2}\right\|}, \frac{1}{1-\left\|\zeta_{2}+\delta_{2}\right\|}, \frac{1}{\gamma_{3}}\right]},  \tag{3.19b}\\
\Psi(y) & :=\left(1+|y|_{V}^{\chi_{1}}\right)\left(1+|y|_{\mathrm{D}(A)}^{\chi_{2}}\right) . \tag{3.19c}
\end{align*}
$$

Combining (3.16), 3.17, 3.18), and 3.19, it follows that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t}|z|_{V}^{2} \leq & -\left(\left(2-\gamma_{1}\right)\left(1-\gamma_{2}\right)-2 \gamma_{3}\right)|\Theta|_{\mathrm{D}(A)}^{2}+2 \gamma_{1}^{-1} C_{\mathrm{rc}}^{2}|\Theta|_{V}^{2} \\
& -2 \lambda|\theta|_{\mathrm{D}\left(A^{\frac{\ell}{2}}\right)}^{2}+\left(\left(2-\gamma_{1}\right)\left(\gamma_{2}^{-1}-1\right)-2 \gamma_{3}\right)|\theta|_{\mathrm{D}(A)}^{2}+2 \gamma_{1}^{-1} C_{\mathrm{rc}}^{2}|\theta|_{V}^{2} \\
& +\widehat{C} \Psi(y)\left(1+|z|_{V}^{\chi_{3}}\right)|z|_{V}^{2} .
\end{aligned}
$$

Next, we choose/fix a triple $\left(\gamma_{1}, \gamma_{2}, \gamma_{3}\right)$, small enough, such that

$$
\begin{aligned}
& \left(\gamma_{1}, \gamma_{2}, \gamma_{3}\right) \in(0,2) \times(0,1) \times \mathbb{R}_{0}, \quad \text { and } \\
& C_{\gamma, 1}:=\left(2-\gamma_{1}\right)\left(1-\gamma_{2}\right)-2 \gamma_{3}>0, \quad C_{\gamma, 2}:=\left(2-\gamma_{1}\right)\left(\gamma_{2}^{-1}-1\right)-2 \gamma_{3}>0 .
\end{aligned}
$$

Let us recall the inequality $|q|_{\mathrm{D}\left(A^{\frac{\ell}{2}}\right)}^{2} \geq \underline{\alpha}_{S, \ell}|q|_{\mathrm{D}(A)}^{2}$, that we have due to 3.1), the inequality $|\Theta|_{\mathrm{D}(A)}^{2} \geq \beta_{S_{\sigma+}}|\Theta|_{V}^{2}$, that we have due to (2.1), and also the inequality $|\theta|_{\mathrm{D}(A)}^{2} \geq \alpha_{1}|\theta|_{V}^{2}$, where $\alpha_{1}>0$ is the first eigenvalue of $A$. These inequalities lead us to

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} t}|z|_{V}^{2} \leq & -\left(C_{\gamma, 1} \beta_{S_{\sigma+}}-2 \gamma_{1}^{-1} C_{\mathrm{rc}}^{2}\right)|\Theta|_{V}^{2}-\left(2 \lambda \underline{\alpha}_{S, \ell}-C_{\gamma, 2}-2 \gamma_{1}^{-1} C_{\mathrm{rc}}^{2} \alpha_{1}^{-1}\right)|\theta|_{\mathrm{D}(A)}^{2} \\
& +\widehat{C} \Psi(y)\left(1+|z|_{V}^{\chi_{3}}\right)|z|_{V}^{2} . \tag{3.20}
\end{align*}
$$

Next note that Assumption 2.7 implies that

$$
\beta_{\bar{S}}^{*}:=\min _{S \geq \bar{S}} \beta_{S_{\sigma+}} \rightarrow+\infty \quad \text { as } \quad \bar{S} \rightarrow+\infty .
$$

Therefore, for any given $\bar{\mu}>0$ we can choose $S$ large enough so that

$$
\begin{equation*}
C_{S}:=C_{\gamma, 1} \beta_{S}^{*}-2 \gamma_{1}^{-1} C_{\mathrm{rc}}^{2} \geq 2 \bar{\mu} \tag{3.21a}
\end{equation*}
$$

and, subsequently, we can choose $\lambda=\lambda(S)$ large enough satisfying

$$
\begin{equation*}
C_{\lambda}:=\alpha_{1}\left(2 \lambda \underline{\alpha}_{S, \ell}-C_{\gamma, 2}-2 \gamma_{1}^{-1} C_{\mathrm{rc}}^{2} \alpha_{1}^{-1}\right) \geq 2 \bar{\mu} \tag{3.21b}
\end{equation*}
$$

Hence, from 3.20 and 3.21, we arrive at the estimate

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t}|z|_{V}^{2} & \leq-2 \bar{\mu}\left(|\Theta|_{V}^{2}+|\theta|_{V}^{2}\right)+\widehat{C} \Psi(y)\left(1+|z|_{V}^{\chi_{3}}\right)|z|_{V}^{2} \\
& \leq-\bar{\mu}|z|_{V}^{2}+\widehat{C} \Psi(y)\left(1+|z|_{V}^{\chi_{3}}\right)|z|_{V}^{2}
\end{aligned}
$$

Using Assumption 2.5. we arrive at

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t}|z|_{V}^{2} \leq-\left(\bar{\mu}-|h(y)|\left(1+|z|_{V}^{\chi_{3}}\right)\right)|z|_{V}^{2}, \quad|z(0)|=\left|z_{0}\right| \tag{3.22a}
\end{equation*}
$$

with

$$
\begin{align*}
& |h(y)|=h(y):=\widehat{C} \Psi(y) \in L_{\mathrm{loc}}^{\mathrm{r}}\left(\mathbb{R}_{0}, \mathbb{R}\right), \quad \mathfrak{r}:=\frac{2}{\chi_{2}}>1,  \tag{3.22b}\\
& |h(y)|_{L^{\mathrm{r}}\left(\left(s, s+\tau_{y}\right), \mathbb{R}\right)}=\widehat{C}|\Psi(y)|_{L^{\mathrm{r}}\left(\left(s, s+\tau_{y}\right), \mathbb{R}\right)} \\
& \quad \leq \widehat{C}\left|1+|y|_{V}^{\chi_{1}}\right|_{L^{\infty}\left(\left(s, s+\tau_{y}\right), \mathbb{R}\right)}\left|1+|y|_{\mathrm{D}(A)}^{\chi_{2}}\right|_{L^{\mathfrak{r}}\left(\left(s, s+\tau_{y}\right), \mathbb{R}\right)} \\
& \quad \leq \widehat{C}\left(1+C_{y}^{\chi_{1}}\right)\left(\tau_{y}^{\frac{1}{\mathbf{r}}}+|y|_{L^{2}\left(\left(s, s+\tau_{y}\right), \mathrm{D}(A)\right)}^{\frac{2}{\mathrm{e}}}\right)=: C_{h} . \tag{3.22c}
\end{align*}
$$

Therefore the norm $\varpi=|z|_{V}^{2}$ satisfies system (3.12), with $h=h(y)$ and $p=\chi_{3} \geq 0$.
In the case $p>0$, we use Proposition 3.10 to conclude that, for any given $\varrho>1$ and $\mu>0$, the norm satisfies

$$
\begin{equation*}
|z(t)|_{V}^{2} \leq \varrho \mathrm{e}^{-\mu(t-s)}|z(s)|_{V}^{2}, \quad \text { for } \quad t \geq s \geq 0, \quad \text { and } \quad|z(0)|_{V}^{2}<R, \quad p>0 \tag{3.23}
\end{equation*}
$$

provided we take $\bar{\mu}$ large enough.
In the case $p=0$, we use Proposition 3.9 to conclude that, for any given $\varrho>1$ and $\mu>0$, the norm satisfies

$$
|z(t)|_{V}^{2} \leq \varrho \mathrm{e}^{-\mu(t-s)}|z(s)|_{V}^{2}, \quad \text { for } \quad t \geq s \geq 0, \quad \text { and } \quad z(0) \in V, \quad p=0
$$

provided we take $\bar{\mu}$ large enough.
In particular (3.23) actually holds for all $p \geq 0$ : we have that

$$
\begin{equation*}
|z(t)|_{V}^{2} \leq \varrho \mathrm{e}^{-\mu(t-s)}|z(s)|_{V}^{2}, \quad \text { for all } \quad t \geq s \geq 0, \quad \text { and all } \quad|z(0)|_{V}^{2}<R \tag{3.24}
\end{equation*}
$$

provided we take $\bar{\mu}$ large enough. That is, provided we take a large enough $S$ and a large enough $\lambda=\lambda(S)>0$. Recalling (3.21), note that $C_{S}$ increases with $S$, and also that, for a fixed $S, C_{\lambda}$ increases with $\lambda$. Finally, note that from Proposition 3.10, we can conclude that it is enough to choose a pair $\left(S_{*}, \lambda_{*}\right) \in \mathbb{N}_{0} \times \mathbb{R}_{0}$ such that

$$
C_{S_{*}} \geq 2 \bar{\mu}, \quad C_{\lambda_{*}} \geq 2 \bar{\mu}, \quad \text { and } \quad \bar{\mu} \geq=\bar{C}_{\left[\mu, \frac{1}{\tau_{y}}, \varrho, \frac{1}{e^{\frac{1}{2}}-1}, \frac{\mathrm{v}+1}{\mathrm{r}-1}, R, C_{h}, \chi_{3}\right.}
$$

For that, using 3.22 , it is enough to choose, firstly $S \geq S^{*}$ with $S_{*}$ in the form

$$
S_{*}=\bar{C}_{\left[R, \mu, \varrho, \frac{1}{e^{\frac{1}{2}}-1}, \frac{1}{\tau_{y}}, \tau_{y}, \chi_{1}, \frac{2+\chi_{2}}{2-\chi_{2}}, \chi_{3}, C_{\mathrm{rc}}, C_{y}\right]},
$$

and subsequently $\lambda \geq \lambda^{*}(S)$ with $\lambda^{*}(S)$ in the form

$$
\left.\lambda^{*}(S)=\bar{C}_{\left[\frac{1}{\alpha_{S}, \ell}, R, \mu, \varrho, \frac{1}{e^{\frac{1}{2}}-1}, \frac{1}{\tau_{y}}, \tau_{y}, \chi_{1}, \frac{2+\chi_{2}}{2-\chi_{2}}, \chi_{3}, C_{\mathrm{rc}}, C_{y}\right.}\right]
$$

We can finish the proof by recalling (3.6) and (3.7).
3.3. Boundedness of the output injection operator. Here we present estimates on the norm of the linear injection operator

$$
\mathfrak{I}_{S}^{[\lambda, \ell]}=-\lambda A^{-1} P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{S}^{\perp}} A^{\ell} P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}} \mathbf{Z}^{W_{S}} \in \mathcal{L}\left(\mathbb{R}^{S_{\sigma}}, H\right), \quad \ell \in[0,2]
$$

Due to 1.3 c we have that $\mathbf{Z}^{W_{S}} \in \mathcal{L}\left(\mathbb{R}^{S_{\sigma}}, \mathcal{W}_{s}\right)$ and we show now that we can write

$$
\left|\mathfrak{I}_{S}^{[\lambda, \ell]}\right|_{\mathcal{L}\left(\mathbb{R}^{\left.S_{\sigma}, H\right)}\right.} \leq \lambda \widetilde{C}_{\mathfrak{J}_{S}}^{[\ell]}\left|\mathbf{Z}^{W_{S}}\right|_{\mathcal{L}\left(\mathbb{R}^{S_{\sigma}}, H\right)},
$$

with

$$
\widetilde{C}_{\mathfrak{J}_{S}}^{[\ell]}:=\left|A^{-1} P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{S}^{\perp}} A^{\ell} P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}}\right|_{\mathcal{L}(H)}<+\infty .
$$

To show such boundedness, we consider the cases $\ell \in[1,2]$ and $\ell \in[0,1]$ separately.
In the case $\ell \in[1,2]$, we have $1-\ell \in[-1,0]$ and $P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{S}}_{S}^{\perp}} \in \mathcal{L}\left(\mathrm{D}\left(A^{1-\ell}\right)\right)$, due to Proposition 3.7. Then we find
$\widetilde{C}_{\mathfrak{J}_{S}}^{\ell \ell]} \leq\left.\left.|\mathbf{1}|_{\mathcal{W}_{S}}\right|_{\mathcal{L}\left(\mathrm{D}\left(A^{1-\ell}\right), \mathrm{D}\left(A^{-1}\right)\right)}\left|P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{S}^{1}}\right|_{\mathcal{L}\left(\mathrm{D}\left(A^{1-\ell}\right)\right)}|\mathbf{1}|_{\widetilde{\mathcal{W}}_{S}}\right|_{\mathcal{L}(H, \mathrm{D}(A))}\left|P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{1}}\right|_{\mathcal{L}(H)}, \quad \ell \in[1,2]$, where we have also used $\left|A^{-1}\right|_{\mathcal{L}\left(\mathrm{D}\left(A^{-1}\right), H\right)}=1=\left|A^{\ell}\right|_{\mathcal{L}\left(\mathrm{D}\left(A^{1}\right), \mathrm{D}\left(A^{1-\ell}\right)\right)}$.

In the case $\ell \in[0,1]$, we have $1-\ell \in[0,1]$ and
$\widetilde{C}_{\widetilde{J}_{S}}^{\ell \ell]} \leq|\mathbf{1}|_{\mathcal{L}(\mathrm{D}(A), H)}\left|P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{S}^{\perp}}\right|_{\mathcal{L}(H)}|\mathbf{1}|_{\mathcal{L}\left(\mathrm{D}\left(A^{1-\ell}\right), H\right)}\left|\mathbf{1}_{\widetilde{\mathcal{W}}_{S}}\right|_{\mathcal{L}(H, \mathrm{D}(A))}\left|P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}}\right|_{\mathcal{L}(H)}, \quad \ell \in[0,1]$.
where we have also used $\left|A^{-1}\right|_{\mathcal{L}(H, \mathrm{D}(A))}=1$.
Next we show that the total "energy" spent by the injection operator is bounded, in case (3.2c) holds true. Indeed, recalling (3.3) we find that

$$
\begin{aligned}
\left|\mathfrak{I}_{S}^{[\lambda, \ell]}(\mathcal{Z} \widehat{y}-w)\right|_{L^{2}\left(\mathbb{R}_{0}, H\right)} & =\left|\mathfrak{I}_{S}^{[\lambda, \ell]} \mathcal{Z} z\right|_{L^{2}\left(\mathbb{R}_{0}, H\right)} \leq \lambda \widetilde{C}_{\mathfrak{J}_{S}}^{[\ell]}\left|P^{\mathcal{W}_{S}} z\right|_{L^{2}\left(\mathbb{R}_{0}, H\right)} \\
& \leq \lambda \widetilde{C}_{\mathfrak{J}_{S}}^{[\ell]}|z|_{L^{2}\left(\mathbb{R}_{0}, H\right)} \leq \lambda \varrho \widetilde{C}_{\mathfrak{J}_{S}}^{[\ell]}|z(0)|_{H}\left(\int_{0}^{+\infty} \mathrm{e}^{-2 \mu t} \mathrm{~d} t\right)^{\frac{1}{2}}
\end{aligned}
$$

which leads us to $\left|\mathfrak{I}_{S}^{[\lambda, \ell]}(\mathcal{Z} \widehat{y}-w)\right|_{L^{2}\left(\mathbb{R}_{0}, H\right)} \leq \lambda \varrho(2 \mu)^{-\frac{1}{2}} \widetilde{C}_{\mathfrak{J}_{S}}^{[\ell]}|z(0)|_{H}$.
3.4. On the existence and uniqueness of solutions for the error. The estimates in Section 3.2 will also hold for Galerkin approximations of system (3.2) as

$$
\begin{align*}
& \dot{z}^{N}+A z^{N}+P_{E_{N}} A_{\mathrm{rc}}(t) z^{N}+P_{E_{N}} \mathfrak{N}_{y}\left(t, z^{N}\right)=P_{E_{N}} \mathfrak{I}_{S}^{[\lambda, \ell]} \mathcal{Z} z^{N}, \quad t \geq 0,  \tag{3.25a}\\
& z^{N}(0)=P_{E_{N}} z_{0} \in V \tag{3.25b}
\end{align*}
$$

where $P_{E_{N}} \in \mathcal{L}(H)$ is the orthogonal projection in $H$ onto the space $E_{N}:=\operatorname{span}\left\{e_{n} \mid\right.$ $1 \leq n \leq N\}$ spanned by the first eigenfunctions of $A$.

Let us fix $\varrho>1, \mu>0$, and $s>0$. We may repeat the estimates in Section 3.2 and arrive to the analogous of 3.22 and 3.24,

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t}\left|z^{N}\right|_{V}^{2} \leq-\left(\bar{\mu}-|h(y)|\left(1+\left|z^{N}\right|_{V}^{\chi_{3}}\right)\right)\left|z^{N}\right|_{V}^{2}, \tag{3.26a}
\end{equation*}
$$

$$
\begin{equation*}
\left|z^{N}(t)\right|_{V}^{2} \leq \varrho \mathrm{e}^{-\mu(t-s)}\left|z^{N}(s)\right|_{V}^{2}, \quad \text { for all } \quad t \geq s \geq 0, \quad\left|z_{0}\right|_{V}^{2}<R \tag{3.26b}
\end{equation*}
$$

provided we take a large enough $S$ and a large enough $\lambda>0$.
Note that $\bar{\mu}, h(y)$, and $\chi_{3}$ are independent of $N$, and that $\left|P_{E_{N}} z_{0}\right|_{V}^{2} \leq\left|z_{0}\right|_{V}^{2}<R$. Hence, $S$ and $\lambda$ can be taken independent of $N$. From 3.26b and 3.25a it follows

$$
\left|z^{N}\right|_{W((0, s), \mathrm{D}(A), H)}^{2}=\left|z^{N}\right|_{L^{2}((0, s), \mathrm{D}(A))}^{2}+\left|\dot{z}^{N}\right|_{L^{2}((0, s), H)}^{2} \leq C
$$

with $C$ independent of $N$. Indeed, proceeding as in [20, Sect. 4.3], multiplying the equation 3.25a by $A z^{N}$,

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} t}\left|z^{N}\right|_{V}^{2} \leq-2\left|z^{N}\right|_{\mathrm{D}(A)}^{2}+2 C_{\mathrm{rc}}\left|z^{N}\right|_{V}\left|z^{N}\right|_{\mathrm{D}(A)}+2\left|\mathfrak{I}_{S}^{[\lambda, \ell]} \mathcal{Z} z^{N}\right|_{H}\left|z^{N}\right|_{\mathrm{D}(A)} \\
&+2\left|\mathfrak{N}_{y}\left(t, z^{N}\right)\right|_{H}\left|z^{N}\right|_{\mathrm{D}(A)} \\
& \leq-\left|z^{N}\right|_{\mathrm{D}(A)}^{2}+3 C_{\mathrm{rc}}^{2}\left|z^{N}\right|_{V}^{2}+3\left|\mathfrak{S}_{S}^{[\lambda, \ell]} \mathcal{Z}\right|_{\mathcal{L}(H)}^{2}\left|z^{N}\right|_{H}^{2}+\widehat{C} \Phi(y)\left(1+\left|z^{N}\right|_{V}^{\chi_{3}}\right)\left|z^{N}\right|_{V}^{2}
\end{aligned}
$$

where we used (3.19) with $\gamma_{3}=\frac{1}{3}$. By 3.26b),

$$
\frac{\mathrm{d}}{\mathrm{~d} t}\left|z^{N}\right|_{V}^{2} \leq-\left|z^{N}\right|_{\mathrm{D}(A)}^{2}+C_{3}
$$

and, after integration,

$$
\left|z^{N}(s)\right|_{V}^{2}+\left|z^{N}\right|_{L^{2}((0, s), \mathrm{D}(A))}^{2} \leq\left|z^{N}(0)\right|_{V}^{2}+s C_{3}
$$

Therefore $\left|z^{N}\right|_{L^{2}((0, s), \mathrm{D}(A))}^{2} \leq C_{4}$, with $C_{4}$ independent of $N$. Using now 3.25a), it follows that $\left|\dot{z}^{N}\right|_{L^{2}((0, s), H)}^{2} \leq C_{5}$, with $C_{5}$ independent of $N$. Hence, there exists a weak limit $z^{\infty} \in W((0, s), \mathrm{D}(A), H)$ so that

$$
z^{N} \underset{L^{2}((0, s), \mathrm{D}(A))}{ } z^{\infty} \quad \text { and } \quad \dot{z}^{N} \xrightarrow[L^{2}((0, s), H)]{ } \dot{z}^{\infty}
$$

Clearly for the linear terms we have

$$
\Phi^{N}:=A z^{N}+A_{\mathrm{rc}}(t) z^{N}-\Im_{S}^{[\lambda, \ell]} \mathcal{Z} z^{N} \xrightarrow[L^{2}((0, s), H)]{ } A z^{\infty}+A_{\mathrm{rc}}(t) z^{\infty}=: \Phi^{\infty}
$$

from which we can derive

$$
A z^{N}+P_{E_{N}} A_{\mathrm{rc}}(t) z^{N}-P_{E_{N}} \mathfrak{I}_{S}^{[\lambda, \ell]} \mathcal{Z} z^{N} \xrightarrow[L^{2}((0, s), H)]{ } A z^{\infty}+A_{\mathrm{rc}}(t) z^{\infty}
$$

due to the facts that $A z^{N}=P_{E_{N}} A z^{N}$, and that for all $h \in L^{2}((0, s), H)$,

$$
\left(P_{E_{N}} \Phi^{N}, h\right)_{L^{2}((0, s), H)}=\left(\Phi^{N}, h\right)_{L^{2}((0, s), H)}-\left(\Phi^{N},\left(1-P_{E_{N}}\right) h\right)_{L^{2}((0, s), H)}
$$

which gives us

$$
\lim _{N \rightarrow+\infty}\left|\left(P_{E_{N}} \Phi^{N}, h\right)_{L^{2}((0, s), H)}\right|_{\mathbb{R}} \leq \lim _{N \rightarrow+\infty}\left|\Phi^{N}\right|_{L^{2}((0, s), H)}\left|\left(1-P_{E_{N}}\right) h\right|_{L^{2}((0, s), H)},
$$

since $\left|\Phi^{N}\right|_{L^{2}((0, s), H)}$ is bounded and $\left|\left(1-P_{E_{N}}\right) h\right|_{\mathcal{L}(H)} \rightarrow 0$. Concerning the existence, it remains to prove that, that the nonlinear term also converges weakly. Actually we can show that it converges strongly

$$
\begin{equation*}
P_{E_{N}} \mathfrak{N}_{y}\left(t, z^{N}\right) \xrightarrow[L^{2}((0, s), H)]{ } \mathfrak{N}_{y}\left(t, z^{\infty}\right) \tag{3.27}
\end{equation*}
$$

In order to show (3.27) we follow arguments from [20, Sect. 4.3]. From Assumption 2.4 we have that

$$
\begin{aligned}
& \left|\mathfrak{N}_{y}\left(t, z^{N}\right)-\mathfrak{N}_{y}\left(t, z^{\infty}\right)\right|_{H}=\left|\mathcal{N}\left(t, y+z^{N}\right)-\mathcal{N}\left(t, y+z^{\infty}\right)\right|_{H} \\
& \leq C_{\mathcal{N}} \sum_{j=1}^{n}\left(\left|y+z^{N}\right|_{V}^{\zeta_{1 j}}\left|y+z^{N}\right|_{\mathrm{D}(A)}^{\zeta_{2 j}}+\left|y+z^{\infty}\right|_{V}^{\zeta_{1 j}}\left|y+z^{\infty}\right|_{\mathrm{D}(A)}^{\zeta_{2 j}}\right)\left|d^{N}\right|_{V}^{\delta_{1 j}}\left|d^{N}\right|_{\mathrm{D}(A)}^{\delta_{2 j}} \\
& =C_{\mathcal{N}} \sum_{j=1}^{n} \sum_{k=1}^{2}\left|w_{k}\right|_{V}^{\zeta_{1 j}}\left|w_{k}\right|_{\mathrm{D}(A)}^{\zeta_{2 j}}\left|d^{N}\right|_{V}^{\delta_{1 j}}\left|d^{N}\right|_{\mathrm{D}(A)}^{\delta_{2 j}}
\end{aligned}
$$

with $d^{N}:=z^{N}-z^{\infty}, w_{1}:=y+z^{N}$, and $w_{2}:=y+z^{\infty}$. Hence we arrive at

$$
\begin{aligned}
& \left|\mathfrak{N}_{y}\left(t, z^{N}\right)-\mathfrak{N}_{y}\left(t, z^{\infty}\right)\right|_{H} \\
& \leq\left.\left.\left.\left. C_{\mathcal{N}} \sum_{j=1}^{n}\left|\left(\sum_{k=1}^{2}\left|w_{k}\right|_{V}^{\zeta_{1 j}}\left|w_{k}\right|_{\mathrm{D}(A)}^{\zeta_{2 j}}\right)\right| d^{N}\right|_{\mathrm{D}(A)} ^{\delta_{2 j}}\right|_{L^{\overline{\zeta_{2 j}+\delta_{2 j}}\left(\mathcal{J}_{s}, \mathbb{R}\right)}}| | d^{N}\right|_{V} ^{\delta_{1 j}}\right|_{L^{\left.\frac{\zeta_{2 j}-\delta_{2 j}}{1-\mathcal{J}_{s}, \mathbb{R}}\right)}},
\end{aligned}
$$

whose right-hand side is similar to an expression we find in [20, Sect. 4.3]. Thus, we can repeat the arguments in [20] to conclude that

$$
\mathfrak{N}_{y}\left(t, z^{N}\right) \xrightarrow[L^{2}((0, s), H)]{ } \mathfrak{N}_{y}\left(t, z^{\infty}\right),
$$

from which we can derive (3.27), due to

$$
\begin{aligned}
& \left|P_{E_{N}} \mathfrak{N}_{y}\left(t, z^{N}\right)-\mathfrak{N}_{y}\left(t, z^{\infty}\right)\right|_{L^{2}((0, s), H)}^{2} \\
& \quad=\left|P_{E_{N}}\left(\mathfrak{N}_{y}\left(t, z^{N}\right)-\mathfrak{N}_{y}\left(t, z^{\infty}\right)\right)\right|_{L^{2}((0, s), H)}^{2}+\left|\left(1-P_{E_{N}}\right) \mathfrak{N}_{y}\left(t, z^{\infty}\right)\right|_{L^{2}((0, s), H)}^{2} \\
& \quad \quad+2\left(P_{E_{N}}\left(\mathfrak{N}_{y}\left(t, z^{N}\right)-\mathfrak{N}_{y}\left(t, z^{\infty}\right)\right),\left(1-P_{E_{N}}\right) \mathfrak{N}_{y}\left(t, z^{\infty}\right)\right)_{L^{2}((0, s), H)}, \\
& \left|P_{E_{N}}\left(\mathfrak{N}_{y}\left(t, z^{N}\right)-\mathfrak{N}_{y}\left(t, z^{\infty}\right)\right)\right|_{L^{2}((0, s), H)}^{2} \leq\left|\left(\mathfrak{N}_{y}\left(t, z^{N}\right)-\mathfrak{N}_{y}\left(t, z^{\infty}\right)\right)\right|_{L^{2}((0, s), H)}^{2},
\end{aligned}
$$

which imply

$$
\begin{aligned}
& \lim _{N \rightarrow+\infty}\left|P_{E_{N}} \mathfrak{N}_{y}\left(t, z^{N}\right)-\mathfrak{N}_{y}\left(t, z^{\infty}\right)\right|_{L^{2}((0, s), H)}^{2} \\
& \quad=\lim _{N \rightarrow+\infty} 2\left(P_{E_{N}}\left(\mathfrak{N}_{y}\left(t, z^{N}\right)-\mathfrak{N}_{y}\left(t, z^{\infty}\right)\right),\left(1-P_{E_{N}}\right) \mathfrak{N}_{y}\left(t, z^{\infty}\right)\right)_{L^{2}((0, s), H)} \\
& \quad \leq \lim _{N \rightarrow+\infty} 2\left|\left(\mathfrak{N}_{y}\left(t, z^{N}\right)-\mathfrak{N}_{y}\left(t, z^{\infty}\right)\right)\right|_{L^{2}((0, s), H)}\left|\left(1-P_{E_{N}}\right) \mathfrak{N}_{y}\left(t, z^{\infty}\right)\right|_{L^{2}((0, s), H)} \\
& \quad=0 .
\end{aligned}
$$

Therefore $z^{\infty}$ solves system (3.2).
Finally, we show the uniqueness of the solution of system 3.2 in $W((0, s), \mathrm{D}(A), H)$. For an arbitrary solution $z$ in $W((0, s), \mathrm{D}(A), H), z(0)=z_{0}$, for $G:=z-z^{\infty}$ we find

$$
\dot{G}+A G+A_{\mathrm{rc}} G+\mathfrak{N}_{y}(z)-\mathfrak{N}_{y}\left(z^{\infty}\right)=\mathfrak{I}_{S}^{[\lambda, \ell]} \mathcal{Z} G, \quad G(0)=0
$$

Observe also that $\mathfrak{N}_{y}(z)-\mathfrak{N}_{y}\left(z^{\infty}\right)=\mathcal{N}(t, y+z)-\mathcal{N}\left(t, y+z^{\infty}\right)$. Again we can repeat the argument in [20, Sect. 4.3], by Assumption 2.4 to conclude that, with $z_{1}=y+z$ and $z_{2}=y+z^{\infty}$,

$$
\begin{aligned}
& 2\left(\left(\mathcal{N}\left(t, z_{1}\right)-\mathcal{N}\left(t, z_{2}\right)\right), A G\right)_{H} \leq|G|_{\mathrm{D}(A)}^{2}+\Phi(t)|G|_{V}^{2}, \\
& \Phi(t):=\bar{C}_{\mathcal{N} 1} \sum_{j=1}^{n}\left(\left|z_{1}\right|_{V}^{\frac{2 \zeta_{1 j}}{1-\delta_{2 j}-\zeta_{2 j}}}+\left|z_{2}\right|_{V}^{\frac{2 \zeta_{1 j}}{1-\delta_{2 j}-\zeta_{2 j}}}+\left|z_{1}\right|_{\mathrm{D}(A)}^{2}+\left|z_{2}\right|_{\mathrm{D}(A)}^{2}\right)|G|_{V}^{\frac{2 \delta_{1 j}}{1-\delta_{2 j}}-2} .
\end{aligned}
$$

By using Assumption 2.3 and the Young inequality, we find

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t}|G|_{V}^{2} & \leq-2|G|_{\mathrm{D}(A)}^{2}+\Phi(t)|G|_{V}^{2}+2 C_{\mathrm{rc}}^{2}|G|_{V}^{2}+2\left|\mathfrak{I}_{S}^{[\lambda, \ell]} \mathcal{Z}\right|_{\mathcal{L}(H)}^{2}|G|_{V}^{2}+2|G|_{\mathrm{D}(A)}^{2} \\
& \leq \Phi_{2}(t)|G|_{V}^{2}
\end{aligned}
$$

with $\Phi_{2}(t):=2 C_{\mathrm{rc}}^{2}+2\left|\mathfrak{J}_{S}^{[\lambda, \ell]} \mathcal{Z}\right|_{\mathcal{L}(H)}^{2}+\Phi(t)$. From $z_{1}=y+z$ and $z_{2}=y+z^{\infty}$, Assumption 2.5, and $\left\{z_{1}, z_{2}\right\} \subset \mathcal{C}([0, s], V) \bigcap L^{2}((0, s), \mathrm{D}(A))$, we see that $\Phi_{2}$ is integrable on $(0, s)$. Hence, by the Gronwall inequality,

$$
|G(t)|_{V}^{2} \leq \mathrm{e}^{\int_{0}^{t} \Phi_{2}(\tau) \mathrm{d} \tau}|G(0)|_{V}^{2}=0, \quad \text { for all } \quad t \in[0, s]
$$

That is, $G=0$ and $z=z^{\infty}+G=z^{\infty}$. We have shown the uniqueness of the solution for $\sqrt[3.2]{ }$ in $W((0, s), \mathrm{D}(A), H)$, for arbitrary $s>0$. In other words, the solution for 3.2 is unique in $W_{\text {loc }}\left(\mathbb{R}_{0}, \mathrm{D}(A), H\right) \supset W\left(\mathbb{R}_{0}, \mathrm{D}(A), H\right)$.
3.5. On the existence and uniqueness of solutions for systems 1.1) and (1.3). Proceeding as in Section 3.4, see also [20, Sect. 4.3], we can show that the solution $y$ for system (1.1), assumed in Assumption 2.5 to exist in $W_{\text {loc }}\left(\mathbb{R}_{0}, \mathrm{D}(A), H\right)$, is unique. Thus from Section 3.4 the solution $z$, given by Theorem 3.1 for the error dynamics, is also unique. Consequently, the solution $\widehat{y}=y+z \in W\left(\mathbb{R}_{0}, \mathrm{D}(A), H\right)$ for 1.3) exists and is unique.

## 4. Parabolic equations evolving in rectangular domains

In order to apply Theorem 3.1 to the case of scalar parabolic equations, it is enough to show that our Assumptions 2.1 2.6 are satisfied, for the operators defined as in Section 1.4. Assumptions 2.1 2.2 are satisfied with $A=-\nu \Delta+1$. Assumption 2.3 is satisfied with $A_{\mathrm{rc}}=a \mathbf{1}+b \cdot \nabla \mathbf{1} \in L^{\infty}\left(\mathbb{R}_{0}, \mathcal{L}(V, H)\right)$, because $a$ and $b$ are both essentially bounded, see (1.8). Assumption 2.4 is proven in [20, Sect. 5.2]. Assumption 2.5 will follow for suitable external forces $f$; see discussion in Section 1.2 and Remark 1.6 Assumption 2.8 is satisfied for outputs as in 1.9.

It remains to show the satisfiability of Assumptions 2.6 2.7 . For this purpose we borrow arguments from [18, Sect. 4] and [19, Sect. 6]. We restrict ourselves to the case of rectangular domains $\Omega^{\times}=X_{j=1}^{d}\left(0, L_{j}\right) \in \mathbb{R}^{d}$.

As set of sensors we take the set of indicators functions

$$
\begin{equation*}
W_{S}:=\left\{1_{\omega_{i}} \mid 1 \leq i \leq S_{\sigma}:=(2 S)^{d}\right\}, \tag{4.1a}
\end{equation*}
$$

where the $\omega_{i}$ s are subrectangles

$$
\begin{equation*}
\omega_{i}=\omega_{i, S}=: \stackrel{d}{X}\left(p_{j}^{i, S}, p_{j}^{i, S}+\frac{r L_{j}}{2 S}\right), \quad p_{j}^{i, S}=\frac{(2 j-1) L_{i}}{4 S}-\frac{r L_{i}}{4 S} . \tag{4.1b}
\end{equation*}
$$

as in [18, Sect. 4], these regions are illustrated in Figure 1, for a planar rectangle $\Omega^{\times}=$ $\left(0, L_{2}\right) \times\left(0, L_{2}\right) \in \mathbb{R}^{2}$, where the total volume (area) covered by the sensors is independent of $S$. In the figure such volume is given by $\frac{1}{16} \operatorname{vol}\left(\Omega^{\times}\right)$, which is $6.25 \%$ of the volume of $\Omega^{\times}, r=\frac{1}{4}$.


Figure 1. The sensor supports as in 4.1b). Case $\Omega^{\times} \subset \mathbb{R}^{d}, d=2$.

The choice of the auxiliary set $\widetilde{W}_{S} \subset \mathrm{D}(A)$ is at our disposal. For example, we can take the Cartesian product eigenfunctions of $A$ as in [18, Sect. 4],

$$
\begin{equation*}
\widetilde{W}_{S}=E_{S}:=\left\{e_{\mathbf{i}} \mid \mathbf{i} \in \widehat{\mathbb{S}}^{d}\right\}, \quad e_{\mathbf{i}}(x):=\underset{j=1}{d} e_{\mathbf{i}_{j}}\left(x_{j}\right), \quad \widehat{\mathbb{S}}:=\{1,2, \ldots, 2 S\} ; \tag{4.2a}
\end{equation*}
$$

or, the more ad-hoc functions as in [19, Sect. 6]

$$
\begin{equation*}
\widetilde{W}_{S}=\mathbf{\Phi}_{S}:=\left\{\Phi_{i} \mid 1 \leq i \leq(2 S)^{d}\right\}, \quad \Phi_{i}(x):=\underset{j=1}{\underset{X}{X}} \sin ^{2}\left(S \frac{x_{j}-p_{j}^{i, S}}{L_{j}}\right), \tag{4.2b}
\end{equation*}
$$

or, we could construct and take the functions

$$
\begin{equation*}
\widetilde{W}_{S}=\mathfrak{A}_{S}:=\left\{A^{-2} 1_{\omega_{i}} \mid 1 \leq i \leq(2 S)^{d}\right\} . \tag{4.2c}
\end{equation*}
$$

From [18, Sect. 4] and [19, Sect. 6] we know that Assumption 2.6 is satisfied for both choices in 4.2, with $\sigma(S):=(2 S)^{d}$.

It remains to show the satisfiability of Assumption 2.7.
4.1. Previous related work. In [18, Sect. 5] it has been shown that a Poincaré-like condition as

$$
\begin{equation*}
\lim _{N \rightarrow+\infty} \inf _{Q \in\left(V \cap \mathcal{O}_{N^{d}}\right)} \frac{|Q\{0\}|_{V}^{2}}{|Q|_{H}^{2}}=+\infty . \tag{4.3}
\end{equation*}
$$

is satisfied for the sensors as indicator functions of the regions

$$
\omega_{i}=\omega_{i, N}=: \stackrel{d}{\underset{j=1}{X}}\left(p_{j}^{i, N}, p_{j}^{i, N}+\frac{r L_{j}}{N}\right), \quad p_{j}^{i, N}=\frac{(2 j-1) L_{i}}{2 N}-\frac{r L_{i}}{2 N} .
$$

Here we prove that the analogous condition in Assumption 2.7 is also satisfied for the subsequence of sets of sensors as in 4.1b.

The proof of 4.3) is given for sensors constructed as in Figure 2, with regions


Figure 2. The sensor supports as in 4.1b. Case $\Omega^{\times} \subset \mathbb{R}^{d}, d=2$.

The proof in [18, Sect. 4] takes the case of $N=1$, corresponding to 1 sensor, as a reference and is based on the observation that the positioning of the actuators in 4.1) gives us a partition of $\Omega^{\times}=\bigcup_{\mathbf{i} \in \widehat{N}^{d}} \Re_{\mathbf{i}}, \widehat{N}=\{1,2, \ldots, N\}$ into rectangles $\Re_{\mathbf{i}}$ which are rescaled copies of the rectangle corresponding to the case of $1^{d}=1$ sensor $1_{\omega \times}=$ $1_{\omega_{(1,1, \ldots, 1), 1}^{\times}}$, with the rescaling factor $N^{-1}$; see one of these copies highlighted, in Figure 2
at the bottom-right corner of the case $N=6$. Then, the Poincaré constant in (4.3) is shown to satisfy, for $N>1$,

$$
\begin{equation*}
\inf _{Q \in\left(V \cap \mathcal{O}_{N^{d}}^{\perp}\right) \backslash\{0\}} \frac{|Q|_{V}^{2}}{|Q|_{H}^{2}} \geq\left(\nu N^{2} D_{0} C_{0}+1\right), \quad D_{0}:=\inf _{Q \in\left(V \cap \mathcal{O}_{\frac{1}{1}}^{\perp}\right) \backslash\{0\}} \frac{|Q|_{V}^{2}}{|Q|_{H}^{2}}, \tag{4.4a}
\end{equation*}
$$

where $D_{0}$ is the Poincaré constant in 4.3), in $\Omega^{\times}$, for the case of 1 sensor. Further $C_{0}$ is a constant satisfying, in the case $N=1$,

$$
\begin{equation*}
C_{0}|h|_{V}^{2} \leq\left|\nabla_{x}(h)\right|_{L^{2}\left(\Omega^{\times}\right)^{d}}^{2}+\left|\left(h, 1_{\omega^{\times}}\right)\right|_{\mathbb{R}}^{2}, \quad \text { for all } \quad h \in H^{1}(\Omega) \tag{4.4b}
\end{equation*}
$$

4.2. Satisfiability of Assumption 2.7. We have mentioned that the proof in 18 , Sect. 4] uses the case $N=1$ as a reference to derive 4.4b. Here we use the case $S=1$, corresponding to $2^{d}$ sensors, as a reference to derive the analogous estimate required in Assumption 2.7 .

Lemma 4.1. For $S=1$ we have an analogous version of 4.4b as

$$
\begin{equation*}
C_{0}|h|_{\mathrm{D}(A)}^{2} \leq\left|\nabla_{x}^{2} h\right|_{L^{2}(\Omega \times)^{d^{2}}}^{2}+\sum_{\mathbf{j} \in\{1,2\}^{d}}\left|\left(h, 1_{\omega_{\mathbf{j}, 1}^{\times}}\right)\right|_{\mathbb{R}}^{2}, \quad \text { for all } \quad h \in H^{2}(\Omega) \tag{4.5}
\end{equation*}
$$

where $\left\{\omega_{\mathbf{j}, 1}^{\times} \mid \mathbf{j} \in\{1,2\}^{d}\right\}=\left\{\omega_{i, S} \mid i \in\left\{1,2, \ldots, 2^{d}\right\}\right\}$.
For the proof we will need some auxiliary results.
Note that, the number of sensors is given by $S_{\sigma}=(2 S)^{d}$, thus $2^{d}$ for $S=1$.
Above, $\nabla_{x}^{2}$ stands for second order derivatives,

$$
\begin{aligned}
\left|\nabla_{x}^{2} h\right|_{L^{2}(\Omega \times)^{2}} & :=\left(\sum_{\mathbf{k} \in K_{d, 2}}\left|\frac{\partial^{\mathbf{k}_{1}} \partial^{\mathbf{k}_{2}} \ldots \partial^{\mathbf{k}_{d}} h}{\partial x_{1}^{k_{1}} \partial x_{2}^{k_{2}} \ldots \partial x_{d}^{k_{d}}}\right|_{L^{2}\left(\Omega^{\times}\right)}^{2}\right)^{\frac{1}{2}}, \\
K_{d, 2} & :=\left\{\mathbf{k} \in\{0,1,2\}^{d} \mid \sum_{s=1}^{d} \mathbf{k}_{s}=2\right\} .
\end{aligned}
$$

Note that the locations as in 4.1b induce a partition of $\Omega^{\times}$with $S^{d}$ rescaled copies of the case $S=1$. See Figure 1, case $S=3$, where a rescaled copy of the case $S=1$ is highlighted at the bottom-right corner.

The following lemma can be found in [13, Ch. 1, Sect. 1.7, Thm. 1.6], written in a slightly different way.
Lemma 4.2. Let $\mathbb{P}_{\times, 1}:=\left\{c_{0}+\sum_{j=1}^{d} a_{j} x_{j} \mid c_{0} \in \mathbb{R}, a=\left(a_{1}, a_{2}, \ldots, a_{d}\right) \in \mathbb{R}^{d}\right\}$ be the set of polynomials of degree at most 1 defined in $\Omega^{\times}$, and consider its orthogonal in $H^{2}\left(\Omega^{\times}\right)$, $\mathbb{P}_{\times, 1}^{\perp, H^{2}}:=\left\{h \in H^{2}\left(\Omega^{\times}\right) \mid(h, p)_{H^{2}\left(\Omega^{\times}\right)}=0\right.$, for all $\left.p \in \mathbb{P}_{\times, 1}\right\}$. Then there exists a constant $C>0$ such that

$$
|h|_{H^{2}\left(\Omega^{\times}\right)}^{2} \leq C\left|\nabla_{x}^{2} h\right|_{L^{2}\left(\Omega^{\times}\right)^{d^{2}}}^{2}, \quad \text { for all } \quad h \in \mathbb{P}_{\times, 1}^{\perp, H^{2}}
$$

Proposition 4.3. Let $\mathbf{J}_{d, 2}:=\left\{\mathbf{j} \in\{1,2\}^{d} \mid \sum_{j=1}^{d} \mathbf{j}_{j} \leq d+1\right\}$. Then, the seminorm $\mathfrak{S}(\cdot):=\left(\sum_{\mathbf{j} \in \mathbf{J}_{d, 2}}\left|\left(\cdot, 1_{\omega_{\mathbf{j}, 1}^{\times}}\right)\right|_{\mathbb{R}}^{2}\right)^{\frac{1}{2}}$ is a norm in $\mathbb{P}_{\times, 1}$.

The proof is given in Section A.7. Note that $\mathbf{J}_{d, 2} \subset \mathbb{S}^{d}$ has cardinality $\# \mathbf{J}_{d, 2}=d+1=$ $\operatorname{dim} \mathbb{P}_{\times, 1}$.

Corollary 4.4. The usual norm $|\cdot|_{H^{2}\left(\Omega^{\times}\right)}$, in $H^{2}\left(\Omega^{\times}\right)$, is equivalent to the norm

$$
\left(\left|\nabla_{x}^{2} \cdot\right|_{L^{2}\left(\Omega^{\times}\right) d^{2}}^{2}+\sum_{\mathbf{j} \in \mathbf{J}_{d, 2}}\left|\left(\cdot, 1_{\omega_{\mathbf{j}, 1}^{\times}}\right)_{L^{2}\left(\Omega^{\times}\right)}\right|_{\mathbb{R}}^{2}\right)^{\frac{1}{2}}
$$

Proof. The proof is standard and can be done by repeating the arguments from the proofs in [13, Ch. 1, Sect. 1.7, Thms. 1.8 and 1.10]. Indeed, it is enough to observe that we have $H^{2}\left(\Omega^{\times}\right)=\mathbb{P}_{\times, 1}^{\perp, H^{2}} \oplus \mathbb{P}_{\times, 1}$, and use Proposition 4.3 .

Proof of Lemma 4.1. Let $h \in H^{2}\left(\Omega^{\times}\right)$. By Corollary 4.4 there exists $C_{1}>0$ such that

$$
\begin{aligned}
\left|\nabla_{x}^{2} h\right|_{L^{2}\left(\Omega^{\times}\right)^{d^{2}}}^{2}+\sum_{\mathbf{j} \in\{1,2\}^{2}}\left|\left(h, 1_{\omega_{\mathbf{j}, 1}^{\times}}\right)\right|_{\mathbb{R}} & \geq\left|\nabla_{x}^{2} h\right|_{L^{2}\left(\Omega^{\times}\right)^{d^{2}}}^{2}+\sum_{\mathbf{j} \in \mathbf{J}_{d, 2}}\left|\left(h, 1_{\omega_{\mathbf{j}, 1}^{\times}}\right)\right|_{\mathbb{R}}^{2} \\
& \geq C_{1}|h|_{H^{2}\left(\Omega^{\times}\right)}^{2} \geq C_{1} C_{2}|h|_{\mathrm{D}(A)}^{2}
\end{aligned}
$$

where $C_{2}$ is a constant satisfying $|h|_{\mathrm{D}(A)}^{2} \leq C_{2}^{-1}|h|_{H^{2}\left(\Omega^{\times}\right)}^{2}$. That is, we may take $C_{0}=$ $C_{1} C_{2}$ in 4.5). Note that such $C_{2}$ can be found as $|h|_{\mathrm{D}(A)}^{2}=|-\nu \Delta h+h|_{L^{2}\left(\Omega^{\times}\right)}^{2} \leq$ $2\left(|-\nu \Delta h|_{L^{2}\left(\Omega^{\times}\right)}^{2}+|h|_{L^{2}\left(\Omega^{\times}\right)}^{2}\right) \leq 2\left(\nu^{2} d|h|_{H^{2}\left(\Omega^{\times}\right)}^{2}+|h|_{H^{2}\left(\Omega^{\times}\right)}^{2}\right)$, that is, we may take $C_{2}^{-1}=$ $2\left(\nu^{2} d+1\right)$.

Proceeding as in [18, Sect. 4], we observe that for a suitable translation $\mathfrak{T}_{\mathbf{i}}$, the injective affine transformation

$$
\Phi_{\mathbf{i}}: \Omega^{\times} \rightarrow \Re_{\mathbf{i}}, \quad x \mapsto z^{\mathbf{i}}:=\frac{x}{S}+\mathfrak{T}_{\mathbf{i}}, \quad \mathbf{i} \in \widehat{\mathbb{S}}^{d}
$$

maps $\Omega^{\times}$onto $\mathfrak{R}_{\mathbf{i}}$, and the sensor regions $\left\{\omega_{\mathbf{j}, 1}^{\times} \mid \mathbf{j} \in\{1,2\}^{d}\right\}$ onto rescaled sensor regions $\left\{\omega_{\mathbf{i}(\mathbf{j}), S}^{\times} \subset \mathfrak{R}_{\mathbf{i}} \mid \mathbf{j} \in\{1,2\}^{d}\right\}$ in the corresponding copy $\mathfrak{R}_{\mathbf{i}}$.

$$
\Phi_{\mathbf{i}}\left(\Omega^{\times}\right)=\mathfrak{R}_{\mathbf{i}}, \quad \Phi\left(\omega_{\mathbf{j}, S}^{\times}\right)=\omega_{\mathbf{i}(\mathbf{j}), S}^{\times} .
$$

From $\mathrm{d} x_{n}=S \mathrm{~d} z_{n}^{\mathbf{i}}$ and $\frac{\partial}{\partial x_{n}}=\frac{1}{S} \frac{\partial}{\partial z_{n}^{\mathbf{i}}}$, for $\mathbf{k} \in K_{d, 2}$ we have $\frac{\partial^{\mathbf{k}_{1}} \partial^{\mathbf{k}_{2}} \ldots \partial^{\mathbf{k}_{d}} Q(x)}{\partial x_{1}^{\mathbf{k}_{1}} \partial x_{2}^{\mathbf{k}_{2}} \ldots \partial x_{d}^{\mathbf{k}_{d}}}=$ $\frac{1}{S^{2}} \frac{\partial^{\mathbf{k}_{1}} \partial^{\mathbf{k}_{2}} \ldots \partial^{\mathbf{k}_{d}} Q(z)}{\partial z_{1}^{\mathbf{k}_{1}} \partial z_{2}^{\mathbf{k}_{\mathbf{2}}} \ldots \partial z_{d}^{\mathbf{i} \mathbf{k}_{d}}}$. Further for $Q \in \mathrm{D}(A) \bigcap \mathcal{O}_{2^{d}}^{\perp}$ we find

$$
\begin{aligned}
& \int_{\Omega^{\times}} \sum_{\mathbf{k} \in K_{d, 2}}^{d}\left(\frac{\partial^{\mathbf{k}_{1}} \partial^{\mathbf{k}_{\mathbf{2}}} \ldots \partial^{\mathbf{k}_{d}} Q(x)}{\partial x_{1}^{\mathbf{k}_{1}} \partial x_{2}^{\mathbf{k}_{2}} \ldots \partial x_{d}^{\mathbf{k}_{d}}}\right)^{2} \mathrm{~d} x=\int_{\mathfrak{R}_{\mathbf{i}}} \sum_{\mathbf{k} \in K_{d, 2}}^{d}\left(\frac{1}{S}\right)^{4}\left(\frac{\partial^{\mathbf{k}_{\mathbf{1}}} \partial^{\mathbf{k}_{2}} \ldots \partial^{\mathbf{k}_{d}} Q(z)}{\left.\partial z_{1}^{\mathbf{\mathbf { k } _ { 1 }} \partial z_{2}^{\mathbf{i}} \mathbf{k}_{2} \ldots \partial z_{d}^{i \mathbf{k}_{d}}}\right)^{2} S^{d} \mathrm{~d} z^{\mathbf{i}},}\right. \\
& \int_{\Omega^{\times}} Q(x)^{2} \mathrm{~d} x=\int_{\mathfrak{R}_{\mathbf{i}}} Q\left(\Phi_{\mathbf{i}}^{-1}(z)\right)^{2} S^{d} \mathrm{~d} z^{\mathbf{i}}, \\
& \int_{\omega} g(x) Q(x) \mathrm{d} x=\int_{\Phi(\omega)} g\left(\Phi_{\mathbf{i}}^{-1}(z)\right) Q\left(\Phi_{\mathbf{i}}^{-1}(z)\right) S^{d} \mathrm{~d} z^{\mathbf{i}}, \text { for all } g \in L^{2}(\omega), \quad \omega \subseteq \Omega^{\times} .
\end{aligned}
$$

which give us,

$$
\begin{aligned}
\left|\nabla_{x}^{2}(Q)\right|_{L^{2}\left(\Omega^{\times}\right)^{2 d}}^{2} & =S^{d-4}\left|\nabla_{z}^{2} Q \circ \Phi_{\mathbf{i}}^{-1}\right|_{L^{2}\left(\Re_{\mathbf{i}}\right)^{d}}^{2}, \\
|Q|_{L^{2}\left(\Omega^{\times}\right)}^{2} & =S^{d}\left|Q \circ \Phi_{\mathbf{i}}^{-1}\right|_{L^{2}\left(\Re_{\mathbf{i}}\right)}^{2}, \\
|Q|_{L^{2}(\omega)}^{2} & =S^{d}\left|Q \circ \Phi_{\mathbf{i}}^{-1}\right|_{L^{2}\left(\Phi_{\mathbf{i}}(\omega)\right)}^{2} .
\end{aligned}
$$

Further, denoting $\left[\omega_{1}^{\times}\right]_{2}:=\left\{1_{\omega_{\mathbf{j}, 1}^{\times}} \mid \mathbf{j} \in\{1,2\}^{d}\right\}$ and $\left[\omega_{\mathbf{i}, S}^{\times}\right]_{2}:=\left\{1_{\omega_{\mathbf{i}(\mathbf{j}), S}^{\times}} \mid \mathbf{j} \in\{1,2\}^{d}\right\}$ and, choosing $g \in\left[\omega_{1}^{\times}\right]_{2}$, we also find

$$
Q \circ \Phi_{\mathbf{i}}^{-1} \in\left[\omega^{\times}\right]_{2}^{\perp} \quad \Longleftrightarrow \quad Q \in\left[\omega_{\mathbf{i}, S}^{\times}\right]_{2}^{\perp} .
$$

Lemma 4.5. For a suitable constant $C>0$, we have

$$
\begin{equation*}
\inf _{Q \in\left(\mathrm{D}(A) \cap \mathcal{O}_{(2 S)^{d}}^{\perp}\right) \backslash\{0\}} \frac{|Q|_{\mathrm{D}(A)}^{2}}{|Q|_{V}^{2}} \geq C D_{0} S^{2}, \quad D_{0}:=\inf _{Q \in\left(\mathrm{D}(A) \cap \mathcal{O}_{2^{d}}^{\perp}\right) \backslash\{0\}} \frac{|Q|_{H^{2}(\Omega)}^{2}}{|Q|_{H^{1}(\Omega)}^{2}} . \tag{4.6}
\end{equation*}
$$

Proof. For arbitrary given $Q \in \mathcal{W}_{S}^{\perp} \bigcap \mathrm{D}(A)$, since the norms $|\cdot|_{\mathrm{D}(A)}^{2}$ and $|\cdot|_{H^{2}\left(\Omega^{\times}\right.}^{2}$ are equivalent (for both Dirichlet and Neumann boundary conditions), and since $\mathfrak{S}(Q)=0$, we find

$$
C_{4}\left|\nabla_{x}^{2} Q\right|_{L^{2}\left(\Omega^{\times}\right)^{d}}^{2} \leq|Q|_{\mathrm{D}(A)}^{2} \geq C_{3}\left|\nabla_{x}^{2} Q\right|_{L^{2}\left(\Omega^{\times}\right)^{d}}^{2}
$$

for suitable constants $C_{3}>0, C_{4}>0$. Furthermore,

$$
\begin{aligned}
|Q|_{\mathrm{D}(A)}^{2} & \geq C_{3}\left|\nabla_{x}^{2} Q\right|_{L^{2}\left(\Omega^{\times}\right)^{d}}^{2}=C_{3} \sum_{\mathbf{i} \in \widehat{\mathbb{S}}^{d}}\left|\nabla_{x}^{2} Q\right|_{L^{2}\left(\Re_{\mathbf{i}}\right)^{d}}^{2} \\
& =C_{3} \sum_{\mathbf{i} \in \widehat{\mathbb{S}}^{d}} S^{4-d}\left|\nabla_{\Phi_{\mathbf{i}}^{-1}(x)}^{2}\left(Q \circ \Phi_{\mathbf{i}}\right)\right|_{L^{2}\left(\Omega^{\times}\right)^{d}}^{2} \geq C_{3} C_{1} S^{4-d} \sum_{\mathbf{i} \in \mathbb{S}^{d}}\left|Q \circ \Phi_{\mathbf{i}}\right|_{H^{2}\left(\Omega^{\times}\right)}^{2} \\
& \geq C_{3} C_{1} D_{0} S^{4-d} \sum_{\mathbf{i} \in \widehat{\mathbb{S}}^{d}}\left|Q \circ \Phi_{\mathbf{i}}\right|_{H^{1}\left(\Omega^{\times}\right)}^{2} \\
& =C_{3} C_{1} D_{0} S^{4-d} \sum_{\mathbf{i} \in \widehat{\mathbb{S}}^{d}}\left(\left|\nabla_{x}^{1} Q \circ \Phi_{\mathbf{i}}\right|_{L^{2}\left(\Omega^{\times}\right)}^{2}+\left|Q \circ \Phi_{\mathbf{i}}\right|_{L^{2}\left(\Omega^{\times}\right)}^{2}\right)
\end{aligned}
$$

with $D_{0}$ as in (4.6). By using the relation

$$
\left|\nabla_{\Phi_{\mathbf{i}}^{-1}(x)}^{1}(Q)\right|_{L^{2}\left(\Omega^{\times}\right)^{d}}^{2}=S^{d-2}\left|\nabla_{x} Q \circ \Phi^{-1}\right|_{L^{2}\left(\mathfrak{R}_{\mathbf{i}}\right)^{d}}^{2}
$$

which we can find in [18, we arrive at

$$
\begin{aligned}
|Q|_{\mathrm{D}(A)}^{2} & \geq C_{3} C_{1} D_{0} S^{4-d} \sum_{\mathbf{i} \in \widehat{\mathbb{S}}^{d}}\left(S^{d-2}\left|\nabla_{x}^{1} Q\right|_{L^{2}\left(\Re_{\mathbf{i}}\right)}^{2}+S^{d}|Q|_{L^{2}\left(\Re_{\mathbf{i}}\right)}^{2}\right) \\
& =C_{3} C_{1} D_{0} S^{2} \sum_{\mathbf{i} \in \widehat{\mathbb{S}}^{d}}\left(\left|\nabla_{x}^{1} Q\right|_{L^{2}\left(\Re_{\mathbf{i}}\right)}^{2}+S^{2}|Q|_{L^{2}\left(\Re_{\mathbf{i}}\right)}^{2}\right) \geq C_{3} C_{1} D_{0} S^{2}|Q|_{H^{1}\left(\Omega^{\times}\right)}^{2} \\
& \geq C_{3} C_{1} C_{5} D_{0} S^{2}|Q|_{V}^{2},
\end{aligned}
$$

which gives us 4.6), with $C=C_{3} C_{1} C_{5}$, and $C_{5}:=\inf _{Q \in V \backslash\{0\}} \frac{|Q|_{H^{1}(\Omega \times)}^{2}}{|Q|_{V}^{2}}$.
Note that (4.6) implies the satisfiability of Assumption 2.7
Note that we have proven the satisfiability of Assumption 2.7 for rectangular domains. We end this section with the following conjecture.

Conjecture 4.6. Assumption 2.7 can be satisfied for smooth domains.
An analogous conjecture has been stated in [18, Sect. 7.3], where we can also find arguments supporting the conjecture.

Finally, we end this section with the following result concerning Remark 3.3 .
Proposition 4.7. For pairwise disjoint sensor rectangular regions $\omega_{i, S}$ as in 4.1b), and auxiliary functions $\widetilde{W}_{S}$ as in 4.2 b , we have that, for any $\theta \in \widetilde{W}_{S} \subset \mathrm{D}(A)$,

$$
|\theta|_{V}^{2}=\left(C_{1} S^{2}+1\right)|\theta|_{H}^{2} \quad \text { and } \quad|\theta|_{\mathrm{D}(A)}^{2}=\left(C_{2} S^{4}+2 C_{1} S^{2}+1\right)|\theta|_{H}^{2},
$$

with $C_{1}=\frac{4 \nu \pi^{2}}{3} \sum_{i=1}^{d} \frac{1}{L_{i}^{2}}$ and $C_{2}=\frac{\nu^{2} 16 \pi^{4}}{3} \sum_{i=1}^{d} \frac{1}{L_{i}^{2}}$.
The proof is given in the Appendix, Section A.8.
As a consequence it follows that the Poincaré-like constant $\underline{\alpha}_{S, \ell}$ in (3.1), satisfies the limit $\lim _{S \rightarrow+\infty} \underline{\alpha}_{S, \ell}=0$, for $\ell \in\{0,1\}$. In these cases it may be necessary to choose firstly $S$ and subsequently $\lambda$ (depending on $S$ ) in Theorem 3.1. see Remark 3.3.

## 5. Numerical simulations

Here we show the results of simulations illustrating the stabilizability result stated in Main Result in the Introduction; see main Theorem 3.1. We consider the following scalar parabolic system as an academic model for the error dynamics; see 1.5).

$$
\begin{gathered}
\frac{\partial}{\partial t} z+(-\nu \Delta+\mathbf{1}) z+a z+b \cdot \nabla z-|z|_{\mathbb{R}}^{3} z+\left(\frac{\partial}{\partial x_{1}} z-2 \frac{\partial}{\partial x_{2}} z\right) z \\
=-\lambda A^{-1} P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{S}^{\perp}} A^{\ell} P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}} \mathbf{Z}^{W_{S}} \mathcal{Z} z, \\
z(0)=z_{0} \in H^{1}(\Omega),\left.\quad \frac{\partial}{\partial \mathbf{n}} z\right|_{\partial \Omega}=0
\end{gathered}
$$

evolving in $V=H^{1}(\Omega)$ under Neumann boundary conditions, where $\Omega=(0,1) \times(0,1) \in$ $\mathbb{R}^{2}$ is the unit square. As parameters we set

$$
\ell=2, \quad \nu=0.1, \quad a=-2+x_{1}-\left|\sin \left(t+x_{1}\right)\right|_{\mathbb{R}}, \quad b=\left[\begin{array}{c}
x_{1}+x_{2} \\
\cos (t) x_{1} x_{2}
\end{array}\right] .
$$

As sensors we take indicator functions $1_{\omega_{i}}=1_{\omega_{i}}(x)$ of rectangular subdomains $\omega_{i} \subset \Omega$ as in Figure 1. Hence, the output $\mathcal{Z} y(t) \in \mathbb{R}^{S_{\sigma}}$ consists of the "averages" of the solution over the same subdomains,

$$
(\mathcal{Z} y(t))_{i}=\left(1_{\omega_{i}}, y(t)\right)_{L^{2}(\Omega)}=\int_{\omega_{i}} y(t), \mathrm{d} \Omega
$$

and the output error is $\mathcal{Z} z(t)=\mathcal{Z} \widehat{y}(t)-\mathcal{Z} y(t) \in \mathbb{R}^{S_{\sigma}}$,

$$
(\mathcal{Z} z(t))_{i}=\left(1_{\omega_{i}}, z(t)\right)_{L^{2}(\Omega)}=\left(1_{\omega_{i}}, \widehat{y}(t)\right)_{L^{2}(\Omega)}-\left(1_{\omega_{i}}, y(t)\right)_{L^{2}(\Omega)} .
$$

Finally, we set the normalized initial condition as

$$
z_{0}=\frac{2-x_{1} x_{2}}{\left|2-x_{1} x_{2}\right|_{V}} \in V .
$$

The number of sensors $S_{\sigma}$ and the parameter $\lambda$, which we know should be both large enough (cf. Main Result in Introduction), will be set later on.

As auxiliary functions we will take the functions in 4.2b).
5.1. Discretization. The following simulations have been performed in matlab and correspond to a piecewise linear (hat functions based) finite element discretization of the equation in the spatial variable. Subsequently, for the time variable and for discrete time instants $t_{j}=k j, j \in \mathbb{N}$ and with time step $k>0$, we use the standard linear approximation for the time derivative, a Crank-Nicolson scheme to approximate the symmetric operator $\mathcal{A}=(-\nu \Delta+\mathbf{1}+a \mathbf{1})$, and a Adams-Bashforth scheme for the remaining terms $\mathcal{R}$, that is, denoting $t_{j}^{*}:=\frac{t_{j}+t_{j+1}}{2}$ we take $\frac{\partial}{\partial t} z\left(t_{j}^{*}\right) \approx \frac{z\left(t_{j+1}\right)-z\left(t_{j}\right)}{k}$, $\mathcal{A}\left(t_{j}^{*}\right) z\left(t_{j}^{*}\right) \approx \frac{\mathcal{A}\left(t_{j}\right) z\left(t_{j}\right)+\mathcal{A}\left(t_{j+1}\right) z\left(t_{j+1}\right)}{2}$ and $\mathcal{R}\left(t_{j}^{*}, z\left(t_{j}^{*}\right)\right) \approx \frac{3 \mathcal{R}\left(t_{j}, z\left(t_{j}\right)\right)-\mathcal{R}\left(t_{j-1}, z\left(t_{j-1}\right)\right)}{2}$.

We will consider the cases where the number $S_{\sigma}$ of sensors belongs to $\{4,9,16\}$. The corresponding triangulations of the spatial domain, used in the simulations, are shown in Figure 3


Figure 3. Locations of sensors in cases $S_{\sigma} \in\{4,9,16\}$.

In the figures below, the symbol "npts $\Omega$ " stands for the number of mesh points in the triangulation of the spatial domain $\Omega$, the symbol " $k$ " stands for the time step, and $T$ stands for the end point of the time interval $[0, T]$ where the simulations have been run in. If the plots in the figures do not include the entire interval $[0, T]$, then it means that the norm $|z|_{V}$ of the error blows up at time $t_{\mathrm{bu}}<T$ near the last plotted time instant.
5.2. Necessity of large $S_{\sigma}$ for error stability. In Figure 4 we see that the free error dynamics (i.e., under no output injection) is blowing up in finite time, namely, at time $t_{\mathrm{bu}} \approx 0.11$. The simulations correspond to the mesh corresponding to 4 sensors. Also in Figure 4 we see that the error norm, for the output injection corresponding to the case of 4 sensors, blows at time $t_{\mathrm{bu}} \approx 9.5$ for $\lambda=0.004$, while it blows up at time $t_{\text {bu }} \approx 10.5$ for larger $\lambda \in\{0.02,0.1,0.5\}$. That is, the blow up time increases due to the the output injection, but such injection is not able to stabilize the error dynamics. In particular, we see that the blow up time seems to converge to a value in the interval $[10.5,11]$ as $\lambda$ increases. Therefore, we can conclude that 4 sensors are likely not able to stabilize the estimation error norm. This confirms the statement of Theorem 3.1 on the necessity of a large enough number of sensors.


Figure 4. The free dynamics and the case of 4 sensors.
In Figure 5 we see that 9 and 16 sensors are able to stabilize the error norm for the considered values of $\lambda$. We also see that, for a fixed number $S_{\sigma}$ of sensors, the exponential stabilization rate increases with $\lambda$, and converges to a bounded value. This means that if we want to achieve a larger stability rate it is not enough to increase $\lambda$; we will need to increase also the number of sensors as stated in Theorem 3.1. This is confirmed in Figure 5 where we see that with 16 sensors we obtain a faster decreasing of the error norm $|z|_{V}$, namely, for $\lambda=0.02$ we find the rates $\mu \approx 5=\frac{1}{2} \frac{150}{15}$ for $S_{\sigma}=9$, and $\mu \approx 11.5 \approx \frac{1}{2} \frac{350}{15}$ for $S_{\sigma}=16$.
5.3. Necessity of large $\lambda$ for error stability. We know that the free error dynamics, with $\lambda=0$, is not stable. Here we show that $\lambda>0$ must be large enough in order to achieve stability of the error dynamics. Indeed in Figure 6 we see that, for small $\lambda$, neither 9 nor 16 sensors are able to stabilize the error dynamics.

The above results show that both $S_{\sigma}$ and $\lambda$ must be taken large enough to achieve the stability of the error dynamics, which agree with the theoretical results.

## 6. FINAL REMARKS

Though the "best" choice of all the parameters involved in the output injection operator $\mathfrak{I}_{S}^{[\lambda, \ell]}$ is not the main focus of this paper. Such choice is (or, may be) important for


Figure 5. The case of 9 and 16 sensors.


Figure 6. The case of small $\lambda$.
applications (e.g., numerical simulations). Here, we just discuss briefly the semiglobal estimatability result presented in this manuscript, from practicability viewpoints, and mention related problems which could be the subject of further investigation.
6.1. On the choice of $S$ and $\lambda$. Let us fix $(\mu, R)$. For a given $\ell$, the estimatability property of the output injection operator $\mathfrak{I}_{S}^{[\lambda, \ell]}$, in Theorem 3.1. depends on the desired exponential decreasing rate $\mu$ and on the upper bound $R$ for the norm of the initial error, simply because the pair $(S, \lambda)$ depends on, and "increases with", $(\mu, R)$. In practice the initial error $z_{0}$ is unknown for us, thus we will not be able to surely choose an appropriate $\mathfrak{I}_{S}^{[\lambda, \ell]}$ stabilizing the nonlinear error dynamics. However, on the other hand, we are sure that it is enough to increase both $S$ and $\lambda$ to find a stabilizing $\Im_{S}^{[\lambda, \ell]}$. Furthermore, the fact that the transient bound $\varrho>1$ will get smaller, for large $S$ and $\lambda$, can be used in applications to decide whether we should (still) increase $S$ and $\lambda$. Namely, we increase $S$ and/or $\lambda$ if (e.g., in simulations) we realize that the error norm is not starting decreasing after a suitable amount of time.

If we knew that the transient bound is $\varrho=1$ then we would know that for large enough $S$ and $\lambda$, the error norm must be strictly decreasing. Hence we would increase $S$ and $\lambda$ if we realize that the norm is not strictly decreasing.
6.2. Strictly decreasing estimate error norm. Let us fix again $(\mu, R)$. We shall see now that we can achieve the optimal transient bound constant $\varrho=1$ when we have $\zeta_{2 i}=0$ for all $i \in\{1,2, \ldots, n\}$ in Assumption 2.4. This case holds for parabolic equations (1.7) in the cases $d \in\{1,2\}$ with $r>1$ and $s \geq 1$, and also in the case $d=3$ with $r \in(1,3]$ and $\widetilde{b}=0$. These facts have been proven in [20, Sects. 5.2 and 5.3]. In particular, in the cases $d \in\{1,2\}$ we can take arbitrary large exponents/degrees for the nonlinearities, $r \in(1,+\infty)$ and $s \in[1,+\infty)$, in 1.7). Note that our simulations have been performed in a two-dimensional domain, and in Figure 5 the estimation error norm is strictly decreasing.

To show that we can take $\varrho=1$ if $\zeta_{2 i}=0, i \in\{1,2, \ldots, n\}$, we observe that in such case we have $\chi_{2}=0$, due to (3.6). Thus we obtain $\Psi(y)=2\left(1+|y|_{V}^{\chi_{1}}\right)$ in (3.19), with $|h(y)|_{L^{\infty}\left(\mathbb{R}_{0}, \mathbb{R}\right)} \leq \widetilde{C}$ and

$$
\frac{\mathrm{d}}{\mathrm{~d} t}|z|_{V}^{2} \leq-\left(\bar{\mu}-|h(y)|\left(1+|z|_{V}^{\chi_{3}}\right)\right)|z|_{V}^{2} \leq-\left(\breve{C}_{1}-\breve{C}_{2}|z|_{V}^{\chi_{3}}\right)|z|_{V}^{2}
$$

in (3.22), with $\breve{C}_{1}:=\bar{\mu}-\widetilde{C}$ and $\breve{C}_{2}=\widetilde{C}$. Recalling (3.21) and the fact that $\bar{\mu}$ can be made arbitrarily large by choosing both $S$ and $\lambda$ large enough, it is clear that for any given $R>0$ and $\mu>0$, we can set both $S$ and $\lambda$ large enough so that $\hat{\mu}:=$ $\breve{C}_{1}-\breve{C}_{2} R^{\frac{\chi_{3}}{2}} \geq \mu$. Then by Proposition 4.3 in [20, we find the following estimate, with transient bound $\varrho=1$,

$$
|z(t)|_{V}^{2} \leq \mathrm{e}^{-\widehat{\mu}(t-s)}|z(s)|_{V}^{2} \leq \mathrm{e}^{-\mu(t-s)}|z(s)|_{V}^{2}, \quad \text { if } \quad|z(0)|_{V}^{2} \leq R
$$

6.3. On the choice of the set of auxiliary functions $\widetilde{W}_{S}$ and $\ell$. The choice of the auxiliary set $\widetilde{W}_{S} \subset \mathrm{D}(A)$ is at our disposal. In Section 4 we have suggested three possible choices, namely, those in 4.2. In Section 5 we have taken only the choice in 4.2b. We did not compare with other possible choices because the "optimal" choice for $W_{S}$ is not the main goal of this paper. However, we must say that, though the operator norm of the oblique projection $P \widetilde{\mathcal{W}}_{S} \mathcal{W}_{S}^{\perp}$ does not play any crucial role in the estimatability result, it plays a role in the norm of the infection operator, as we have seen in Section 3.3. A large operator norm of the oblique projection can influence negatively the practicability of the observer in applications (e.g., leading to the need of taking a very small time step $k$ in simulations), as shown/discussed through numerical results presented in 18 , Sect. 6] (in there, for choices as spans of eigenfunctions, cf. 4.2a). By this reason, it could be interesting to investigate the performance of the feedback for different choices of $\widetilde{W}_{S}$ (e.g., those in 4.2 ), or even try to define and investigate the "optimal choice".

In our simulations we have taken only the border case $\ell=2$ for the power $\ell$ of the diffusion taken in the injection operator $\mathfrak{I}_{S}^{[\lambda, \ell]}$. Another point that could be investigated is the performance of the observer for different values of $\ell$.
6.4. On the time step. In Table 1 we see that the $H$-norm of the output injection at initial time, for some pairs $\left(S_{\sigma}, \lambda\right)$. This is the reason we took a small time step as $k=10^{-4}$. Note that such norm increases with $\lambda$, so for larger $\lambda$ we may need to take a smaller time step to capture (or, accurately approximate) the effect of the output injection on the dynamics. A very small time step may be impracticable for real world applications, thus it could be interesting to investigate, in a future work, whether an appropriate choice of $\widetilde{W}_{S}$ and/or $\ell$ allows us to take larger $k$.

| $\left(S_{\sigma}, \lambda\right)$ | $(4,0.5)$ | $(9,0.02)$ | $(16,0.02)$ |
| :--- | :--- | :--- | :--- |
| $\left\|\mathfrak{I}_{S}^{[\lambda, \ell]} z_{0}\right\|_{H}$ | 3537.9599 | 747.3875 | 2594.0443 |

Table 1. Norm of output injection at initial time.

## Appendix

A.1. Proof of Proposition 3.5. With $\widehat{y}_{1}:=y+z_{1}$ and $\widehat{y}_{2}:=y+z_{2}$, we write

$$
\begin{aligned}
\mathfrak{N}_{y}\left(t, z_{1}\right)-\mathfrak{N}_{y}\left(t, z_{2}\right) & =\mathcal{N}\left(t, \widehat{y}_{1}\right)-\mathcal{N}(t, y)-\left(\mathcal{N}\left(t, \widehat{y}_{2}\right)-\mathcal{N}(t, y)\right) \\
& =\mathcal{N}\left(t, \widehat{y}_{1}\right)-\mathcal{N}\left(t, \widehat{y}_{2}\right)=\mathcal{N}\left(t, y+z_{1}\right)-\mathcal{N}\left(t, y+z_{2}\right),
\end{aligned}
$$

which leads us to $\widehat{y}_{1}-\widehat{y}_{2}=z_{1}-z_{2}=: d$ and, using Lemma 3.4 ,

$$
\begin{aligned}
& \left.2\left(\mathfrak{N}_{y}\left(t, z_{1}\right)-\mathfrak{N}_{y}\left(t, z_{2}\right), A\left(z_{1}-z_{2}\right)\right)_{H}=2\left(\mathcal{N}\left(t, \widehat{y}_{1}\right)-\mathcal{N}\left(t, \widehat{y}_{2}\right), A\left(\widehat{y}_{1}-\widehat{y}_{2}\right)\right)\right)_{H} \\
& \leq \widehat{\gamma}_{0}|d|_{\mathrm{D}(A)}^{2}+\left(1+\bar{\gamma}_{0}^{-\frac{1+\left\|\delta_{2}\right\|}{1-\left\|\delta_{2}\right\|}}\right) \bar{C}_{\mathcal{N} 1} \sum_{j=1}^{n}|d|_{V}^{\frac{2 \delta_{1 j}}{1-\delta_{2 j}}} \sum_{k=1}^{2}\left|\widehat{y}_{k}\right|_{V}^{\frac{2 \varsigma_{1 j}}{1-\delta_{2 j}}}\left|\widehat{y}_{k}\right|_{\mathrm{D}(A)}^{\frac{2 \zeta_{2 j}}{1-\delta_{2 j}}}
\end{aligned}
$$

Therefore (3.4) follows with $\widetilde{C}_{\mathfrak{N} 1}=\bar{C}_{\mathcal{N} 1}$.
By setting $z_{2}=0$ in (3.4), we obtain for each $\widetilde{\gamma}_{0}>0$,

$$
\begin{align*}
& 2\left(\mathfrak{N}_{y}\left(t, z_{1}\right), A z_{1}\right)_{H}  \tag{A.1}\\
& \leq \widetilde{\gamma}_{0}\left|z_{1}\right|_{\mathrm{D}(A)}^{2}+\left(1+\widetilde{\gamma}_{0}^{-\frac{1+\left\|\delta_{2}\right\|}{1-\left\|\delta_{2}\right\|}}\right) \widetilde{C}_{\mathcal{N} 1} \sum_{j=1}^{n}\left|z_{1}\right|_{V}^{\frac{2 \delta_{1 j}}{1-\delta_{2 j}}} \sum_{l=0}^{1}\left|y+l z_{1}\right|_{V}^{\frac{2 \zeta_{1 j}}{1-\delta_{2 j}}}\left|y+l z_{1}\right|_{\mathrm{D}(A)}^{\frac{2 \varsigma_{2 j}}{1-\delta_{j}}}
\end{align*}
$$

Now, for simplicity we fix $j$, and set

$$
r=r_{j}:=\frac{2 \delta_{1 j}}{1-\delta_{2 j}} \geq 2, \quad p=p_{j}:=\frac{2 \zeta_{1 j}}{1-\delta_{2 j}} \quad \text { and } \quad q=q_{j}:=\frac{2 \zeta_{2 j}}{1-\delta_{2 j}}<2 .
$$

Note that $p \geq 0, r \geq 2$, and $q \in[0,2)$ due to the relations $\delta_{1 j}+\delta_{2 j} \geq 1$ and $\zeta_{2 j}+\delta_{2 j}<1$, in Assumption 2.4

We consider first the case $q \neq 0$. By the triangle inequality and [16, Prop. 2.6], we obtain

$$
\begin{align*}
\Upsilon_{j} & :=\left|z_{1}\right|_{V}^{r} \sum_{l=0}^{1}\left|y+l z_{1}\right|_{V}^{p}\left|y+l z_{1}\right|_{\mathrm{D}(A)}^{q} \\
= & \left|z_{1}\right|_{V}^{r}|y|_{V}^{p}|y|_{\mathrm{D}(A)}^{q}+\left|z_{1}\right|_{V}^{r}\left|y+z_{1}\right|_{V}^{p}\left|y+z_{1}\right|_{\mathrm{D}(A)}^{q} \\
\leq & \left|z_{1}\right|_{V}^{r}|y|_{V}^{p}|y|_{\mathrm{D}(A)}^{q} \\
& +\left|z_{1}\right|_{V}^{r}\left(1+2^{p-1}\right)\left(1+2^{q-1}\right)\left(|y|_{V}^{p}+\left|z_{1}\right|_{V}^{p}\right)\left(\left.|y|\right|_{\mathrm{D}(A)} ^{q}+\left|z_{1}\right|_{\mathrm{D}(A)}^{q}\right) . \tag{A.2}
\end{align*}
$$

Setting $D_{p, q}:=\left(1+2^{p-1}\right)\left(1+2^{q-1}\right)$ and using the Young inequality, the last term satisfies for each $\gamma_{2}>0$,

$$
\begin{aligned}
D_{p, q}^{-1} \mathcal{T}_{j}: & =\left|z_{1}\right|_{V}^{r}\left(|y|_{V}^{p}+\left|z_{1}\right|_{V}^{p}\right)\left(|y|_{\mathrm{D}(A)}^{q}+\left|z_{1}\right|_{\mathrm{D}(A)}^{q}\right) \\
\leq & \left|z_{1}\right|_{V}^{r}\left(|y|_{V}^{p}+\left|z_{1}\right|_{V}^{p}\right)\left|z_{1}\right|_{\mathrm{D}(A)}^{q}+\left|z_{1}\right|_{V}^{r}\left(|y|_{V}^{p}+\left|z_{1}\right|_{V}^{p}\right)|y|_{\mathrm{D}(A)}^{q} \\
\leq & \gamma_{2}^{\frac{2}{q}}\left|z_{1}\right|_{\mathrm{D}(A)}^{2}+\gamma_{2}^{-\left(1-\frac{q}{2}\right)^{-1}}\left(\left|z_{1}\right|_{V}^{r}\left(|y|_{V}^{p}+\left|z_{1}\right|_{V}^{p}\right)\right)^{\left(1-\frac{q}{2}\right)^{-1}} \\
& +\left|z_{1}\right|_{V}^{r}\left(|y|_{V}^{p}+\left|z_{1}\right|_{V}^{p}\right)\left(1+\left.|y|\right|_{\mathrm{D}(A)} ^{\| \frac{2 \zeta_{2}}{1-\delta_{2}}} \|\right. \\
&
\end{aligned}
$$

which implies, since $1-\frac{q}{2}=\frac{2-q}{2}$,

$$
\begin{aligned}
D_{p, q}^{-1} \mathcal{T}_{j} \leq & \gamma_{2}^{\frac{2}{q}}\left|z_{1}\right|_{\mathrm{D}(A)}^{2}+\gamma_{2}^{-\frac{2}{2-q}}\left|z_{1}\right|_{V}^{\frac{2 r}{2-q}}\left(|y|_{V}^{p}+\left|z_{1}\right|_{V}^{p}\right)^{\frac{2}{2-q}} \\
& +\left|z_{1}\right|_{V}^{r}\left(|y|_{V}^{p}+\left|z_{1}\right|_{V}^{p}\right)\left(1+|y|_{\mathrm{D}(A)}^{\left\|\frac{2 \zeta_{2}}{1 \delta_{2}}\right\|}\right) \\
\leq & \gamma_{2}^{\frac{2}{q}}\left|z_{1}\right|_{\mathrm{D}(A)}^{2}+\gamma_{2}^{-\frac{2}{2-q}}\left(1+2^{\frac{2}{2-q}-1}\right)\left|z_{1}\right|_{V}^{\frac{2 r}{2-q}}\left(|y|_{V}^{\frac{2 p}{2-q}}+\left|z_{1}\right|_{V}^{\frac{2 p}{2-q}}\right) \\
& +\left|z_{1}\right|_{V}^{r}\left(|y|_{V}^{p}+\left|z_{1}\right|_{V}^{p}\right)\left(1+|y|_{\mathrm{D}(A)}^{\left\|\frac{2 \zeta_{2}}{1-\delta_{2}^{2}}\right\|}\right) \\
\leq & \gamma_{2}^{\frac{2}{q}}\left|z_{1}\right|_{\mathrm{D}(A)}^{2} \\
& +D_{q, \gamma_{2}}\left(\left|z_{1}\right|_{V}^{\frac{2 r}{2-q}}|y|_{V}^{\frac{2 p}{2-q}}+\left|z_{1}\right|_{V}^{\frac{2(r+p)}{2-q}}+\left|z_{1}\right|_{V}^{r}|y|_{V}^{p}+\left|z_{1}\right|_{V}^{r+p}\right)\left(1+|y|_{\mathrm{D}(A)}^{\left\|\frac{2 \zeta_{2}}{1-\delta_{2}}\right\|}\right)
\end{aligned}
$$

with

$$
D_{q, \gamma_{2}}:=1+\gamma_{2}^{-\frac{2}{2-q}}\left(1+2^{\frac{2}{2-q}-1}\right),
$$

and then

$$
\begin{align*}
& D_{q, \gamma_{2}}^{-1}\left(D_{p, q}^{-1} \mathcal{T}_{j}-\gamma_{2}^{\frac{2}{q}}\left|z_{1}\right|_{\mathrm{D}(A)}^{2}\right)\left(1+|y|_{\mathrm{D}(A)}^{\left\|\frac{2 \varsigma_{2}}{1-\left(\delta_{2}\right.}\right\|}\right)^{-1} \\
& \leq\left(\left|z_{1}\right|_{V}^{\frac{2 r}{2-q}-2}|y|_{V}^{\frac{2 p}{2-q}}+\left|z_{1}\right|_{V}^{\frac{2(r+p)}{2-q}-2}+\left|z_{1}\right|_{V}^{r-2}|y|_{V}^{p}+\left|z_{1}\right|_{V}^{r+p-2}\right)\left|z_{1}\right|_{V}^{2} \tag{A.3a}
\end{align*}
$$

Observe that

$$
\begin{align*}
\frac{2 p}{2-q} & =\frac{4 \zeta_{1 j}}{2-2 \delta_{2 j}-2 \zeta_{2 j}} \leq \frac{2\left\|\zeta_{1}\right\|}{1-\left\|\delta_{2}+\zeta_{2}\right\|}  \tag{A.4a}\\
\frac{2(r+q+p)-4}{2-q} & =\frac{4\left(\delta_{1 j}+\zeta_{1 j}+\zeta_{2 j}\right)-4\left(1-\delta_{2 j}\right)}{2-2 \delta_{2 j}-2 \zeta_{2 j}} \leq \frac{2\left\|\delta_{1}+\delta_{2}+\zeta_{1}+\zeta_{2}\right\|-2}{1-\left\|\delta_{2}+\zeta_{2}\right\|} \tag{A.4b}
\end{align*}
$$

and that $\frac{2 c}{2-q} \geq c \Longleftrightarrow 0 \geq-q c$. Thus, we obtain that $\frac{2 c}{2-q} \geq c$ for all $c \geq 0$, and

$$
\frac{2 p}{2-q} \geq p \quad \text { and } \quad \frac{2(r+q+p)-4}{2-q} \geq \frac{2(r+p)-4}{2-q} \geq(r+p)-2 \geq r-2 \geq 0,
$$

which together with A.3 give us

$$
\begin{aligned}
& D_{q, \gamma_{2}}^{-1}\left(D_{p, q}^{-1} \mathcal{T}_{j}-\gamma_{2}^{\frac{2}{q}}\left|z_{1}\right|_{\mathrm{D}(A)}^{2}\right)\left(1+|y|_{\mathrm{D}(A)}^{\left\|\frac{2 \zeta_{2}}{1-\delta_{2}}\right\|}\right)^{-1} \\
& \leq\left(\left|z_{1}\right|_{V}^{\frac{2(r+q)-4}{2-q}}+\left|z_{1}\right|_{V}^{\frac{2(r+q+p)-4}{2-q}}+\left|z_{1}\right|_{V}^{r-2}+\left|z_{1}\right|_{V}^{r+p-2}\right)\left(2+|y|_{V}^{\frac{2 p}{2-q}}+|y|_{V}^{p}\right)\left|z_{1}\right|_{V}^{2} \\
& \leq\left(4+4\left|z_{1}\right|_{V}^{\frac{2\left\|\delta_{1}+\delta_{2}+\zeta_{1}+\zeta_{2}\right\|-2}{1-\left\|\delta_{2}+\zeta_{2}\right\|}}\right)\left(4+2|y|_{V}^{\frac{2\left\|\zeta_{1}\right\|}{1-\left\|\delta_{2}+\zeta_{2}\right\|}}\right)\left|z_{1}\right|_{V}^{2}
\end{aligned}
$$

which implies

$$
\begin{align*}
& \mathcal{T}_{j} \leq D_{p, q} \gamma_{2}^{\frac{2}{q}}\left|z_{1}\right|_{\mathrm{D}(A)}^{2}+\widehat{D}\left(1+|y|_{V}^{\chi_{1}}\right)\left(1+|y|_{\mathrm{D}(A)}^{\chi_{2}}\right)\left(1+\left|z_{1}\right|_{V}^{\chi_{3}}\right)\left|z_{1}\right|_{V}^{2}  \tag{A.5a}\\
& D_{p, q}:=\left(1+2^{p-1}\right)\left(1+2^{q-1}\right), \quad \widehat{D}:=16 D_{p, q} D_{q, \gamma_{2}}  \tag{A.5b}\\
& \chi_{1}:=\frac{2\left\|\zeta_{1}\right\|}{1-\left\|\delta_{2}+\zeta_{2}\right\|} \geq 0, \quad \chi_{2}:=\left\|\frac{2 \zeta_{2}}{1-\delta_{2}}\right\| \geq 0, \quad \chi_{3}:=\frac{2\left\|\delta_{1}+\delta_{2}+\zeta_{1}+\zeta_{2}\right\|-2}{1-\left\|\delta_{2}+\zeta_{2}\right\|} \geq 0 . \tag{A.5c}
\end{align*}
$$

For the first term on the right-hand side of A.2), we also obtain

$$
\begin{equation*}
\mathcal{F}_{j}:=\left|z_{1}\right|_{V}^{r}|y|_{V}^{p}|y|_{\mathrm{D}(A)}^{q} \leq\left(1+|y|_{V}^{\chi_{1}}\right)\left(1+|y|_{\mathrm{D}(A)}^{\chi_{2}}\right)\left(1+\left|z_{1}\right|_{V}^{\chi_{3}}\right)\left|z_{1}\right|_{V}^{2} \tag{A.6}
\end{equation*}
$$

because $p \leq \chi_{1}, 0<q \leq \chi_{2}$, and $r-2 \leq \chi_{3}$. See A.4).

Therefore, by A.2, A.5 , and A.6), it follows that for all $\gamma_{2} \in(0,1]$,

$$
\begin{align*}
& \Upsilon_{j} \leq \mathcal{F}_{j}+\mathcal{T}_{j} \\
& \leq D_{p, q} \gamma_{2}^{\frac{2}{q}}\left|z_{1}\right|_{\mathrm{D}(A)}^{2}+(1+\widehat{D})\left(1+|y|_{V}^{\chi_{1}}\right)\left(1+|y|_{\mathrm{D}(A)}^{\chi_{2}}\right)\left(1+\left|z_{1}\right|_{V}^{\chi_{3}}\right)\left|z_{1}\right|_{V}^{2}, \\
& \leq D_{p, q} \gamma_{2}^{\frac{2}{\chi_{2}}}\left|z_{1}\right|_{\mathrm{D}(A)}^{2}+(1+\widehat{D})\left(1+|y|_{V}^{\chi_{1}}\right)\left(1+|y|_{\mathrm{D}(A)}^{\chi_{2}}\right)\left(1+\left|z_{1}\right|_{V}^{\chi_{3}}\right)\left|z_{1}\right|_{V}^{2},  \tag{A.7a}\\
& \text { for } q \neq 0, \quad \text { with } \quad \gamma_{2} \leq 1 \text {. } \tag{A.7b}
\end{align*}
$$

Note that $\gamma_{2}^{\frac{2}{q}} \leq \gamma_{2}^{\frac{2}{\chi_{2}}}$ because $\gamma_{2} \leq 1$ and $0<q \leq \chi_{2}$.
Finally, we consider the case $q=0$. We find

$$
\begin{align*}
\Upsilon_{j} & :=\left|z_{1}\right|_{V}^{r} \sum_{l=0}^{1}\left|y+l z_{1}\right|_{V}^{p}=\left|z_{1}\right|_{V}^{r}|y|_{V}^{p}+\left|z_{1}\right|_{V}^{r}\left|y+z_{1}\right|_{V}^{p} \\
& \leq\left|z_{1}\right|_{V}^{r}\left(\left(2+2^{p-1}\right)|y|_{V}^{p}+\left(1+2^{p-1}\right)\left|z_{1}\right|_{V}^{p}\right) \\
& \leq\left(2+2^{p-1}\right)\left(1+|y|_{V}^{p}\right)\left(\left|z_{1}\right|_{V}^{r-2}+\left|z_{1}\right|_{V}^{r+p-2}\right)\left|z_{1}\right|_{V}^{2} \\
& \leq\left(2+2^{p-1}\right)\left(2+|y|_{V}^{\chi_{1}}\right)\left(2+2\left|z_{1}\right|_{V}^{\chi_{3}}\right)\left|z_{1}\right|_{V}^{2} \\
& \leq 8\left(1+2^{p-1}\right)\left(1+|y|_{V}^{\chi_{1}}\right)\left(1+\left|z_{1}\right|_{V}^{\chi_{3}}\right)\left|z_{1}\right|_{V}^{2}, \quad q=0 .  \tag{A.8a}\\
\text { for } q & =0, \quad \text { with } \quad \gamma_{2} \leq 1, \tag{A.8b}
\end{align*}
$$

with $\chi_{1}$ and $\chi_{3}$ as in A.5), where we have used A.4).
Now we observe that

$$
\begin{align*}
\left(1+2^{p-1}\right) & \leq D_{p, q}=\left(1+2^{p-1}\right)\left(1+2^{q-1}\right) \\
& \leq\left(1+2^{\frac{\left\|2 \zeta_{1}+\delta_{2}\right\|-1}{1-\left\|\delta_{2}\right\|}}\right)\left(1+2^{\frac{\left\|2 \zeta_{2}+\delta_{2}\right\|-1}{1-\left\|\delta_{2}\right\|}}\right)=: \widetilde{D}_{1}  \tag{A.9a}\\
\widehat{D} & =16 D_{p, q} D_{q, \gamma_{2}} \leq 16 \widetilde{D}_{1}\left(1+2^{\frac{2 q}{2-q}}\left(1+\gamma_{2}^{-\frac{2}{2-q}}\right)\right. \\
& \leq 16 \widetilde{D}_{1}\left(2+2^{\frac{2\left\|\zeta_{2}\right\|}{1-\left\|\delta_{2}+\zeta_{2}\right\|}}\right)\left(2+\gamma_{2}^{-\frac{1}{1-\left\|\delta_{2}+\zeta_{2}\right\|}}\right) \\
& \leq 32 \widetilde{D}_{1}\left(2+2^{\frac{2\left\|\zeta_{2}\right\|}{1-\left\|\delta_{2}+\zeta_{2}\right\|}}\right)\left(1+\gamma_{2}^{-\frac{1}{1-\left\|\delta_{2}+\zeta_{2}\right\|}}\right)=\widetilde{D}_{2}\left(1+\gamma_{2}^{-\chi_{4}}\right),  \tag{A.9b}\\
\widetilde{D}_{2} & :=32 \widetilde{D}_{1}\left(2+2^{\frac{2\left\|\zeta_{2}\right\|}{1-\| \delta_{2}+\zeta_{2} \pi}}\right)  \tag{A.9c}\\
\chi_{4} & :=\frac{1}{1-\left\|\delta_{2}+\zeta_{2}\right\|}>1 . \tag{A.9d}
\end{align*}
$$

Further, from $\widetilde{D}_{2}>64 \widetilde{D}_{1} \geq 8\left(1+2^{p-1}\right)$ and from from A.7, A.8), and A.9p, we conclude that for both cases, $q>0$ and $q=0$, we have

$$
\begin{align*}
& \Upsilon_{j} \leq \vartheta\left|z_{1}\right|_{\mathrm{D}(A)}^{2}+\widetilde{D}_{2}\left(1+\gamma_{2}^{-\chi_{4}}\right)\left(1+|y|_{V}^{\chi_{1}}\right)\left(1+|y|_{\mathrm{D}(A)}^{\chi_{2}}\right)\left(1+\left|z_{1}\right|_{V}^{\chi_{3}}\right)\left|z_{1}\right|_{V}^{2},  \tag{A.10a}\\
& \text { for all } \gamma_{2} \in(0,1], \quad \text { with } \vartheta:= \begin{cases}\widetilde{D}_{1} \gamma_{2}^{\frac{2}{\chi_{2}}}, & \text { for } \chi_{2}>0 \\
0, & \text { for } \chi_{2}=0\end{cases}  \tag{A.10b}\\
& \text { and } \widetilde{D}_{1}=\bar{C}_{\left[\left\|\zeta_{1}\right\|,\left\|\zeta_{2}\right\|, \frac{1}{1-\left\|\delta_{2}+\zeta_{2}\right\|}\right]}, \quad \widetilde{D}_{2}=\bar{C}_{\left[\left\|\zeta_{1}\right\|,\left\|\zeta_{2}\right\|, \frac{1}{1-\left\|\delta_{2}+\zeta_{2}\right\|}\right]} . \tag{A.10c}
\end{align*}
$$

Now, from A.1 and A.10, for all $\widetilde{\gamma}_{0}>0$ and $\gamma_{2} \in(0,1]$, and with

$$
\chi_{5}:=\frac{1+\left\|\delta_{2}\right\|}{1-\left\|\delta_{2}\right\|}>1,
$$

we derive that

$$
\begin{align*}
& 2\left(\mathfrak{N}_{y}\left(t, z_{1}\right), A z_{1}\right)_{H} \leq \widetilde{\gamma}_{0}\left|z_{1}\right|_{\mathrm{D}(A)}^{2}+\left(1+\widetilde{\gamma}_{0}^{-\chi_{5}}\right) \widetilde{C}_{\mathcal{N} 1} \sum_{j=1}^{n} \Upsilon_{j}, \\
& \leq\left(\widetilde{\gamma}_{0}+n \vartheta\left(1+\widetilde{\gamma}_{0}^{-\chi_{5}}\right) \widetilde{C}_{\mathcal{N} 1}\right)\left|z_{1}\right|_{\mathrm{D}(A)}^{2}  \tag{A.11}\\
& \quad+n \widetilde{D}_{2}\left(1+\gamma_{2}^{-\chi_{4}}\right)\left(1+\widetilde{\gamma}_{0}^{-\chi_{5}}\right) \widetilde{C}_{\mathcal{N} 1}\left(1+|y|_{V}^{\chi_{1}}\right)\left(1+|y|_{\mathrm{D}(A)}^{\chi_{2}}\right)\left(1+\left|z_{1}\right|_{V}^{\chi_{3}}\right)\left|z_{1}\right|_{V}^{2}
\end{align*}
$$

For an arbitrary $\widehat{\gamma}_{0}>0$, we can choose

$$
\gamma_{2}=\left(\frac{\widehat{\gamma}_{0}}{(n+1) \widetilde{D}_{1} \widetilde{C}_{\mathcal{N} 1}\left(1+\widetilde{\gamma}_{0}^{-\chi_{5}}\right)+\widehat{\gamma}_{0}}\right)^{\frac{\chi_{2}}{2}} \leq 1,
$$

and $\widetilde{\gamma}_{0}=\frac{\widehat{\gamma}_{0}}{n+1}$. Note that, in particular,

$$
\widetilde{D}_{1} \gamma_{2}^{\frac{2}{\chi_{2}}}\left(1+\widetilde{\gamma}_{0}^{-\chi_{5}}\right) \widetilde{C}_{\mathcal{N} 1}<\gamma_{2}^{\frac{2}{\chi_{2}}}\left(\widetilde{D}_{1} \widetilde{C}_{\mathcal{N} 1}\left(1+\widetilde{\gamma}_{0}^{-\chi_{5}}\right)+\frac{\widehat{\gamma}_{0}}{n+1}\right)=\frac{\widehat{\gamma}_{0}}{n+1}, \quad \text { for } \quad \chi_{2}>0
$$

and thus for the coefficient of $\left|z_{1}\right|_{\mathrm{D}(A)}^{2}$ in A.11, we find

$$
\begin{cases}\widetilde{\gamma}_{0}+n \vartheta\left(1+\widetilde{\gamma}_{0}^{-\chi_{5}}\right) \widetilde{C}_{\mathcal{N} 1}<\frac{\widehat{\gamma}_{0}}{n+1}+n \frac{\widehat{\gamma}_{0}}{n+1}=\widehat{\gamma}_{0}, & \text { if } \chi_{2}>0 ;  \tag{A.12}\\ \widetilde{\gamma}_{0}+n \vartheta\left(1+\widetilde{\gamma}_{0}^{-\chi_{5}}\right) \widetilde{C}_{\mathcal{N} 1}=\frac{\hat{\gamma}_{0}}{n+1}<\widehat{\gamma}_{0}, & \text { if } \chi_{2}=0\end{cases}
$$

Observe, next, that

$$
\begin{equation*}
1+\widetilde{\gamma}_{0}^{-\chi_{5}}=1+(n+1)^{\chi_{5}} \widehat{\gamma}_{0}^{-\chi_{5}} \tag{A.13}
\end{equation*}
$$

and

$$
\begin{aligned}
1+\gamma_{2}^{-\chi_{4}} & =2, \quad \text { if } \quad \chi_{2}=0 \\
1+\gamma_{2}^{-\chi_{4}} & \leq 1+\left((n+1) \widetilde{D}_{1} \widetilde{C}_{\mathcal{N} 1}\left(1+\widetilde{\gamma}_{0}^{-\chi_{5}}\right)+\widehat{\gamma}_{0}\right)^{\frac{\chi_{2} \chi_{4}}{2}} \widehat{\gamma}_{0}^{-\frac{\chi_{2} \chi_{4}}{2}} \\
& \leq 1+\left(1+2^{\frac{\chi_{2} \chi_{4}}{2}-1}\right)\left(\left((n+1) \widetilde{D}_{1} \widetilde{C}_{\mathcal{N} 1}\left(1+\widetilde{\gamma}_{0}^{-\chi_{5}}\right)\right)^{\frac{\chi_{2} \chi_{4}}{2}}+\widehat{\gamma}_{0}^{\frac{\chi_{2} \chi_{4}}{2}}\right) \widehat{\gamma}_{0}^{-\frac{\chi_{2} x_{4}}{2}} \\
& \leq \widehat{C}_{1}+\widehat{C}_{2}\left(1+\widetilde{\gamma}_{0}^{-\chi_{5}}\right)^{\frac{\chi_{2} \chi_{4}}{2}} \widehat{\gamma}_{0}^{-\frac{\chi_{2} \chi_{4}}{2}}, \quad \text { if } \quad \chi_{2}>0 ;
\end{aligned}
$$

with

$$
\begin{aligned}
& \widehat{C}_{1}:=1+\left(1+2^{\frac{\chi_{2} \chi_{4}}{2}-1}\right)=\bar{C}_{\left[\left\|\zeta_{2}\right\|, \frac{1}{1-\left\|\delta_{2}\right\|}, \frac{1}{1-\left\|\delta_{2}+\zeta_{2}\right\|}\right]}, \\
& \widehat{C}_{2}:=\left(1+2^{\frac{\chi_{2} \chi_{4}}{2}-1}\right)\left((n+1) \widetilde{D}_{1} \widetilde{C}_{\mathcal{N} 1}\right)^{\frac{\chi_{2} \chi_{4}}{2}}=\bar{C}_{\left[n, \widetilde{C}_{\mathcal{N} 1},\left\|\zeta_{1}\right\|,\left\|\zeta_{2}\right\|, \frac{1}{1-\left\|\delta_{2}\right\|}\right]} .
\end{aligned}
$$

Since $\widehat{C}_{1} \geq 2$ holds for $\chi_{2} \geq 0$ we can write

$$
1+\gamma_{2}^{-\chi_{4}} \leq \widehat{C}_{1}+\widehat{C}_{2}\left(1+\widetilde{\gamma}_{0}^{-\chi_{5}}\right)^{\frac{\chi_{2} \chi_{4}}{2}} \widehat{\gamma}_{0}^{-\frac{\chi_{2} \chi_{4}}{2}}, \quad \text { for } \quad \chi_{2} \geq 0
$$

Further, we see that

$$
1+\gamma_{2}^{-\chi_{4}} \leq \widehat{C}_{1}+\widehat{C}_{2}\left(1+2^{\frac{\chi_{2} \chi_{4}}{2}-1}\right)\left(\widehat{\gamma}_{0}^{-\frac{\chi_{2} \chi_{4}}{2}}+\widetilde{\gamma}_{0}^{-\frac{\chi_{5} \chi_{2} \chi_{4}}{2}} \widehat{\gamma}_{0}^{-\frac{\chi_{2} \chi_{4}}{2}}\right)
$$

and

$$
\begin{aligned}
& \widetilde{\gamma}_{0}^{-\frac{x_{5} x_{2} \chi_{4}}{2}} \widehat{\gamma}_{0}^{-\frac{x_{2} x_{4}}{2}}=(n+1)^{\frac{x_{5} \chi_{2} x_{4}}{2}} \widehat{\gamma}_{0}^{-\frac{\left(x_{5}+1\right) x_{2} x_{4}}{2}} \\
& \widehat{\gamma}_{0}^{-\frac{x_{2} \chi_{4}}{2}} \leq 1+\widehat{\gamma}_{0}^{-\frac{\left(x_{5}+1\right) x_{2} \chi_{4}}{2}}
\end{aligned}
$$

from which we obtain

$$
\begin{align*}
1+\gamma_{2}^{-\chi_{4}} & \leq \widehat{C}_{1}+\widehat{C}_{2}\left(1+2^{\frac{\chi_{2} \chi_{4}}{2}-1}\right)\left(1+\left(1+(n+1)^{\frac{x_{5} x_{2} \chi_{4}}{2}}\right) \widehat{\gamma}_{0}^{-\frac{\left(x_{5}+1\right) \chi_{2} \chi_{4}}{2}}\right) \\
& \leq \widehat{C}_{3}+\widehat{C}_{4} \widehat{\gamma}_{0}^{-\frac{\left(x_{5}+1\right) \chi_{2} \chi_{4}}{2}} \tag{A.14a}
\end{align*}
$$

with

$$
\begin{equation*}
\widehat{C}_{3}:=\widehat{C}_{1}+\widehat{C}_{2}\left(1+2^{\frac{\chi_{2} \chi_{4}}{2}-1}\right), \quad \widehat{C}_{4}:=\widehat{C}_{2}\left(1+2^{\frac{\chi_{2} \chi_{4}}{2}-1}\right)\left(1+(n+1)^{\frac{\chi_{5} \chi_{2} \chi_{4}}{2}}\right) \tag{A.14b}
\end{equation*}
$$

Therefore A.13 and A.14 lead us to

$$
\begin{equation*}
\left(1+{\left.\gamma_{2}^{-\chi_{4}}\right)\left(1+\widetilde{\gamma}_{0}^{-\chi_{5}}\right) \leq \widehat{C}_{5}\left(1+\widehat{\gamma}_{0}^{-\chi_{5}}\right)\left(1+\widehat{\gamma}_{0}^{-\frac{\left(\chi_{5}+1\right) \chi_{2} \chi_{4}}{2}}\right), ~, ~, ~}\right. \tag{A.15a}
\end{equation*}
$$

with, recalling that $\chi_{2} \chi_{4} \geq 0$,

$$
\begin{equation*}
\widehat{C}_{5}:=(n+1)^{\chi_{5}}+\widehat{C}_{3}+\widehat{C}_{4}=\bar{C}_{\left[n, \widetilde{C}_{\mathcal{N} 1},\left\|\zeta_{1}\right\|,\left\|\zeta_{2}\right\|, \frac{1}{1-\left\|\delta_{2}\right\|}, \frac{1}{1-\left\|\zeta_{2}+\delta_{2}\right\|}\right]} . \tag{A.15b}
\end{equation*}
$$

Hence A.11, A.12, and A.15 give us

$$
\begin{aligned}
& 2\left(\mathfrak{N}_{y}\left(t, z_{1}\right), A z_{1}\right)_{H} \leq \widehat{\gamma}_{0}\left|z_{1}\right|_{\mathrm{D}(A)}^{2} \\
& +\widetilde{C}_{\mathfrak{N} 2}\left(1+\widehat{\gamma}_{0}^{-\chi_{5}}\right)\left(1+\widehat{\gamma}_{0}^{-\frac{\left(\chi_{5}+1\right) \chi_{2} \chi_{4}}{2}}\right)\left(1+|y|_{V}^{\chi_{1}}\right)\left(1+\left.|y|\right|_{\mathrm{D}(A)} ^{\chi_{2}}\right)\left(1+\left|z_{1}\right|_{V}^{\chi_{3}}\right)\left|z_{1}\right|_{V}^{2}
\end{aligned}
$$

with $\widetilde{C}_{\mathfrak{N} 2}:=n \widetilde{D}_{2} \widehat{C}_{5} \widetilde{C}_{\mathcal{N} 1}=\bar{C}_{\left[n, \widetilde{C}_{\mathcal{N} 1},\left\|\zeta_{1}\right\|,\left\|\zeta_{2}\right\|, \frac{1}{1-\left\|\delta_{2}\right\|}, \frac{1}{1-\left\|\zeta_{2}+\delta_{2}\right\|}\right]}$. This ends the proof of Proposition 3.5
A.2. Proof of Proposition 3.6. Recall that $\mathrm{D}\left(A^{\xi}\right) \hookrightarrow H$, for $\xi \geq 0$, and $H=$ $\widetilde{\mathcal{W}}_{S} \oplus \mathcal{W}_{S}^{\perp}$. We prove firstly that $\mathcal{W}_{S}$ and $\mathcal{W}_{S}^{\perp} \cap \mathrm{D}\left(A^{\xi}\right)$ are closed subspaces of $\mathrm{D}\left(A^{\xi}\right)$. Clearly $\widetilde{\mathcal{W}}_{S}$ is closed, because it is finite-dimensional. Let now be an arbitrary sequence $\left(h_{n}\right)_{n \in \mathbb{N}_{0}}$ in $\mathcal{W}_{S}^{\perp} \cap \mathrm{D}\left(A^{\xi}\right)$ and a vector $\bar{h} \in \mathrm{D}\left(A^{\xi}\right)$, so that $\left|h_{n}-\bar{h}\right|_{\mathrm{D}\left(A^{\xi}\right)} \rightarrow 0$, as $n \rightarrow+\infty$. Since $\left|h_{n}-\bar{h}\right|_{H} \leq C\left|h_{n}-\bar{h}\right|_{\mathrm{D}\left(A^{\xi}\right)}$, for a suitable constant $C>0$, it follows that $\left|h_{n}-\bar{h}\right|_{H} \rightarrow 0$, and since $\mathcal{W}_{S}^{\perp}$ is closed in $H$, it follows that $\bar{h} \in \mathcal{W}_{S}^{\perp}$. Thus $\bar{h} \in \mathcal{W}_{S}^{\perp} \cap \mathrm{D}\left(A^{\xi}\right)$, and we can conclude that $\mathcal{W}_{S}^{\perp} \cap \mathrm{D}\left(A^{\xi}\right)$ is a closed subspace of $V$. Next we observe that $\mathrm{D}\left(A^{\xi}\right)=\widetilde{\mathcal{W}}_{S} \oplus\left(\mathcal{W}_{S}^{\perp} \cap \mathrm{D}\left(A^{\xi}\right)\right)$, which is a straightforward consequence of $H=\widetilde{\mathcal{W}}_{S} \oplus \mathcal{W}_{S}^{\perp}$. To show that the oblique projection $P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp} \cap \mathrm{D}\left(A^{\xi}\right)}$ in $\mathrm{D}\left(A^{\xi}\right)$ coincides with the restriction $\left.P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}}\right|_{\mathrm{D}\left(A^{\xi}\right)}$ of the oblique projection $P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}}$ in $H$, it is enough to observe that by definition of a projection we have that

$$
\begin{gathered}
\left.P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp} \cap \mathrm{D}\left(A^{\xi}\right)}\right|_{\mathrm{D}\left(A^{\xi}\right)} w_{1}=w_{1}=P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}} w_{1}, \quad \text { for all } \quad w_{1} \in \widetilde{\mathcal{W}}_{S}, \\
\left.P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp} \cap \mathrm{D}\left(A^{\xi}\right)}\right|_{\mathrm{D}\left(A^{\xi}\right)} w_{2}=0=P_{\widetilde{\mathcal{W}}_{S}^{\prime}}^{\mathcal{W}_{S}^{\prime}} w_{2}, \quad \text { for all } \quad w_{2} \in \mathcal{W}_{S}^{\perp} \cap \mathrm{D}\left(A^{\xi}\right) .
\end{gathered}
$$

Finally, we have $P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp} \cap \mathrm{D}\left(A^{\xi}\right)} \in \mathcal{L}\left(\mathrm{D}\left(A^{\xi}\right)\right)$ because (oblique) projections are continuous, see [4, Sect. 2.4, Thm. 2.10].
A.3. Proof of Proposition 3.7. It is clear that $\left.P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{1}}\right|^{\mathrm{D}\left(A^{-\xi}\right)}$ is an extension of the oblique projection $P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{s}^{\perp}} \in \mathcal{L}(H)$ to $\mathrm{D}\left(A^{-\xi}\right) \supseteq H$, because for $z \in H$ we have that $\left\langle\left. P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}^{\frac{1}{S}}}\right|^{\mathrm{D}\left(A^{-\xi}\right)} z, w\right\rangle_{\mathrm{D}\left(A^{-\xi}\right), \mathrm{D}\left(A^{\xi}\right)}=\left(z, \mathcal{P}_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}} w\right)_{H}=\left(P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{S}^{\perp}} z, w\right)_{H}$, where for the last identity we have used $P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{S}^{\perp}}=\left(\mathcal{P}_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{1}}\right)^{*}$; see [21, Lem. 3.8]. By the relation (3.8) and Proposition 3.7 it follows the inequality $\left.\left|P_{\mathcal{W}_{S}}^{\widetilde{\mathcal{W}}_{S}^{\perp}}\right|^{\mathrm{D}\left(A^{-\xi}\right)}\right|_{\mathcal{L}\left(\mathrm{D}\left(A^{-\xi}\right)\right)} \leq\left.\left|P_{\widetilde{\mathcal{W}}_{S}}^{\mathcal{W}_{S}^{\perp}}\right|_{\mathrm{D}\left(A^{\xi}\right)}\right|_{\mathcal{L}\left(\mathrm{D}\left(A^{\xi}\right)\right)}<+\infty$, and afterwards the same relation (3.8) gives us the converse inequality. Hence we obtain the stated norm identity. Finally, by definition of the adjoint operator we also have the stated adjoint identity.
A.4. Proof of Proposition 3.8. Observe that, since $\mathfrak{s} \in(0,1)$, we have that $g(\tau):=$ $-\eta_{1} \tau+\eta_{2} \tau^{\mathfrak{s}}$ satisfies $g(0)=0, \lim _{\tau \rightarrow+\infty} g(\tau)=-\infty$, and $\left.\frac{\mathrm{d}}{\mathrm{d} \tau}\right|_{\tau=\tau_{0}} g(\tau)=-\eta_{1}+\mathfrak{s} \eta_{2} \tau_{0}^{\mathfrak{s s}-1}$, for $\tau_{0}>0$. In particular, $g$ is differentiable at each $\tau_{0}>0$. Furthermore, $\left.\frac{\mathrm{d}}{\mathrm{d} \tau}\right|_{\tau=\tau_{0}} g(\tau)>$ $0 \Longleftrightarrow \tau_{0}^{\mathfrak{s}-1}>\eta_{1}\left(\mathfrak{s} \eta_{2}\right)^{-1} \Longleftrightarrow \tau_{0}^{1-\mathfrak{s}}<\left(\mathfrak{s} \eta_{2}\right) \eta_{1}^{-1} \Longleftrightarrow \tau_{0}<\left(\mathfrak{s} \eta_{2}\right)^{\frac{1}{1-\mathfrak{s}}} \eta_{1}^{-\frac{1}{1-\mathfrak{s}}}$. Thus $g(\tau)$ strictly increases only if $\tau \in(0, \bar{\tau})$, with $\bar{\tau}:=\left(\mathfrak{s} \eta_{2}\right)^{\frac{1}{1-\mathfrak{s}}} \eta_{1}^{-\frac{1}{1-\mathfrak{s}}}=\left(\mathfrak{s} \eta_{2}\right)^{\frac{1}{1-\mathfrak{s}}} \eta_{1}^{\frac{1}{\mathfrak{s}-1}}$. Analogously we find that $\left.\frac{\mathrm{d}}{\mathrm{d} \tau}\right|_{\tau=\tau_{0}} g(\tau)>0 \Longleftrightarrow \tau_{0}>\bar{\tau}$. Necessarily, the maximum is attained at $\bar{\tau}>0$, and can be computed as

$$
-\eta_{1} \bar{\tau}+\eta_{2} \bar{\tau}^{\mathfrak{s}}=-\eta_{1}\left(\mathfrak{s} \eta_{2}\right)^{\frac{1}{1-\mathfrak{s}}} \eta_{1}^{\frac{1}{\mathfrak{s}-1}}+\eta_{2}\left(\left(\mathfrak{s} \eta_{2}\right)^{\frac{1}{1-\mathfrak{s}}} \eta_{1}^{\frac{1}{\mathfrak{s}-1}}\right)^{\mathfrak{s}}=\eta_{2}^{\frac{1}{1-\mathfrak{s}}} \eta_{1}^{\frac{\mathfrak{s}}{\mathfrak{s}-1}}\left(-\mathfrak{s}^{\frac{1}{1-\mathfrak{s}}}+\mathfrak{s}^{\frac{\mathfrak{s}}{1-\mathfrak{s}}}\right)
$$

Thus, $-\eta_{1} \bar{\tau}+\eta_{2} \bar{\tau}^{\mathfrak{s}}=(1-\mathfrak{s}) \mathfrak{s}^{\frac{\mathfrak{s}}{1-\mathfrak{s}}} \eta_{2}^{\frac{1}{1-\mathfrak{s}}} \eta_{1}^{\frac{\mathfrak{s}}{\mathfrak{s}-1}}$, which finishes the proof.
A.5. Proof of Proposition 3.9, For the sake of simplicity we shall omit the subscript in the usual norm in $\mathbb{R}$, that is, $|\cdot|:=|\cdot|_{\mathbb{R}}$. The solution of (3.11) is given by

$$
\begin{equation*}
v(t)=\mathrm{e}^{-\bar{\mu}(t-s)+\int_{s}^{t}|h(\tau)| \mathrm{d} \tau} v(s), \quad t \geq s \geq 0, \quad v(0)=v_{0} \tag{A.16}
\end{equation*}
$$

Observe that the exponent satisfies, using 3.9,

$$
\begin{align*}
-\bar{\mu}(t-s)+\int_{s}^{t}|h(\tau)| \mathrm{d} \tau & \leq-\bar{\mu}(t-s)+(t-s)^{\frac{\mathrm{r}-1}{\mathrm{r}}}\left(\int_{s}^{t}|h(\tau)|^{\mathfrak{r}} \mathrm{d} \tau\right)^{\frac{1}{\mathrm{r}}} \\
& \leq-\bar{\mu}(t-s)+(t-s)^{\frac{\mathrm{r}-1}{\mathrm{r}}}\left(\int_{s}^{s+T\left\lceil\frac{t-s}{T}\right\rceil}|h(\tau)|^{\mathfrak{r}} \mathrm{d} \tau\right)^{\frac{1}{\mathrm{r}}} \\
& \leq-\bar{\mu}(t-s)+(t-s)^{\frac{\mathrm{r}-1}{\mathrm{r}}}\left(\left\lceil\frac{t-s}{T}\right\rceil C_{h}^{\mathrm{r}}\right)^{\frac{1}{\mathrm{r}}} \tag{A.17}
\end{align*}
$$

where $\lceil r\rceil \in \mathbb{N}_{0}$ is the positive integer defined by

$$
\begin{equation*}
r \leq\lceil r\rceil<r+1 \tag{A.18}
\end{equation*}
$$

From A.17) and A.18, it follows that

$$
\begin{align*}
-\bar{\mu}(t-s)+\int_{s}^{t}|h(\tau)| \mathrm{d} \tau & \leq-\bar{\mu}(t-s)+(t-s)^{\frac{\mathrm{r}-1}{\mathrm{r}}}\left(\frac{t-s}{T}+1\right)^{\frac{1}{\mathrm{r}}} C_{h} \\
& \leq T^{-\frac{1}{\mathrm{v}}}\left(-\bar{\mu} T^{\frac{1}{\mathrm{r}}}+C_{h}\right)(t-s)+(t-s)^{\frac{\mathrm{r}-1}{\tau}} C_{h} \tag{A.19}
\end{align*}
$$

where we have used $\left(\frac{t-s}{T}+1\right)^{\frac{1}{r}} \leq\left(\frac{t-s}{T}\right)^{\frac{1}{r}}+1$, since $\mathfrak{r}>1$, see [16, Proposition 2.6].
By (3.10), we have that

$$
\begin{equation*}
\widehat{\mu}:=T^{-\frac{1}{\mathfrak{r}}}\left(\bar{\mu} T^{\frac{1}{\mathrm{r}}}-C_{h}\right) \geq \max \left\{2 \frac{\mathrm{r}-1}{\mathfrak{r}}\left(\frac{C_{h}^{\mathrm{r}}}{\mathrm{r} \log (\varrho)}\right)^{\frac{1}{\mathrm{r}-1}}, 2 \mu\right\}>0 \tag{A.20}
\end{equation*}
$$

from which, together with A.19 and Proposition 3.8, we obtain

$$
\begin{align*}
-\bar{\mu}(t-s)+\int_{s}^{t}|h(\tau)| \mathrm{d} \tau & \leq-\frac{1}{2} \widehat{\mu}(t-s)-\frac{1}{2} \widehat{\mu}(t-s)+(t-s)^{\frac{\mathfrak{r}-1}{\mathfrak{r}}} C_{h} \\
& \leq-\frac{\widehat{\mu}}{2}(t-s)+\frac{1}{\mathfrak{r}}\left(\frac{\mathfrak{r}-1}{\mathfrak{r}}\right)^{\mathfrak{r}-1} C_{h}^{\mathfrak{r}}\left(\frac{\widehat{\mu}}{2}\right)^{1-\mathfrak{r}} \tag{A.21}
\end{align*}
$$

because by Proposition 3.8 , with $\mathfrak{s}=\frac{\mathfrak{r}-1}{\mathfrak{r}}$ and $\eta_{1}=\frac{\widehat{\mu}}{2}, \eta_{2}=C_{h}$,

$$
\max _{t-s \geq 0}\left\{-\eta_{1}(t-s)+(t-s)^{\frac{\mathfrak{r}-1}{\mathfrak{r}}} \eta_{2}\right\}=(1-\mathfrak{s})^{\frac{\mathfrak{s}}{1-\mathfrak{s}}} \eta_{2}^{\frac{1}{1-\mathfrak{s}}} \eta_{1}^{\frac{\mathfrak{s}}{\mathfrak{s}-1}}=\frac{1}{\mathfrak{r}}\left(\frac{\mathfrak{r}-1}{\mathfrak{r}}\right)^{\mathfrak{r}-1} \eta_{2}^{\mathfrak{r}} \eta_{1}^{1-\mathfrak{r}}
$$

Therefore, from A.16, A.20, and A.21, we derive that

$$
|v(t)| \leq \mathrm{e}^{\frac{C_{h}^{\mathfrak{r}}}{\mathfrak{r}}\left(\frac{2(\mathfrak{r}-1)}{\mathrm{r}}\right)^{\mathrm{r}-1} \widehat{\mu}^{1-\mathrm{r}}} \mathrm{e}^{-\frac{\hat{\mu}}{2}(t-s)}|v(s)| \leq \varrho \mathrm{e}^{-\mu(t-s)}|v(s)|,
$$

which gives us A.16. Indeed, observe that

$$
\begin{aligned}
\mathrm{e}^{\frac{C_{h}^{\mathfrak{r}}}{\mathfrak{r}}\left(\frac{2(\mathfrak{r}-1)}{\mathfrak{r}}\right)^{\mathfrak{r}-1} \hat{\mu}^{1-\mathfrak{r}}} \leq \varrho \quad \Longleftrightarrow \quad \frac{C_{h}^{\mathfrak{r}}}{\mathfrak{r}}\left(\frac{2(\mathfrak{r}-1)}{\mathfrak{r}}\right)^{\mathfrak{r}-1} \widehat{\mu}^{1-\mathfrak{r}} \leq \log (\varrho) \\
\Longleftrightarrow \frac{C_{h}^{\mathfrak{r}}}{\mathfrak{r} \log (\varrho)}\left(\frac{2(\mathfrak{r}-1)}{\mathfrak{r}}\right)^{\mathfrak{r}-1} \leq \widehat{\mu}^{\mathfrak{r}-1} \quad \Longleftrightarrow \quad \widehat{\mu} \geq\left(\frac{C_{h}^{\mathfrak{r}}}{\mathfrak{r} \log (\varrho)}\right)^{\frac{1}{\mathfrak{r}-1}} \frac{2(\mathfrak{r}-1)}{\mathfrak{r}},
\end{aligned}
$$

and the last inequality follows from A.20, which also gives us $\frac{\widehat{\mu}}{2}>\mu$.
A.6. Proof of Proposition 3.10. We shall use a fixed point argument, through the contraction principle, in the closed subset

$$
\mathcal{Z}_{\varrho,\left|\varpi_{0}\right|}^{\mu_{0}}:=\left\{g \in L^{\infty}\left(\mathbb{R}_{0}, \mathbb{R}\right)| | \mathrm{e}^{\mu_{0} t} g(t)|\leq \varrho| \varpi_{0} \mid\right\}
$$

of the Banach space

$$
\mathcal{Z}_{0}^{\mu}:=\left\{g \in L^{\infty}\left(\mathbb{R}_{0}, \mathbb{R}\right) \mid \mathrm{e}^{\mu_{0}(\cdot)} g \in L^{\infty}\left(\mathbb{R}_{0}, \mathbb{R}\right)\right\}, \quad|g|_{\mathcal{Z}^{\mu_{0}}}:=\sup _{t \geq 0}\left|\mathrm{e}^{\mu_{0} t} g(t)\right|
$$

We show now that, since (3.14) holds true, the mapping

$$
\Psi: \mathcal{Z}_{\varrho,\left|\varpi_{0}\right|}^{\mu_{0}} \rightarrow \mathcal{Z}_{\varrho,\left|\varpi_{0}\right|}^{\mu_{0}}, \quad \breve{\varpi} \mapsto \varpi
$$

where $\varpi$ solves

$$
\begin{equation*}
\dot{\varpi}=-(\bar{\mu}-|h|) \varpi+|h||\breve{\varpi}|^{p} \breve{\varpi}, \quad \varpi(0)=\varpi_{0}, \tag{A.22}
\end{equation*}
$$

is well defined and is a contraction in $\mathcal{Z}_{\varrho,\left|\varpi_{0}\right|}^{\mu_{0}}$.
We look at A.22 as a perturbation of the nominal linear system

$$
\begin{equation*}
\dot{v}=-(\bar{\mu}-|h|) v, \quad v(0)=v_{0}=\varpi_{0} \in \mathbb{R} . \tag{A.23}
\end{equation*}
$$

Note that (3.14) implies that

$$
\bar{\mu} \geq \max \left\{2 \frac{\mathfrak{r}-1}{\mathfrak{r}}\left(\frac{C_{h}^{\mathfrak{r}}}{\mathfrak{r} \log \left(\varrho^{\frac{1}{2}}\right)}\right)^{\frac{1}{\mathfrak{r}-1}}, 4 \mu_{0}\right\}-T^{-\frac{1}{\mathbf{r}}} C_{h}
$$

which we use together with Proposition 3.9 to conclude that the solution

$$
v(t)=: \mathcal{S}(t, s) v(s)
$$

of A.23 satisfies

$$
\begin{equation*}
|v(t)|=|\mathcal{S}(t, s) v(s)| \leq \varrho^{\frac{1}{2}} \mathrm{e}^{-2 \mu_{0}(t-s)}|v(s)|, \quad t \geq s \geq 0, \quad v(0)=v_{0} \tag{A.24}
\end{equation*}
$$

By the Duhamel formula we have that the solution $w$ of A.22 is given as

$$
\begin{equation*}
\varpi(t)=\mathcal{S}(t, s) \varpi(s)+\int_{s}^{t} \mathcal{S}(t, \tau)|h(\tau)||\breve{\varpi}(\tau)|^{p} \breve{\varpi}(\tau) \mathrm{d} \tau, \quad \varpi=\Psi(\breve{\varpi}) . \tag{A.25}
\end{equation*}
$$

(s) Step 1: $\Psi$ maps $\mathcal{Z}_{\varrho,\left|\varpi_{0}\right|}^{\mu_{0}}$ into itself, if $\left|\varpi_{0}\right|<\varrho R$. We observe that A.24 and A.25 give us the estimate

$$
\begin{equation*}
|\varpi(t)| \leq \varrho^{\frac{1}{2}} \mathrm{e}^{-2 \mu_{0} t}\left|\varpi_{0}\right|+\int_{0}^{t} \varrho^{\frac{1}{2}} \mathrm{e}^{-2 \mu_{0}(t-\tau)}|h(\tau)||\breve{\varpi}(\tau)|^{p+1} \mathrm{~d} \tau . \tag{A.26}
\end{equation*}
$$

Next, we also find, since $\breve{\varpi} \in \mathcal{Z}_{\varrho} \mathcal{U}^{\mu_{0}}{ }_{\varpi_{0}} \mid$,

$$
\begin{align*}
& \int_{0}^{t} \mathrm{e}^{-2 \mu_{0}(t-\tau)}|h(\tau)||\breve{\varpi}(\tau)|^{p+1} \mathrm{~d} \tau \leq \varrho^{p+1}\left|\varpi_{0}\right|^{p+1} \int_{0}^{t} \mathrm{e}^{-2 \mu_{0}(t-\tau)} \mathrm{e}^{-\mu_{0}(p+1) \tau}|h(\tau)| \mathrm{d} \tau \\
& \leq \varrho^{p+1}\left|\varpi_{0}\right|^{p+1} \mathrm{e}^{-\mu_{0} t} \int_{0}^{t} \mathrm{e}^{-\mu_{0}(t-\tau)} \mathrm{e}^{-\mu_{0} p \tau}|h(\tau)| \mathrm{d} \tau \\
& \leq \varrho^{p+1}\left|\varpi_{0}\right|^{p+1} \mathrm{e}^{-\mu_{0} t}\left(\int_{0}^{t} \mathrm{e}^{-\frac{\mathfrak{r}}{\mathfrak{r}-1} \mu_{0}(t-\tau)} \mathrm{d} \tau\right)^{\frac{\mathfrak{r}-1}{\mathfrak{r}}}\left(\int_{0}^{t} \mathrm{e}^{-\mathfrak{r} \mu_{0} p \tau}|h(\tau)|^{\mathfrak{r}} \mathrm{d} \tau\right)^{\frac{1}{\mathfrak{r}}} \\
& \leq \varrho^{p+1}\left|\varpi_{0}\right|^{p+1} \mathrm{e}^{-\mu_{0} t}\left(\frac{\mathfrak{r}-1}{\mathfrak{r} \mu_{0}}\right)^{\frac{\mathfrak{r}-1}{\mathfrak{r}}}\left(\sum_{i=1}^{\left\lceil\frac{t}{T}\right\rceil} \mathrm{e}^{-\mathfrak{r} \mu_{0} p(i-1) T} \int_{(i-1) T}^{i T}|h(\tau)|^{\mathfrak{r}} \mathrm{d} \tau\right)^{\frac{1}{\mathfrak{r}}} \\
& \leq \varrho^{p+1}\left|\varpi_{0}\right|^{p+1} \mathrm{e}^{-\mu_{0} t}\left(\frac{\mathfrak{r}-1}{\mathfrak{r} \mu_{0}}\right)^{\frac{\mathfrak{r}-1}{\mathfrak{r}}} C_{h}\left(\sum_{i=1}^{\left\lceil\frac{t}{T}\right\rceil} \mathrm{e}^{-\mathfrak{r} \mu_{0} p(i-1) T}\right)^{\frac{1}{\mathfrak{r}}} \\
& \leq \varrho^{p+1}\left|\varpi_{0}\right|^{p+1} C_{h}\left(\frac{1}{1-\mathrm{e}^{-\mathfrak{r} \mu_{0} p T}}\right)^{\frac{1}{\tau}}\left(\frac{\mathfrak{r}-1}{\mathfrak{r} \mu_{0}}\right)^{\frac{\mathrm{r}-1}{\mathfrak{r}}} \mathrm{e}^{-\mu_{0} t} . \tag{A.27}
\end{align*}
$$

By combining A.26 with A.27, we arrive at

$$
\begin{align*}
\mathrm{e}^{\mu_{0} t}|\varpi(t)| & \leq \varrho^{\frac{1}{2}} \mathrm{e}^{-\mu_{0} t}\left|\varpi_{0}\right|+\varrho^{p+\frac{3}{2}}\left|\varpi_{0}\right|^{p+1} C_{h}\left(\frac{1}{1-\mathrm{e}^{-\mathfrak{r} \mu_{0} p T}}\right)^{\frac{1}{\mathfrak{r}}}\left(\frac{\mathfrak{r}-1}{\mathfrak{r}}\right)^{\frac{\mathfrak{r}-1}{\mathfrak{r}}} \mu_{0}^{\frac{1-\mathfrak{r}}{\mathfrak{r}}} \\
& \leq \varrho^{\frac{1}{2}}\left(1+\varrho^{p+1}\left|\varpi_{0}\right|^{p} C_{h}\left(\frac{1}{1-e^{-\mathfrak{r} \mu_{0} p T}}\right)^{\frac{1}{\mathfrak{r}}}\left(\frac{\mathfrak{r}-1}{\mathfrak{r}}\right)^{\frac{\mathfrak{r}-1}{\mathfrak{r}}} \mu_{0}^{\frac{1-\mathfrak{r}}{\mathfrak{r}}}\right)\left|\varpi_{0}\right| \tag{A.28}
\end{align*}
$$

Next we use (3.13) and $\left|\varpi_{0}\right| \leq \varrho R$ to obtain

$$
\begin{equation*}
\frac{1}{1-\mathrm{e}^{-\mathrm{r} \mu_{0} p T}} \leq \frac{1}{1-\mathrm{e}^{-\mu_{0} p T}} \leq 2, \tag{A.29a}
\end{equation*}
$$

and

$$
\begin{align*}
& 1+\varrho^{p+1}\left|\varpi_{0}\right|^{p} C_{h}\left(\frac{1}{1-e^{-\boldsymbol{r} \mu_{0} p T}}\right)^{\frac{1}{\mathfrak{r}}}\left(\frac{\mathfrak{r}-1}{\mathfrak{r}}\right)^{\frac{\mathfrak{r}-1}{\mathfrak{r}}} \mu_{0}^{\frac{1-\mathfrak{r}}{\mathfrak{r}}} \leq 1+\varrho^{2 p+1} R^{p} C_{h}\left(\frac{\mathfrak{r}-1}{\mathfrak{r}}\right)^{\frac{\mathfrak{r}-1}{\mathfrak{r}}} \mu_{0}^{\frac{1-\mathfrak{r}}{\mathfrak{r}}} 2^{\frac{1}{\mathfrak{r}}} \\
& \leq 1+\varrho^{2 p+1} R^{p} C_{h}\left(\frac{\mathfrak{r}-1}{\mathfrak{r}}\right)^{\frac{\mathfrak{r}-1}{\mathfrak{r}}} 2^{\frac{1}{\mathfrak{r}}}\left(\frac{\varrho^{2 p+1} R^{p} C_{h}}{\varrho^{\frac{1}{2}}-1}\right)^{-1} 2^{-\frac{1}{\mathfrak{r}}}\left(\frac{\mathfrak{r}-1}{\mathfrak{r}}\right)^{\frac{1-\mathfrak{r}}{\mathfrak{r}}}=1+\left(\frac{1}{\varrho^{\frac{1}{2}}-1}\right)^{-1} \\
& =\varrho^{\frac{1}{2}} \text {. } \tag{A.29b}
\end{align*}
$$

From A.28 and A.29, we find $\mathrm{e}^{\mu_{0} t}|\varpi(t)| \leq \varrho\left|\varpi_{0}\right|$, hence $\varpi=\Psi(\breve{\varpi}) \in \mathcal{Z}_{\varrho}^{\mu_{0},\left|\varpi_{0}\right|}$.
(s) Step 2: $\Psi$ is a contraction in $\mathcal{Z}_{\varrho,\left|\varpi_{0}\right|}^{\mu_{0}}$, if $\left|\varpi_{0}\right|<\varrho R$. For an arbitrary given $\left(\breve{\varpi}_{1}, \breve{\varpi}_{2}\right) \in$ $\mathcal{Z}_{\varrho,\left|\varpi_{0}\right|}^{\mu_{0}} \times \mathcal{Z}_{\varrho,\left|\varpi_{0}\right|}^{\mu_{0}}$, we have that the difference

$$
D:=\Psi\left(\breve{\varpi}_{1}\right)-\Psi\left(\breve{\varpi}_{2}\right)
$$

solves

$$
\dot{D}=-(\bar{\mu}-|h|) D+|h|\left(\left|\breve{\varpi}_{1}\right|^{p} \breve{\varpi}_{1}-\left|\breve{\varpi}_{2}\right|^{p} \breve{\varpi}_{2}\right), \quad D(0)=0
$$

By the Duhamel formula and the Mean Value Theorem, we obtain

$$
\begin{align*}
|D(t)| & =|\mathcal{S}(t, 0) D(0)|+\left|\int_{0}^{t} \mathcal{S}(t, \tau)\right| h(\tau)| |\left|\breve{\varpi}_{1}\right|^{p} \breve{\varpi}_{1}-\left|\breve{\varpi}_{2}\right|^{p} \breve{\varpi}_{2}|\mathrm{~d} \tau| \\
& \leq \varrho^{\frac{1}{2}}(p+1) \int_{0}^{t} \mathrm{e}^{-\mu_{0}(t-\tau)}|h(\tau)|\left(\left|\breve{\varpi}_{1}(\tau)\right|^{p}+\left|\breve{\varpi}_{2}(\tau)\right|^{p}\right)\left|\breve{\varpi}_{1}(\tau)-\breve{\varpi}_{2}(\tau)\right| \mathrm{d} \tau \\
& \leq \varrho^{\frac{1}{2}}(p+1)\left|\breve{\varpi}_{1}-\breve{\varpi}_{2}\right|_{\mathcal{Z}_{e, \mid \varpi_{0}}^{\mu_{0}} \mid} \mathrm{e}^{-\mu_{0} t} \int_{0}^{t}|h(\tau)|\left(\left|\breve{\varpi}_{1}(\tau)\right|^{p}+\left|\breve{\varpi}_{2}(\tau)\right|^{p}\right) \mathrm{d} \tau \\
& \leq 2 \varrho^{p+\frac{1}{2}}(p+1)\left|\varpi_{0}\right|^{p}\left|\breve{\varpi}_{1}-\breve{\varpi}_{2}\right|_{\mathcal{Z}_{e,\left|\varpi_{0}\right|}^{\mu_{0}}} \mathrm{e}^{-\mu_{0} t} \int_{0}^{t} \mathrm{e}^{-\mu_{0} \tau p}|h(\tau)| \mathrm{d} \tau \tag{A.30}
\end{align*}
$$

Note that

$$
\begin{align*}
& \int_{0}^{t} \mathrm{e}^{-\mu_{0} \tau p}|h(\tau)| \mathrm{d} \tau=\int_{0}^{t} \mathrm{e}^{-\frac{\mathrm{r}-1}{\mathrm{r}} \mu_{0} \tau p} \mathrm{e}^{-\frac{1}{\mathrm{r}} \mu_{0} \tau p}|h(\tau)| \mathrm{d} \tau \\
& \leq\left(\int_{0}^{t} \mathrm{e}^{-\mu_{0} \tau p} \mathrm{~d} \tau\right)^{\frac{\mathrm{r}-1}{\tau}}\left(\int_{0}^{t} \mathrm{e}^{-\mu_{0} \tau p}|h(\tau)|^{\mathrm{r}} \mathrm{~d} \tau\right)^{\frac{1}{\mathrm{r}}} \\
& \leq\left(\mu_{0} p\right)^{\frac{1-\mathrm{r}}{\tau}}\left(\sum_{i=1}^{\left\lceil\frac{t}{T}\right\rceil} \mathrm{e}^{-\mu_{0} p(i-1) T} \int_{(i-1) T}^{i T}|h(\tau)|^{\mathfrak{r}} \mathrm{d} \tau\right)^{\frac{1}{\mathrm{r}}} \\
& \leq\left(\mu_{0} p\right)^{\frac{1-\mathrm{r}}{\tau}} C_{h}\left(\sum_{i=1}^{\left\lceil\frac{t}{T}\right\rceil} \mathrm{e}^{-\mu_{0} p(i-1) T}\right)^{\frac{1}{\tau}} \leq\left(\mu_{0} p\right)^{\frac{1-\mathrm{r}}{\tau}} C_{h}\left(\frac{1}{1-\mathrm{e}^{-\mu_{0} p T}}\right)^{\frac{1}{\mathrm{r}}} . \tag{A.31}
\end{align*}
$$

From A.30 and A.31,

$$
\mathrm{e}^{\mu_{0} t}|D(t)| \leq 2 \varrho^{p+\frac{1}{2}}(p+1) p^{\frac{1-\mathrm{r}}{\mathrm{r}}} C_{h}\left(\frac{1}{1-\mathrm{e}^{-\mu_{0} p T}}\right)^{\frac{1}{r}}\left|\varpi_{0}\right|^{p} \mu_{0}^{\frac{1-\mathrm{r}}{\mathrm{r}}}\left|\breve{\varpi}_{1}-\breve{\varpi}_{2}\right|_{\mathcal{Z}_{Q, \mid \varpi_{0}}^{\mu_{0}}}
$$

which together $\left|\varpi_{0}\right| \leq \varrho R$ and $\mu_{0} \geq \frac{\log (2)}{p T}$, see (3.13), give us $\frac{1}{1-\mathrm{e}^{-\mu_{0} p T}} \leq 2$ and

$$
\begin{align*}
& \mathrm{e}^{\mu_{0} t}|D(t)| \leq 2^{\frac{\mathrm{r}+1}{\mathrm{r}}} \varrho^{2 p+\frac{1}{2}}(p+1) p^{\frac{1-\mathrm{r}}{\mathrm{r}}} C_{h} R^{p} \mu_{0}^{\frac{1-\mathrm{r}}{\mathrm{r}}}\left|\breve{\varpi}_{1}-\breve{\varpi}_{2}\right|_{\mathcal{Z}_{\varrho,\left|\varpi_{0}\right|}^{\mu_{0}}} \\
& \leq 2^{\frac{r+1}{r}} \varrho^{2 p+\frac{1}{2}}(p+1) p^{\frac{1-\tau}{r}} C_{h} R^{p} \mu_{0}^{\frac{1-\tau}{r}}\left|\breve{\varpi}_{1}-\breve{\varpi}_{2}\right|_{\mathcal{Z}_{e, \mid \varpi_{0}}^{\mu_{0}}} \\
& \leq 2^{\frac{\mathrm{r}+1}{\mathrm{r}}} \varrho^{2 p+\frac{1}{2}}(p+1) p^{\frac{1-\mathrm{r}}{\mathfrak{r}}} C_{h} R^{p}\left(2^{\frac{\mathrm{r}+1}{\mathrm{r}-1}}\left(\varrho^{2 p+\frac{1}{2}} C_{h} \frac{p+1}{p} R^{p} c\right)^{\frac{\mathrm{r}}{\mathrm{r}-1}} p^{\frac{1}{\mathrm{r}-1}}\right)^{\frac{1-\mathrm{r}}{\mathfrak{r}}}\left|\breve{\varpi}_{1}-\breve{\varpi}_{2}\right|_{\mathcal{Z}_{e, \mid \varpi_{0}} \mu_{0} \mid} \\
& \leq c^{-1} p^{\frac{1-\mathrm{r}}{r}}\left(\left(\frac{1}{p}\right)^{\frac{\mathrm{r}}{\mathrm{r}-1}} p^{\frac{1}{\mathrm{r}-1}}\right)^{\frac{1-\mathrm{r}}{\tau}}\left|\breve{\varpi}_{1}-\breve{\varpi}_{2}\right|_{\mathcal{Z}_{e,\left|w_{0}\right|}^{\mu_{0}}}=c^{-1}\left|\breve{\varpi}_{1}-\breve{\varpi}_{2}\right|_{\mathcal{Z}_{e, \mid \varpi_{0}}^{\mu_{0}} \mid} \tag{A.32}
\end{align*}
$$

with $c>1$ as in 3.13. Therefore, A.32 implies that

$$
\left|\Psi\left(\breve{\varpi}_{1}\right)-\Psi\left(\breve{\varpi}_{2}\right)\right|_{\mathcal{Z}_{e,\left|\varpi_{0}\right|}^{\mu_{0}}}=|D|_{\mathcal{Z}_{e,\left|\varpi_{0}\right|}^{\mu_{0}}} \leq c^{-1}|d|_{\mathcal{Z}_{e,\left|\varpi_{0}\right|}^{\mu_{0}}}=c^{-1}\left|\breve{\varpi}_{1}-\breve{\varpi}_{2}\right|_{\mathcal{Z}_{e,\left|\varpi_{0}\right|}^{\mu_{0}} \mid}
$$

which shows that $\Psi$ is a contraction.
(s) Step 3: Existence of a solution in $\mathcal{Z}_{\varrho,\left|\varpi_{0}\right|}^{\mu_{0}}$, if $\left|\varpi_{0}\right|<\varrho R$. By the contraction mapping principle, there exists a fixed point for $\Psi$ in $\mathcal{Z}_{\varrho,\left|\varpi_{0}\right|}^{\mu_{0}}$. Such fixed point is a solution for (3.12).
(S) Step 4: Uniqueness of the solution in $L^{\infty}\left(\mathbb{R}_{0}, \mathbb{R}\right)$. The uniqueness follows from the fact that the right-hand side of 3.12 is locally Lipschitz.
(s) Step 5: Estimate (3.15) holds true. Fix $s \geq 0$ and note that $\widetilde{h}(\tau):=h(\tau+s)$ also satisfies (3.9), with $C_{\widetilde{h}} \leq C_{h}$.

Let $\varpi_{\underline{s}}:=\left.\varpi\right|_{\mathbb{R}_{s}}$ be the restriction to $\mathbb{R}_{s}=[s,+\infty)$ of the solution $\varpi \in \mathcal{Z}_{\varrho,\left|\varpi_{0}\right|}^{\mu_{0}}$ of (3.12), and observe that $z(\tau):=\varpi_{\underline{s}}(\tau+s)$ solves

$$
\frac{\mathrm{d}}{\mathrm{~d} \tau} z=-(\bar{\mu}-|\widetilde{h}|) z+|\widetilde{h}||z|^{p} z, \quad z(0)=z_{0}, \quad \tau \geq 0
$$

If $\left|\varpi_{0}\right|<R$ it follows that $\left|z_{0}\right|=|\varpi(s)| \leq \varrho \mathrm{e}^{-\mu_{0} s}\left|\varpi_{0}\right| \leq \varrho R$. Then, by Step 3 we have that $z \in \mathcal{Z}_{\varrho,\left|z_{0}\right|}^{\mu_{0}}$, which implies that for $t \geq s$,

$$
|\varpi(t)|=\left|\varpi_{\underline{s}}(s+t-s)\right|=|z(t-s)| \leq \varrho \mathrm{e}^{-\mu_{0}(t-s)}|z(0)|=\varrho \mathrm{e}^{-\mu_{0}(t-s)}|\varpi(s)|,
$$

which gives us 3.15).
The proof is finished.
A.7. Proof of Proposition 4.3. Let us denote by $\left.\tau^{i}=\left(\tau_{1}^{i}, \tau_{2}^{i}, \ldots, \tau_{d}^{i}\right)\right) \in \mathbb{R}^{d}$ the unit vector whose coordinates are $\tau_{i}^{2}=1$ and $\tau_{j}^{i}=0$ for $j \neq i$. Observe that $\mathbf{J}_{d, 2}$ has exactly $d+1$ vectors. The only element in $\mathbf{J}_{d, 2}$ with $\sum_{j=1}^{d} \mathbf{j}_{j}=d$ is $\mathbf{1}^{d}:=(1,1, \ldots, 1)$. All the other elements in $\mathbf{J}_{d, 2}$ are of the form $\mathbf{1}^{d}+\tau^{i}, i=1,2, \ldots, d$.

Let now $p \in \mathbb{P}_{\times, 1}$ such that $\mathfrak{S}(p)=0$, which implies that

$$
\left|\left(p, 1_{\omega_{1^{d}, 1}^{\times}}\right)\right|_{\mathbb{R}}=0, \quad \text { and } \quad\left|\left(p, 1_{\omega_{1^{d}+\tau^{i}, 1}^{\times}}\right)\right|_{\mathbb{R}}=0, \text { for all } i=\{1,2, \ldots, d\},
$$

that is, with $\omega_{*}:=\omega_{1^{d}, 1}$

$$
\int_{\omega_{*}} p(x) \mathrm{d} x=0, \quad \text { and } \quad \int_{\omega_{*}} p\left(x-\tau^{i}\right) \mathrm{d} x=0 . \quad 1 \leq i \leq d
$$

Denoting $\mathfrak{L}_{a} x:=\sum_{i=1}^{d} a_{i} x_{i}$, and $p(x)=: a_{0}+\mathfrak{L}_{a} x$, we obtain

$$
\int_{\omega_{*}} c_{0}+\mathfrak{L}_{a} x \mathrm{~d} x=0, \quad \text { and } \quad \int_{\omega_{*}} c_{0}+\mathfrak{L}_{a}\left(x-\tau^{i}\right) \mathrm{d} x=0, \quad 1 \leq i \leq d
$$

which implies

$$
\begin{equation*}
\int_{\omega_{*}} c_{0}+\mathfrak{L}_{a} x \mathrm{~d} x=0, \quad \text { and } \quad \int_{\omega_{*}} \mathfrak{L}_{a} \tau^{i} \mathrm{~d} x=0, \quad 1 \leq i \leq d \tag{A.33}
\end{equation*}
$$

Note that for fixed $i$ we have

$$
\int_{\omega_{*}} \mathfrak{L}_{a} \tau^{i} \mathrm{~d} x=0 \quad \Longleftrightarrow \quad \int_{\omega_{*}} a_{i} \mathrm{~d} x=0 \quad \Longleftrightarrow \quad a_{i}=0
$$

which together with A.33 leads us to $a_{i}=0,1 \leq i \leq d$, and $c_{0}=0$.
We have just shown that $p \in \mathbb{P}_{\times, 1}$ and $\mathfrak{S}(p)=0$ imply that $p=0$. Therefore, we can conclude that $\mathfrak{S}(\cdot)$ is a norm on $\mathbb{P}_{\times, 1}$.
A.8. Proof of Proposition 4.7. Let $\theta=\sum_{k=1}^{S_{\sigma}} \theta_{k} \Phi_{k} \in \widetilde{\mathcal{W}}_{S}$, with the auxiliary functions $\Phi_{i}$ as in 4.2b). Then, after a translation, for the $H$-norm we find that

$$
\left|\Phi_{k}\right|_{H}^{2}=\underset{j=1}{\underset{X}{X}}\left|\sin ^{2}\left(\frac{S \pi x_{j}}{L_{j}}\right)\right|_{L^{2}\left(\left(0, \frac{L_{j}}{S}\right), \mathbb{R}\right)}^{2}=\left(\frac{3}{8 S}\right)^{d} \stackrel{d}{\underset{j=1}{X} L_{j}}
$$

and, with $L^{\times}:=\times_{j=1}^{d} L_{j}$, since the $\Phi_{i}$ S are pairwise orthogonal, we arrive at

$$
|\theta|_{H}^{2}=\sum_{k=1}^{S_{\sigma}} \theta_{k}^{2}\left|\Phi_{k}\right|_{H}^{2}=S^{-d}\left(\frac{3}{8}\right)^{d} L^{\times} \sum_{k=1}^{S_{\sigma}} \theta_{k}^{2}
$$

Next, for the $V$-norm we find

$$
|\theta|_{V}^{2}=\sum_{k=1}^{S_{\sigma}} \theta_{k}^{2}\left|\Phi_{k}\right|_{V}^{2}=\nu \sum_{k=1}^{S_{\sigma}} \theta_{k}^{2}\left|\nabla \Phi_{k}\right|_{L^{2}(\Omega)^{d}}^{2}+|\theta|_{H}^{2}
$$

and, due to

$$
\begin{aligned}
\left|\nabla \Phi_{k}\right|_{L^{2}(\Omega)^{d}}^{2} & =\sum_{i=1}^{d}\left|\frac{S \pi}{L_{i}} \sin \left(\frac{2 S \pi x_{i}}{L_{i}}\right)\right|_{L^{2}\left(\left(0, \frac{L_{i}}{S}\right), \mathbb{R}\right)}^{2} \underset{i \neq j=1}{\times}\left|\sin ^{2}\left(\frac{S \pi x_{j}}{L_{j}}\right)\right|_{L^{2}\left(\left(0, \frac{L_{j}}{S}\right), \mathbb{R}\right)}^{2} \\
& =\sum_{i=1}^{d}\left(\frac{S \pi}{L_{i}}\right)^{2} \frac{L_{i}}{2 S} \underset{i \neq j=1}{\times} \frac{3}{8 S} L_{j}=\sum_{i=1}^{d}\left(\frac{S \pi}{L_{i}}\right)^{2} \frac{4}{3} \underset{j=1}{\times} \frac{3}{8 S} L_{j} \\
& =\left(\frac{3}{8 S}\right)^{d} L^{\times} \frac{4 S^{2} \pi^{2}}{3} \sum_{i=1}^{d} \frac{1}{L_{i}^{2}}=S^{2-d}\left(\frac{3}{8}\right)^{d} \frac{4 \pi^{2}}{3} L^{\times} \sum_{i=1}^{d} \frac{1}{L_{i}^{2}},
\end{aligned}
$$

we obtain

$$
|\theta|_{V}^{2}=\nu L^{\times} \sum_{k=1}^{S_{\sigma}} \theta_{k}^{2} S^{2-d}\left(\frac{3}{8}\right)^{d} \frac{4 \pi^{2}}{3} \sum_{i=1}^{d} \frac{1}{L_{i}^{2}}+|\theta|_{H}^{2}=\left(S^{2} \frac{4 \nu \pi^{2}}{3} \sum_{i=1}^{d} \frac{1}{L_{i}^{2}}+1\right)|\theta|_{H}^{2},
$$

That is,

$$
|\theta|_{V}^{2}=\left(C_{1} S^{2}+1\right)|\theta|_{H}^{2}, \quad \text { with } \quad C_{1}:=\frac{4 \nu \pi^{2}}{3} \sum_{i=1}^{d} \frac{1}{L_{i}^{2}} .
$$

Finally, for the $\mathrm{D}(A)$-norm we find

$$
\begin{aligned}
|\theta|_{\mathrm{D}(A)}^{2} & =|-\nu \Delta \theta+\theta|_{H}^{2}=\nu^{2}|\Delta \theta|_{H}^{2}+2 \nu\left|\nabla \Phi_{k}\right|_{L^{2}(\Omega)^{d}}^{2}+|\theta|_{H}^{2} \\
& =\nu^{2}|\Delta \theta|_{H}^{2}+2|\theta|_{V}^{2}-|\theta|_{H}^{2}=\nu^{2}|\Delta \theta|_{H}^{2}+\left(2 C_{1} S^{2}+1\right)|\theta|_{H}^{2}
\end{aligned}
$$

and from

$$
\begin{aligned}
\left|\Delta \Phi_{k}\right|_{H}^{2} & =\sum_{i=1}^{d}\left|2\left(\frac{S \pi}{L_{i}}\right)^{2} \cos \left(\frac{2 S \pi x_{i}}{L_{i}}\right)\right|_{L^{2}\left(\left(0, \frac{L_{i}}{S}\right), \mathbb{R}\right)}^{2} \stackrel{d}{\underset{i \neq j=1}{\times}\left|\sin ^{2}\left(\frac{S \pi x_{j}}{L_{j}}\right)\right|_{L^{2}\left(\left(0, \frac{L_{j}}{S}\right), \mathbb{R}\right)}^{2}} \\
& =\sum_{i=1}^{d} 4\left(\frac{S \pi}{L_{i}}\right)^{4} \frac{L_{i}}{2 S} \stackrel{d}{\times} \frac{3}{\times \neq j=1} 8 L_{j}=\sum_{i=1}^{d}\left(\frac{S \pi}{L_{i}}\right)^{4} \frac{16}{3} \underset{j=1}{\times} \frac{3}{8 S} L_{j} \\
& =\left(\frac{3}{8 S}\right)^{d} L^{\times} \frac{16 S^{4} \pi^{4}}{3} \sum_{i=1}^{d} \frac{1}{L_{i}^{2}}=S^{4-d}\left(\frac{3}{8}\right)^{d} \frac{16 \pi^{4}}{3} L^{\times} \sum_{i=1}^{d} \frac{1}{L_{i}^{2}}
\end{aligned}
$$

we obtain

$$
|\Delta \theta|_{H}^{2}=\sum_{k=1}^{S_{\sigma}} \theta_{k}^{2}\left|\Delta \Phi_{k}\right|_{H}^{2}=S^{4-d}\left(\frac{3}{8}\right)^{d} \frac{16 \pi^{4}}{3} \sum_{i=1}^{d} \frac{1}{L_{i}^{2}} L^{\times} \sum_{k=1}^{S_{\sigma}} \theta_{k}^{2}=S^{4} \frac{16 \pi^{4}}{3} \sum_{i=1}^{d} \frac{1}{L_{i}^{2}}|\theta|_{H}^{2},
$$

hence

$$
|\theta|_{\mathrm{D}(A)}^{2}=\left(C_{2} S^{4}+2 C_{1} S^{2}+1\right)|\theta|_{H}^{2}, \quad \text { with } \quad C_{2}:=\frac{\nu^{2} 16 \pi^{4}}{3} \sum_{i=1}^{d} \frac{1}{L_{i}^{2}}
$$

which finishes the proof.

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## References

[1] T. Ahmed-Ali, F. Giri, M. Krstic, F. Lamnabhi-Lagarrigue, and L. Burlion. Adaptive observer for a class of parabolic pdes. IEEE Trans. Automat. Control, 61(10):3083-3090, 2016. doi:10.1109/TAC.2015.2500237.
[2] A. I. Astrovskii and I. V. Gaishun. State estimation for linear timevarying observation systems. Differ. Equ., 55(3):363-373, 2019. doi:10.1134/ S0012266119030108.
[3] A. Azouani, E. Olson, and E. S. Titi. Continuous data assimilation using general interpolant observables. J. Nonlinear Sci., 24(2):277-304, 2014. doi:10.1007/ s00332-013-9189-y.
[4] H. Brezis. Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext. Springer, 2011. doi:10.1007/978-0-387-70914-7.
[5] J.-M. Buchot, J.-P. Raymond, and J. Tiago. Coupling estimation and control for a two dimensional Burgers type equation. ESAIM Control Optim. Calc. Var., 21(2):535-560, 2015. doi:10.1051/cocv/2014037.
[6] H. Feng and B.-Z. Guo. New unknown input observer and output feedback stabilization for uncertain heat equation. Automatica J. IFAC, 86:1-10, 2017. doi:10.1016/j.automatica.2017.08.004.
[7] N. Fujii. Feedback stabilization of distributed parameter systems by a functional observer. SIAM J. Control Optim., 18(2):108-120, 1980. doi:10.1137/0318009,
[8] L. Jadachowski, T. Meurer, and A. Kugi. State estimation for parabolic PDEs with varying parameters on 3-dimensional spatial domains. In Proceedings of the 18th World Congress IFAC, Milano, Italy, pages 13338-13343, August-September 2011. doi:10.3182/20110828-6-IT-1002.02964.
[9] L. Jadachowski, T. Meurer, and A. Kugi. State estimation for parabolic PDEs with reactive-convective non-linearities. In Proceedings of the 2013 European Control Conference (ECC), Zurich, Switzerland, pages 1603-1608, July 2013. doi:10. 23919/ECC.2013.6669588.
[10] W. Kang and E. Fridman. Distributed stabilization of Korteweg-deVries-Burgers equation in the presence of input delay. Automatica J. IFAC, 100:260-263, 2019. doi:10.1016/j.automatica.2018.11.025.
[11] T. Meurer. On the extended Luenberger-type observer for semilinear distributedparameter systems. IEEE Trans. Automat. Control, 58(7):1732-1743, 2013. doi: 10.1109/TAC. 2013.2243312
[12] T. Meurer and A. Kugi. Tracking control for boundary controlled parabolic PDEs with varying parameters: Combining backstepping and differential flatness. Automatica J. IFAC, 45:1182-1194, 2009. doi:10.1016/j.automatica.2009.01.006.
[13] J. Nečas. Les Méthodes Directes en Théorie des Équations Elliptiques. Masson \& Cie Éditeurs, 1967. doi:10.1007/978-3-642-10455-8.
[14] E. Olson and E. S. Titi. Determining modes for continuous data assimilation in 2D turbulence. J. Stat. Phys., 113(5-6):799-840, 2003. doi:10.1023/A: 1027312703252.
[15] Y. Orlov, A. Pisano, A. Pilloni, and E. Usai. Output feedback stabilization of coupled reaction-diffusion processes with constant parameters. SIAM J. Control Optim., 55(6):4112-4155, 2017. doi:10.1137/15M1034325.
[16] D. Phan and S. S. Rodrigues. Gevrey regularity for Navier-Stokes equations under Lions boundary conditions. J. Funct. Anal., 272(7):2865-2898, 2017. doi:10.1016/ j.jfa.2017.01.014.
[17] K. Ramdani, M. Tucsnak, and J. Valein. Detectability and state estimation for linear age-structured population diffusion models. ESAIM: M2AN, 50(6):17311761, 2016. doi:10.1051/m2an/2016002
[18] S. S. Rodrigues. Oblique projection exponential dynamical observer for nonautonomous linear parabolic-like equations. SIAM J. Control Optim., 59(1):464-488, 2021. doi:10.1137/19M1278934.
[19] S. S. Rodrigues. Oblique projection output-based feedback exponential stabilization of nonautonomous parabolic equations. Automatica J. IFAC (accepted), 2021. RICAM Report no. 2020-33, 2021. URL: https://www.ricam.oeaw.ac.at/ publications/ricam-reports/.
[20] S. S. Rodrigues. Semiglobal exponential stabilization of nonautonomous semilinear parabolic-like systems. Evol. Equ. Control Theory, 9(3):635-672, 2020. doi:10. 3934/eect. 2020027.
[21] S. S. Rodrigues and K. Sturm. On the explicit feedback stabilisation of onedimensional linear nonautonomous parabolic equations via oblique projections. IMA J. Math. Control Inform., 37(1):175-207, 2020. doi:10.1093/imamci/ dny045.
[22] M.Y. Wu. A note on stability of linear time-varying systems. IEEE Trans. Automat. Control, 19(2):162, 1974. doi:10.1109/TAC.1974.1100529.
[23] X.-W. Zhang and H.-N. Wu. Switching state observer design for semilinear parabolic pde systems with mobile sensors. J. Franklin Inst., 357(2):1299-1317, 2020. doi: 10.1016/j.jfranklin.2019.11.028.


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