# Stochastic variational principles for dissipative equations with advected quantities 

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#### Abstract

This paper presents symmetry reduction for material stochastic Lagrangian systems with advected quantities whose configuration space is a Lie group. Such variational principles yield deterministic as well as stochastic constrained variational principles for dissipative equations of motion in spatial representation. The general theory is presented for the finite dimensional situation. In infinite dimensions we obtain partial differential equations and stochastic partial differential equations. When the Lie group is, for example, a diffeomorphism group, the general result is not directly applicable but the setup and method suggest rigorous proofs valid in infinite dimensions which lead to similar results. We apply this technique to the compressible Navier-Stokes equation and to magnetohydrodynamics for charged viscous compressible fluids. A stochastic Kelvin-Noether theorem is presented. We derive, among others, the classical deterministic dissipative equations from purely variational and stochastic principles, without any appeal to thermodynamics.


## 1 Introduction

The goal of this paper is to develop a Lagrangian symmetry reduction process for a large class of stochastic systems with advected parameters. The general theory, which yields both deterministic and stochastic constrained variational principles and deterministic, as well as stochastic reduced equations of motion, is developed for finite dimensional systems. The resulting abstract equations then serve as a template for the study

[^0]of infinite dimensional stochastic systems, for which the rigorous analysis has to be carried out separately. The examples of the compressible Navier-Stokes equations and dissipative compressible magnetohydrodynamics equations, as well as their randomly perturbed counterparts and are treated in detail. We recover with our method the classical dissipative fluid and magnetohydrodynamic equations without any appeal to thermodynamical considerations, except for the form of the internal energy density.

The dynamics of many conservative physical systems can be described geometrically taking advantage of the intrinsic symmetries in their material description. These symmetries induce Noether conserved quantities and allow for the elimination of unknowns, producing an equivalent system consisting of new equations of motion in spaces with less variables and a non-autonomous ordinary differential equation, called the reconstruction equation. This geometric procedure is known as reduction, a method that is ubiquitous in symplectic, Poisson, and Dirac geometry and has wide applications in theoretical physics, quantum and continuum mechanics, control theory, and various branches of engineering. For example, in continuum mechanics, the passage from the material (Lagrangian) to the spatial (Eulerian) or convective (body) description is a reduction procedure. Of course, depending on the problem, one of the three representations may be preferable. However, it is often the case that insight from the other two representations, although apparently more intricate, leads to a deeper understanding of the physical phenomenon under consideration and is useful in the description of the dynamics.

A simple example in which the three descriptions are useful and serve different purposes is free rigid body dynamics (e.g., [68, Section 15]). If one is interested in the motion of the attitude matrix, the material picture is appropriate. The classical free rigid body dynamics result, obtained by applying Hamilton's standard variational principle on the tangent bundle of the proper rotation group $S O(3)$, states that the attitude matrix describes a geodesic of a left invariant Riemannian metric on $S O(3)$, characterized by the mass distribution of the body. However, as shown already by Euler, the equations of motion simplify considerably in the convective (or body) picture because the total energy of the rotating body, which in this case is just kinetic energy, is invariant under left translations. The convective description takes place on the Lie algebra $\mathfrak{s o}(3)$ of $S O(3)$ and is given by the classical Euler equations for a free rigid body, after implementing the Lie algebra isomorphism of $\mathbb{R}^{3}$ with $\mathfrak{s o}(3)$ given by the cross product operation. Finally, the spatial description comes into play, because the spatial angular momentum is conserved during the motion and is hence used in the description of the rigid body motion.

The present paper uses exclusively Lagrangian mechanics, where variational principles play a fundamental role since they produce the equations of motion. In continuum mechanics, the variational principle used in the material description is the standard Hamilton principle producing curves in the configuration space of the problem that are critical points of the action functional. However, in the spatial and convective
representations, if the configuration space of the problem is a Lie group, the induced variational principle requires the use of constrained variations, a fundamental result of Poincaré [78]; the resulting equations of motion are called today the Euler-Poincaré equations (69], [68, Section 13.5], [16]). These equations have been vastly extended to include the motion of advected quantities $([15,49,50)$ as well as affine ( 40$]$ ) and noncommutative versions thereof that naturally appear in models of complex materials with internal structure ( 41,43$]$ ) and whose geometric description has led to the solution of a long-standing controversy in the nematodynamics of liquid crystals (45, 46]). Euler-Poincaré equations have also very important generalizations to problems whose configuration space is an arbitrary manifold and the Lagrangian is invariant under a Lie group action ([17) as well as its extension to higher order Lagrangians ([36, 32, 33). Lagrange-Poincaré equations turn out to model the motion of spin systems (35]), long molecules ([24, 34]), free boundary fluids and elastic bodies ([38]), as well as charged and Yang-Mills fluids (43). There are also Lagrange-Poincaré theorems for field theory ([12, 13, 14, 42, 25]) and non-holonomic systems ([18). Lagrange-Poincaré equations also have interesting applications to Riemannian cubics and splines ([76), the representation of images ([11), certain classes of textures in condensed matter ([39), and some control ([44) and optimization ([37) problems.

Variational principles play an important role in the design of structure preserving numerical algorithms. One discretizes both spatially and temporally such that the symmetry structure of the problem is preserved. Integrators based on a discrete version of Hamilton's principle are called variational integrators ([7]). The resulting equations of motion are the discrete Euler-Lagrange equations and the associated algorithm for classical conservative systems is both symplectic as well as momentum-preserving and manifests very good long time energy behavior; see [62, 63] for additional information. There are versions of such variational integrators for certain forced ([54]), controlled ([77]), constrained holonomic ([64, [66]), non-holonomic ([55]), non-smooth ([29, [22]), multiscale ( $[65,81]$ ), and stochastic ([7]) systems. In the presence of symmetry, these systems can be reduced. However, today a general theory of discrete reduction in all of these cases is still missing and is currently being developed. If the configuration space is a Lie group, the first discretization of symmetric Lagrangian systems appears in [73], motivated by problems in complete integrability; for an in-depth analysis of such problems see [80].

All the above mentioned systems, both in the smooth and discrete versions, should have various stochastic analogues, depending on what phenomenon is modeled. The basic idea is to start with variational principles, motivated by Feynman's path integral approach to quantum mechanics and also by stochastic optimal control. The latter has its origins in the foundational work of Bismut (5, 6]) in the late seventies and in recent developments by Lázaro-Camí and Ortega ([58, 59, 60, 61). Non-holonomic systems have been studied in the same spirit ([47). All of this work investigated mainly stochastic perturbations of Hamiltonian systems. A very recent approach on
the Lagrangian side, in Euler-Poincaré form, has been developed in [21] and [48], where both the position and the momentum of the system are (independently) randomly perturbed, as well as the Lagrangian.

The stochastic version of Euler-Poincaré reduction introduced in [2] is closer in spirit to Feynman's viewpoint and, particularly, to the approach initiated in the eighties by Zambrini (c.f. 84] and references therein as well as [83, (74]). It uses as a main tool the notion of generalized (or mean-value) derivative in order to remove the contribution of the martingale part of the stochastic Lagrangian paths. This derivative has been introduced in stochastic dynamics by E. Nelson ([75]). We also refer to [1, 2, 19, 56] and references therein for various extensions on infinite dimensional spaces and applications of this derivative in stochastic Euler-Poincaré reduction.

The crucial idea is that the generalized derivative contains a contraction term induced by noise (stochastic force) which gives rise to a second order operator (such as the Laplacian) in the velocity equation of the stochastic model in continuum mechanics. Then, the stochastic reduction procedure leads to characterizations of various partial differential equations whose viscous term only appears in relation with the Laplacian, such as the incompressible the Navier-Stokes or the viscous Camassa-Holm equations. This stochastic Euler-Poincaré reduction is formulated on the group of volume preserving diffeomorphisms and the Lagrangian variables correspond to semimartingales.

The theory of reduction of variational principles of mechanical systems with advected parameters, leading to Euler-Poincaré equations coupled with advection equations, and hence associated to semidirect products, has been been developed in [49]. For continuum mechanical models, this method is particularly useful to characterize several kinds of evolutionary partial differential equations arising in conservative compressible fluids, such as the compressible Euler and ideal MHD equations (see, e.g., [49, Section 7]). Therefore, a first natural question arises whether it is possible to find a stochastic Euler-Poincaré reduction method that would characterize equations with viscous terms in compressible fluids, such as the compressible Navier-Stokes equation or the viscous compressible MHD equation. The main difficulty is that the generalized derivative, alluded to above, is not capable by itself to generate these viscosity terms since they do not appear only in connection with the generators of the underlying stochastic Lagrangian paths as in the case of incompressible fluids. The second natural question, amplifying the first one, is whether one can formulate a stochastic reduction procedure that would lead to interesting stochastic partial differential equations, appropriate for applications to continuum mechanics.

It is well known ([49) that the Euler-Poincaré formulation naturally leads to Kelvin circulation theorems. The classical Kelvin Circulation Theorem for barotropic ideal fluids states that the circulation of the velocity around a closed loop moving with the fluid is constant in time. This statement is intimately connected to Poisson geometric properties of Euler's ideal fluid equations (it characterizes the symplectic leaves in the phase space of Euler's equations; see [70]) and has important applications, for example,
in the Lyapunov stability analysis of stationary solutions (see, e.g., [3, 4, 51, 50]). For more general fluids, this theorem fails; instead of the vanishing of the time derivative of the circulation around a closed loop moving with the fluid, there is an explicit right hand side, responsible for generating circulation, involving advected quantities and the potential energy of the material. These identities are also known under the same name. For a general abstract formulation and a large class of examples of such Kelvin Circulation Theorems, see [49], [50]. In addition, these Kelvin Circulation identities are equivalent to reformulations of the equations of motion that turn out to be convenient for the qualitative study of the fluid. It is natural hence to seek for a counterpart of such Kelvin-Noether identities appearing in stochastic Euler-Poincaré reduction.

The main purpose in the paper is to solve the questions mentioned above. We summarize now the main achievements of the paper.
(1) We introduce a contraction matrix for the stochastic Lagrangian paths, which is different from the generalized derivative described above. This contraction matrix gives rise to a contraction force term in the action functional, capable to access separately, via reduction, each viscosity term, introduced usually by physical considerations, in the continuum mechanical model. In particular, we deduce the compressible Navier-Stokes and the viscous compressible MHD equations (Section (5) only from our stochastic variational principle, without any appeal to thermodynamics.
(2) We study random action functionals, by introducing an additional stochastic force. Various stochastic partial differential equations, such as stochastic (both compressible and incompressible) Navier-Stokes or Euler equations and stochastic viscous MHD equations, are deduced from our stochastic reduction procedure.
(3) We derive Euler-Poincaré equations for stochastic processes defined on semidirect product Lie algebras and give the associated deterministic constrained variational principle when the stochastic force (in the action functional) vanishes. In other words, we develop the semidirect Euler-Poincaré reduction for a large class of stochastic systems.
(4) We prove a stochastic version of the Kelvin-Noether Circulation Theorem for our stochastic reduction procedure. Compared with the result in [49, our (stochastic) evolution equations also depend on some martingales and some viscosity terms, in addition to the usual advected quantities (Section 4).

As discussed earlier, the generalized derivative only produces a trace term on the contraction part of the associated stochastic Lagrangian path. In order to obtain different viscosity terms in the models of continuum mechanics, we have to investigate in more detail the effect of the contraction induced by the martingale term. To do this, we introduce a contraction matrix, which carries much more information, involving
each entry in the matrix, and not just their sum (as is the case for the generalized derivative).

Moreover, partially inspired by [21] and 48], we also consider random perturbations of the action functionals so that the corresponding critical points satisfy a stochastic differential equation (a stochastic partial differential equation in the infinite dimensional case). Therefore, our action functionals have integrands that consist of three parts: the Lagrangian, a contraction force, and a stochastic force, which model the Lagrangian structure, the viscosity, and the stochastic (martingale) nature of the action.

Plan of the paper. In Section 2, we recall some basic probability notions necessary for the rest of the paper and give the crucial definition of the contraction matrix and martingale part for group valued semimartingales. Section 3 contains the first main result of the paper, namely the stochastic semidirect product Euler-Poincaré reduction for finite dimensional Lie groups, both in left and right-invariant versions. We give the deterministic variational principle and the reduced equations of motion as well as their random deformations. In Section 4 we derive a stochastic Kelvin-Noether theorem. Section 5 presents the second main result of the paper, the reduction from the material to the spatial representation in infinite dimensions, which applies to the compressible Navier-Stokes equation and to the stochastic compressible magnetohydrodynamics equations. The stochastic reduction process recovers the standard deterministic equations in Eulerian representation as well as their random deformations.

## 2 The derivative for semimartingales

In [2], we gave the notion of generalized derivative for semimartingales taking values on some topological groups. In this section, we decompose a $G$-valued semimartingale (when the dimension of $G$ is finite, see, e.g. [26]) into its velocity part, martingale part, and contraction part (matrix), which is crucial for our stochastic reduction procedure.

### 2.1 Some probability notions

We review in this subsection some basic notions of stochastic analysis on Euclidean spaces. We recall the concepts omitting the proofs, which can be found, for example, in 53 .

We denote $\mathbb{R}^{+}:=[0, \infty[$. Let $(\Omega, \mathcal{P}, \mathbb{P})$ be a probability space. Suppose we are given a family $\left(\mathcal{P}_{t}\right)_{t \in \mathbb{R}^{+}}$of sub- $\sigma$-algebras of $\mathcal{P}$ which is non-decreasing (namely, $\mathcal{P}_{s} \subset \mathcal{P}_{t}$ for $0 \leq s \leq t$ ) and right-continuous, i.e., $\cap_{\epsilon>0} \mathcal{P}_{t+\epsilon}=\mathcal{P}_{t}$ for all $t \in \mathbb{R}^{+}$. We then say that the probability space is endowed with a non-decreasing filtration $\left(\mathcal{P}_{t}\right)_{t \in \mathbb{R}^{+}}$. A stochastic process $X: \mathbb{R}^{+} \times \Omega \rightarrow \mathbb{R}$ is $\left(\mathcal{P}_{t}\right)$-adapted if $X(t, \cdot): \Omega \rightarrow \mathbb{R}^{+}$is $\mathcal{P}_{t^{-}}$-measurable for every $t \geq 0$. Typically, a filtration describes the past history of a process: one starts with a
process $X$ and defines $\mathcal{P}_{t}$ to be the sigma-algebra generated by all sets $X(s, \cdot)^{-1}(B)$, with $0 \leq s \leq t$ and $B$ a Borel subset in $\mathbb{R}$. Then the process $X$ is automatically $\left(\mathcal{P}_{t}\right)$-adapted.

A stochastic process $M: \mathbb{R}^{+} \times \Omega \rightarrow \mathbb{R}$ is a ( $\mathbb{R}$-valued) martingale with respect to $\left(\mathcal{P}_{t}\right)_{t \geq 0}$ if
(i) $\mathbb{E}\left|M_{\omega}(t)\right|<\infty$ for all $t \geq 0$;
(ii) $M_{\omega}(t)$ is $\left(\mathcal{P}_{t}\right)$-adapted;
(iii) $\left.\mathbb{E}_{s}\left(M_{\omega}(t)\right)\right)=M_{\omega}(s)$ a.s. for all $0 \leq s<t$.

In the above definition, $\mathbb{E}$ denotes the expectation of the random variable with respect to the probability measure $\mathbb{P} ; \mathbb{E}_{s}\left(M_{\omega}(t)\right):=\mathbb{E}\left[M_{\omega}(t) \mid \mathcal{P}_{s}\right]$, for each $s \geq 0$, is the conditional expectation of the random variable $M_{\omega}(t), t>s$, relative to the $\sigma$-algebra $\left(\mathcal{P}_{s}\right)$, i.e., $\Omega \ni \omega \mapsto \mathbb{E}_{s}\left[M_{\omega}(t)\right] \in \mathbb{R}$ is a $\mathcal{P}_{s}$-measurable function satisfying

$$
\mathbb{E}\left[\mathbb{E}_{s}\left[M_{\omega}(t)\right] \chi_{A}(\omega)\right]=\mathbb{E}\left[M_{\omega}(t) \chi_{A}(\omega)\right], \quad \forall A \in \mathcal{P}_{s}
$$

where $\chi_{A}$ is the characteristic function of the set $A$. Thus, condition (iii) is equivalent to $\mathbb{E}\left[\left(M_{\omega}(t)-M_{\omega}(s)\right) \chi_{A}(\omega)\right]=0$ for all $A \in \mathcal{P}_{s}$ and all $t, s \in \mathbb{R}$ satisfying $t>s \geq 0$.

In this paper we shall only consider processes defined on compact time intervals $[0, T]$ which have continuous sample paths (i.e., continuous with respect to the time variable $t$ for almost all $\omega \in \Omega$ ).

If a martingale $M_{\omega}(\cdot): \mathbb{R}^{+} \rightarrow \mathbb{R}$ is continuous for a.s. $\omega \in \Omega$ and $\mathbb{E}\left[M_{\omega}(t)^{2}\right]<\infty$ for all $t \geq 0$, we say that $M$ has a quadratic variation $\left\{\llbracket M_{\omega}, M_{\omega} \rrbracket_{t} \mid t \in[0, T]\right\}$ if $M_{\omega}^{2}(t)-\llbracket M_{\omega}, M_{\omega} \rrbracket_{t}$ is a martingale with respect to $\left(\mathcal{P}_{t}\right)_{t \geq 0}$, and $\llbracket M_{\omega}, M_{\omega} \rrbracket .: \mathbb{R}^{+} \rightarrow \mathbb{R}$ is a continuous, non-decreasing process with $\llbracket M_{\omega}, M_{\omega} \rrbracket_{0}=0$ for a.s. $\omega \in \Omega$. Such a process is unique and coincides with the following limit (convergence in probability),

$$
\lim _{n \rightarrow \infty} \sum_{t_{i}, t_{i+1} \in \sigma_{n}}\left(M_{\omega}\left(t_{i+1}\right)-M_{\omega}\left(t_{i}\right)\right)^{2}
$$

where $\sigma_{n}$ is a partition of the interval $[0, t]$ and the mesh converges to zero as $n \rightarrow \infty$. Actually, the definition of the quadratic variation requires only right-continuity of $M$.

Moreover, for two martingales $M$ and $N$, under the same assumptions and conventions as given above, one can also define their covariation

$$
\llbracket M_{\omega}, N_{\omega} \rrbracket_{t}:=\lim _{n \rightarrow \infty} \sum_{t_{i}, t_{i+1} \in \sigma_{n}}\left(M_{\omega}\left(t_{i+1}\right)-M_{\omega}\left(t_{i}\right)\right)\left(N_{\omega}\left(t_{i+1}\right)-N_{\omega}\left(t_{i}\right)\right),
$$

which extends the notion of quadratic variation. Clearly,

$$
2 \llbracket M_{\omega}, N_{\omega} \rrbracket_{t}=\llbracket M_{\omega}+N_{\omega}, M_{\omega}+N_{\omega} \rrbracket_{t}-\llbracket M_{\omega}, M_{\omega} \rrbracket_{t}-\llbracket N_{\omega}, N_{\omega} \rrbracket_{t} .
$$

More generally, one can consider local martingales. A stopping time is a random variable $\tau: \Omega \rightarrow \mathbb{R}^{+}$such that for all $t \geq 0,\{\omega \in \Omega \mid \tau(\omega) \leq t\} \in \mathcal{P}_{t}$. Then, a
stochastic process $M$ is a local martingale if there exists a sequence of stopping times $\left\{\tau_{n} \mid n \geq 1\right\}$, such that $\lim _{n \rightarrow \infty} \tau_{n}(\omega)=\infty$ a.s., and $M_{\omega}^{n}(t):=M_{\omega}\left(t \wedge \tau_{n}(\omega)\right)$ is a square integrable martingale for all $n \geq 1$, where $t \wedge \tau_{n}(\omega):=\min \left(t, \tau_{n}(\omega)\right)$. Thus, for a local martingale $M$, we define $\llbracket M_{\omega}, M_{\omega} \rrbracket_{t}:=\llbracket M_{\omega}^{n}, M_{\omega}^{n} \rrbracket_{t}$ if $t \leq \tau_{n}(\omega)$.

A real-valued Brownian motion is a martingale $W(t)$ with continuous sample paths, $t \in \mathbb{R}^{+}$, such that $W^{2}(t)-t$ is a martingale; or, equivalently, such that $\llbracket W_{\omega}, W_{\omega} \rrbracket_{t}=t$ for a.s. $\omega \in \Omega$.

A stochastic process $X: \Omega \times[0, T] \rightarrow \mathbb{R}$ is a (local) semimartingale with respect to the non-decreasing filtration $\left(\mathcal{P}_{t}\right)_{t \geq 0}$ if, for every $t \geq 0$, it can be decomposed into a sum

$$
X_{\omega}(t)=X_{\omega}(0)+M_{\omega}(t)+A_{\omega}(t)
$$

where $M$ is a local martingale with respect to $\left(\mathcal{P}_{t}\right)_{t \geq 0}$ such that $M_{\omega}(0)=0$ and $A$ is a càdlàg $\left(\mathcal{P}_{t}\right)_{t \geq 0}$-adapted process of locally bounded variation with $A_{\omega}(0)=0$ a.s. (càdlàg $=$ "continue à droite, limite à gauche" means, by definition, that $A$ is rightcontinuous with left limits at each $t \geq 0$; however, we consider only processes that are continuous in the time variable $t$, which is a standing assumption throughout this paper).

For a (local) semimartingale we define $\llbracket X_{\omega}, X_{\omega} \rrbracket_{t}:=\llbracket M_{\omega}, M_{\omega} \rrbracket_{t}$.
Martingales and, in particular, Brownian motion, are not (a.s.) differentiable in time (unless they are constant); therefore, one cannot integrate with respect to martingales as one does with respect to functions of bounded variation. We recall the definition of the two most commonly used stochastic integrals, the Itô and the Stratonovich integrals.

If $X$ and $Y$ are real-valued semimartingales with continuous sample paths such that for some $T>0$,

$$
\mathbb{E}\left[\int_{0}^{T}\left|X_{\omega}(t)\right|^{2} d t+\int_{0}^{T}\left|Y_{\omega}(t)\right|^{2} d t\right]<\infty
$$

the Itô stochastic integral in the time interval $[0, t], 0<t \leq T$, with respect to $Y$ is defined as the limit in probability (if the limit exists) of the sums

$$
\int_{0}^{t} X_{\omega}(s) d Y_{\omega}(s)=\lim _{n \rightarrow \infty} \sum_{t_{i}, t_{i+1} \in \sigma_{n}} X_{\omega}\left(t_{i}\right)\left(Y_{\omega}\left(t_{i+1}\right)-Y_{\omega}\left(t_{i}\right)\right)
$$

where $\sigma_{n}$ is a partition of the interval $[0, t]$ with mesh converging to zero as $n \rightarrow \infty$.
The Stratonovich stochastic integral is defined by

$$
\int_{0}^{t} X_{\omega}(s) \delta Y_{\omega}(s)=\lim _{n \rightarrow \infty} \sum_{t_{i}, t_{i+1} \in \sigma_{n}} \frac{\left(X_{\omega}\left(t_{i}\right)+X_{\omega}\left(t_{i+1}\right)\right)}{2}\left(Y_{\omega}\left(t_{i+1}\right)-Y_{\omega}\left(t_{i}\right)\right)
$$

whenever such limit exists.

These integrals do not coincide, in general, even though $X$ is a process with continuous sample paths (due to the lack of differentiability of the paths of $Y$ ). The Itô and the Stratonovich integrals are related by

$$
\begin{equation*}
\int_{0}^{t} X_{\omega}(s) \delta Y_{\omega}(s)=\int_{0}^{t} X_{\omega}(s) d Y_{\omega}(s)+\frac{1}{2} \int_{0}^{t} d \llbracket X_{\omega}, Y_{\omega} \rrbracket_{s} \tag{2.1}
\end{equation*}
$$

If $f \in C^{2}(\mathbb{R})$, Itô's formula states that

$$
\begin{equation*}
f\left(X_{\omega}(t)\right)=f\left(X_{\omega}(0)\right)+\int_{0}^{t} f^{\prime}\left(X_{\omega}(s)\right) d X_{\omega}(s)+\frac{1}{2} \int_{0}^{t} f^{\prime \prime}\left(X_{\omega}(s)\right) d \llbracket X_{\omega}, X_{\omega} \rrbracket_{s} \tag{2.2}
\end{equation*}
$$

This formula, for Stratonovich integrals, reads,

$$
f\left(X_{\omega}(t)\right)=f\left(X_{\omega}(0)\right)+\int_{0}^{t} f^{\prime}\left(X_{\omega}(s)\right) \delta X_{\omega}(s)
$$

One advantage of Stratonovich integrals is that they allow the use of the same rules as those of the standard deterministic differential calculus. On the other hand an Itô integral with respect to a martingale $M$ is again a martingale (under the integrability condition $\left.\mathbb{E}\left[\int_{0}^{T}\left|X_{\omega}(t)\right|^{2} d \llbracket M_{\omega}, M_{\omega} \rrbracket_{t}\right]<\infty\right)$, a very important property. For example, we have, as an immediate consequence, that $\mathbb{E}_{s}\left[\int_{s}^{t} X_{\omega}(r) d M_{\omega}(r)\right]=0$ for all $0 \leq s<t$. This property does not hold for Stratonovich integrals.

In higher dimensions, the difference between the Stratonovich and the Itô integral in Itô's formula is given in terms of the Hessian of $f$ (see Subsection (2.2). In fact, suppose that $X$ is an $\mathbb{R}^{d}$-valued semimartingale; then Itô's formula in $d$-dimensions (see also (2.2)) states that, for every $f \in C^{2}\left(\mathbb{R}^{d}\right)$,
$f\left(X_{\omega}(t)\right)=f\left(X_{\omega}(0)\right)+\sum_{i=1}^{d} \int_{0}^{t} \partial_{i} f\left(X_{\omega}(s)\right) d X_{\omega}^{i}(s)+\frac{1}{2} \sum_{i, j=1}^{d} \int_{0}^{t} \partial_{i, j}^{2} f\left(X_{\omega}(s)\right) d \llbracket X_{\omega}^{i}, X_{\omega}^{j} \rrbracket_{s}$

$$
\begin{equation*}
=f\left(X_{\omega}(0)\right)+\sum_{i=1}^{d} \int_{0}^{t} \partial_{i} f\left(X_{\omega}(s)\right) \delta X_{\omega}^{i}(s) \tag{2.3}
\end{equation*}
$$

For independent Brownian motions $W^{i}, i=1, \ldots, k$, we have

$$
\begin{equation*}
d \llbracket W_{\omega}^{i}, W_{\omega}^{j} \rrbracket_{t}=\delta_{i j} d t \tag{2.4}
\end{equation*}
$$

where $\delta_{i j}$ denotes the Kronecker delta symbol. As the covariation of semimartingales is determined by their martingale parts, the following identities hold (see, e.g., [53]),

$$
\begin{equation*}
d \llbracket W_{\omega}^{i}, \iota \rrbracket_{t}=0 \quad \forall i=1, \ldots, d, \quad d \llbracket \iota, \iota \rrbracket_{t}=0 \tag{2.5}
\end{equation*}
$$

where $\iota(t)=t$ is the identity (deterministic) function.

### 2.2 The generalized derivative and martingale part for (topological) group valued semimartingales

Let $G$ denote a topological group, endowed with a Banach manifold structure (possibly infinite dimensional) whose underlying topology is the given one, such that all left (or right) translations $L_{g}$ (resp. $R_{g}$ ) by arbitrary $g \in G$ are smooth maps, where $L_{g} h:=g h$, $R_{g} h:=h g$, for all $g, h \in G$. Given a vector $v \in T_{e} G$, we denote by $v^{L}$ (resp. $v^{R}$ ) the left (resp. right) invariant vector field whose value at the neutral element $e$ of $G$ is $v$, i.e., $v^{L}(g):=T_{e} L_{g} v$ (resp. $v^{R}(g):=T_{e} R_{g} v$ ), where $T_{e} L_{g}: T_{e} G \rightarrow T_{g} G$ is the tangent map (derivative) of $L_{g}$ (and similarly for $R_{g}$ ). The operation $\left[v_{1}, v_{2}\right]:=\left[v_{1}^{L}, v_{2}^{L}\right](e)$, for any $v_{1}, v_{2} \in T_{e} G$, defines a (left) Lie bracket on $T_{e} G$. In this paper, we denote by $\mathfrak{g}$ the Lie algebra of $G$, which is the set of left invariant vector fields on $G$. When working with right invariant vector fields, we shall still use, formally, the left Lie bracket defined above, i.e., we shall never work with right Lie algebras; the bracket defined by right invariant vector fields is equal to the negative of the left Lie bracket defined above. Denote, as usual, by $\operatorname{ad}_{u} v:=[u, v]$ the adjoint action of $T_{e} G$ on itself and by $\operatorname{ad}_{u}^{*}: T_{e}^{*} G \rightarrow T_{e}^{*} G$ its dual map (the coadjoint action of $T_{e} G$ on its dual $T_{e}^{*} G$ ).

Suppose that $\nabla$ is a left invariant linear connection on $G$, i.e., $\nabla_{v_{1}^{L}} v_{2}^{L}$ is a left invariant vector field, for any $v_{1}, v_{2} \in T_{e} G$. Then we define $\nabla_{v_{1}} v_{2}:=\nabla_{v_{1}^{L}} v_{2}^{L}(e)$ for all $v_{1}, v_{2} \in T_{e} G$. If right translation is smooth, in all the definitions above, we can replace left translation by right translation in a similar way. We also assume that the left invariant connection $\nabla$ is torsion free, namely

$$
\nabla_{v_{1}} v_{2}-\nabla_{v_{2}} v_{1}=\left[v_{1}, v_{2}\right], \quad \text { for all } \quad v_{1}, v_{2} \in T_{e} G
$$

For a fixed $g_{1} \in G$, let $T_{g_{2}} L_{g_{1}}: T_{g_{2}} G \rightarrow T_{g_{1} g_{2}} G$ be the tangent map (or derivative) of $L_{g_{1}}$ at the point $g_{2} \in G$.

Let $G$ be endowed with a left invariant linear torsion free connection $\nabla$. The corresponding Hessian $\operatorname{Hess} f(g): T_{g} G \times T_{g} G \rightarrow \mathbb{R}$ of $f \in C^{2}(G)$ at $g \in G$ is defined by

$$
\begin{equation*}
\operatorname{Hess} f(g)\left(v_{1}, v_{2}\right):=\tilde{v}_{1} \tilde{v}_{2} f(g)-\nabla_{\tilde{v}_{1}} \tilde{v}_{2} f(g), \quad v_{1}, v_{2} \in T_{g} G, \tag{2.6}
\end{equation*}
$$

where $\tilde{v}_{i}, i=1,2$, are arbitrary smooth vector fields on $G$ such that $\tilde{v}_{i}(g)=v_{i}$. Since the connection is torsion free, Hess $f(g)$ is a symmetric $\mathbb{R}$-bilinear form on each $T_{g} G$. In addition, Hess $f=\nabla^{2} f=\nabla \mathbf{d} f$ (see, e.g., [26]) is the covariant derivative associated with $\nabla$ of the one-form $\mathbf{d} f$, where $\mathbf{d}$ denotes the exterior differential.

Given a probability space $(\Omega, \mathcal{P}, \mathbb{P})$ endowed with a non-decreasing filtration $\left(\mathcal{P}_{t}\right)_{t \geq 0}$, a semimartingale with values in $G$ (with respect to $\left.\left(\mathcal{P}_{t}\right)_{t \geq 0}\right)$ is a $\mathcal{P}_{t}$-adapted stochastic process $g: \Omega \times \mathbb{R}^{+} \rightarrow G$ such that, for every function $f \in C^{2}(G), f \circ g: \Omega \times \mathbb{R}^{+} \rightarrow \mathbb{R}$ is a real-valued semimartingale (on $(\Omega, \mathcal{P}, \mathbb{P})$ ), as introduced in subsection 2.1 (see, e.g., [26] for the case of finite dimensional Lie groups).

A semimartingale with values in $G$ is a $\nabla$-(local) martingale if

$$
t \longmapsto f\left(g_{\omega}(t)\right)-f\left(g_{\omega}(0)\right)-\frac{1}{2} \int_{0}^{t} \operatorname{Hess} f\left(g_{\omega}(s)\right) d \llbracket g_{\omega}, g_{\omega} \rrbracket_{s} d s
$$

is a real-valued (local) martingale for any $f \in C^{2}(G)$, where $\llbracket g_{\omega}, g_{\omega} \rrbracket_{t}$ is the quadratic variation of $g_{\omega}$. If $G$ is a finite dimensional Lie group, then we have the following expression

$$
d \llbracket g_{\omega}, g_{\omega} \rrbracket_{t}:=d \llbracket \int_{0} \mathbf{P}_{s}^{-1} \delta g_{\omega}(s), \int_{0} \mathbf{P}_{s}^{-1} \delta g_{\omega}(s) \rrbracket_{t}
$$

where $\mathbf{P}_{t}: T_{g_{\omega}(0)} G \rightarrow T_{g_{\omega}(t)} G$ is the (stochastic) parallel translation along the (stochastic) curve $t \mapsto g_{\omega}(t)$ associated with the connection $\nabla$; see, e.g., [26] or [53]. Moreover, for some infinite dimensional groups $G$ (for example the diffeomorphism group on a torus), the quadratic variation is also well defined; we refer the reader to [2, 19] for details (see also Section 5 of this paper).

For a $G$-valued semimartingale $g_{\omega}(\cdot)$, suppose there exist an integer $m>0$ and $\mathcal{P}_{t}$-adapted processes $\mathbf{v}: \Omega \times \mathbb{R}^{+} \rightarrow T_{e} G, \mathbf{w}^{i}: \Omega \times \mathbb{R}^{+} \rightarrow T_{e} G, M^{i}: \Omega \times \mathbb{R}^{+} \rightarrow \mathbb{R}$, $1 \leq i \leq m$, such that $M^{i}$ is a ( $\mathbb{R}$-valued) martingale with continuous sample paths, and for every $f \in C^{2}(G)$,

$$
\begin{align*}
& f\left(g_{\omega}(t)\right)=f\left(g_{\omega}(0)\right)+\sum_{i=1}^{m} \int_{0}^{t}\left\langle\mathbf{d} f\left(g_{\omega}(s)\right), T_{e} L_{g_{\omega}(s)} \mathbf{w}_{\omega}^{i}(s)\right\rangle d M_{\omega}^{i}(s) \\
& \left.+\frac{1}{2} \sum_{i, j=1}^{m} \int_{0}^{t} \operatorname{Hess} f\left(g_{\omega}(s)\right)\right)\left(T_{e} L_{g_{\omega}(s)} \mathbf{w}_{\omega}^{i}(s), T_{e} L_{g_{\omega}(s)} \mathbf{w}_{\omega}^{j}(s)\right) d \llbracket M_{\omega}^{i}, M_{\omega}^{j} \rrbracket_{s}  \tag{2.7}\\
& +\frac{1}{2} \sum_{i, j=1}^{m} \int_{0}^{t}\left\langle\mathbf{d} f\left(g_{\omega}(s)\right), T_{e} L_{g_{\omega}(t)}\left(\nabla_{\mathbf{w}_{\omega}^{i}(s)} \mathbf{w}_{\omega}^{j}(s)\right)\right\rangle d \llbracket M_{\omega}^{i}, M_{\omega}^{j} \rrbracket_{s} \\
& +\frac{1}{2} \sum_{i=1}^{m} \int_{0}^{t}\left\langle\mathbf{d} f\left(g_{\omega}(s)\right), d \llbracket \mathbf{w}_{\omega}^{i}, M_{\omega}^{i} \rrbracket_{s}\right\rangle+\int_{0}^{t}\left\langle\mathbf{d} f\left(g_{\omega}(s)\right), T_{e} L_{g_{\omega}(s)} \mathbf{v}_{\omega}(s)\right\rangle d s .
\end{align*}
$$

For such a $G$-valued semimartingale $g_{\omega}$, having the form (2.7) above, the following is true,

$$
\begin{equation*}
d g_{\omega}(t)=T_{e} L_{g_{\omega}(t)}\left(\sum_{i=1}^{m} \mathbf{w}_{\omega}^{i}(t) \delta M_{\omega}^{i}(t)+\mathbf{v}_{\omega}(t) d t\right) \tag{2.8}
\end{equation*}
$$

Here $\delta$ denotes the Stratonovich integral (of the tangent vectors in $G$ ).
Note that although for a given left invariant connection $\nabla$, the choice of $\left\{\left(\mathbf{w}_{\omega}^{i}, M_{\omega}^{i}\right) \mid\right.$ $1 \leq i \leq m\}$ in (2.8) may not be unique, the decomposition into the martingale
part (which is $\sum_{i=1}^{m} \mathbf{w}_{\omega}^{i}(t) d M_{\omega}^{i}(t)$ ), and the drift part without contraction (which is $\left.T_{e} L_{g_{\omega}(t)} \mathbf{v}_{\omega}(s) d t\right)$ in (2.8) is unique. Then we define the velocity derivative of $g_{\omega}(\cdot)$ by

$$
\begin{equation*}
\frac{\mathscr{D} g_{\omega}(t)}{d t}:=T_{e} L_{g_{\omega}(t)} \mathbf{v}_{\omega}(t) \tag{2.9}
\end{equation*}
$$

and the stochastic differential with respect to the martingale part of $g_{\omega}(\cdot)$ by

$$
\begin{equation*}
d^{\Delta} g_{\omega}(t):=\sum_{i=1}^{m} T_{e} L_{g_{\omega}(t)}\left(\mathbf{w}_{\omega}^{i}(t) d M_{\omega}^{i}(t)\right) \tag{2.10}
\end{equation*}
$$

where $d M_{\omega}^{i}(t)$ denotes the Itô integral with respect to the martingale $M_{\omega}^{i}(t)$. Note that the two terms above do not depend on the choice of the left invariant connection $\nabla$.

In order to obtain the viscous terms in the associated stochastic Euler-Poincaré equation, we need to make a more detailed analysis of the contraction part of the semimartingale (or stochastic Lagrangian path) $g_{\omega}(\cdot)$. For a given left invariant connection $\nabla$ on $G$ and some fixed choice $\left\{\left(\mathbf{w}_{\omega}^{i}, M_{\omega}^{i}\right) \mid 1 \leq i \leq m\right\}$, where both $\mathbf{w}_{\omega}^{i}$ and $M_{\omega}^{i}$ are $\mathcal{P}_{t}$-adapted processes and $M_{\omega}^{i}$ are real valued martingales with continuous sample paths, we define the contraction matrix $\frac{\mathbf{D}^{\nabla,\left(w_{\omega}^{i}, M_{\omega}^{i}\right)_{i=1}^{m} g_{\omega}(t)}}{d t}$ as the following $T_{g_{\omega}(t)} G$-valued $m \times m$ matrix:

$$
\begin{align*}
\left(\frac{\mathbf{D}^{\nabla,\left(\mathbf{w}_{\omega}^{i}, M_{\omega}^{i}\right)_{i=1}^{m} g_{\omega}(t)}}{d t}\right)_{i, j}: & =T_{e} L_{g_{\omega}(t)}\left(\nabla_{\mathbf{w}_{\omega}^{i}(t)} \mathbf{w}_{\omega}^{j}(t) \frac{d \llbracket M_{\omega}^{i}, M_{\omega}^{j} \rrbracket_{t}}{d t}\right.  \tag{2.11}\\
& \left.+\frac{d \llbracket \mathbf{w}_{\omega}^{i}, M_{\omega}^{i} \rrbracket_{t}}{d t} 1_{\{i=j\}}\right), \quad 1 \leq i, j \leq m .
\end{align*}
$$

Therefore, we can split the differential of a $G$-valued semimartingale into the velocity part, the Hessian (second order ) term, the martingale part, and the contraction part (more accurately the contraction matrix). Intuitively, the velocity part could be seen as the direction where the particles flow, the martingale part represents their random fluctuations, while the contraction part describes the contraction effect from the noise.

The term $\left(\frac{\mathbf{D}^{\nabla,\left(\mathbf{w}_{\omega}^{i}, M_{\omega}^{i}\right)_{i=1}^{m} g_{\omega}(t)}}{d t}\right)_{i, j}$ corresponds to the contraction between the noises in vectors $\mathbf{w}_{\omega}^{i}$ and $\mathbf{w}_{\omega}^{j}$. Thus, the contraction matrix describes explicitly the behavior of the noises interaction along different vector fields (directions) $\left\{\mathbf{w}_{\omega}^{i}\right\}_{i=1}^{m}$.

Let

$$
\operatorname{Sum}\left(\frac{\mathbf{D}^{\nabla,\left(\boldsymbol{w}_{\omega}^{i}, M_{\omega}^{i}\right)_{i=1}^{m}} g_{\omega}(t)}{d t}\right):=\sum_{i, j=1}^{m}\left(\frac{\mathbf{D}^{\nabla,\left(\boldsymbol{w}_{\omega}^{i}, M_{\omega}^{i}\right)_{i=1}^{m} g_{\omega}(t)}}{d t}\right)_{i, j} \in T_{g_{\omega}(t)} G
$$

denote the sum of all entries of the matrix $\frac{\mathbf{D}^{\nabla,\left(\mathbf{w}_{\omega}^{i}, M_{\omega}^{i}\right)_{i=1}^{m} g_{\omega}(t)}}{d t}$; for each fixed $t$ this is a $T_{g_{\omega}(t)} G$-valued random variable.

Then it is easy to verify that for a $G$-valued semimartingale of the form (2.8) and any $f \in C^{2}(G)$, the process

$$
\begin{aligned}
N_{t}^{f} & \left.\left.:=f\left(g_{\omega}(t)\right)-f\left(g_{\omega}(0)\right)\right)-\frac{1}{2} \int_{0}^{t} \operatorname{Hess} f\left(g_{\omega}(s)\right)\right) d \llbracket g_{\omega}, g_{\omega} \rrbracket_{s} \\
& -\frac{1}{2} \int_{0}^{t}\left\langle\mathbf{d} f\left(g_{\omega}(s)\right), \operatorname{Sum}\left(\frac{\mathbf{D}^{\nabla,\left(\mathbf{w}_{\omega}^{i}, M_{\omega}^{i}\right)_{i=1}^{m} g_{\omega}(s)}}{d s}\right)\right\rangle-\int_{0}^{t}\left\langle\mathbf{d} f\left(g_{\omega}(s)\right), \frac{\mathscr{D} g_{\omega}(s)}{d s}\right\rangle d s
\end{aligned}
$$

is a real-valued local martingale.
We remark that by (2.9)-(2.11), the terms $\frac{\mathscr{Q}}{d t}, d^{\Delta}, \frac{\mathrm{D}^{\nabla,\left(w_{\omega}^{i}, M_{\omega}^{i}\right)_{i=1}^{m}}}{d t}$ are well defined for semimartingales with values in a finite dimensional Lie group as well as in some infinite dimensional groups (the diffeomorphism group on a torus for example); see, e.g., 2] or Section 5 below.

In the stochastic Euler-Poincaré reduction introduced in Section 3, the martingale part and the contraction part generate, respectively, the martingale term and the viscosity term in associated (stochastic) Euler-Poincaré equation.

Moreover, when $G$ is a finite dimensional compact Lie group, for a $G$-valued semimartingale $g_{\omega}(\cdot)$ of the form (2.8), we have the following equalities (see, e.g., [26])

$$
\begin{align*}
\frac{D^{\nabla} g_{\omega}(t)}{d t} & :=\mathbf{P}_{t}\left(\lim _{\epsilon \rightarrow 0} \mathbb{E}_{t}\left[\frac{\eta_{\omega}(t+\epsilon)-\eta_{\omega}(t)}{\epsilon}\right]\right) \\
& =\frac{1}{2} \operatorname{Sum}\left(\frac{\mathbf{D}^{\nabla,\left(\boldsymbol{w}_{\omega}^{i}, M_{\omega}^{i}\right)_{i=1}^{m} g_{\omega}(t)}}{d t}\right)+\frac{\mathscr{D} g_{\omega}(t)}{d t}, \tag{2.12}
\end{align*}
$$

where $\mathbf{P}_{t}: T_{e} G \rightarrow T_{g_{\omega}(t)} G$ is the stochastic parallel translation associated to $\nabla, \mathbb{E}_{t}[\cdot]=$ $\mathbb{E}\left[\cdot \mid \mathcal{P}_{t}\right]$ denotes the conditional expectation, and

$$
\eta_{\omega}(t)=\int_{0}^{t} \mathbf{P}_{s}^{-1} \delta g_{\omega}(s) \in T_{e} G
$$

Therefore, according to the definition, if a $G$-valued semimartingale $g_{\omega}(t)$ satisfies $\frac{D^{\nabla} g_{\omega}(t)}{d t}=0$, then $g_{\omega}(t)$ is a $\nabla$-martingale.

In fact, $\frac{D^{\nabla}}{d t}$ is the generalized derivative in [2], which is a generalization for groupvalued semimartingales of those in [19, ,75, 83, ,74, 84]; it contains a single term formed by the sum of all elements in the contraction matrix. The generalized derivative is sufficient to generate the viscosity terms (second order differential terms) in some partial differential equations through the stochastic reduction procedure. This is the case, for example, for the incompressible Navier-Stokes equation; see, e.g., [2, 19, 75, 83, 74, 84. However, for a large class of equations in fluid mechanics, the viscous terms do not depend only on such kind of contraction terms; see, e.g., the compressible

Navier-Stokes equation or the viscous MHD equation in Section 5. This is one of our motivations to introduce the decomposition of $\frac{D^{\nabla}}{d t}$ above.

The generalized derivative coincides with the drift of a diffusion processes. It was commonly used since the beginning of Stochastic Analysis but was first associated with a dynamical interpretation, as a mean velocity, in the context of Nelson's Stochastic Mechanics 75].

Given a $\mathbb{R}^{m}$-valued martingale $M_{\omega}(t)=\left(M_{\omega}^{1}(t), \ldots, M_{\omega}^{m}(t)\right), t \in[0, T]$, which has a continuous sample path, (non-random) vectors $H_{i} \in T_{e} G, 1 \leq i \leq m$, and a $\mathcal{P}_{t^{-}}$ adapted, $T_{e} G$-valued semi-martingale $u_{\omega}: \Omega \times[0, T] \rightarrow T_{e} G$, consider the following Stratonovich SDE on $G$,

$$
\left\{\begin{array}{l}
d g_{\omega}(t)=T_{e} L_{g_{\omega}(t)}\left(\sum_{i=1}^{m} H_{i} \delta M_{\omega}^{i}(t)+u_{\omega}(t) d t\right)  \tag{2.13}\\
g_{\omega}(0)=e
\end{array}\right.
$$

As explained in [26], given the connection $\nabla$, the difference (contraction term) between the Itô and Stratonovich integrals has the following form

$$
\begin{aligned}
& \sum_{i=1}^{m}\left(T_{e} L_{g_{\omega}(t)} H_{i} \delta M_{\omega}^{i}(t)-T_{e} L_{g_{\omega}(t)} H_{i} d M_{\omega}^{i}(t)\right) \\
& =\frac{1}{2} \sum_{i=1}^{m} d \llbracket\left(T_{e} L_{g_{\omega}(t)} H_{i}\right), M_{\omega}^{i}(t) \rrbracket_{t} \\
& =\frac{1}{2} \sum_{i, j=1}^{m} T_{e} L_{g_{\omega}(t)}\left(\nabla_{H_{i}} H_{j}\right) d \llbracket M_{\omega}^{i}, M_{\omega}^{j} \rrbracket_{t} .
\end{aligned}
$$

Therefore, equation (2.13) is equivalent to

$$
\left\{\begin{array}{l}
d g_{\omega}(t)=T_{e} L_{g_{\omega}(t)}\left(\sum_{i=1}^{m} H_{i} d M_{\omega}^{i}(t)+\frac{1}{2} \sum_{i, j=1}^{m} \nabla_{H_{i}} H_{j} d \llbracket M_{\omega}^{i}, M_{\omega}^{j} \rrbracket_{t}+u_{\omega}(t) d t\right),  \tag{2.14}\\
g_{\omega}(0)=e .
\end{array}\right.
$$

If $G$ is a finite dimensional Lie group, there exists a unique strong solution for (2.13) (c.f. [53], [26]) and hence also for (2.14). When $G$ is the diffeomorphism group on a torus and $u$ is less regular, a weak solution to (2.13) still exists ([2], [19]) under suitable conditions on $H_{i}$.

Applying Itô's formula to the solution $g_{\omega}(t)$ of (2.13) (see [26] for the case where $G$ is finite dimensional and [2, Section 4.2] or Section 5elow, for the case where $G$ is
the diffeomorphism group on a torus), for every $f \in C^{2}(G)$ we have,

$$
\begin{aligned}
f\left(g_{\omega}(t)\right) & =f\left(g_{\omega}(0)\right)+\sum_{i=1}^{m} \int_{0}^{t}\left\langle\mathbf{d} f\left(g_{\omega}(s)\right), T_{e} L_{g_{\omega}(s)} H_{i}\right\rangle d M_{\omega}^{i}(s) \\
& +\frac{1}{2} \int_{0}^{t} \operatorname{Hess} f\left(g_{\omega}(s)\right) d \llbracket g_{\omega}, g_{\omega} \rrbracket_{s}+\int_{0}^{t}\left\langle\mathbf{d} f\left(g_{\omega}(s)\right), T_{e} L_{g_{\omega}(s)} u_{\omega}(s)\right\rangle d s \\
& +\frac{1}{2} \sum_{i, j=1}^{m} \int_{0}^{t}\left\langle\mathbf{d} f\left(g_{\omega}(s)\right), T_{e} L_{g_{\omega}(s)} \nabla_{H_{i}} H_{j}\right\rangle d \llbracket M_{\omega}^{i}, M_{\omega}^{j} \rrbracket_{s}
\end{aligned}
$$

Actually, this last equality, valid for each $f \in C^{2}(G)$, is a characterization of the solution of the stochastic differential equation (2.13) (or (2.14)), in a weak sense.

Clearly, by the definition (2.9) and (2.10), we have

$$
\begin{align*}
& \frac{\mathscr{D} g_{\omega}(t)}{d t}=T_{e} L_{g_{\omega}(t)} u_{\omega}(t), \\
& d^{\Delta} g_{\omega}(t)=\sum_{i=1}^{m}\left(T_{e} L_{g_{\omega}(t)} H_{i}\right) d M_{\omega}^{i}(t),  \tag{2.15}\\
& \left(\frac{\mathbf{D}^{\nabla,\left(H_{i}, M_{\omega}^{i}\right)_{i=1}^{m}} g_{\omega}(t)}{d t}\right)_{i, j}=T_{e} L_{g_{\omega}(t)}\left(\nabla_{H_{i}} H_{j}\right) \frac{d \llbracket M_{\omega}^{i}, M_{\omega}^{j} \rrbracket_{t}}{d t} .
\end{align*}
$$

## 3 Stochastic semidirect product Euler-Poincaré reduction

In this section, partially inspired by [2], [21], [48], we extend the deterministic semidirect product Euler-Poincaré reduction, formulated and developed in [49], to the stochastic setting. By such a reduction, we obtain a large class of partial differential equations and stochastic partial differential equations with various viscosity terms; see Section 5 below.

### 3.1 Left invariant version

Let $U$ be a vector space and $U^{*}$ its dual, also denote by $\langle\cdot, \cdot\rangle_{U}: U^{*} \times U \rightarrow \mathbb{R}$ the (weak) duality pairing. Suppose that $G$ is a group endowed with a manifold structure making it into a topological group whose left translation is smooth. As discussed in subsection 2.2, the tangent space $T_{e} G$ to $G$ at the identity element $e \in G$ is (isomorphic to) a Lie algebra. Assume that $G$ has a left representation on $U$; therefore, there are naturally induced left representations of the group $G$ and the Lie algebra $T_{e} G$ on $U$ and $U^{*}$. All
actions will be denoted by concatenation. Let $\langle\cdot, \cdot\rangle_{T_{e} G}: T_{e}^{*} G \times T_{e} G \rightarrow \mathbb{R}$ be the (weak) duality pairing between $T_{e}^{*} G$ and $T_{e} G$. Define the operator $\diamond: U \times U^{*}: \rightarrow T_{e}^{*} G$ by

$$
\begin{equation*}
\langle a \diamond \alpha, v\rangle_{T_{e} G}:=-\langle v \alpha, a\rangle_{U}=\langle\alpha, v a\rangle_{U}, \quad v \in T_{e} G, \quad a \in U, \quad \alpha \in U^{*} \tag{3.1}
\end{equation*}
$$

In fact, $a \diamond \alpha$ is the value at $(a, \alpha)$ of the momentum map $U \times U^{*} \rightarrow T_{e}^{*} G$ of the cotangent lifted action induced by the left representation of $G$ on $U$.

Let $\mathscr{S}(G)$ denote the collection of $G$-valued semimartingales with smooth coefficients defined on the time interval $[0, T]$. Let $\mathscr{M}_{m}:=\left\{\left(a_{i, j}\right)_{i, j=1}^{m} \mid a_{i, j} \in T_{e} G\right\}$ be the vector space of all $m \times m, T_{e} G$-valued matrices. Define $\mathscr{M}:=\cup_{m=1}^{\infty} \mathscr{M}_{m}$. In order to define the contraction matrix for $g_{\omega} \in \mathscr{S}(G)$ having the form (2.8), we need to fix a pair $\left\{\left(\mathbf{w}_{\omega}^{i}, M_{\omega}^{i}\right) \mid 1 \leq i \leq m\right\}$ in the martingale part of (2.8) (the first term of the right hand side of (2.8), i.e., the Itô integral). The hypotheses on this set of pairs remain the same: $\mathbf{w}_{\omega}^{i}$ and $M_{\omega}^{i}$ are $\mathcal{P}_{t^{-}}$adapted processes and $M_{\omega}^{i}$ are real valued martingales with continuous sample paths, for all $i=1, \ldots, m$. We denote by $\left(g_{\omega}, \mathbf{w}_{\omega}^{i}, M_{\omega}^{i}\right)_{i=1}^{m}$ an element in $\mathscr{S}(G)$ with a fixed choice $\left\{\left(\mathbf{w}_{\omega}^{i}, M_{\omega}^{i}\right) \mid 1 \leq i \leq m\right\}$ in (2.8). Let $\mathscr{\mathscr { S } ( G )}$ be the collection of all these triples.

Given a (left invariant) linear connection $\nabla$ on $G$, a point $\alpha_{0} \in U^{*}$, a random (Lagrangian) function $l: \Omega \times[0, T] \times T_{e} G \times U^{*} \rightarrow \mathbb{R}$ such that $l_{\omega}(t)$ is $\mathcal{P}_{t}$-adapted for each $t \in[0, T]$, a (viscosity force) function $p: \mathscr{M} \times \mathscr{M} \times T_{e} G \rightarrow \mathbb{R}$, a (stochastic force) function $q:[0, T] \times T_{e} G \times U^{*} \rightarrow T_{e}^{*} G$, vectors $V_{i} \in T_{e} G$ (which are non-random), $1 \leq i \leq k$, and an $\mathbb{R}^{k}$-valued martingale $N_{\omega}(t)$, we define a stochastic action functional $J^{\nabla, \alpha_{0}, l, p, q,\left(V_{i}, N_{\omega}^{i}\right)_{i=1}^{k}}: \widetilde{\mathscr{S}(G)} \times \widetilde{\mathscr{S}(G)} \times \mathscr{S}(G) \rightarrow \mathbb{R}$ by

$$
\begin{align*}
& J^{\nabla, \alpha_{0}, l, p, q,\left(V_{i}, N_{\omega}^{i}\right)_{i=1}^{k}}\left(\left(g_{\omega}^{1}, \mathbf{w}_{\omega}^{1, i}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}},\left(g_{\omega}^{2}, \mathbf{w}_{\omega}^{2, i}, M_{\omega}^{2, i}\right)_{i=1}^{m_{2}}, g_{\omega}^{3}\right) \\
&:= \int_{0}^{T} l_{\omega}\left(t, T_{g_{\omega}^{1}(t)} L_{g_{\omega}^{1}(t)^{-1}} \frac{\mathscr{D} g_{\omega}^{1}(t)}{d t}, \alpha_{\omega}(t)\right) d t \\
&+\int_{0}^{T} p\left(T_{g_{\omega}^{1}(t)} L_{g_{\omega}^{1}(t)^{-1}} \frac{\mathbf{D}^{\nabla,\left(\mathbf{w}_{\omega}^{1, i}, M_{\omega}^{i, 1}\right)_{i=1}^{m_{1}}} g_{\omega}^{1}(t)}{d t}, T_{g_{\omega}^{2}(t)^{2}} L_{g_{\omega}^{2}(t)^{-1}} \frac{\mathbf{D}^{\nabla,\left(\mathbf{w}_{\omega}^{2, i}, M_{\omega}^{i, 2}\right)_{i=1}^{m_{2}}} g_{\omega}^{2}(t)}{d t},\right. \\
&\left.\quad T_{g_{\omega}^{1}(t)} L_{g_{\omega}^{1}(t)^{-1}} \frac{\mathscr{D} g_{\omega}^{1}(t)}{d t}\right) d t  \tag{3.2}\\
&+\int_{0}^{T}\left\langle q\left(T_{g_{\omega}^{1}(t)} L_{g_{\omega}^{1}(t)^{-1}} \frac{\mathscr{D} g_{\omega}^{1}(t)}{d t}, \alpha_{\omega}(t)\right), T_{g_{\omega}^{1}(t)} L_{g_{\omega}^{1}(t)^{-1}} d^{\Delta} g_{\omega}^{1}(t)\right\rangle \\
&-\sum_{i=1}^{k} \int_{0}^{T}\left\langle q\left(T_{g_{\omega}^{1}(t)} L_{g_{\omega}^{1}(t)^{-1}} \frac{\mathscr{D} g_{\omega}^{1}(t)}{d t}, \alpha_{\omega}(t)\right), V_{i} d N_{\omega}^{i}(t)\right\rangle
\end{align*}
$$

where $\left(g_{\omega}^{1}, \mathbf{w}_{\omega}^{1, i}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}} \in \widetilde{\mathscr{S}(G)},\left(g_{\omega}^{2}, \mathbf{w}_{\omega}^{2, i}, M_{\omega}^{2, i}\right)_{i=1}^{m_{2}} \in \widetilde{\mathscr{S}(G)}, T_{g_{\omega}^{1}(t)} L_{g_{\omega}^{1}(t)-1} d^{\Delta} g_{\omega}^{1}(t)$ corresponds to the Itô integral on the vector space $T_{e} G$, and

$$
\begin{equation*}
\alpha_{\omega}(t):=g_{\omega}^{3}(t)^{-1} \alpha_{0} \tag{3.3}
\end{equation*}
$$

Remark 3.1. We explain intuitively why we want the action functional $J^{\nabla, \alpha_{0}, l, p, q,\left(V_{i}, N_{\omega}^{i}\right)_{i=1}^{k}}$ to have the form (3.2). When the Lagrangian $l(v)=\frac{1}{2}\langle v, v\rangle_{\mathbb{R}^{d}}, v \in \mathbb{R}^{d}$, is the kinetic energy, for a stochastic Lagrangian path $d g_{\omega}(t)=d M_{\omega}(t)+u_{\omega}(t) d t=d^{\Delta} g(t)+\frac{\mathscr{D} g_{\omega}(t)}{d t} d t$ with $M_{\omega}(t)$ being a $\mathbb{R}^{d}$-valued martingale, we can formally write the kinetic energy as follows

$$
\begin{aligned}
\int_{0}^{T} l\left(T_{g_{\omega}(t)} L_{g_{\omega}(t)^{-1}} d g_{\omega}(t)\right)= & \int_{0}^{T} \frac{1}{2}\left\langle T_{g_{\omega}(t)} L_{g_{\omega}(t)^{-1}} \frac{\mathscr{D} g_{\omega}(t)}{d t}, T_{g_{\omega}(t)^{\prime}} L_{g_{\omega}(t)^{-1}} \frac{\mathscr{D} g_{\omega}(t)}{d t}\right\rangle d t \\
& +\int_{0}^{T}\left\langle T_{g_{\omega}(t)} L_{g_{\omega}(t)^{-1}} \frac{\mathscr{D} g_{\omega}(t)}{d t}, T_{g_{\omega}(t)} L_{g_{\omega}(t)^{-1}} d^{\Delta} g_{\omega}(t)\right\rangle \\
& +\frac{1}{2} \int_{0}^{T}\left\langle T_{g_{\omega}(t)} L_{g_{\omega}(t)^{-1}} \frac{d^{\Delta} g_{\omega}(t)}{d t}, T_{g_{\omega}(t)} L_{g_{\omega}(t)^{-1}} \frac{d^{\Delta} g_{\omega}(t)}{d t}\right\rangle \\
& :=I_{1}+I_{2}+I_{3} .
\end{aligned}
$$

Here $I_{1}$ represents the kinetic energy of the velocity: it is the action functional in the deterministic case, based on which a standard Euler-Poincaré equation is obtained via the reduction procedure. The summand $I_{2}$ contains a stochastic differential for the martingale part of $d g_{\omega}(t)$ and we can interpret it as the Itô integral with respect to this martingale. Concerning $I_{3}$, since it is not well-defined (it is almost-everywhere infinite), we drop this term in the action functional.

Besides the kinetic energy, we could also add some extra terms of the form

$$
\sum_{i=1}^{k} \int_{0}^{T}\left\langle T_{g_{\omega}(t)} L_{g_{\omega}(t)^{-1}} \frac{\mathscr{D} g_{\omega}(t)}{d t}, V_{i} d N_{\omega}^{i}(t)\right\rangle
$$

which represents the external stochastic fluctuation for the velocity.
Therefore, we define an action functional as follows

$$
\begin{aligned}
J\left(g_{\omega}(\cdot)\right)= & \int_{0}^{T} \frac{1}{2}\left\langle T_{g_{\omega}(t)} L_{g_{\omega}(t)^{-1}} \frac{\mathscr{D} g_{\omega}(t)}{d t}, T_{g_{\omega}(t)} L_{g_{\omega}(t)^{-1}} \frac{\mathscr{D} g_{\omega}(t)}{d t}\right\rangle d t \\
& +\int_{0}^{T}\left\langle T_{g_{\omega}(t)} L_{g_{\omega}(t)^{-1}} \frac{\mathscr{D} g_{\omega}(t)}{d t}, T_{g_{\omega}(t)} L_{g_{\omega}(t)^{-1}} d^{\Delta} g_{\omega}(t)\right\rangle \\
& -\sum_{i=1}^{k} \int_{0}^{T}\left\langle T_{g_{\omega}(t)} L_{g_{\omega}(t)^{-1}} \frac{\mathscr{D} g_{\omega}(t)}{d t}, V_{i} d N_{\omega}^{i}(t)\right\rangle
\end{aligned}
$$

which, when we add the viscous term (defined by a viscosity force $q$ and the contraction matrix for $g_{\omega}$ ), is a particular case of (3.2) for $q(v, a)=v, \forall v \in \mathbb{R}^{d}, a \in U^{*}$ (we use here the identification of $T_{e}^{*} G$ with $\left.T_{e} G\right)$.

From now on, we write $J^{\nabla,\left(V_{i}, N_{\omega}^{i}\right)_{i=1}^{k}}$ for $J^{\nabla, \alpha_{0}, l, p, q,\left(V_{i}, N_{\omega}^{i}\right)_{i=1}^{k}}$ for simplicity. In order to characterize the critical points of the action functional and to derive the corresponding

Euler-Poincaré equation, it is necessary to consider a variation for $\left(g_{\omega}^{1}, \mathbf{w}_{\omega}^{1, i}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}} \in$ $\widetilde{\mathscr{S}(G)}$ and $\left(g_{\omega}^{2}, \mathbf{w}_{\omega}^{2, i}, M_{\omega}^{2, i}\right)_{i=1}^{m_{2}} \in \widetilde{\mathscr{S}(G)}$.

For every $\varepsilon \in[0,1)$ and $\mathcal{P}_{t}$-adapted process $\mathcal{g}: \Omega \times[0, T] \rightarrow T_{e} G$ satisfying $\mathcal{g}_{\omega}(0)=$ $g_{\omega}(T)=0$ and $g_{\omega}(\cdot) \in C^{1}\left([0,1] ; T_{e} G\right)$ a.s., let $e_{\omega, \varepsilon, g}(\cdot) \in C^{1}([0, T] ; G)$ be the unique solution of the (random) time-dependent ordinary differential equation on $G$

$$
\left\{\begin{array}{l}
\frac{d}{d t} e_{\omega, \varepsilon, g}(t)=\varepsilon T_{e} L_{e_{\omega, e, g}(t)} \dot{g}_{\omega}(t)  \tag{3.4}\\
e_{\omega, \varepsilon, g}(0)=e
\end{array}\right.
$$

where $\dot{\mathscr{g}}_{\omega}(t)$ denotes the derivative with respect to the time variable $t$. Note that this system implies $e_{\omega, 0, g}(t)=e$ a.s. for all $t \in[0, T]$.

From now on, in this section, we assume that $G$ is a finite dimensional Lie group endowed with a left invariant linear connection $\nabla$ and $U$ is a finite dimensional left $G$-representation space.

We first give the following lemma concerning the variations induced by $e_{\omega, \varepsilon, \mathscr{g}}$ on a semimartingale $g_{\omega} \in \mathscr{S}(G)$.

Lemma 3.2. Suppose $g_{\omega} \in \mathscr{S}(G)$ has the form (2.8) and let

$$
g_{\omega, \varepsilon, g}^{i}(t):=g_{\omega}^{i}(t) e_{\omega, \varepsilon, g}(t), \quad t \in[0, T], \quad \varepsilon \in[0,1) .
$$

Then we have

$$
\begin{equation*}
d g_{\omega, \varepsilon, \mathscr{g}}(t)=T_{e} L_{g_{\omega, \varepsilon, g}(t)}\left(\sum_{i=1}^{m} \operatorname{Ad}_{e_{\omega, \varepsilon, g}^{-1}(t)} \mathbf{w}_{\omega}^{i}(t) \delta M_{\omega}^{i}(t)+\operatorname{Ad}_{e_{\bar{\omega}, \varepsilon, \mathscr{g}}(t)} \mathbf{v}_{\omega}(t) d t+\varepsilon \dot{g_{\omega}}(t) d t\right) . \tag{3.5}
\end{equation*}
$$

Proof. By Itô's formula and recalling that the Leibniz rule holds for Stratonovich integrals, we have

$$
\begin{aligned}
& d g_{\omega, \varepsilon, \mathcal{g}}(t)=T_{e} L_{g_{\omega, \varepsilon, \mathcal{g}}(t)}\left(\sum_{i=1}^{m} \operatorname{Ad}_{e_{\bar{\omega}, \varepsilon, \mathscr{g}}^{-1}(t)} \mathbf{w}_{\omega}^{i}(t) \delta M_{\omega}^{i}(t)+\operatorname{Ad}_{e_{\bar{\omega}, \varepsilon, \mathscr{G}}^{-1}(t)} \mathbf{v}_{\omega}(t) d t\right. \\
& \left.+T_{e_{\omega, \varepsilon, \mathcal{g}}(t)} L_{e_{\omega, \varepsilon, \mathscr{g}}^{-1}(t)} \dot{e}_{\omega, \varepsilon, \mathscr{g}}(t) d t\right) \\
& =T_{e} L_{g_{\omega, \varepsilon, g}(t)}\left(\sum_{i=1}^{m} \operatorname{Ad}_{e_{\omega, \varepsilon, g}^{-1}(t)} \mathbf{w}_{\omega}^{i}(t) \delta M_{\omega}^{i}(t)+\operatorname{Ad}_{e_{\bar{\omega}, \varepsilon, g}^{-1}(t)} \mathbf{v}_{\omega}(t) d t+\varepsilon \dot{g}_{\omega}(t) d t\right),
\end{aligned}
$$

where the last equality is due to (3.4).

Based on (3.5), it is natural to consider $\left(g_{\omega, \varepsilon, g}, \operatorname{Ad}_{e_{\omega, \varepsilon, g}^{-1}(t)} \mathbf{w}_{\omega}^{i}, M_{\omega}^{i}\right)_{i=1}^{m}$ as a deformation for $\left(g_{\omega}, \mathbf{w}_{\omega}^{i}, M_{\omega}^{i}\right)_{i=1}^{m}$ with $g_{\omega} \in \mathscr{S}(G)$ having the expression (2.8). Meanwhile, using definitions (2.9)-(2.11), it is easy to verify that

$$
\begin{align*}
& T_{g_{\omega, \varepsilon, g}(t)} L_{g_{\omega, \varepsilon, g}(t)^{-1}} \frac{\mathscr{D} g_{\omega, \varepsilon, g}(t)}{d t}=\operatorname{Ad}_{e_{\omega, \bar{\epsilon}, g}^{-1}(t)} \mathbf{v}_{\omega}(t)+\varepsilon \dot{\dot{g}_{\omega}}(t) \\
& T_{g_{\omega, \varepsilon, g}(t)} L_{g_{\omega, \varepsilon, g}(t)^{-1}} d^{\Delta} g_{\omega, \varepsilon, g}(t)=\sum_{i=1}^{m}\left(\operatorname{Ad}_{e_{\omega, \varepsilon, g}^{-1}(t)} \mathbf{w}_{\omega}^{i}(t)\right) d M_{\omega}^{i}(t) \\
& \left(T_{g_{\omega, \varepsilon, g}(t)} L_{g_{\omega}, \varepsilon, \mathcal{g}}(t)^{-1} \frac{\mathbf{D}^{\nabla,\left(\operatorname{Ad}_{e_{\omega, \varepsilon, \mathcal{G}}^{-1}(t)} \mathbf{w}^{i}, M_{\omega}^{i}\right)_{i=1}^{m}} g_{\omega, \varepsilon, g}(t)}{d t}\right)_{i, j}  \tag{3.6}\\
& =\left(\nabla_{\operatorname{Ad}_{e_{\bar{\omega}, \varepsilon, g}^{-1}(t)} \mathbf{w}_{\omega}^{i}(t)} \operatorname{Ad}_{e_{\bar{\omega}, \varepsilon, \mathscr{g}}^{-1}(t)} \mathbf{w}_{\omega}^{j}(t)\right) \frac{d \llbracket M_{\omega}^{i}, M_{\omega}^{j} \rrbracket_{t}}{d t}+\frac{d \llbracket \operatorname{Ad}_{e_{\bar{\omega}, \varepsilon, \mathscr{g}}^{1}, \mathbf{w}^{i}}, M_{\omega}^{i} \rrbracket_{t}}{d t} 1_{\{i=j\}} .
\end{align*}
$$

Remark 3.3. Although by now we assume that $G$ is a finite dimensional Lie group, by the arguments in [2, Section 4.2] we know that (3.6) still holds when $G$ is the diffeomorphism group on torus, see, e.g., (5.17) below. Hence Theorem 3.5 stated below still holds for the diffeomorphism group on the torus (see Section [5).

Now we define the critical point for action functional based on the variations we introduced above. We say that $\left(\left(g_{\omega}^{1}, \mathbf{w}_{\omega}^{1, i}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}},\left(g_{\omega}^{2}, \mathbf{w}_{\omega}^{2, i}, M_{\omega}^{2, i}\right)_{i=1}^{m_{2}}, g_{\omega}^{3}\right) \in \widetilde{\mathscr{S}(G)} \times$ $\widetilde{\mathscr{S}(G)} \times \mathscr{S}(G)$ is a critical point of $J^{\nabla,\left(V_{i}, N_{\omega}^{i}\right)_{i=1}^{k}}$ if for every $\mathcal{P}_{t^{\prime}}$-adapted process $g_{\omega}$ satisfying $g_{\omega}(\cdot) \in C^{1}\left([0, T] ; T_{e} G\right)$ and $g_{\omega}(0)=g_{\omega}(T)=0$ a.s., we have

$$
\begin{equation*}
\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0} J^{\nabla,\left(V_{i}, N_{\omega}^{i}\right)_{i=1}^{k}}\left(\left(g_{\omega, \varepsilon, g}^{1}, \operatorname{Ad}_{e_{\omega, \varepsilon, \mathscr{g}}} \mathbf{w}_{\omega}^{1, i}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}},\left(g_{\omega, \varepsilon, g}^{2}, \operatorname{Ad}_{e_{\omega, \varepsilon, \mathcal{g}}^{-1}} \mathbf{w}_{\omega}^{2, i}, M_{\omega}^{2, i}\right)_{i=1}^{m_{2}}, g_{\omega, \varepsilon, g}^{3}\right)=0 \tag{3.7}
\end{equation*}
$$

where

$$
\begin{equation*}
g_{\omega, \varepsilon, \mathcal{g}}^{i}(t):=g_{\omega}^{i}(t) e_{\omega, \varepsilon, g}(t), \quad t \in[0, T], \quad i=1,2,3, \quad \varepsilon \in[0,1) . \tag{3.8}
\end{equation*}
$$

We emphasize the particular form of these deformations in the Lie group: they correspond to developments along (random) directions $g_{\omega}(t)$.

Remark 3.4. As will be seen in the applications presented in Section 5. the reason why we choose three different semimartingales in the variational principle (3.7) is that the viscosity constants in different equations may be different.

Fixing (non-random) $\left\{H_{i}^{j}\right\}_{i=1}^{m_{j}} \in T_{e} G, j=1,2,3$, as well as $\mathbb{R}^{m_{j}}$-valued martingales $M_{\omega}^{j}(t)=\left(M_{\omega}^{j, 1}(t), \ldots, M_{\omega}^{j, m_{j}}(t)\right), j=1,2,3$, we consider $\left(g_{\omega}^{j}, H_{i}^{j}, M_{\omega}^{j, i}\right)_{i=1}^{m_{j}} \in \widetilde{\mathscr{S}(G)}$,
$j=1,2,3$, where $g_{\omega}^{j}$ are the solutions of the following SDEs on $G$,

$$
\left\{\begin{array}{l}
d g_{\omega}^{j}(t)=T_{e} L_{g_{\omega}^{j}(t)}\left(\sum_{i=1}^{m_{j}} H_{i}^{j} \delta M_{\omega}^{j, i}(t)+u_{\omega}(t) d t\right)  \tag{3.9}\\
g_{\omega}^{j}(0)=e
\end{array}\right.
$$

and where $u_{\omega}$ is a $\mathcal{P}_{t}$-adapted, $T_{e} G$-valued semimartingale. Note that $u_{\omega}(\cdot)$ is not given a priori and is the same for $j=1,2,3$; we shall see below that it is the solution of a certain (stochastic) equation when $\left(\left(g_{\omega}^{1}, H_{i}^{1}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}},\left(g_{\omega}^{2}, H_{i}^{2}, M_{\omega}^{2, i}\right)_{i=1}^{m_{2}}, g_{\omega}^{3}\right)$ is a critical point for $J^{\nabla,\left(H_{i}^{1}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}}}$.

### 3.2 Stochastic variational principle for stochastic differential equations

In the theorem below we use the functional derivative notation. Let $V$ be (a possibly infinite dimensional) vector space and $V^{*}$ a space in weak duality $\langle\cdot, \cdot\rangle: V^{*} \times V \rightarrow \mathbb{R}$ with $V$; in finite dimensions, $V^{*}$ is the usual dual vector space, but in infinite dimensions it rarely is the topological dual. If $f: V \rightarrow \mathbb{R}$ is a smooth function, then the functional derivative $\frac{\delta f}{\delta a} \in V^{*}$, if it exists, is defined by $\lim _{\varepsilon \rightarrow 0} \frac{f(a+\varepsilon b)-f(a)}{\varepsilon}=\left\langle\frac{\delta f}{\delta a}, b\right\rangle$ for all $a, b \in V$.

In this section, we assume that $l, p, q$ in the action functional $J^{\nabla,\left(V_{i}, N_{\omega}^{i}\right)_{i=1}^{k} \text { are smooth }}$ with respect to all variables, except, of course, $\omega \in \Omega$.

Thus, in the theorem below, $\frac{\delta l}{\delta u} \in T_{e}^{*} G, \frac{\delta l}{\delta \alpha} \in U, \frac{\delta p}{\delta \xi_{1}}, \frac{\delta p}{\delta \xi_{2}} \in \mathscr{M}^{*}$, and $\frac{\delta p}{\delta u} \in T_{e}^{*} G$ are the partial functional derivatives of $l: \Omega \times[0, T] \times T_{e} G \times U^{*} \rightarrow \mathbb{R}$ and $p: \mathscr{M} \times \mathscr{M} \times T_{e} G \rightarrow \mathbb{R}$. Recall that here $\mathscr{M}_{m}:=\left\{\left(a_{i, j}\right)_{i, j=1}^{m}, a_{i, j} \in T_{e} G\right\}$ and $\mathscr{M}:=\cup_{m=1}^{\infty} \mathscr{M}_{m}$.

Theorem 3.5. Let $l: \Omega \times[0, T] \times T_{e} G \times U^{*} \rightarrow \mathbb{R}, p: \mathscr{M} \times \mathscr{M} \times T_{e} G \rightarrow \mathbb{R}, q:$ $[0, T] \times T_{e} G \times U^{*} \rightarrow T_{e}^{*} G$ such that $\frac{\delta l_{\omega}}{\delta u}$ is non-random and $l_{\omega}(t)$ is $\mathcal{P}_{t}$-adapted. Suppose that the semimartingales $g_{\omega}^{j}(\cdot), j=1,2,3$, have the form (3.9).
(i) Then $\left(\left(g_{\omega}^{1}, H_{i}^{1}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}},\left(g_{\omega}^{2}, H_{i}^{2}, M_{\omega}^{2, i}\right)_{i=1}^{m_{2}}, g_{\omega}^{3}\right)$ is a critical point of $J^{\nabla,\left(H_{i}^{1}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}}}$ (given in (3.2)) if and only if the $\mathcal{P}_{t^{-}}$-adapted process $u_{\omega}(t)$ coupled with the $\mathcal{P}_{t^{-}}$ adapted process $\alpha_{\omega}(t)$ (which is defined by (3.3)) satisfies the following (stochastic) semidirect product Euler-Poincaré equation for stochastic particle paths:

$$
\left\{\begin{array}{l}
d\left(\frac{\delta l_{\omega}}{\delta u}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)+\frac{\delta p}{\delta u}\left(\tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right)\right)  \tag{3.10}\\
=\sum_{i=1}^{m_{1}} \operatorname{ad}_{H_{i}}^{*} q\left(u_{\omega}(t), \alpha_{\omega}(t)\right) d M_{\omega}^{1, i}(t)+\operatorname{ad}_{u_{\omega}(t)}^{*}\left(\frac{\delta l_{\omega}}{\delta u}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)\right) d t \\
\quad+\left(\frac{\delta l_{\omega}}{\delta \alpha}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)\right) \diamond \alpha_{\omega}(t) d t+\operatorname{ad}_{u_{\omega}(t)}^{*}\left(\frac{\delta p}{\delta u}\left(\tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right)\right) d t \\
\quad+K_{\omega}\left(t, \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right) d t, \\
d \alpha_{\omega}(t)=-\sum_{i=1}^{m_{3}} H_{i}^{3} \alpha_{\omega}(t) d M_{\omega}^{3, i}(t) \\
\quad+\frac{1}{2} \sum_{i, k=1}^{m_{3}} H_{k}^{3}\left(H_{i}^{3} \alpha_{\omega}(t)\right) d \llbracket M_{\omega}^{3, i}, M_{\omega}^{3, k} \rrbracket_{t}-u_{\omega}(t) \alpha_{\omega}(t) d t .
\end{array}\right.
$$

Here the operation $\diamond$ is given by formula (3.1), $\tilde{H}_{\omega, j}(t) \in \mathscr{M}_{m_{j}}, j=1,2$, is the $m_{j} \times m_{j}$ matrix whose entries are given by

$$
\begin{equation*}
\left(\tilde{H}_{\omega, j}(t)\right)_{i, k}=\left(\nabla_{H_{i}^{j}} H_{k}^{j}\right) \frac{d \llbracket M_{\omega}^{j, i}, M_{\omega}^{j, k} \rrbracket_{t}}{d t}, \quad 1 \leq i, k \leq m_{j}, \tag{3.11}
\end{equation*}
$$

the operator $K_{\omega}:[0, T] \times \mathscr{M} \times \mathscr{M} \times T_{e} G \rightarrow T_{e}^{*} G$ is defined for every $\omega \in \Omega$ by

$$
\begin{equation*}
\left.\left\langle K_{\omega}\left(t, A_{1}, A_{2}, u\right), v\right\rangle=-\sum_{j=1}^{2}\left\langle\frac{\delta p}{\delta \xi_{j}}\left(A_{1}, A_{2}, u\right), B_{\omega, j}(t, v)\right)\right\rangle, \quad \forall t \in[0, T] \tag{3.12}
\end{equation*}
$$

1 where $A_{j} \in \mathscr{M}_{m_{j}}, j=1,2, u, v \in T_{e} G$, and $B_{\omega, j}(t, v) \in \mathscr{M}_{m_{j}}$ is the $m_{j} \times m_{j}$ matrix whose entries are

$$
\begin{equation*}
\left(B_{\omega, j}(t, v)\right)_{i, k}:=\left(\nabla_{H_{i}^{j}}\left(\operatorname{ad}_{v} H_{k}^{j}\right)+\nabla_{\mathrm{ad}_{v} H_{i}^{j}} H_{k}^{j}\right) \frac{d \llbracket M_{\omega}^{j, i}, M_{\omega}^{j, k} \rrbracket_{t}}{d t}, 1 \leq i, k \leq m_{j}, t \in[0, T] . \tag{3.13}
\end{equation*}
$$

(ii) The first equation in (3.10) is equivalent to the stochastic dissipative EulerPoincaré variational principle

$$
\begin{align*}
& \left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0}\left(\int_{0}^{T} l_{\omega}\left(t, u_{\omega, \varepsilon}(t), \alpha_{\omega, \varepsilon}(t)\right) d t+\int_{0}^{T} p\left(\tilde{H}_{\omega, 1, \varepsilon}(t), \tilde{H}_{\omega, 2, \varepsilon}(t), u_{\omega, \varepsilon}(t)\right) d t\right.  \tag{3.14}\\
& \left.+\int_{0}^{T}\left\langle q\left(t, u_{\omega, \varepsilon}(t), \alpha_{\omega, \varepsilon}(t)\right), d \beta_{\omega, \varepsilon}(t)\right\rangle-\sum_{i=1}^{m_{1}}\left(\int_{0}^{T}\left\langle q\left(t, u_{\omega, \varepsilon}(t), \alpha_{\omega, \varepsilon}(t)\right), H_{i}^{1}\right\rangle d M_{\omega}^{1, i}(t)\right)\right)=0
\end{align*}
$$

on $T_{e} G \times U^{*}$, for variations of the form

$$
\left\{\begin{array}{l}
\left.\frac{d u_{\omega, \varepsilon}(t)}{d \varepsilon}\right|_{\varepsilon=0}=\dot{v}_{\omega}(t)+a d_{u_{\omega}(t)} v_{\omega}(t)  \tag{3.15}\\
\left.\frac{d \alpha_{\omega, \varepsilon}(t)}{d \varepsilon}\right|_{\varepsilon=0}=-v_{\omega}(t) \alpha_{\omega}(t) \\
\left.\frac{d \tilde{H}_{\omega, j, \varepsilon}(t)}{d \varepsilon}\right|_{\varepsilon=0}=-B_{\omega, j}\left(t, v_{\omega}(t)\right), j=1,2 \\
\left.\frac{d \beta_{\omega, \varepsilon}(t)}{d \varepsilon}\right|_{\varepsilon=0}=-\sum_{i=1}^{m_{1}} \int_{0}^{t} a d_{v_{\omega}(s)} H_{i}^{1} d M_{\omega}^{1, i}(s) \\
u_{\omega, 0}(t)=u_{\omega}(t), \alpha_{\omega, 0}(t)=\alpha_{\omega}(t), \beta_{\omega, 0}(t)=\sum_{i=1}^{m_{1}} \int_{0}^{t} H_{i}^{1} d M_{\omega}^{1, i}(s), \tilde{H}_{\omega, j, 0}(t)=\tilde{H}_{\omega, j}(t),
\end{array}\right.
$$

[^1]where $v_{\omega}(t)$ is an $\mathcal{P}_{t^{-}}$-adapted process such that $v_{\omega} \in C^{1}\left([0, T] ; T_{e} G\right)$ and $v_{\omega}(0)=0$, $v_{\omega}(T)=0$ a.s.. (Note that this variational principle is constrained and stochastic.)

Proof. (i) Step 1. We start by proving that $\alpha_{\omega}(t)=g_{\omega}^{3}(t)^{-1} \alpha_{0}$ satisfies the second equation in (3.10).

Since $d\left(\left(g^{3}(t)\right)^{-1} g^{3}(t)\right)=0$, we have

$$
d\left(g_{\omega}^{3}(t)\right)^{-1}=-T_{e} R_{\left(g_{\omega}^{3}(t)\right)^{-1}} T_{g_{\omega}^{3}(t)} L_{\left(g_{\omega}^{3}(t)\right)^{-1}} d g_{\omega}^{3}(t),
$$

so replacing $d g_{\omega}^{3}(t)$ by its expression in (3.9) we obtain,

$$
\left\{\begin{array}{l}
d\left(g_{\omega}^{3}(t)\right)^{-1}=T_{e} R_{\left(g_{\omega}^{3}(t)\right)^{-1}}\left(\sum_{i=1}^{m_{3}}-H_{i}^{3} \delta M_{\omega}^{3, i}(t)-u_{\omega}(t) d t\right)  \tag{3.16}\\
g_{\omega}^{3}(0)^{-1}=e
\end{array}\right.
$$

We now derive the stochastic differential equation satisfied by $\alpha_{\omega}(t)$ :

$$
\begin{align*}
d \alpha_{\omega}(t) & =d\left(g_{\omega}^{3}(t)^{-1} \alpha_{0}\right)=\left[-T_{g_{\omega}^{3}(t)} L_{g_{\omega}^{3}(t)^{-1}} d g_{\omega}^{3}(t)\right] g_{\omega}^{3}(t)^{-1} \alpha_{0}  \tag{3.17}\\
& =-\sum_{i=1}^{m_{3}} H_{i}^{3}\left(g_{\omega}^{3}(t)^{-1} \alpha_{0}\right) \delta M_{\omega}^{3, i}(t)-u_{\omega}(t)\left(g_{\omega}^{3}(t)^{-1} \alpha_{0}\right) d t
\end{align*}
$$

Since we assume $U^{*}$ to be a finite dimensional vector space, the difference between the Stratonovich and Itô integrals (see (2.1)) yields

$$
\begin{aligned}
\sum_{i=1}^{m_{3}} & \left(H_{i}^{3}\left(g_{\omega}^{3}(t)^{-1} \alpha_{0}\right)\right) \delta M_{\omega}^{3, i}(t) \\
& =\sum_{i=1}^{m_{3}}\left(\left(H_{i}^{3}\left(g_{\omega}^{3}(t)^{-1} \alpha_{0}\right)\right) d M_{\omega}^{3, i}(t)+\frac{1}{2} d \llbracket H_{i}^{3}\left(g_{\omega}^{3}(\cdot)^{-1} \alpha_{0}\right), M_{\omega}^{3, i} \rrbracket_{t}\right) .
\end{aligned}
$$

By the same procedure as in (3.17), the (local) martingale part of $H_{i}^{3}\left(g_{\omega}^{3}(\cdot)^{-1} \alpha_{0}\right)$ is equal to $-\sum_{k=1}^{m_{3}} \int_{0} H_{k}^{3} H_{i}^{3}\left(g_{\omega}^{3}(t)^{-1} \alpha_{0}\right) d M_{\omega}^{3, k}(t)$. Therefore, by (2.4) and (2.5) we derive

$$
\sum_{i=1}^{m_{3}} d \llbracket H_{i}^{3}\left(g_{\omega}^{3}(\cdot)^{-1} \alpha_{0}\right), M_{\omega}^{3, i} \rrbracket_{t}=-\sum_{i, k=1}^{m_{3}} H_{k}^{3}\left(H_{i}^{3}\left(g_{\omega}^{3}(t)^{-1} \alpha_{0}\right)\right) d \llbracket M_{\omega}^{3, i}, M_{\omega}^{3, k} \rrbracket_{t} .
$$

Using (3.17) we have,

$$
\begin{align*}
& d \alpha_{\omega}(t)=-\sum_{i=1}^{m_{3}} H_{i}^{3} \alpha_{\omega}(t) d M_{\omega}^{3, i}(t)  \tag{3.18}\\
& \quad+\frac{1}{2} \sum_{i, k=1}^{m_{3}} H_{k}^{3}\left(H_{i}^{3} \alpha_{\omega}(t)\right) d \llbracket M_{\omega}^{3, i}, M_{\omega}^{3, k} \rrbracket_{t}-u_{\omega}(t) \alpha_{\omega}(t) d t
\end{align*}
$$

which is the second equation in (3.10).
Step 2. Now we prove the first equation in (3.10). Recall from (3.4) that, for every $\mathcal{P}_{t}$-adapted process $g_{\omega}$ satisfying $g_{\omega}(\cdot) \in C^{1}\left([0,1] ; T_{e} G\right)$ and $g_{\omega}(0)=g_{\omega}(T)=0$ a.s., $e_{\omega, \varepsilon, g}(\cdot) \in C^{1}([0, T] ; G)$ a.s. uniquely solves the following (random) ordinary differential equation on $G$

$$
\frac{d}{d t} e_{\omega, \varepsilon, g}(t)=\varepsilon T_{e} L_{e_{\omega, \varepsilon, g}(t)} \dot{g}_{\omega}(t), \quad e_{\omega, \varepsilon, g}(0)=e .
$$

By [2, Lemma 3.1], we have

$$
\begin{equation*}
\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0} e_{\omega, \varepsilon, g}(t)=g_{\omega}(t),\left.\quad \frac{d}{d \varepsilon}\right|_{\varepsilon=0} e_{\omega, \varepsilon, g}(t)^{-1}=-g_{\omega}(t), \text { a.s.. } \tag{3.19}
\end{equation*}
$$

Since this computation is important in the proof, for the sake of completeness, we recall it below. Denoting by $\frac{D}{D t}$ and $\frac{D}{D \epsilon}$ the covariant derivatives, induced by $\nabla$ on $G$, along curves parametrized by $t$ and $\varepsilon$, respectively. Since the torsion vanishes, Gauss Lemma yields

$$
\begin{align*}
\frac{D}{D t} \frac{d}{d \varepsilon} e_{\omega, \varepsilon, g}(t) & =\frac{D}{D \varepsilon} \frac{d}{d t} e_{\omega, \varepsilon, g}(t)=\frac{D}{D \varepsilon}\left(\varepsilon T_{e} L_{e_{\omega, \varepsilon, g}(t)} \dot{g}_{\omega}(t)\right)  \tag{3.20}\\
& =T_{e} L_{e_{\omega, \varepsilon, g}(t)} \dot{g}_{\omega}(t)+\varepsilon \frac{D}{D \varepsilon}\left(T_{e} L_{e_{\omega, \varepsilon, g}(t)} \dot{g}_{\omega}(t)\right)
\end{align*}
$$

Taking $\varepsilon=0$ and since $e_{\omega, 0, g}(t)=e$ for all $t$, we obtain $\left.\frac{D}{D t} \frac{d}{d \varepsilon}\right|_{\varepsilon=0} e_{\omega, \varepsilon, g}(t)=\dot{g}_{\omega}(t)$. Moreover $\left.t \mapsto \frac{d}{d \varepsilon}\right|_{\varepsilon=0} e_{\omega, \varepsilon, g_{\omega}}(t)$ is a curve in the vector space $T_{e} G$ and hence $\left.\frac{d}{d t} \frac{d}{d \varepsilon}\right|_{\varepsilon=0} e_{\omega, \varepsilon, g}(t)=$ $\left.\frac{D}{D t} \frac{d}{d \varepsilon}\right|_{\varepsilon=0} e_{\omega, \varepsilon, g}(t)=\dot{g}_{\omega}(t)$. The first equality in (3.19) is then a consequence of $g_{\omega}(0)=0$ and $\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0} e_{\omega, \varepsilon, g}(0)=0$. Finally, since

$$
\frac{d}{d \varepsilon} e_{\omega, \varepsilon, g}(t)^{-1}=-T_{e} R_{e_{\omega, \varepsilon, g}^{-1}(t)} T_{e_{\omega, \varepsilon, g}(t)} L_{e_{\omega, \varepsilon, g}^{-1}(t)} \frac{d}{d \varepsilon} e_{\omega, \varepsilon, g}(t)
$$

the second equality in (3.19) follows from the first.
Note that due to (3.19) we have $\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0} \operatorname{Ad}_{e_{\bar{\omega}, \varepsilon, \varepsilon, g}^{1}(t)} v=-\operatorname{ad}_{g_{\omega}(t)} v, v \in T_{e} G$. Combining this with (3.6) we have

$$
\begin{gather*}
\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0}\left(T_{g_{\omega, \varepsilon, g}^{1}(t)} L_{g_{\omega, \varepsilon, g}^{1}(t)^{-1}} \frac{\mathscr{D} g_{\omega, \varepsilon, g}^{1}(t)}{d t}\right)  \tag{3.21}\\
=\dot{g}_{\omega}(t)+\operatorname{ad}_{u_{\omega}(t) g_{\omega}}(t) \\
\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0}\left(T_{g_{\omega, \varepsilon, g}}(t) L_{g_{\omega, \varepsilon, g}}(t)^{-1} d^{\Delta} g_{\omega, \varepsilon, g}^{1}(t)\right) \\
=-\sum_{i=1}^{m_{1}} \operatorname{ad}_{g_{\omega}(t)} H_{i}^{1} d M_{\omega}^{1, i}(t) \tag{3.22}
\end{gather*}
$$

$$
\begin{align*}
& \left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0}\left(T_{g_{\omega, \varepsilon, g}^{j}(t)} L_{g_{\omega, \varepsilon, g}^{j}(t)^{-1}} \frac{\mathbf{D}^{\nabla,\left(H_{\omega, i}^{j, \varepsilon}, M_{\omega}^{j, i}\right)_{i=1}^{m_{j}}} g_{\omega, \varepsilon, g}^{j}(t)}{d t}\right)_{k, m}  \tag{3.23}\\
& =-\left(\nabla_{\operatorname{ad}_{g_{\omega}(t)} H_{m}^{j}} H_{k}^{j}+\nabla_{H_{m}^{j}}\left(\operatorname{ad}_{g_{\omega}(t)} H_{k}^{j}\right)\right) \frac{d \llbracket M_{\omega}^{j, k}, M_{\omega}^{j, m} \rrbracket_{t}}{d t} \\
& =-B_{\omega, j}\left(t, g_{\omega}(t)\right), \quad j=1,2,
\end{align*}
$$

where $g_{\omega, \varepsilon, g}^{j}(t)=g_{\omega}^{j}(t) e_{\omega, \varepsilon, g}(t), H_{\omega, i}^{j, \varepsilon}(t):=\operatorname{Ad}_{e_{\omega, \varepsilon, g}^{-1}(t)} H_{i}^{j}, B_{\omega, j}(t, \cdot)$ is defined by (3.13) and we have applied the property $\llbracket H_{\omega, i}^{j, \varepsilon}, M_{\omega}^{j, i} \rrbracket \equiv 0$ since $H_{\omega, i}^{j, \varepsilon}(\cdot)$ is a process with bounded variation.

Since $g_{\omega, \varepsilon, \mathcal{g}}^{j}(t):=g_{\omega}^{j}(t) e_{\omega, \varepsilon, \mathcal{g}}(t)$ and $e_{\omega, 0, g}(t)=e$ for all $t \in[0, T]$, we conclude $g_{\omega, 0, g}^{j}(t)=g_{\omega}^{j}(t)$, for all $t \in[0, T], j=1,2,3$. Therefore,

$$
\begin{align*}
\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0} g_{\omega, \varepsilon, \mathscr{g}}^{3}(t)^{-1} \alpha_{0} & =-g_{\omega}^{3}(t)^{-1}\left(\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0} g_{\omega, \varepsilon, \mathscr{g}}^{3}(t)\right) g_{\omega}^{3}(t)^{-1} \alpha_{0} \\
& =-\left(\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0} e_{\omega, \varepsilon, \mathscr{g}}(t)\right) g_{\omega}^{3}(t)^{-1} \alpha_{0} \stackrel{(3.19)}{=}-g_{\omega}(t) g_{\omega}^{3}(t)^{-1} \alpha_{0}  \tag{3.24}\\
& =-g_{\omega}(t) \alpha_{\omega}(t) .
\end{align*}
$$

Based on (3.2), (3.21)-(3.24) and noting that $\left.d^{\Delta} g_{\omega, \varepsilon, g}^{1}(t)\right|_{\varepsilon=0}=\sum_{i=1}^{m_{1}} H_{i}^{1} d M_{\omega}^{1, i}(t)$,
we have

$$
\begin{align*}
& \left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0} J^{\nabla,\left(H_{i}^{1}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}}}\left(\left(g_{\omega, \varepsilon, g}^{1}, H_{\omega, i}^{1, \varepsilon}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}},\left(g_{\omega, \varepsilon, g}^{2}, H_{\omega, i}^{2, \varepsilon}, M_{\omega}^{2, i}\right)_{i=1}^{m_{2}}, g_{\omega, \varepsilon, g}^{3}\right)  \tag{3.25}\\
& =\int_{0}^{T}\left\langle\frac{\delta l_{\omega}}{\delta u}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right),\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0}\left(T_{g_{\omega, \varepsilon, \mathcal{G}}^{1}(t)} L_{g_{\omega, \varepsilon, \mathcal{G}}^{1}(t)^{-1}} \frac{\mathscr{D} g_{\omega, \varepsilon, \mathscr{g}}^{1}(t)}{d t}\right)\right\rangle d t \\
& +\int_{0}^{T}\left\langle\frac{\delta p}{\delta u}\left(\tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right),\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0}\left(T_{g_{\omega, \varepsilon, g}}(t) L_{g_{\omega, \varepsilon, g}^{1}(t)^{-1}} \frac{\mathscr{D} g_{\omega, \varepsilon, g}^{1}(t)}{d t}\right)\right\rangle d t \\
& +\sum_{j=1}^{2}\left\langle\frac{\delta p}{\delta \xi_{j}}\left(\tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right),\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0}\left(T_{g_{\omega, \varepsilon, g}^{j}(t)} L_{g_{\omega, \varepsilon, g}^{j}(t)-1} \frac{\mathbf{D}^{\nabla,\left(H_{\omega, i}^{j,}, M_{\omega}^{j, i}\right)_{i=1}^{m_{j}}}}{d t}\right)\right\rangle \\
& +\int_{0}^{T}\left\langle\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0}\left(g_{\omega, \varepsilon, g}^{3}(t)^{-1} \alpha_{0}\right), \frac{\delta l_{\omega}}{\delta \alpha}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)\right\rangle d t \\
& +\int_{0}^{T}\left\langle q\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right),\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0}\left(T_{g_{\omega, \varepsilon, \mathcal{G}}^{1}(t)} L_{g_{\omega, \varepsilon, g}^{1}(t)^{-1}} d^{\Delta} g_{\omega, \varepsilon, \mathcal{G}}^{1}(t)\right)\right\rangle \\
& =\int_{0}^{T}\left\langle\frac{\delta l_{\omega}}{\delta u}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right), \dot{g}_{\omega}(t)+\operatorname{ad}_{u_{\omega}(t) g_{\omega}}(t)\right\rangle d t \\
& -\int_{0}^{T}\left\langle g_{\omega}(t) \alpha_{\omega}(t), \frac{\delta l_{\omega}}{\delta \alpha}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)\right\rangle d t \\
& +\int_{0}^{T}\left\langle\frac{\delta p}{\delta u}\left(\tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right), \dot{g}_{\omega}(t)+\operatorname{ad}_{u_{\omega}(t) g_{\omega}}(t)\right\rangle d t \\
& -\sum_{j=1}^{2}\left\langle\frac{\delta p}{\delta \xi_{j}}\left(\tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right), B_{\omega, j}\left(t, g_{\omega}(t)\right)\right\rangle d t \\
& -\sum_{i=1}^{m_{1}} \int_{0}^{T}\left\langle q\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right), \operatorname{ad}_{g_{\omega}(t)} H_{i} d M_{\omega}^{1, i}(t)\right\rangle \\
& =\int_{0}^{T}\left\langle-d\left(\frac{\delta l_{\omega}}{\delta u}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)+\frac{\delta p}{\delta u}\left(\tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right)\right)\right. \\
& +\operatorname{ad}_{u_{\omega}(t)}^{*}\left(\frac{\delta l_{\omega}}{\delta u}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)\right) d t+\operatorname{ad}_{u_{\omega}(t)}^{*}\left(\frac{\delta p}{\delta u}\left(\tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right)\right) d t \\
& +\left(\frac{\delta l_{\omega}}{\delta \alpha}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)\right) \diamond \alpha_{\omega}(t) d t+K_{\omega}\left(t, \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right) d t \\
& \left.+\sum_{i=1}^{m_{1}} \operatorname{ad}_{H_{i}}^{*}\left(q\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)\right) d M_{\omega}^{1, i}(t), g_{\omega}(t)\right\rangle,
\end{align*}
$$

where the first equality follows from the property $T_{g_{\omega}^{1}(t)} L_{g_{\omega}^{1}(t)^{-1}} d^{\Delta} g_{\omega}^{1}(t)=\sum_{i=1}^{m_{1}} H_{i}^{1} d M_{\omega}^{1, i}(t)$ (which implies that the term depending on derivatives of $q$ vanishes), the last equality is obtained by applying the following equation

$$
\begin{aligned}
0= & \frac{\delta l_{\omega}}{\delta u}\left(T, u_{\omega}(T), \alpha_{\omega}(T)\right) g_{\omega}(T)-\frac{\delta l_{\omega}}{\delta u}\left(0, u_{\omega}(0), \alpha_{\omega}(0)\right) g_{\omega}(0) \\
& =\int_{0}^{T}\left\langle d\left(\frac{\delta l_{\omega}}{\delta u}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)\right), g_{\omega}(t)\right\rangle+\int_{0}^{T}\left\langle\frac{\delta l_{\omega}}{\delta u}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right), \dot{g}_{\omega}(t)\right\rangle
\end{aligned}
$$

(note that $g_{\omega}(\cdot)$ has bounded variation and $\frac{\delta l_{\omega}}{\delta u}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)$ is a semimartingale because $\frac{\delta l_{\omega}}{\delta u}$ is differentiable with respect to variable $t$ and is $\mathcal{P}_{t^{-}}$-adapted, so $d \llbracket \frac{\delta l_{\omega}}{\delta u}, g_{\omega} \rrbracket_{t}=$ 0 ), and by applying the definitions of $\mathrm{ad}^{*}, \diamond($ see (3.1) $)$, and $K$ (see (3.12)).

Since $\mathcal{g}_{\omega}(\cdot)$ is a $\mathcal{P}_{t^{\prime}}$-adapted arbitrary process, $\left(\left(g_{\omega}^{1}, H_{i}^{1}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}},\left(g_{\omega}^{2}, H_{i}^{2}, M_{\omega}^{2, i}\right)_{i=1}^{m_{2}}, g_{\omega}^{3}\right)$ is a critical point of $J^{\nabla,\left(H_{i}^{1}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}}}$ if and only if $u_{\omega}$ satisfies the first equation in (3.10).

This proves statement (i).
(ii) The expressions of the variations (3.15) have been already found in the previous computations. Applying the same methods in the variational procedure (3.25) we obtain (3.14).

Remark 3.6. As we shall see in Section [5, the conclusion of Theorem [3.5 still holds when $G$ is the diffeomorphism group of a torus and the action of $G$ on $U^{*}$ is the pull back map.

If $G$ is a finite dimensional Lie group and $U$ a finite dimensional vector space, then (3.10) is a actually an SDE. However, when $G$ is the diffeomorphism group, as illustrated in Section 5 (3.10) is a system of SPDEs.

Remark 3.7. The variation (3.15) is a stochastic version of Lin's constrained variational principle (see, e.g., [49, Theorem 1.2]). In fact, if we take $p=q=0$ and $H_{i}^{j}=0$, (3.15) is the deterministic constrained variational principle in [49, Theorem 1.2].

Remark 3.8. For simplicity, in Theorem 3.5 we assume that the contraction force $p$ and stochastic force $q$ are independent of the advection space $U^{*}$. In fact, following the same procedure in the proof of Theorem 3.5, we could also characterize the critical points of an action functional even if $p$ and $q$ depend on $U^{*}$.

### 3.3 Stochastic variational principle for ordinary differential equations

If we take $q=0$, and take the expectation in (3.3), through the stochastic reduction procedure in Theorem 3.5, we obtain a system of ODEs for the drift of the underlying stochastic paths (not the SDE in (3.10)).

Let $l, p$ be the same terms in Theorem 3.5 such that $l:[0, T] \times T_{e} G \times U^{*} \rightarrow \mathbb{R}$ is non-random. We define $\tilde{J}^{\nabla}: \widetilde{\mathscr{S}(G)} \times \widetilde{\mathscr{S}(G)} \times \mathscr{S}(G) \rightarrow \mathbb{R}$ by

$$
\begin{aligned}
& \tilde{J}^{\nabla}\left(\left(g_{\omega}^{1}, \mathbf{w}_{\omega}^{1, i}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}},\left(g_{\omega}^{2}, \mathbf{w}_{\omega}^{2, i}, M_{\omega}^{2, i}\right)_{i=1}^{m_{2}}, g_{\omega}^{3}\right) \\
& :=\int_{0}^{T} l\left(t, T_{g_{\omega}^{1}(t)} L_{g_{\omega}^{1}(t)^{-1}} \frac{\mathscr{D} g_{\omega}^{1}(t)}{d t}, \tilde{\alpha}(t)\right) d t+\int_{0}^{T} p\left(T_{g_{\omega}^{1}(t)} L_{g_{\omega}^{1}(t)^{-1}} \frac{\mathbf{D}^{\nabla,\left(\mathbf{w}_{\omega}^{1, i}, M_{\omega}^{i, 1}\right)_{i=1}^{m_{1}}} g_{\omega}^{1}(t)}{d t},\right. \\
& \left.T_{g_{\omega}^{2}(t)} L_{g_{\omega}^{2}(t)^{-1}} \frac{\mathbf{D}^{\nabla,\left(\mathbf{w}_{\omega}^{2, i}, M_{\omega}^{i, 2}\right)_{i=1}^{m_{2}} g_{\omega}^{2}(t)}}{d t}, T_{g_{\omega}^{1}(t)} L_{g_{\omega}^{1}(t)^{-1}} \frac{\mathscr{D} g_{\omega}^{1}(t)}{d t}\right) d t
\end{aligned}
$$

where $g_{\omega}^{j}, j=1,2,3$ are $G$-valued semimartingales with form (2.8), and $\tilde{\alpha}(t):=$ $\mathbb{E}\left[\alpha_{\omega}(t)\right]=\mathbb{E}\left[\tilde{g}_{\omega}^{3}(t)^{-1} \alpha_{0}\right] \in U^{*}$ is non-random. The action functional $\tilde{J}^{\nabla}$ can be viewed as a deterministic counterpart of (3.2), where $q=0, \alpha_{\omega}(t)$ is replaced by $\tilde{\alpha}(t)$, and there is no external stochastic force term (stochastic integral term).

Suppose also that deformations are of the form (3.7) with $g_{\omega}$ non-random (we write $\mathcal{g}$ for $g_{\omega}$ in this section); then we can also define the critical point of $\tilde{J}^{\nabla}$ in the same way of that in (3.7). To further simplify notation, we drop the index $\omega$ on some of the variables in the statement of the theorem below; for example, we write $u$ and $\llbracket M^{j, i}, M^{j, k} \rrbracket_{t}$ for $u_{\omega}$ and $\llbracket M_{\omega}^{j, i}, M_{\omega}^{j, k} \rrbracket_{t}$, respectively, when such functions are deterministic.

Theorem 3.9. (Stochastic reduction with deterministic drift and deformations) Let the semimartingales $g_{\omega}^{j}(\cdot), j=1,2,3$, have the form (3.9) with $u \in C^{1}\left([0, T] ; T_{e} G\right)$ and $\llbracket M^{j, i}, M^{j, k} \rrbracket_{t}, 1 \leq j \leq 3,1 \leq i, k \leq m_{j}$ being non-random.
(i) Then $\left(\left(g_{\omega}^{1}, H_{i}^{1}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}},\left(g_{\omega}^{2}, H_{i}^{2}, M_{\omega}^{2, i}\right)_{i=1}^{m_{2}}, g_{\omega}^{3}\right)$ is a critical point of $\tilde{J}^{\nabla}$ if and only if $u(t)$ coupled with $\tilde{\alpha}(t)$ satisfies the following (ordinary differential) equation

$$
\left\{\begin{array}{l}
d\left(\frac{\delta l}{\delta u}(t, u(t), \tilde{\alpha}(t))+\frac{\delta p}{\delta u}\left(\tilde{H}_{1}(t), \tilde{H}_{2}(t), u(t)\right)\right)  \tag{3.26}\\
\quad=\operatorname{ad}_{u(t)}^{*}\left(\frac{\delta l}{\delta u}(t, u(t), \tilde{\alpha}(t))\right) d t+\operatorname{ad}_{u(t)}^{*}\left(\frac{\delta p}{\delta u}\left(\tilde{H}_{1}(t), \tilde{H}_{2}(t), u(t)\right)\right) d t \\
\quad+\left(\frac{\delta l}{\delta \alpha}(t, u(t), \tilde{\alpha}(t))\right) \diamond \tilde{\alpha}(t) d t+K\left(t, \tilde{H}_{1}(t), \tilde{H}_{2}(t), u(t)\right) d t \\
d \tilde{\alpha}(t)=\frac{1}{2} \sum_{i, k=1}^{m_{3}} H_{k}^{3}\left(H_{i}^{3} \tilde{\alpha}(t)\right) d \llbracket M^{3, i}, M^{3, k} \rrbracket_{t}-u(t) \tilde{\alpha}(t) d t
\end{array}\right.
$$

where $\tilde{H}_{1}(t) \in \mathscr{M}_{m_{1}}, \tilde{H}_{2}(t) \in \mathscr{M}_{m_{2}}, \diamond, K$ are the same terms as in Theorem 3.5 (except that we omit the subscript $\omega$ in order to emphasize that these terms are non-random here).
(ii) The first equation in (3.26) is equivalent to the following stochastic variational principle

$$
\begin{equation*}
\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0}\left(\int_{0}^{T} l\left(t, u_{\varepsilon}(t), \tilde{\alpha}_{\varepsilon}(t)\right) d t+\int_{0}^{T} p\left(\tilde{H}_{1, \varepsilon}(t), \tilde{H}_{2, \varepsilon}(t), u_{\varepsilon}(t)\right) d t\right)=0 \tag{3.27}
\end{equation*}
$$

on $T_{e} G \times U^{*}$ for variations of the form

$$
\left\{\begin{array}{l}
\left.\frac{d u_{\varepsilon}(t)}{d \varepsilon}\right|_{\varepsilon=0}=\dot{v}(t)+\operatorname{ad}_{u(t)} v(t) \\
\left.\frac{d \tilde{\alpha}_{\varepsilon}(t)}{d \varepsilon}\right|_{\varepsilon=0}=-v(t) \tilde{\alpha}(t), \\
\left.\frac{d \tilde{H}_{j, \varepsilon}(t)}{d \varepsilon}\right|_{\varepsilon=0}=-B_{j}(t, v(t)), \quad j=1,2 \\
u_{0}(t)=u(t), \tilde{\alpha}_{0}(t)=\tilde{\alpha}(t), \quad \tilde{H}_{j, 0}(t)=\tilde{H}_{j}(t)
\end{array}\right.
$$

where $v \in C^{1}\left([0, T] ; T_{e} G\right)$ with $v(0)=0, v(T)=0$ is non-random and $B_{j}(t, v)$ is defined by (3.13). Note that this variational principle is constrained and deterministic.

Proof. (i) Since $H_{i}^{3}, u$ are non-random and the action of $T_{e} G$ on $U^{*}$ is linear, we have

$$
\begin{aligned}
& \mathbb{E}\left[H_{j}^{3}\left(H_{i}^{3} \alpha_{\omega}(t)\right)\right]=H_{j}^{3}\left(H_{i}^{3}\left(\mathbb{E}\left[\alpha_{\omega}(t)\right]\right)\right)=H_{j}^{3}\left(H_{i}^{3}(\tilde{\alpha}(t))\right), \\
& \mathbb{E}\left[u(t) \alpha_{\omega}(t)\right]=u(t) \mathbb{E}\left[\alpha_{\omega}(t)\right]=u(t) \tilde{\alpha}(t)
\end{aligned}
$$

Then taking the expectation on both side of (3.18), we arrive to the second equation of (3.26).

Note that $e_{\omega, \varepsilon, g}$ is non-random since $g_{\omega}$ is non-random; from (3.24) we obtain

$$
\begin{aligned}
& \left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0} \mathbb{E}\left[g_{\omega, \varepsilon, g}^{3}(t)^{-1} \alpha_{0}\right]=\mathbb{E}\left[\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0} g_{\omega, \varepsilon, \mathfrak{g}}^{3}(t)^{-1} \alpha_{0}\right] \\
& =-\mathbb{E}\left[g(t) g_{\omega}^{3}(t)^{-1} \alpha_{0}\right]=-\boldsymbol{g}(t) \mathbb{E}\left[g_{\omega}^{3}(t)^{-1} \alpha_{0}\right]=-\boldsymbol{g}(t) \tilde{\alpha}(t)
\end{aligned}
$$

Based on this and following the same procedure of (3.25) (note that here $q=0$ ), we have the first equation of (3.26).
(ii) By the same steps of (3.15) we derive (3.27).

Remark 3.10. As we will see in Section 5, for the case that $G$ is a diffeomorphism group, the system of (3.26) is a PDE with viscosity term.

### 3.4 Right invariant version

Due to relative sign changes in the equations of motion and the dissipative constrained variational principle, with a view to applications for the spatial representation in continuum mechanics, we give below the right invariant version of Theorem 3.5.

Suppose that $G$ acts on the right on a vector space $U$ (we will write the action of $g \in G$ on $u \in U$ by $u g$ and similarly for the induced infinitesimal $\mathfrak{g}$-representation).

Thus, let $g_{\omega}$ be a $G$-valued semimartingale of the form

$$
\begin{equation*}
d g_{\omega}(t)=T_{e} R_{g_{\omega}(t)}\left(\sum_{i=1}^{m} \mathbf{w}_{\omega}^{i}(t) \delta M_{\omega}^{i}(t)+\mathbf{v}_{\omega}(t) d t\right) \tag{3.28}
\end{equation*}
$$

where $T_{e} R_{g_{\omega}(t)}$ denotes the differential of the right translation $R_{g_{\omega}(t)}$ at the point $e$. For a fixed right invariant connection $\nabla$ on $G$ and $\left(\mathbf{w}^{i}, M_{\omega}^{i}\right)_{i=1}^{m}$, where $\mathbf{w}_{\omega}^{i}$ and $M_{\omega}^{i}$ are $\mathcal{P}_{t}$-adapted processes and $M_{\omega}^{i}$ are real valued martingales with continuous sample paths for all $i=1, \ldots, m$, we define

$$
\begin{aligned}
& \left(\frac{\mathbf{D}^{\nabla,\left(\mathbf{w}_{\omega}^{i}, M_{\omega}^{i}\right)_{i=1}^{m} g_{\omega}(t)}}{d t}\right)_{i, j}:=T_{e} R_{g_{\omega}(t)}\left(\nabla_{\mathbf{w}_{\omega}^{i}(t)} \mathbf{w}_{\omega}^{j}(t) \frac{d \llbracket M_{\omega}^{i}, M_{\omega}^{j} \rrbracket_{t}}{d t}\right. \\
& \left.\quad+\frac{d \llbracket \mathbf{w}_{\omega}^{i}, M_{\omega}^{i} \rrbracket_{t}}{d t} 1_{\{i=j\}}\right), \quad 1 \leq i, j \leq m
\end{aligned}
$$

The terms $\frac{\mathscr{O} g_{\omega}(t)}{d t}$ and $d^{\Delta} g_{\omega}(t)$ are defined similarly as in the left invariant case (see the defining formulas (2.15)).

With $l: \Omega \times[0, T] \times T_{e} G \times U^{*} \rightarrow \mathbb{R}, p: \mathscr{M} \times \mathscr{M} \times T_{e} G \rightarrow \mathbb{R}, q:[0, T] \times T_{e} G \times U^{*} \rightarrow$ $T_{e}^{*} G, V_{i} \in T_{e} G, N_{\omega}^{i}, 1 \leq i \leq k$, satisfying the same conditions as those in subsections 3.1 and 3.2 (for the left invariant case), the action functional $J^{\nabla,\left(V_{i}, N_{\omega}^{i}\right)_{i=1}^{k}}$ is defined for the right invariant case by

$$
\begin{align*}
& J^{\nabla,\left(V_{i}, N_{\omega}^{i}\right)_{i=1}^{k}}\left(\left(g_{\omega}^{1}, \mathbf{w}_{\omega}^{1, i}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}},\left(g_{\omega}^{2}, \mathbf{w}_{\omega}^{2, i}, M_{\omega}^{2, i}\right)_{i=1}^{m_{2}}, g_{\omega}^{3}\right)  \tag{3.29}\\
&:= \int_{0}^{T} l_{\omega}\left(t, T_{g_{\omega}^{1}(t)} R_{g_{\omega}^{1}(t)^{-1}} \frac{\mathscr{D} g_{\omega}^{1}(t)}{d t}, \alpha_{\omega}(t)\right) d t \\
&+\int_{0}^{T} p\left(T_{g_{\omega}^{1}(t)} R_{g_{\omega}^{1}(t)^{-1}} \frac{\mathbf{D}^{\nabla,\left(\mathbf{w}_{\omega}^{1, i}, M_{\omega}^{i, 1}\right)_{i=1}^{m_{1}}} g_{\omega}^{1}(t)}{d t}, T_{g_{\omega}^{2}(t)} R_{g_{\omega}^{2}(t)^{-1}} \frac{\mathbf{D}^{\nabla,\left(\mathbf{w}_{\omega}^{2, i}, M_{\omega}^{i, 2}\right)_{i=1}^{m_{2}}} g_{\omega}^{2}(t)}{d t}\right. \\
&\left.T_{g_{\omega}^{1}(t)} R_{g_{\omega}^{1}(t)^{-1}} \frac{\mathscr{D} g_{\omega}^{1}(t)}{d t}\right) d t \\
&+\int_{0}^{T}\left\langle q\left(t, T_{g_{\omega}^{1}(t)} R_{g_{\omega}^{1}(t)^{-1}} \frac{\mathscr{D} g_{\omega}^{1}(t)}{d t}, \alpha_{\omega}(t)\right), T_{g_{\omega}^{1}(t)} R_{g_{\omega}^{1}(t)^{-1}} d^{\Delta} g_{\omega}^{1}(t)\right\rangle \\
&-\sum_{i=1}^{k} \int_{0}^{T}\left\langle q\left(t, T_{g_{\omega}^{1}(t)} R_{g_{\omega}^{1}(t)^{-1}} \frac{\mathscr{D} g_{\omega}^{1}(t)}{d t}, \alpha_{\omega}(t)\right), V_{i} d N_{\omega}^{i}(t)\right\rangle
\end{align*}
$$

and

$$
\begin{equation*}
\alpha_{\omega}(t):=\alpha_{0} g_{\omega}^{3}(t)^{-1} \tag{3.30}
\end{equation*}
$$

As for the left invariant case, for every (random) $\mathcal{P}_{t}$-adapted process $\mathscr{g}_{\omega}(\cdot)$ such that $\mathscr{g}_{\omega} \in C^{1}\left([0, T] ; T_{e} G\right), g_{\omega}(0)=g_{\omega}(T)=0$ a.s., and $\varepsilon \in[0,1)$, let $e_{\omega, \varepsilon, g}(\cdot)$ be the unique solution of the (random) time-dependent ordinary differential equation on $G$

$$
\left\{\begin{array}{l}
\frac{d}{d t} e_{\omega, \varepsilon, \mathfrak{g}}(t)=\varepsilon T_{e} R_{e_{\omega, \varepsilon, g}(t)} \dot{g}_{\omega}(t)  \tag{3.31}\\
e_{\omega, \varepsilon, g}(0)=e
\end{array}\right.
$$

Define

$$
\begin{equation*}
g_{\omega, \varepsilon, g}^{j}(t):=e_{\omega, \varepsilon, g}(t) g_{\omega}^{j}(t), \quad j=1,2,3 \tag{3.32}
\end{equation*}
$$

With such deformations of $g_{\omega}^{j}(\cdot)$, we can consider (right invariant) critical points of $J^{\nabla,\left(V_{i}, N_{\omega}^{i}\right)_{i=1}^{k}}$ as in (3.7).

In the procedure leading to Theorem 3.5 and its proof, we can interchange all left actions and left translation operators by their right counterparts to obtain Theorem 3.11 and 3.12 below, so we omit the proof here.

Theorem 3.11. Assume that $G$ is a finite dimensional Lie group endowed with a right invariant linear connection $\nabla, H_{i}^{j} \in T_{e} G, 1 \leq i \leq m_{j}$, and $M_{\omega}^{j}=\left\{M_{\omega}(t)^{j, i}\right\}_{i=1}^{m_{j}}, j=$ $1,2,3$ is an $\mathbb{R}^{m_{j}}$-valued martingale. Suppose that the semimartingales $g_{\omega}^{j}(\cdot) \in \mathscr{S}(G)$, $j=1,2,3$, have the following form,

$$
\left\{\begin{array}{l}
d g_{\omega}^{j}(t)=T_{e} R_{g_{\omega}^{j}(t)}\left(\sum_{i=1}^{m_{j}} H_{i}^{j} \delta M_{\omega}^{j, i}(t)+u_{\omega}(t) d t\right),  \tag{3.33}\\
g_{\omega}^{j}(0)=e,
\end{array}\right.
$$

where $u_{\omega}$ is a $\mathcal{P}_{t}$-adapted, $T_{e} G$-valued semimartingale.
(i) If $\frac{\delta l_{\omega}}{\delta u}$ is non-random and $l_{\omega}(t)$ is $\mathcal{P}_{t}$-adapted, then $\left(\left(g_{\omega}^{1}, H_{i}^{1}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}},\left(g_{\omega}^{2}, H_{i}^{2}, M_{\omega}^{2, i}\right)_{i=1}^{m_{2}}, g_{\omega}^{3}\right)$ is a critical point of $J^{\nabla,\left(H_{i}^{1}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}}}$ defined by (3.29) if and only if the $\mathcal{P}_{t}$-adapted, $T_{e} G$ valued semimartingale $u_{\omega}(\cdot)$ coupled with $\mathcal{P}_{t}$-adapted, $U^{*}$-valued semimartingale $\alpha_{\omega}(\cdot)$ satisfies the following equation

$$
\left\{\begin{array}{l}
d\left(\frac{\delta l_{\omega}}{\delta u}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)+\frac{\delta p}{\delta u}\left(\tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right)\right)  \tag{3.34}\\
=-\sum_{i=1}^{m_{1}} \operatorname{ad}_{H_{i}}^{*} q\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right) d M_{\omega}^{1, i}(t)-\operatorname{ad}_{u_{\omega}(t)}^{*}\left(\frac{\delta l_{\omega}}{\delta u}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)\right) d t \\
\quad-\operatorname{ad}_{u_{\omega}(t)}^{*}\left(\frac{\delta p}{\delta u}\left(\tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right)\right) d t+\left(\frac{\delta l_{\omega}}{\delta \alpha}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)\right) \diamond \alpha_{\omega}(t) d t \\
\quad-K_{\omega}\left(t, \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right) d t \\
d \alpha_{\omega}(t)=-\sum_{i=1}^{m_{3}} \alpha_{\omega}(t) H_{i}^{3} d M_{\omega}^{3, i}(t)+\frac{1}{2} \sum_{i, k=1}^{m_{3}}\left(\alpha_{\omega}(t) H_{i}^{3}\right) H_{k}^{3} d \llbracket M_{\omega}^{3, i}, M_{\omega}^{3, k} \rrbracket_{t}-\alpha_{\omega}(t) u_{\omega}(t) d t,
\end{array}\right.
$$

where $\tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t)$, and $K_{\omega}$ are defined by (3.11), and (3.12) respectively.
(ii) The first equation in (3.34) is equivalent to the stochastic dissipative Euler-Poincaré variational principle

$$
\begin{align*}
& \left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0}\left(\int_{0}^{T} l_{\omega}\left(t, u_{\omega, \varepsilon}(t), \alpha_{\omega, \varepsilon}(t)\right) d t+\int_{0}^{T} p\left(\tilde{H}_{\omega, 1, \varepsilon}(t), \tilde{H}_{\omega, 2, \varepsilon}(t), u_{\omega, \varepsilon}(t)\right) d t\right.  \tag{3.35}\\
& \left.+\int_{0}^{T}\left\langle q\left(t, u_{\omega, \varepsilon}(t), \alpha_{\omega, \varepsilon}(t)\right), d \beta_{\omega, \varepsilon}(t)\right\rangle-\sum_{i=1}^{m_{1}}\left(\int_{0}^{T}\left\langle q\left(t, u_{\omega, \varepsilon}(t), \alpha_{\omega, \varepsilon}(t)\right), H_{i}^{1}\right\rangle d M_{\omega}^{1, i}(t)\right)\right)=0
\end{align*}
$$

on $T_{e} G \times U^{*}$, for variations of the form

$$
\left\{\begin{array}{l}
\left.\frac{d u_{\omega, \varepsilon}(t)}{d \varepsilon}\right|_{\varepsilon=0}=\dot{v}_{\omega}(t)-a d_{u_{\omega}(t)} v_{\omega}(t),  \tag{3.36}\\
\left.\frac{d \alpha_{\omega, \varepsilon}(t)}{d \varepsilon}\right|_{\varepsilon=0}=-\alpha_{\omega}(t) v_{\omega}(t), \\
\left.\frac{d \tilde{H}_{\omega, j, \varepsilon}(t)}{d \varepsilon}\right|_{\varepsilon=0}=B_{\omega, j}\left(t, v_{\omega}(t)\right), j=1,2 \\
\left.\frac{d \beta_{\omega, \varepsilon}(t)}{d \varepsilon}\right|_{\varepsilon=0}=-\sum_{i=1}^{m_{1}} \int_{0}^{t} a d_{v_{\omega}(s)} H_{i}^{1} d M_{\omega}^{1, i}(s) \\
u_{\omega, 0}(t)=u_{\omega}(t), \alpha_{\omega, 0}(t)=\alpha_{\omega}(t), \beta_{\omega, 0}(t)=\sum_{i=1}^{m_{1}} \int_{0}^{t} H_{i}^{1} d M_{\omega}^{1, i}(s), \tilde{H}_{\omega, j, 0}(t)=\tilde{H}_{\omega, j}(t),
\end{array}\right.
$$

where $v_{\omega}(t)$ is an $\mathcal{P}_{t}$-adapted process such that $v \in C^{1}\left([0, T] ; T_{e} G\right)$ and $v(0)=0$, $v(T)=0$ a.s..

The right invariant version for the deterministic action functional in Theorem 3.9 is the following:

$$
\begin{aligned}
& \tilde{J}^{\nabla}\left(\left(g_{\omega}^{1}, \mathbf{w}_{\omega}^{1, i}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}},\left(g_{\omega}^{2}, \mathbf{w}_{\omega}^{2, i}, M_{\omega}^{2, i}\right)_{i=1}^{m_{2}}, g_{\omega}^{3}\right) \\
& :=\int_{0}^{T} l\left(t, T_{g_{\omega}^{1}(t)} R_{g_{\omega}^{1}(t)^{-1}} \frac{\mathscr{D} g_{\omega}^{1}(t)}{d t}, \tilde{\alpha}(t)\right) d t+\int_{0}^{T} p\left(T_{g_{\omega}^{1}(t)} R_{g_{\omega}^{1}(t)^{-1}} \frac{\mathbf{D}^{\nabla,\left(\mathbf{w}_{\omega}^{1, i}, M_{\omega}^{i, 1}\right)_{i=1}^{m_{1}} g_{\omega}^{1}(t)}}{d t},\right. \\
& \left.T_{g_{\omega}^{2}(t)} R_{g_{\omega}^{2}(t)^{-1}} \frac{\mathbf{D}^{\nabla,\left(\mathbf{w}_{\omega}^{2, i}, M_{\omega}^{i, 2}\right)_{i=1}^{m_{2}}} g_{\omega}^{2}(t)}{d t}, T_{g_{\omega}^{1}(t)} R_{g_{\omega}^{1}(t)^{-1}} \frac{\mathscr{D} g_{\omega}^{1}(t)}{d t}\right) d t .
\end{aligned}
$$

Here $l$ is non-random and $\tilde{\alpha}(t):=\mathbb{E}\left[\alpha_{\omega}(t)\right] \in U^{*}$ and $\alpha_{\omega}(t):=\alpha_{0} g_{\omega}^{3}(t)^{-1}$.
Theorem 3.12. Suppose that the semimartingales $g_{\omega}^{j}(\cdot), j=1,2,3$, have the form (3.33) with $u \in C^{1}\left([0, T] ; T_{e} G\right)$ and $\llbracket M^{j, i}, M^{j, k} \rrbracket_{t}, 1 \leq j \leq 3,1 \leq i, k \leq m_{j}$ being nonrandom (we write $u, \llbracket M^{j, i}, M^{j, k} \rrbracket_{t}$ for $u_{\omega}$ and $\llbracket M_{\omega}^{j, i}, M_{\omega}^{j, k} \rrbracket_{t}$ in this theorem). Consider deformations of the form (3.32) with $g$ non-random (we write $\mathcal{g}$ for $g_{\omega}$ in this theorem because it is non-random).
(i) Then $\left(\left(g_{\omega}^{1}, H_{i}^{1}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}},\left(g_{\omega}^{2}, H_{i}^{2}, M_{\omega}^{2, i}\right)_{i=1}^{m_{2}}, g_{\omega}^{3}\right)$ is a critical point of $\tilde{J}^{\nabla}$ if and only if $u(t)$ coupled with $\tilde{\alpha}(t)$ satisfies the following (ordinary differential) equation

$$
\left\{\begin{array}{l}
d\left(\frac{\delta l}{\delta u}(t, u(t), \tilde{\alpha}(t))+\frac{\delta p}{\delta u}\left(\tilde{H}_{1}(t), \tilde{H}_{2}(t), u(t)\right)\right)  \tag{3.37}\\
=-\operatorname{ad}_{u(t)}^{*}\left(\frac{\delta l}{\delta u}(t, u(t), \tilde{\alpha}(t))\right) d t-\operatorname{ad}_{u(t)}^{*}\left(\frac{\delta p}{\delta u}\left(\tilde{H}_{1}(t), \tilde{H}_{2}(t), u(t)\right)\right) d t \\
\quad+\left(\frac{\delta l}{\delta \alpha}(t, u(t), \tilde{\alpha}(t))\right) \diamond \tilde{\alpha}(t) d t-K\left(t, \tilde{H}_{1}(t), \tilde{H}_{2}(t), u(t)\right) d t, \\
d \tilde{\alpha}(t)=\frac{1}{2} \sum_{i, k=1}^{m_{3}}\left(\tilde{\alpha}(t) H_{i}^{3}\right) H_{k}^{3} d \llbracket M_{\omega}^{3, i}, M_{\omega}^{3, k} \rrbracket_{t}-\tilde{\alpha}(t) u(t) d t,
\end{array}\right.
$$

where $\tilde{H}_{1}(t), \tilde{H}_{2}(t), \diamond, K$ are the same terms as in Theorem 3.9.
(ii) The first equation in (3.37) is equivalent to the following stochastic variational principle

$$
\begin{equation*}
\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0}\left(\int_{0}^{T} l\left(t, u_{\varepsilon}(t), \tilde{\alpha}_{\varepsilon}(t)\right) d t+\int_{0}^{T} p\left(\tilde{H}_{1, \varepsilon}(t), \tilde{H}_{2, \varepsilon}(t), u_{\varepsilon}(t)\right) d t\right)=0 \tag{3.38}
\end{equation*}
$$

on $T_{e} G \times U^{*}$ for variations of the form

$$
\left\{\begin{array}{l}
\left.\frac{d u_{\varepsilon}(t)}{d \varepsilon}\right|_{\varepsilon=0}=\dot{v}(t)-\operatorname{ad}_{u(t)} v(t) \\
\left.\frac{d \tilde{\alpha}_{\varepsilon}(t)}{d \varepsilon}\right|_{\varepsilon=0}=-v(t) \tilde{\alpha}(t) \\
\left.\frac{d \tilde{H}_{j, \varepsilon}(t)}{d \varepsilon}\right|_{\varepsilon=0}=B_{j}(t, v(t)), \quad j=1,2 \\
u_{0}(t)=u(t), \tilde{\alpha}_{0}(t)=\tilde{\alpha}(t), \quad \tilde{H}_{j, 0}(t)=\tilde{H}_{j}(t)
\end{array}\right.
$$

where $v \in C^{1}\left([0, T] ; T_{e} G\right)$ with $v(0)=0, v(T)=0$ is non-random.

## 4 Stochastic Kelvin-Noether theorem

In this section we study a (stochastic) version of the Kelvin-Noether Theorem which holds for solutions of stochastic Euler-Poincaré equations with advection terms (see (3.10)).

Let $G, U^{*}, l, q$ be as in Section 3. Suppose $\mathscr{C}$ is a manifold and $G$ acts on the left on $\mathscr{C}$. Let $\mathscr{K}: \mathscr{C} \times U^{*} \rightarrow T_{e}^{* *} G$ be an equivariant map, i.e.,

$$
\begin{equation*}
\left\langle\mathscr{K}\left(g^{-1} c, g^{-1} \alpha\right), \mu\right\rangle=\left\langle\mathscr{K}(c, \alpha), \operatorname{Ad}_{g^{-1}}^{*} \mu\right\rangle, \quad c \in \mathscr{C}, \alpha \in U^{*}, g \in G, \mu \in T_{e}^{*} G, \tag{4.1}
\end{equation*}
$$

where $\langle$,$\rangle denotes the weak pairing between T_{e}^{* *} G$ and $T_{e}^{*} G$.
As explained before, we identify the Lie algebra $\mathfrak{g}$ with the tangent space $T_{e} G$ at the unit element. As usual (see, e.g., [49, Section 4]), we define the Kelvin-Noether quantity $I: \mathscr{C} \times T_{e} G \times U^{*} \rightarrow \mathbb{R}$ by

$$
\begin{equation*}
I(c, u, \alpha):=\left\langle\mathscr{K}(c, \alpha), \frac{\delta l}{\delta u}(u, \alpha)\right\rangle, \quad c \in \mathscr{C}, u \in T_{e} G, \alpha \in U^{*} . \tag{4.2}
\end{equation*}
$$

Now we are ready to state the Kelvin-Noether Theorem for the solutions of (3.10).
Theorem 4.1. Suppose $u_{\omega}(t)$ satisfies the first equation in (3.10) with $\frac{\delta p}{\delta u}\left(\tilde{H}_{\omega, 1}(t)\right.$, $\left.\tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right) \equiv 0, \alpha_{\omega}(t)=g_{\omega}^{3}(t)^{-1} \alpha_{0}$ and $\frac{\delta l}{\delta u}$ non-random. Let the semimartingales
$g_{\omega}^{j}(\cdot) \in \mathscr{S}(G), j=1,2,3$, be defined by (3.9) using this $u_{\omega}(t)$. For a fixed $c_{0} \in \mathscr{C}$, set $c_{\omega}(t):=g_{\omega}^{3}(t)^{-1} c_{0}, I_{\omega}(t):=I\left(c_{\omega}(t), u_{\omega}(t), \alpha_{\omega}(t)\right)$. We have

$$
\begin{align*}
d I_{\omega}(t)=\langle & \mathscr{K}\left(c_{\omega}(t), \alpha_{\omega}(t)\right),\left(\sum_{i=1}^{m_{1}} \operatorname{ad}_{H_{i}^{1}}^{*} q_{\omega}(t) d M_{\omega}^{1, i}(t)-\sum_{i=1}^{m_{3}} \operatorname{ad}_{H_{i}^{3}}^{*} \frac{\delta l_{\omega}}{\delta u}(t) d M_{\omega}^{3, i}(t)\right)  \tag{4.3}\\
& +\left(\frac{\delta l_{\omega}}{\delta \alpha}(t) \diamond \alpha_{\omega}(t)+K_{\omega}\left(t, \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right)\right. \\
& +\frac{1}{2} \sum_{i, k=1}^{m_{3}} \operatorname{ad}_{H_{i}^{3}}^{*} \operatorname{ad}_{H_{k}^{3}}^{*}\left(\frac{\delta l_{\omega}}{\delta u}(t)\right) \frac{d \llbracket M_{\omega}^{3, i}, M_{\omega}^{3, k} \rrbracket_{t}}{d t} \\
& \left.\left.\left.-\sum_{i=1}^{m_{1}} \sum_{k=1}^{m_{3}} \operatorname{ad}_{H_{k}^{3}}^{*} \operatorname{ad}_{H_{i}^{1}}^{*} q_{\omega}(t) \frac{d \llbracket M_{\omega}^{1, i}, M_{\omega}^{3, k} \rrbracket_{t}}{d t}\right) d t\right\rangle\right]
\end{align*}
$$

where $\frac{\delta l_{\omega}}{\delta u}(t):=\tilde{\delta l}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right), \frac{\delta l_{\omega}}{\delta \alpha}(t):=\frac{\delta l_{\omega}}{\delta \alpha}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right), q_{\omega}(t):=q\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)$, and $\tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t) \in \mathscr{M}, K_{\omega}:[0, T] \times \mathscr{M} \times \mathscr{M} \times T_{e} G \rightarrow T_{e}^{*} G$ are defined by (3.11) and (3.12), respectively.

In particular, if $\frac{\delta l_{\omega}}{\delta u}(t)=q_{\omega}(t), m_{1}=m_{3}=m$, and $H_{i}^{1}=H_{i}^{3}=H_{i}, M_{\omega}^{1, i}(t)=$ $M_{\omega}^{3, i}(t), 1 \leq i \leq m$, we have

$$
\begin{align*}
d I_{\omega}(t)=\langle & \mathscr{K}\left(c_{\omega}(t), \alpha_{\omega}(t)\right), \frac{\delta l_{\omega}}{\delta \alpha}(t) \diamond \alpha_{\omega}(t) d t+K_{\omega}\left(t, \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right) d t \\
& \left.-\frac{1}{2} \sum_{i, k=1}^{m} \operatorname{ad}_{H_{i}}^{*} \operatorname{ad}_{H_{k}}^{*} q_{\omega}(t) d \llbracket M_{\omega}^{3, i}, M_{\omega}^{3, k} \rrbracket_{t}\right\rangle . \tag{4.4}
\end{align*}
$$

Proof. Due to (4.1) we have

$$
\begin{align*}
I_{\omega}(t) & =\left\langle\mathscr{K}\left(g_{\omega}^{3}(t)^{-1} c_{0}, g_{\omega}^{3}(t)^{-1} \alpha_{0}\right), \frac{\delta l}{\delta u}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)\right\rangle \\
& =\left\langle\mathscr{K}\left(c_{0}, \alpha_{0}\right), \operatorname{Ad}_{g_{\omega}^{3}(t)^{-1}}^{*} \frac{\delta l}{\delta u}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)\right\rangle . \tag{4.5}
\end{align*}
$$

By Itô's formula, for any $T_{e}^{*} G$-valued semimartingale $\beta_{\omega}(\cdot)$, we have

$$
\begin{align*}
& d \operatorname{Ad}_{g_{\omega}^{3}(t)^{-1}}^{*} \beta_{\omega}(t) \\
&= \operatorname{Ad}_{g_{\omega}^{3}(t)^{-1}}^{*}\left(-\operatorname{ad}_{g_{\omega}^{3}(t)^{-1} \delta g_{\omega}^{3}(t)}^{*} \beta_{\omega}(t)+\delta \beta_{\omega}(t)\right) \\
&= \operatorname{Ad}_{g_{\omega}^{3}(t)^{-1}}^{*}\left(-\sum_{i=1}^{m_{3}} \operatorname{ad}_{H_{i}^{3}}^{*} \beta_{\omega}(t) d M_{\omega}^{3, i}(t)-\operatorname{ad}_{u_{\omega}(t)}^{*} \beta_{\omega}(t) d t\right.  \tag{4.6}\\
&\left.+d \beta_{\omega}(t)+\frac{1}{2} \sum_{i, k=1}^{m_{3}} \operatorname{ad}_{H_{i}^{3}}^{*} \operatorname{ad}_{H_{k}^{3}}^{*} \beta_{\omega}(t) d \llbracket M_{\omega}^{3, i}, M_{\omega}^{3, k} \rrbracket_{t}-\sum_{i=1}^{m_{3}} \operatorname{ad}_{H_{i}^{3}}^{*} d \llbracket M_{\omega}^{3, i}, \beta_{\omega} \rrbracket_{t}\right) .
\end{align*}
$$

Combining (4.6) with (3.9) and (3.10) yields

$$
\begin{aligned}
& d\left(\operatorname{Ad}_{g_{\omega}^{3}(t)^{-1}}^{*} \frac{\delta l_{\omega}}{\delta u}(t)\right) \\
& =\operatorname{Ad}_{g_{\omega}^{3}(t)^{-1}}^{*}\left(-\sum_{i=1}^{m_{3}} \operatorname{ad}_{H_{i}^{3}}^{*} \frac{\delta l_{\omega}}{\delta u}(t) d M_{\omega}^{3, i}(t)-\operatorname{ad}_{u_{\omega}(t)}^{*} \frac{\delta l_{\omega}}{\delta u}(t) d t+\sum_{i=1}^{m_{1}} \operatorname{ad}_{H_{i}^{1}}^{*} q_{\omega}(t) d M_{\omega}^{1, i}(t)\right. \\
& \quad+\operatorname{ad}_{u_{\omega}(t)}^{*} \frac{\delta l_{\omega}}{\delta u}(t) d t+\frac{\delta l_{\omega}}{\delta \alpha}(t) \diamond \alpha_{\omega}(t) d t+K_{\omega}\left(t, \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right) d t \\
& \left.\quad+\frac{1}{2} \sum_{i, k=1}^{m_{3}} \operatorname{ad}_{H_{i}^{3}}^{*} \operatorname{ad}_{H_{k}^{3}}^{*}\left(\frac{\delta l_{\omega}}{\delta u}(t)\right) d \llbracket M_{\omega}^{3, i}, M_{\omega}^{3, k} \rrbracket_{t}-\sum_{i=1}^{m_{1}} \sum_{k=1}^{m_{3}} \operatorname{ad}_{H_{k}^{3}}^{*} \operatorname{ad}_{H_{i}^{1}}^{*} q_{\omega}(t) d \llbracket M_{\omega}^{1, i}, M_{\omega}^{3, k} \rrbracket_{t}\right) .
\end{aligned}
$$

Here we also apply the assumption $\frac{\delta p}{\delta u}\left(\tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right) \equiv 0$. Putting this into (4.5) and applying (4.2), we arrive at

$$
\begin{aligned}
d I_{\omega}(t) & =\left\langle\mathscr{K}\left(c_{0}, \alpha_{0}\right), d\left(\operatorname{Ad}_{g_{\omega}^{3}(t)^{-1}}^{*} \frac{\delta l_{\omega}}{\delta u}(t)\right)\right\rangle \\
& =\left\langle\mathscr{K}\left(c_{0}, \alpha_{0}\right), \operatorname{Ad}_{g_{\omega}^{3}(t)^{-1}}^{*}\left(\sum_{i=1}^{m_{1}} \operatorname{ad}_{H_{i}^{1}}^{*} q_{\omega}(t) d M_{\omega}^{1, i}(t)-\sum_{i=1}^{m_{3}} \operatorname{ad}_{H_{i}^{3}}^{*} \frac{\delta l_{\omega}}{\delta u}(t) d M_{\omega}^{3, i}(t)\right.\right. \\
& +\frac{\delta l_{\omega}}{\delta a}(t) \diamond \alpha_{\omega}(t) d t+K_{\omega}\left(t, \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right) d t \\
& \left.\left.+\frac{1}{2} \sum_{i, k=1}^{m_{3}} \operatorname{ad}_{H_{i}^{3}}^{*} \operatorname{ad}_{H_{k}^{3}}^{*}\left(\frac{\delta l_{\omega}}{\delta u}(t)\right) d \llbracket M_{\omega}^{3, i}, M_{\omega}^{3, k} \rrbracket_{t}-\sum_{i=1}^{m_{1}} \sum_{k=1}^{m_{3}} \operatorname{ad}_{H_{k}^{3}}^{*} \operatorname{ad}_{H_{i}^{1}}^{*} q_{\omega}(t) d \llbracket M_{\omega}^{1, i}, M_{\omega}^{3, k} \rrbracket_{t}\right)\right\rangle \\
& =\left\langle\mathscr{K}\left(c_{\omega}(t), \alpha_{\omega}(t)\right),\left(\sum_{i=1}^{m_{1}} \operatorname{ad}_{H_{i}^{1}}^{*} q_{\omega}(t) d M_{\omega}^{1, i}(t)-\sum_{i=1}^{m_{3}} \operatorname{ad}_{H_{i}^{3}}^{*} \frac{\delta l_{\omega}}{\delta u}(t) d M_{\omega}^{3, i}(t)\right)\right. \\
& +\frac{\delta l_{\omega}}{\delta a}(t) \diamond \alpha_{\omega}(t) d t+K_{\omega}\left(t, \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right) d t \\
& \left.+\frac{1}{2} \sum_{i, k=1}^{m_{3}} \operatorname{ad}_{H_{i}^{3}}^{*} \operatorname{ad}_{H_{k}^{3}}^{*}\left(\frac{\delta l_{\omega}}{\delta u}(t)\right) d \llbracket M_{\omega}^{3, i}, M_{\omega}^{3, k} \rrbracket_{t}-\sum_{i=1}^{m_{1}} \sum_{k=1}^{m_{3}} \operatorname{ad}_{H_{k}^{3}}^{*} \operatorname{ad}_{H_{i}^{1}}^{*} q_{\omega}(t) d \llbracket M_{\omega}^{1, i}, M_{\omega}^{3, k} \rrbracket_{t}\right\rangle
\end{aligned}
$$

which proves (4.3).
If $\frac{\delta l_{\omega}}{\delta u}(t)=q_{\omega}(t), m_{1}=m_{3}=m$, and $H_{i}^{1}=H_{i}^{3}=H_{i}, M_{\omega}^{1, i}(t)=M_{\omega}^{3, i}(t), 1 \leq i \leq m$, then $\operatorname{ad}_{H_{i}}^{*} \frac{\delta l_{\omega}}{\delta u}(t)=\operatorname{ad}_{H_{i}}^{*} q_{\omega}(t)$ which, combined with (4.3), yields (4.4).

Remark 4.2. If $H_{i}^{j}=0,1 \leq i \leq m_{j}, j=1,2,3$, then (4.3) becomes

$$
d I_{\omega}(t)=\left\langle\mathscr{K}\left(c_{\omega}(t), \alpha_{\omega}(t)\right), \frac{\delta l_{\omega}}{\delta a}(t) \diamond \alpha_{\omega}(t) d t\right\rangle,
$$

thus recovering [49, Theorem 4.1] for the deterministic Euler-Poincaré equation with advection term.

Remark 4.3. As we will see in Section 5, the term $K_{\omega}\left(t, \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right)$ usually corresponds to some viscosity terms of the system.

## 5 Application to PDEs and SPDEs in fluid mechanics

We begin by recalling, from [23] and [67], the necessary standard facts about the group of diffeomorphisms on a smooth compact boundaryless $n$-dimensional manifold $M$. Then, when we present the compressible Navier-Stokes and MHD equations in the periodic case, we shall take $M=\mathbb{T}^{3}$, the usual three dimensional flat torus.

Let $M$ be a smooth compact boundaryless $n$-dimensional manifold. Define

$$
G^{s}:=\left\{g: M \rightarrow M \text { is a bijection } \mid g, g^{-1} \in H^{s}(M, M)\right\}
$$

where $H^{s}(M, M)$ denotes the manifold of Sobolev maps of class $s>1+\frac{n}{2}$ from $M$ to itself. The condition $s>\frac{n}{2}$ suffices to ensure the manifold structure of $H^{s}(M, M)$; only for such regularity class does the notion of an $H^{s}$-map from $M$ to itself make intrinsic sense. If $s>1+\frac{n}{2}$ (the additional regularity is needed in order to ensure that all elements of $G^{s}$ are $C^{1}$ and hence the inverse function theorem is applicable), then $G^{s}$ is an open subset in $H^{s}(M, M)$, so it is a $C^{\infty}$ Hilbert manifold. Moreover, it is a group under composition of diffeomorphisms maps, right translation by any element is smooth, left translation and inversion are only continuous, and $G^{s}$ is a topological group (relative to the underlying manifold topology) (see [23], [67]); thus, $G^{s}$ is not a Lie group. Since $G^{s}$ is an open subset of $H^{s}(M, M)$, the tangent space $T_{e} G^{s}$ to the identity $e: M \rightarrow M$ coincides with the Hilbert space $\mathfrak{X}^{s}(M)$ of $H^{s}$ vector fields on $M$. Denote by $\mathfrak{X}(M)$ the Lie algebra of $C^{\infty}$ vector fields on $M$. The failure of $G^{s}$ to be a Lie group is mirrored by the fact that $\mathfrak{X}^{s}(M)$ is not a Lie algebra: the usual Jacobi-Lie bracket of vector fields, i.e., $[X, Y][f]=X[Y[f]]-Y[X[f]]$ for any $X, Y \in \mathfrak{X}(M)$ and $f \in C^{\infty}(M)$, where $X[f]:=\mathbf{d} f(X)$ is the differential of $f$ in the direction $X$, loses a derivative for finite differentiability class of vector fields and thus $[\cdot, \cdot]: \mathfrak{X}^{s}(M) \times \mathfrak{X}^{s}(M) \rightarrow \mathfrak{X}^{s-1}(M)$ is not an operation on $\mathfrak{X}^{s}(M)$. In general, the tangent space $T_{\eta} G^{s}$ at an arbitrary $\eta \in G^{s}$ is $T_{\eta} G^{s}=\left\{U: M \rightarrow T M\right.$ of class $\left.H^{s} \mid U(m) \in T_{\eta(m)} M\right\}$. If $s>1+\frac{n}{2}$ and $X \in \mathfrak{X}^{s}(M)$, then its global (since $M$ is compact) flow $\mathbb{R} \ni t \mapsto F_{t} \in G^{s}$ exists and is a $C^{1}$-curve in $G^{s}$ (see, e.g., [67, Theorem 2.4.2]). The candidate of what should have been the Lie group exponential map is $\exp : T_{e} G^{s}=\mathfrak{X}^{s}(M) \ni X \mapsto F_{1} \in G^{s}$, where $F_{t}$ is the flow of $X$; however, exp does not cover a neighborhood of the identity and it is not $C^{1}$. Therefore, all classical proofs in the theory of finite dimensional Lie groups
based on the exponential map, break down for $G^{s}$. From now on, we shall always assume $s>1+\frac{n}{2}$.

Since right translation is smooth, each $X \in \mathfrak{X}(M)$ induces a $C^{\infty}$ right invariant vector field $X^{R} \in \mathfrak{X}\left(G^{s}\right)$ on $G^{s}$, defined by $X^{R}(\eta):=X \circ \eta$. With this notation, we have the identity $\left[X^{R}, Y^{R}\right](e)=[X, Y]$, for any $X, Y \in \mathfrak{X}(M)$. This is the analogue of saying that $\mathfrak{X}(M)$ is the "right Lie algebra" of $G^{s}$.

Assume, in addition, that $M$ is connected, oriented, and Riemannian; denote by $\langle\cdot, \cdot\rangle$ the Riemannian metric. Let $\mu_{g}$ be the Riemannian volume form on $M$, whose expression in local coordinates $\left(x^{1}, \ldots, x^{n}\right)$ is $\mu_{g}=\sqrt{\operatorname{det}\left(g_{i j}\right)} d x^{1} \wedge \cdots \wedge d x^{n}$, where $g_{i j}:=\left\langle\partial / \partial x^{i}, \partial / \partial x^{j}\right\rangle$ for all $i, j=1, \ldots n$. Let $K: T(T M) \rightarrow T M$ be the connector of the Levi-Civita connection (with Christoffel symbols $\Gamma_{j k}^{i}$ ) defined by the Riemannian metric on $M$; in local coordinates, this intrinsic object, which is a vector bundle map $K: T(T M) \rightarrow T M$ covering the canonical vector bundle projection $\tau_{M}: T M \rightarrow M$, has the expression $K\left(x^{i}, u^{i}, v^{i}, w^{i}\right)=\left(x^{i}, w^{i}+\Gamma_{j k}^{i} u^{j} v^{k}\right)$. The connector $K$ satisfies hence the identity $\tau_{M} \circ K=\tau_{M} \circ \tau_{T M}$ and has the property that the vector bundles $\tau_{T M}$ : $T(T M) \rightarrow T M$ and $\operatorname{ker} K \oplus \operatorname{ker} T \tau_{M} \rightarrow T M$ are isomorphic. These two properties characterize the connector. Conversely, the connector $K$ determines $\nabla$ by the formula: $\nabla_{X} Y:=K \circ T Y \circ X$ for any $X, Y \in \mathfrak{X}(M)$.

The Riemannian structure on $M$ induces the weak $L^{2}$, or hydrodynamic, metric $\langle\langle\cdot \cdot \cdot\rangle\rangle_{\eta}$ on $G^{s}$ given by

$$
\left\langle\left\langle U_{\eta}, V_{\eta}\right\rangle\right\rangle_{\eta}:=\int_{M}\left\langle U_{\eta}(m), V_{\eta}(m)\right\rangle_{\eta(m)} d \mu_{g}(m)
$$

for any $\eta \in G^{s}, U_{\eta}, V_{\eta} \in T_{\eta} G^{s}$. This means that the association $\eta \mapsto\langle\langle\cdot, \cdot\rangle\rangle_{\eta}$ from $G^{s}$ to the vector bundle of symmetric covariant two-tensors on $G^{s}$ is smooth but that for every $\eta \in G^{s}$, the map $T_{\eta} G^{s} \ni U_{\eta} \mapsto\left\langle\left\langle U_{\eta}, \cdot\right\rangle\right\rangle \in T_{\eta}^{*} G^{s}$, where $T_{\eta}^{*} G^{s}$ denotes the linear continuous functionals on $T G^{s}$, is only injective and not, in general, surjective. This weak metric is not right invariant (because of the Jacobian appearing in the change of variables formula in the integral).

The usual proof for finite dimensional Lie groups showing the existence of a unique Levi-Civita connection associated to a Riemannian metric breaks down, because $\langle\langle\cdot, \cdot\rangle\rangle$ is weak; the proof would only show uniqueness. However, $K^{0}: T\left(T G^{s}\right) \rightarrow T G^{s}$ given by $K^{0}\left(\mathcal{Z}_{U_{\eta}}\right):=K \circ \mathcal{Z}_{U_{\eta}}$, where $\mathcal{Z}_{U_{\eta}} \in T_{U_{\eta}}\left(T G^{s}\right)$, is a connector for the vector bundle $\tau_{G^{s}}: T G^{s} \rightarrow G^{s}$ (since $\tau_{G^{s}} \circ K^{0}=\tau_{G^{s}} \circ \tau_{T G^{s}}$ and the vector bundles $\tau_{T G^{s}}$ : $T\left(T G^{s}\right) \rightarrow T G^{s}$ and $\operatorname{ker} K^{0} \oplus \operatorname{ker} T \tau_{G^{s}} \rightarrow T G^{s}$ are isomorphic). Here, $\mathcal{Z}_{U_{\eta}} \in T_{U_{\eta}}\left(T G^{s}\right)$ means that $\mathcal{Z}_{U_{\eta}}: M \rightarrow T(T M)$ satisfies $\tau_{T M}\left(\mathcal{Z}_{U_{\eta}}(m)\right) \in T_{\eta(m)} M$. The covariant derivative $\nabla^{0}$ on $G^{s}$ is defined by $\nabla_{\mathcal{X}}^{0} \mathcal{Y}:=K^{0} \circ T \mathcal{Y} \circ \mathcal{X}$, for any $\mathcal{X}, \mathcal{Y} \in \mathfrak{X}\left(G^{s}\right)$. This is the Levi-Civita connection associated to the weak metric $\langle\langle\cdot, \cdot\rangle\rangle$ since it is torsion free $\left(\nabla_{\mathcal{X}}^{0} \mathcal{Y}-\nabla_{\mathcal{Y}}^{0} \mathcal{X}=[\mathcal{X}, \mathcal{Y}]\right.$, for all $\mathcal{X}, \mathcal{Y} \in \mathfrak{X}^{s}\left(G^{s}\right)$, where $[\mathcal{X}, \mathcal{Y}]$ is the Jacobi-Lie bracket of vector fields on $\left.G^{s}\right)$ and $\langle\langle\cdot, \cdot\rangle\rangle$-compatible $(\mathcal{Z}[\langle\mathcal{X}, \mathcal{Y}\rangle\rangle]=\left\langle\left\langle\nabla_{\mathcal{Z}} \mathcal{X}, \mathcal{Y}\right\rangle\right\rangle+\left\langle\left\langle\mathcal{X}, \nabla_{\mathcal{Z}} \mathcal{Y}\right\rangle\right\rangle$, for all $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathfrak{X}^{s}\left(G^{s}\right)$ ); see [23, Theorem 9.1]. Uniqueness of such a connection
follows from the weak non-degeneracy of the metric $\langle\langle\cdot, \cdot\rangle\rangle$. There is an explicit formula for right-invariant covariant derivatives on diffeomorphism groups (see, e.g., 31, page $6]$ ). For $\nabla^{0}$ this formula is

$$
\begin{equation*}
\left(\nabla_{\mathcal{X}}^{0} \mathcal{Y}\right)(\eta):=\frac{\partial}{\partial t}\left(\mathcal{Y}\left(\eta_{t}\right) \circ \eta_{t}^{-1}\right) \circ \eta+\left(\nabla_{X^{\eta}} Y^{\eta}\right) \circ \eta, \tag{5.1}
\end{equation*}
$$

where $\nabla$ denotes the Levi-Civita connection on $M, \mathcal{X}, \mathcal{Y} \in \mathfrak{X}\left(G^{s}\right), X^{\eta}:=\mathcal{X} \circ \eta^{-1}, Y^{\eta}:=$ $\mathcal{Y} \circ \eta^{-1} \in \mathfrak{X}^{s}(M)$, and $t \mapsto \eta_{t}$ is a $C^{1}$ curve in $G^{s}$ such that $\eta_{0}=\eta$ and $\left.\frac{d}{d t}\right|_{t=0} \eta_{t}=\mathcal{X}(\eta)$; this formula is identical to [72, (3.1)]. Note that each term on the right hand side of this formula is only of class $H^{s-1}$ and, nevertheless, their sum is of class $H^{s}$ because of the abstract definition of the covariant derivative on $G^{s}$. A similar phenomenon occurs with the geodesic spray $T G^{s} \rightarrow T\left(T G^{s}\right)$ of the weak Riemannian metric; its existence and smoothness for boundaryless $M$ was proved in [23], i.e., the geodesic spray is in $\mathfrak{X}\left(T G^{s}\right)$. If $M$ has a boundary, this statement is false.

The discussion above shows that one cannot apply the theorems of Section 3 to the infinite dimensional group $G^{s}$ directly. For infinite dimensional problems, they serve only as a guideline and direct proofs are needed, which is what we do below. However, for each important formula, we shall point out the analogue in the finite dimensional abstract setting of Section 3 which inspired the result, that still holds for the model presented here.

### 5.1 Stochastic semidirect product Euler-Poincaré reduction for $G^{s}$.

We formulate now the theory presented in Section 3 for the infinite dimensional group $G^{s}$. From now on we consider the case $M=\mathbb{T}^{3}$. We focus on the following type of SDEs on $G^{s}$,

$$
\left\{\begin{array}{l}
d g_{\omega}(t, \theta)=\sum_{i=1}^{m} H_{i}\left(g_{\omega}(t, \theta)\right) \delta M_{\omega}^{i}(t)+u_{\omega}\left(t, g_{\omega}(t, \theta)\right) d t \\
g_{\omega}(0, \theta)=\theta, \quad \theta \in \mathbb{T}^{3}
\end{array}\right.
$$

where $H_{j} \in \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)$ is non-random, $\left\{M_{\omega}^{i}(t)\right\}_{i=1}^{m}$ is a $\mathbb{R}^{m}$-valued martingale with continuous sample paths on a probability space $(\Omega, \mathcal{P}, \mathbb{P})$ with respect to the filtration $\mathcal{P}_{t}$, $u: \Omega \times[0, T] \rightarrow \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)$ is such that $u_{\omega}(t, x)$ is a ( $\mathcal{P}_{t^{-}}$-adapted) semimartingale for every $x \in \mathbb{T}^{3}$.

In particular, here we take the constant vector fields $H_{1}=H_{1, \nu}=\sqrt{2 \nu}(1,0,0)$, $H_{2}=H_{2, \nu}=\sqrt{2 \nu}(0,1,0), H_{3}=H_{3, \nu}=\sqrt{2 \nu}(0,0,1)$ on $\mathbb{T}^{3}$, where $\nu \geq 0$ is a (viscosity) constant. This is understood in the trivialization $T \mathbb{T}^{3}=\mathbb{T}^{3} \times \mathbb{R}^{3}$, so $H_{1}, H_{2}, H_{3}: \mathbb{T}^{3} \rightarrow$
$\mathbb{R}^{3}$ are constant maps. Let $g_{\omega}^{\nu, M}(t, \theta)$ be the solution to the following SDE,

$$
\left\{\begin{align*}
d g_{\omega}^{\nu, M}(t, \theta) & =\sum_{i=1}^{3} H_{i, \nu} d M_{\omega}^{i}(t)+u_{\omega}\left(t, g_{\omega}^{\nu, M}(t, \theta)\right) d t  \tag{5.2}\\
& =\sqrt{2 \nu} d M_{\omega}(t)+u_{\omega}\left(t, g_{\omega}^{\nu, M}(t, \theta)\right) d t \\
g_{\omega}^{\nu, M}(0, \theta) & =\theta,
\end{align*}\right.
$$

where $M_{\omega}=\left(M_{\omega}^{1}, M_{\omega}^{2}, M_{\omega}^{3}\right)$ is an $\mathbb{R}^{3}$-valued martingale with continuous sample paths.
By the standard theory of stochastic flows (see, e.g., [57] and standard embedding theorems), if $u_{\omega}$ is regular enough (with respect to the space variable), i.e., $u_{\omega} \in$ $C\left([0, T] ; \mathfrak{X}^{s^{\prime}}\left(\mathbb{T}^{3}\right)\right)$ for some $s^{\prime}>s$ large enough, then $g_{\omega}^{\nu, M}(t, \cdot) \in G^{s}$ for every $t \in[0, T]$. From now on, for simplicity, we always assume $u_{\omega}$ to be regular enough.

As in [49, Section 6], let $U^{*}$ be some linear space which can be a space of functions, densities, or differential forms on $\mathbb{T}^{3}$. The action of $G^{s}$ on $U^{*}$ is the pull back map and the action of the "Lie algebra" $T_{e} G^{s}$ on $U^{*}$ is the Lie derivative.

If we take $\alpha_{0}=A_{0}(\theta) \cdot \mathbf{d} \theta:=\sum_{i=1}^{3} A_{0, i}(\theta) \mathbf{d} \theta_{i}$ to be a $C^{2}$ one-form on $\mathbb{T}^{3}$, we derive the following result (see also an equivalent expression in [27], equations (32)-(34) for the deterministic case). Formula (5.4) below is the analogue of the second equation in (3.34), derived here by hand for the infinite dimensional group $G^{s}$.

Proposition 5.1. Let $g_{\omega}^{\nu}(t)$ be given by (5.2) with $M_{\omega}=W_{\omega}$, where $W_{\omega}$ is a standard $\mathbb{R}^{3}$-valued Brownian motion (i.e., $W_{\omega}=\left(W_{\omega}^{1}, W_{\omega}^{2}, W_{\omega}^{3}\right)$ with $W_{\omega}^{i}, 1 \leq i \leq 3$, independent $\mathbb{R}$-valued Brownian motions). Define
$\alpha_{\omega}(t, \theta):=\left(\alpha_{0} g_{\omega}^{\nu}(t, \cdot)^{-1}\right)(\theta)=\left(\left(g_{\omega}^{\nu}(t, \cdot)^{-1}\right)^{*} \alpha_{0}\right)(\theta):=A_{\omega}(t, \theta) \cdot \mathbf{d} \theta:=\sum_{i=1}^{3} A_{\omega, i}(t, \theta) \mathbf{d} \theta_{i}$,
where $\left(g_{\omega}^{\nu}(t, \cdot)^{-1}\right)^{*}$ denotes the pull back map by $g_{\omega}^{\nu}(t, \cdot)^{-1}$, and

$$
\tilde{\alpha}(t, \theta):=\mathbb{E}\left[\alpha_{\omega}(t, \theta)\right]:=\tilde{A}(t, \theta) \cdot \mathbf{d} \theta:=\sum_{i=1}^{3} \tilde{A}_{i}(t, \theta) \mathbf{d} \theta_{i}
$$

Then $A_{\omega}$ satisfies the following SPDE,

$$
\begin{align*}
d A_{\omega, i}(t, \theta)= & -\sum_{j=1}^{3} \sqrt{2 \nu} \partial_{j} A_{\omega, i}(t, \theta, \omega) d W_{\omega}^{j}(t) \\
& -\sum_{j=1}^{3}\left(u_{\omega, j}(t, \theta) \partial_{j} A_{\omega, i}(t, \theta)+A_{\omega, j}(t, \theta) \partial_{i} u_{\omega, j}(t, \theta)\right) d t  \tag{5.3}\\
& +\nu \Delta A_{\omega, i}(t, \theta) d t, \quad i=1,2,3
\end{align*}
$$

where we use the notation $u_{\omega}(t):=\left(u_{\omega, 1}(t), u_{\omega, 2}(t), u_{\omega, 3}(t)\right)$ and $\partial_{j}$ and $\Delta$ stand for the partial derivative and the Laplacian with respect to the space variable $\theta$ of $A_{\omega}(t, \theta)$,
respectively. Equation (5.3) can also be expressed as

$$
\begin{align*}
d A_{\omega}(t, \theta)= & -\sqrt{2 \nu} \nabla A_{\omega}(t, \theta) \cdot d W_{\omega}(t)  \tag{5.4}\\
& -\left(u_{\omega}(t, \theta) \times \operatorname{curl} A_{\omega}(t, \theta)-\nabla\left(u_{\omega}(t, \theta) \cdot A_{\omega}(t, \theta)\right)\right) d t+\nu \Delta A_{\omega}(t) d t
\end{align*}
$$

(the term $d A_{\omega}(t, \theta)$ above denotes the Ito differential of $A_{\omega}(t, \theta)$ with respect to the time variable).

Moreover, if $u_{\omega}$ is non-random (in this case we write $u$ for $u_{\omega}$ ), we have

$$
\left\{\begin{array}{l}
\partial_{t} \tilde{A}(t, \theta)=(u(t, \theta) \times \operatorname{curl} \tilde{A}(t, \theta)-\nabla(u(t, \theta) \cdot \tilde{A}(t, \theta)))+\nu \Delta \tilde{A}(t, \theta)  \tag{5.5}\\
\tilde{A}(0, \theta)=A_{0}(\theta)
\end{array}\right.
$$

Proof. We use the methods in [20, Lemma 4.1] and [20, Proposition 4.2]. It is not hard to see that, for the $C^{2}$ (note that we assume $u$ to be regular) spatial process $A_{\omega, i}(t, \theta)$, there exist adapted spatial processes $h_{\omega, i j}(t, \theta)$ and $z_{\omega, i}(t, \theta), 1 \leqslant i, j \leqslant 3$, such that,

$$
\begin{equation*}
d A_{\omega, i}(t, \theta)=\sum_{j=1}^{3} h_{\omega, i j}(t, \theta) d W_{\omega}^{j}(t)+z_{\omega, i}(t, \theta) d t, \quad i=1,2,3 . \tag{5.6}
\end{equation*}
$$

We compute below the expressions of $h_{\omega, i j}(t, \theta)$ and $z_{\omega, i}(t, \theta)$.
Notice that by the definition of $\alpha_{\omega}(t, \theta),\left(g_{\omega}^{\nu}(t, \theta)\right)^{*} \alpha_{\omega}(t, \theta)=\alpha(0, \theta)$ is a constant with respect to the time variable, and

$$
\left(g_{\omega}^{\nu}(t, \theta)\right)^{*} \alpha_{\omega}(t, \theta)=\sum_{j=1}^{3}\left(\sum_{i=1}^{3} A_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta), \omega\right) V_{\omega, i j}(t, \theta)\right) d \theta_{j},
$$

where the process $V_{\omega, i j}(t, \theta):=\partial_{j} g_{\omega, i}^{\nu}(t, \theta)$ (here we use the notation $g_{\omega}^{\nu}(t)=\left(g_{\omega, 1}^{\nu}(t)\right.$, $\left.\left.g_{\omega, 2}^{\nu}(t), g_{\omega, 3}^{\nu}(t)\right)\right)$. We get for each $1 \leq j \leq 3$,

$$
\begin{equation*}
d\left(\sum_{i=1}^{3} A_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right) V_{\omega, i j}(t, \theta)\right)=0 \tag{5.7}
\end{equation*}
$$

By (5.6) and the generalized Itô formula for spatial processes (see [57. Theorem 3.3.1.]), we have

$$
\begin{align*}
& d A_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right)=\sum_{j=1}^{3}\left(h_{\omega, i j}\left(t, g_{\omega}^{\nu}(t, \theta)\right)+\sqrt{2 \nu} \partial_{j} A_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right)\right) d W_{\omega}^{j}(t) \\
& \quad+\left(z_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right)+\nu \Delta A_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right)\right.  \tag{5.8}\\
& \left.\quad+\sum_{j=1}^{3}\left(u_{\omega, j}\left(t, g_{\omega}^{\nu}(t, \theta)\right) \partial_{j} A_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right)+\sqrt{2 \nu} \partial_{j} h_{\omega, i j}\left(t, g_{\omega}^{\nu}(t, \theta)\right)\right)\right) d t .
\end{align*}
$$

By the theory of the stochastic flows in [57, Theorem 3.3.3.], from (5.2) we obtain

$$
\begin{equation*}
d V_{\omega, i j}(t, \theta)=\sum_{k=1}^{3} \partial_{k} u_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right) V_{\omega, k j}(t, \theta) d t \tag{5.9}
\end{equation*}
$$

In particular, the martingale part of the above equality vanishes due to the fact that the diffusion coefficients in (5.2) are constant.

According to (5.8) and (5.9), for each $1 \leq j \leq 3$, the Itô differential with respect to the time variable is as follows

$$
\begin{aligned}
& d\left(\sum_{i=1}^{3} A_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right) V_{\omega, i j}(t, \theta)\right) \\
& \quad=\sum_{i, k=1}^{3}\left(h_{\omega, i k}\left(t, g_{\omega}^{\nu}(t, \theta)\right)+\sqrt{2 \nu} \partial_{k} A_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right)\right) V_{\omega, i j}(t, \theta) d W_{\omega}^{k}(t) \\
& \quad+\sum_{i=1}^{3}\left(\sum_{k=1}^{3}\left(u_{\omega, k}\left(t, g_{\omega}^{\nu}(t, \theta)\right) \partial_{k} A_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right)+\sqrt{2 \nu} \partial_{k} h_{\omega, i k}\left(t, g_{\omega}^{\nu}(t, \theta)\right)\right)\right. \\
& \left.\quad+z_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right)+\nu \Delta A_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right)\right) V_{\omega, i j}(t, \theta) d t \\
& \quad+\sum_{i, k=1}^{3} A_{\omega, k}\left(t, g_{\omega}^{\nu}(t, \theta)\right) \partial_{i} u_{\omega, k}\left(t, g_{\omega}^{\nu}(t, \theta)\right) V_{\omega, i j}(t, \theta) d t
\end{aligned}
$$

Hence from (5.7), we derive for each $1 \leq j, m \leq 3$,

$$
\begin{equation*}
\sum_{i=1}^{3}\left(h_{\omega, i m}\left(t, g_{\omega}^{\nu}(t, \theta)\right)+\sqrt{2 \nu} \partial_{m} A_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right)\right) V_{\omega, i j}(t, \theta)=0 \tag{5.10}
\end{equation*}
$$

and

$$
\begin{align*}
& \sum_{i=1}^{3}\left(z_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right)+\nu \Delta A_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right)\right. \\
& \quad+\sum_{k=1}^{3}\left(u_{\omega, k}\left(t, g_{\omega}^{\nu}(t, \theta)\right) \partial_{k} A_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right)+\sqrt{2 \nu} \partial_{k} h_{\omega, i k}\left(t, g_{\omega}^{\nu}(t, \theta)\right)\right.  \tag{5.11}\\
& \left.\left.\quad \quad \quad+A_{\omega, k}\left(t, g_{\omega}^{\nu}(t, \theta)\right) \partial_{i} u_{\omega, k}\left(t, g_{\omega}^{\nu}(t, \theta)\right)\right)\right) V_{\omega, i j}(t, \theta)=0
\end{align*}
$$

As $\left\{V_{i j}(t, \theta, \omega)\right\}_{1 \leqslant i, j \leqslant 3}$ is a non-degenerate matrix-valued process (see, e.g., 57]), from (5.10) we deduce that, for each $1 \leq i, j \leq 3$,

$$
h_{\omega, i j}\left(t, g_{\omega}^{\nu}(t, \theta)\right)=-\sqrt{2 \nu} \partial_{j} A_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right) .
$$

Since $g_{\omega}^{\nu}(t, \theta)$ is invertible in $\theta$ we can derive the expression for $h_{\omega, i j}$ not only as a function of $g_{\omega}^{\nu}(t, \theta)$ but also as a function (at the origin) of $(t, \theta)$. Indeed, noticing that, $\omega$-almost surely, $\theta \mapsto g_{\omega}^{\nu}(t, \theta)$ is a diffeomorphism for each fixed $t$, we get

$$
\begin{equation*}
h_{\omega, i j}(t, \theta)=-\sqrt{2 \nu} \partial_{j} A_{\omega, i}(t, \theta), \quad \forall \theta \in \mathbb{T}^{3} \tag{5.12}
\end{equation*}
$$

which is the expression for $h_{\omega, i j}(t, \theta)$.
Since $\left\{V_{\omega, i j}(t, \theta)\right\}_{1 \leqslant i, j \leqslant 3}$ is non-degenerate, by (5.11), for each $1 \leq i \leq 3$,

$$
\begin{aligned}
z_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right)= & -\nu \Delta A_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right)-\sum_{k=1}^{3}\left(u_{\omega, k}\left(t, g_{\omega}^{\nu}(t, \theta)\right) \partial_{k} A_{\omega, i}\left(t, g_{\omega}^{\nu}(t, \theta)\right)\right) \\
& \left.-\sqrt{2 \nu} \partial_{k} h_{\omega, i k}\left(t, g_{\omega}^{\nu}(t, \theta)\right)+A_{\omega, k}\left(t, g_{\omega}^{\nu}(t, \theta)\right) \partial_{i} u_{\omega, k}\left(t, g_{\omega}^{\nu}(t, \theta)\right)\right) .
\end{aligned}
$$

We use (5.12) in the above equation and the fact that $\theta \mapsto g_{\omega}(t, \theta)$ is a diffeomorphism for each fixed $t, \omega$-almost surely and we obtain the expression for $z_{\omega, i}(t, \theta)$, namely,

$$
\begin{align*}
z_{\omega, i}(t, \theta)= & \nu \Delta A_{\omega, i}(t, \theta) \\
& -\sum_{k=1}^{3}\left(u_{\omega, k}(t, \theta) \partial_{k} A_{\omega, i}(t, \theta)+A_{\omega, k}(t, \theta) \partial_{i} u_{\omega, k}(t, \theta)\right), \quad \forall \theta \in \mathbb{T}^{3} . \tag{5.13}
\end{align*}
$$

Combining (5.6), (5.12), and (5.13) proves (5.3). We can check that (5.3) is equivalent to (5.4) by direct computation.

If $u_{\omega}(t, \cdot)=u(t, \cdot)$ is non-random, then it is easy to verify that

$$
\begin{aligned}
\mathbb{E}\left[u(t, \theta) \times \operatorname{curl} A_{\omega}(t, \theta)\right] & =u(t, \theta) \times \mathbb{E}\left[\operatorname{curl} A_{\omega}(t, \theta)\right] \\
& =u(t, \theta) \times \operatorname{curl\mathbb {E}}\left[A_{\omega}(t, \theta)\right]=u(t, \theta) \times \operatorname{curl} \tilde{A}_{\omega}(t, \theta) . \\
\mathbb{E}\left[\nabla\left(u_{\omega}(t, \theta) \cdot A_{\omega}(t, \theta)\right)\right] & =\nabla\left(u(t, \theta) \cdot \mathbb{E}\left[A_{\omega}(t, \theta)\right]\right) \\
& =\nabla(u(t, \theta) \cdot \tilde{A}(t, \theta)) .
\end{aligned}
$$

So taking the expectation of the two sides of equation (5.4), (5.5) follows.
Remark 5.2. In Proposition 5.1, $U^{*}$ is taken to be a space of differential forms on $\mathbb{T}^{3}$. Note that the action of $G^{s}$ on $U^{*}$ is the pull back map and the action of the "Lie algebra" $T_{e} G^{s}$ on $U^{*}$ is the Lie derivative. Then, for $H_{1, \nu}=\sqrt{2 \nu}(1,0,0), H_{2, \nu}=\sqrt{2 \nu}(0,1,0)$, $H_{3, \nu}=\sqrt{2 \nu}(0,0,1)$, and $\alpha_{\omega}(t, \theta)=A_{\omega}(t, \theta) \cdot \mathbf{d} \theta$, we have

$$
\begin{aligned}
& \sum_{i=1}^{3} \alpha_{\omega}(t) H_{i, \nu} d W_{\omega}^{i}(t)=\sqrt{2 \nu}\left(\nabla A_{\omega}(t, \theta) \cdot d W_{\omega}(t)\right) \cdot \mathbf{d} \theta \\
& \sum_{i=1}^{3} \alpha_{\omega}(t) H_{i, \nu} H_{i, \nu}=\nu \Delta A_{\omega}(t, \theta) \cdot \mathbf{d} \theta \\
& \alpha_{\omega}(t) u_{\omega}(t)=\left(u_{\omega}(t, \theta) \times \operatorname{curl} A_{\omega}(t, \theta)-\nabla\left(u_{\omega}(t, \theta) \cdot A_{\omega}(t, \theta)\right)\right) \cdot \mathbf{d} \theta
\end{aligned}
$$

which implies that (5.4) is just the second equation of (3.34).
In the same way, we can verify that the second equation of (3.37) is (5.5).
Remark 5.3. By the same procedure as in the proof of Proposition 5.1, if $\alpha_{0}$ is replaced by another term, such as a function or a density, we can still prove the corresponding evolution equation for $\alpha_{\omega}(t):=\alpha_{0} g_{\omega}^{\nu}(t, \cdot)^{-1}=\left(g_{\omega}^{\nu}(t, \cdot)^{-1}\right)^{*} \alpha_{0}$.

For example, if $\alpha_{0}=\beta_{0}: \mathbb{T}^{3} \rightarrow \mathbb{R}$ is a $C^{2}$ function, then $\alpha_{\omega}(t, \theta)$ satisfies the following stochastic transport equation,

$$
\left\{\begin{array}{l}
d \alpha_{\omega}(t, \theta)=-\sqrt{2 \nu} \nabla \alpha_{\omega}(t, \theta) \cdot d W_{\omega}(t)-u_{\omega}(t, \theta) \cdot \nabla \alpha_{\omega}(t, \theta) d t+\nu \Delta \alpha_{\omega}(t, \theta) d t \\
\alpha_{\omega}(0, \theta)=\beta_{0}(x)
\end{array}\right.
$$

This equation has been studied in [30] which illustrates that the added stochastic force (noise) can turn an ill-posed ODE into a well-posed one.

If $\alpha_{0}=D_{0}(\theta) d^{3} \theta$ is a density (volume form), write $\alpha_{\omega}(t, \theta)=D_{\omega}(t, \theta) d^{3} \theta$. Then $D_{\omega}(t, \theta)$ satisfies the following equation

$$
\left\{\begin{array}{l}
d D_{\omega}(t, \theta)=-\sqrt{2 \nu} \nabla D_{\omega}(t, \theta) \cdot d W_{\omega}(t)-\operatorname{div}\left(D_{\omega} u_{\omega}\right)(t, \theta) d t+\nu \Delta D_{\omega}(t, \theta) d t  \tag{5.14}\\
D_{\omega}(0, \theta)=D_{0}(\theta)
\end{array}\right.
$$

Assume that $u_{\omega}(\cdot)=u(\cdot)$ is non-random and $\alpha_{0}=D_{0}(\theta) d^{3} \theta$ is a probability measure, let $\tilde{\alpha}(t):=\mathbb{E}\left[\alpha_{\omega}(t)\right]:=\tilde{D}(t, \theta) d^{3} \theta$. Then $\tilde{D}(t, \theta)$ satisfies the following forward Kolmogorov equation (or Fokker-Planck equation),

$$
\left\{\begin{array}{l}
d \tilde{D}(t, \theta)=-\operatorname{div}(\tilde{D} u)(t, \theta) d t+\nu \Delta \tilde{D}(t, \theta) d t  \tag{5.15}\\
\tilde{D}(0, \theta)=D_{0}(\theta)
\end{array}\right.
$$

Moreover, let $\hat{g}_{\omega}^{\nu}(t, \theta)$ be the process satisfying

$$
d \hat{g}_{\omega}^{\nu}(t, \theta)=\sqrt{2 \nu} d W_{\omega}(t)+u\left(t, \hat{g}_{\omega}^{\nu}(t, \theta)\right) d t
$$

whose initial distribution is $D_{0}(\theta) d^{3} \theta$. Then for every $t \in[0, T]$, the distribution of $\hat{g}_{\omega}^{\nu}(t, \theta)$ is of the form $\tilde{D}(t, \theta) d^{3} \theta$, where $\tilde{D}(t, \theta)$ satisfies (5.15).

Remark 5.4. By carefully tracking the proof of Proposition 5.1, if we take $M_{\omega}$ in (5.2) to be a general $\mathbb{R}^{3}$-valued martingale, equation (5.4) becomes

$$
\begin{aligned}
d A_{\omega}(t, \theta)= & -\sqrt{2 \nu} \nabla A_{\omega}(t, \theta) \cdot d M_{\omega}(t)-u_{\omega}(t, \theta) \times \operatorname{curl} A_{\omega}(t, \theta) d t \\
& +\nabla\left(u_{\omega}(t, \theta) \cdot A_{\omega}(t, \theta)\right) d t+\nu \sum_{i, j=1}^{3} \partial_{i} \partial_{j} A_{\omega}(t) d \llbracket M_{\omega}^{i}, M_{\omega}^{j} \rrbracket_{t} .
\end{aligned}
$$

For each $\mathcal{P}_{t^{-}}$adapted process $v$ such that $v_{\omega}(\cdot, \cdot) \in C^{1}\left([0,1] ; \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)\right)$ (for $s$ large enough) with $v_{\omega}(0, \theta)=v_{\omega}(T, \theta)=0$ a.s., the deformation (3.31) (for right invariant systems) in the formulation here is determined by the following stochastic flows $e_{\omega, \varepsilon, v}(t, \cdot) \in G^{s}$ (see e.g., [2], [19] for the deterministic counterpart)

$$
\left\{\begin{array}{l}
\frac{d e_{\omega, \varepsilon, v}(t, \theta)}{d t}=\varepsilon \dot{v}_{\omega}\left(t, e_{\omega, \varepsilon, v}(t, \theta)\right)  \tag{5.16}\\
e_{\omega, \varepsilon, v}(0, \theta)=\theta
\end{array}\right.
$$

Setting $g_{\omega, \varepsilon, v}^{\nu, M}(t, \theta):=e_{\omega, \varepsilon, v}\left(t, g_{\omega}^{\nu, M}(t, \theta)\right)$, where $g_{\omega}^{\nu, M}$ is the solution to (5.2), and using such deformations, we can also define the critical point for an action functional in the same way as in (3.7), Section 3 ,

By the analysis in [2, Section 4.2] (especially (4.5)-(4.6) in [2]) for the (infinite dimensional group) $G^{s}$, we have

$$
d g_{\omega, \varepsilon, v}^{\nu, M}(t)=T_{e} R_{g_{\omega, \varepsilon, v}^{\nu, M}(t)}\left(\sum_{i=1}^{3} H_{\omega, i, \nu, \varepsilon}(t) \delta M_{\omega}^{i}(t)+\operatorname{Ad}_{e_{\omega, \varepsilon, v}(t)} u_{\omega}(t) d t+\varepsilon \dot{v}_{\omega}(t) d t\right)
$$

where $H_{\omega, i, \nu, \varepsilon}(t)=\operatorname{Ad}_{e_{\omega, \varepsilon, v}(t)} H_{i, \nu}$. Based on the above equation for $g_{\omega, \varepsilon, v}^{\nu, M}$ and according to the definition of $\frac{\mathscr{V}}{d t}$ and $\frac{\mathbf{D}^{\nabla_{0},\left(H_{i, \nu, \varepsilon}^{\nu}, M_{\omega}^{i}\right)_{i=1}^{3}}}{d t}$ (especially using the right-invariant version of (2.15) and (3.6)), it is easy to verify that

$$
\begin{align*}
& T_{g_{\omega}^{\nu, M}(t, \theta)} R_{g_{\omega}^{\nu, M}(t, \theta)^{-1}} \frac{\mathscr{D} g_{\omega}^{\nu, M}(t, \theta)}{d t}=u_{\omega}(t, \theta), \\
& \left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0}\left(T_{g_{\omega, \varepsilon, v}^{\nu, M}(t, \theta)} R_{g_{\omega, \varepsilon, v}^{\nu, M}(t, \theta)-1} \frac{\mathscr{D} g_{\omega, \varepsilon, v}^{\nu, M}(t, \theta)}{d t}\right)  \tag{5.17}\\
& =\left(\operatorname{ad}_{v_{\omega}(t)} u_{\omega}(t)\right)(\theta)=-\left[v_{\omega}(t, \cdot), u_{\omega}(t, \cdot)\right](\theta), \\
& T_{g_{\omega}^{\nu, M}(t, \theta)} R_{g_{\omega}^{\nu, M}(t, \theta)^{-1}}\left(d^{\Delta} g_{\omega}^{\nu, M}(t, \theta)\right)=\sqrt{2 \nu} d M_{\omega}(t), \\
& \left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0}\left(T_{g_{\omega, \varepsilon, v}^{\nu, M}(t, \theta)} R_{g_{\omega, \varepsilon, v}^{\nu, M}(t, \theta)^{-1}} d^{\Delta} g_{\omega, \varepsilon, v}^{\nu, M}(t, \theta)\right)  \tag{5.18}\\
& =\sum_{i=1}^{3}\left(\operatorname{ad}_{v_{\omega}(t)} H_{i, \nu}\right)(\theta) d M_{\omega}^{i}(t)=\sum_{i=1}^{3} \sqrt{2 \nu} \partial_{i} v_{\omega}(t, \theta) d M_{\omega}^{i}(t),
\end{align*}
$$

$$
\begin{aligned}
& T_{g_{\omega}^{\nu, M}(t, \theta)} R_{g_{\omega}^{\nu, M}(t, \theta)^{-1}}\left(\frac{\mathbf{D}^{\nabla^{0},\left(H_{i, \nu}, M_{\omega}^{i}\right)_{i=1}^{3}}}{d t}\right)_{i, j}^{3}=\nabla_{H_{i, \nu}}^{0} H_{j, \nu} \frac{d \llbracket M_{\omega}^{i}, M_{\omega}^{j} \rrbracket_{t}}{d t}=0 \\
& \left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0} T_{g_{\omega, \varepsilon, v}^{\nu, M}(t, \theta)} R_{g_{\omega, e, v}^{\nu, M}(t, \theta)^{-1}}\left(\frac{\mathbf{D}^{\nabla^{0},\left(H_{i, \nu, \varepsilon}, M_{\omega}^{i}\right)_{i=1}^{3}}}{d t}\right)_{i, j} \\
& \quad=\left(\nabla_{\mathrm{ad}}^{v_{\omega}(t)} 0\right. \\
& \left.\left.\quad=2 \nu \partial_{i, \nu} H_{j, \nu}+\nabla_{H_{i, \nu}}^{0} v_{\omega}(t, \theta) \frac{d \llbracket M_{\omega}^{i}, M_{\omega}^{j} \rrbracket_{t}(t)}{d t} H_{j, \nu}\right)\right) \frac{d \llbracket M_{\omega}^{i}, M_{\omega}^{j} \rrbracket_{t}}{d t}
\end{aligned}
$$

where $\nabla^{0}$ denotes the connection on $\mathfrak{X}\left(G^{s}\right)$ defined by (5.1) (in particular, we apply the property that $\nabla_{X}^{0} Y(\theta)=\sum_{i, j=1}^{3} X_{i}(\theta) \partial_{i} Y_{j}(\theta) \partial_{j}$ for every $X=\sum_{i=1}^{3} X_{i}(\theta) \partial_{i}, Y=$ $\sum_{i=1}^{3} Y_{i}(\theta) \partial_{i} \in \mathfrak{X}\left(\mathbb{T}^{3}\right)$ because the Christoffel symbols are zero, $\mathbb{T}^{3}$ being the flat torus).

Based on these formulas, the procedure on the variational principle in the proof of Theorem 3.11 and 3.12 also holds for the infinite dimensional group $G^{s}$ needed here. Hence the first equation of (3.34) (also the first equation of (3.37)) remains true for $G^{s}$.

Combining all the conclusions above, we deduce that Theorem 3.11 and 3.12 still hold for the infinite dimensional group $G^{s}$.

### 5.2 Compressible Navier-Stokes equation

Suppose $\nabla^{0}$ is the connection on $\mathfrak{X}\left(G^{s}\right)$ defined by (5.1). Let $U^{*}$ denote the vector space of all probability densities on $\mathbb{T}^{3}$ and define $\alpha_{0}:=D_{0}(\theta) d^{3} \theta \in U^{*}$. Let $\mathscr{M}_{m}\left(G^{s}\right)$ be the collection of all $m \times m \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)$-valued matrices and define $\mathscr{M}\left(G^{s}\right):=\cup_{m=1}^{\infty} \mathscr{M}_{m}\left(G^{s}\right)$.

As in [49, we take the dual space $\left(\mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)\right)^{*}$ of $\mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)$ to be the vector space $\Omega^{1}\left(\mathbb{T}^{3}\right)$ of all differential one-forms on $\mathbb{T}^{3}$ (here we fix the volume measure to be the Lebesgue measure on $\mathbb{T}^{3}$ ).

We define the Lagrangian $l: \Omega \times[0, T] \times \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right) \times U^{*} \rightarrow \mathbb{R}$ by

$$
\begin{gathered}
l_{\omega}(t, u, \alpha):=\int_{\mathbb{T}^{3}}\left(\frac{D(\theta)}{2}|u(\theta)|^{2}-D(\theta) e_{\omega}(t, D(\theta))\right) d^{3} \theta \\
\forall u \in \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right), \quad \forall \alpha=D(\theta) d^{3} \theta \in U^{*}
\end{gathered}
$$

where $e_{\omega}(t, D)$ is the fluid's specific internal energy, and the $\mathcal{P}_{t}$-adapted pressure $p_{\omega}(t)$ is given by $\mathbf{d} e_{\omega}(t)=-p_{\omega}(t) \mathbf{d}\left(\frac{1}{D}\right)$ (d denotes the space differential). See [49, Section 7] for more details on such Lagrangians. Here we use a random version, since the pressure may depend on the randomness of the system.

Then $\frac{\delta l}{\delta u}(t, u, \alpha)=u D \in \Omega_{1}\left(\mathbb{T}^{3}\right)$ is non-random, independent of $t$, and

$$
\left\langle\frac{\delta l}{\delta u}(t, u, \alpha), v\right\rangle=\int_{\mathbb{T}^{3}}\langle u(\theta), v(\theta)\rangle D(\theta) d^{3} \theta, \quad \forall u, v \in \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right), \alpha=D(\theta) d^{3} \theta \in U^{*} .
$$

Define the contraction force $\tilde{p}: \mathscr{M}\left(G^{s}\right) \times \mathscr{M}\left(G^{s}\right) \times \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right) \rightarrow \mathbb{R}$ by

$$
\begin{gather*}
\tilde{p}(A, B, u):=\frac{1}{2} \int_{\mathbb{T}^{3}} u(\theta) \cdot \operatorname{Tr}(A)(\theta) d^{3} \theta+\frac{1}{2} \sum_{i, j=1}^{m} \int_{\mathbb{T}^{3}} \mathbf{P}_{i}(u(\theta)) \mathbf{P}_{j}\left((B)_{i, j}(\theta)\right) d^{3} \theta  \tag{5.20}\\
\forall A \in \mathscr{M}_{n}\left(G^{s}\right), B \in \mathscr{M}_{m}\left(G^{s}\right), \quad \forall u \in \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)
\end{gather*}
$$

where $\operatorname{Tr}: \mathscr{M}\left(G^{s}\right) \rightarrow \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)$ is the trace operator and $\mathbf{P}_{i}: \mathbb{R}^{3} \rightarrow \mathbb{R}$ is the projection operator defined by

$$
\mathbf{P}_{i}\left(x_{1}, x_{2}, x_{3}\right):= \begin{cases}x_{i}, & \text { if } 1 \leq i \leq 3 \\ 0, & \text { if } i>3\end{cases}
$$

We take the stochastic force $q:[0, T] \times \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right) \times U^{*} \rightarrow\left(\mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)\right)^{*}$ to be $q(t, u, \alpha):=\langle u, \cdot\rangle$.
With $\nabla^{0}, l, \tilde{p}, q, \alpha_{0}$ given above, we define an action functional $\mathbf{J}^{\nu}:=J^{\nabla^{0},\left(H_{i, \nu}, W_{\omega}^{i}\right)_{i=1}^{3}}$ according to (3.29) as follows

$$
\begin{align*}
& \mathbf{J}^{\nu}\left(\left(g_{\omega}^{1}, \mathbf{w}_{i}^{1}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}},\left(g_{\omega}^{2}, \mathbf{w}_{i}^{1}, M_{\omega}^{2, i}\right)_{i=1}^{m_{1}}, g_{\omega}^{3}\right) \\
&:= \int_{0}^{T} \int_{\mathbb{T}^{3}}\left(\frac{1}{2}\left|w_{\omega}(t, \theta)\right|^{2} D_{\omega}(t, \theta)-D_{\omega}(t, \theta) e\left(D_{\omega}(t, \theta)\right)\right) d^{3} \theta d t \\
&+\int_{0}^{T} \tilde{p}\left(\frac{\mathbf{D}^{\nabla^{0},\left(\mathbf{w}_{i}^{1}, M_{\omega}^{1, i}\right) i_{i=1}^{m_{1}}} g_{\omega}^{1}(t)}{d t}, \frac{\mathbf{D}^{\nabla^{0},\left(\mathbf{w}_{i}^{2}, M_{\omega}^{2, i}\right) i_{i=1}^{m_{2}}} g_{\omega}^{2}(t)}{d t}, w_{\omega}(t)\right) d t  \tag{5.21}\\
&+\int_{0}^{T} \int_{\mathbb{T}^{3}}\left\langle w_{\omega}(t, \theta), d \Xi_{\omega}(t, \theta)\right\rangle d^{3} \theta \\
& \quad-\sum_{i=1}^{3} \sqrt{2 \nu} \int_{0}^{T} \int_{\mathbb{T}^{3}} w_{\omega, i}(t, \theta) d^{3} \theta d W_{\omega}^{i}(t),
\end{align*}
$$

where $w_{\omega}(t, \cdot):=T_{g_{\omega}^{1}(t)} R_{g_{\omega}^{1}(t)^{-1}}\left(\frac{\mathscr{O 1 _ { \omega } ^ { 1 } ( t )}}{d t}\right)=\left(w_{\omega, 1}(t, \cdot), w_{\omega, 2}(t, \cdot), w_{\omega, 3}(t, \cdot)\right), d \Xi_{\omega}(t, \cdot):=$ $T_{g_{\omega}^{1}(t)} R_{g_{\omega}^{1}(t)^{-1}}\left(d^{\Delta} g_{\omega}^{1}(t)\right), D_{\omega}(t, \theta) d^{3} \theta=\left(g_{\omega}^{3}(t, \cdot)^{-1}\right)^{*} \alpha_{0}$, and $W_{\omega}$ is a standard $\mathbb{R}^{3}$-valued Brownian motion.

Let $g_{\omega}^{\nu}$ be the solution of (5.2) with $\nu>0$ and $M_{\omega}=W_{\omega}$ the same $\mathbb{R}^{3}$-valued Brownian motion as in the definition of $\mathbf{J}^{\nu}$ above. In particular, $g_{\omega}^{0}$ is a solution of (5.2) with the same $u_{\omega}$ and $\nu=0$, which in fact is an ODE for each fixed $\omega \in \Omega$.

Suppose also that $\tilde{g}_{\omega}^{\nu}$ is a solution of the following SDE,

$$
\left\{\begin{array}{l}
d \tilde{g}_{\omega}^{\nu}(t, \theta)=\sum_{i=1}^{3} H_{i, \nu} d \tilde{W}_{\omega}(t)+u_{\omega}\left(t, \tilde{g}_{\omega}^{\nu}(t, \theta)\right) d t  \tag{5.22}\\
\tilde{g}_{\omega}^{\nu}(0, \theta)=\theta
\end{array}\right.
$$

where $\tilde{W}_{\omega}$ is an standard $\mathbb{R}$-valued Brownian motion, $H_{i, \nu}, 1 \leq i \leq 3$, and $u_{\omega}$ are the same as in (5.2). From now on, we use the notation $\left(\tilde{W}_{\omega}^{1}, \tilde{W}_{\omega}^{2}, \tilde{W}_{\omega}^{3}\right)=\left(\tilde{W}_{\omega}, \tilde{W}_{\omega}, \tilde{W}_{\omega}\right)$ to
denote an $\mathbb{R}^{3}$-valued Brownian motion with three equal components $\tilde{W}_{\omega}^{1}=\tilde{W}_{\omega}^{2}=\tilde{W}_{\omega}^{3}$, the same $\mathbb{R}$-valued Brownian motion.

We can therefore characterize the critical points of $\mathbf{J}^{\nu}$ as follows.
Theorem 5.5. (SPDE case) $\left(\left(g_{\omega}^{\nu}, H_{i, \nu}, W_{\omega}^{i}\right)_{i=1}^{3},\left(\tilde{g}_{\omega}^{\mu}, H_{i, \mu}, \tilde{W}_{\omega}^{i}\right)_{i=1}^{3}, g_{\omega}^{0}\right)$ is a critical point of $\mathbf{J}^{\nu}$ if and only if $\left(u_{\omega}, D_{\omega}\right)$ satisfies the following stochastic compressible NavierStokes equation,

$$
\begin{cases}d u_{\omega}(t)=-u_{\omega}(t) \cdot \nabla u_{\omega}(t) d t-\frac{1}{D_{\omega}(t)}( & \sqrt{2 \nu} \nabla u_{\omega}(t) \cdot d W_{\omega}(t)-\nu \Delta u_{\omega}(t) d t  \tag{5.23}\\ d D_{\omega}(t)=-\operatorname{div}\left(u_{\omega}(t) D_{\omega}(t)\right) d t, & \left.-\mu \nabla \operatorname{div} u_{\omega}(t)+\nabla p_{\omega}(t) d t\right)\end{cases}
$$

where $D_{\omega}(t, \theta) d^{3} \theta:=\left(g_{\omega}^{0}(t, \cdot)^{-1}\right)^{*}\left(D_{0}(\theta) d^{3} \theta\right)$.
Theorem 5.6. (PDE case) Let

$$
\begin{aligned}
& \mathbf{J}\left(\left(g_{\omega}^{1}, \mathbf{w}_{i}^{1}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}},\left(g_{\omega}^{2}, \mathbf{w}_{i}^{1}, M_{\omega}^{2, i}\right)_{i=1}^{m_{1}}, g_{\omega}^{3}\right) \\
&:= \int_{0}^{T} \int_{\mathbb{T}^{3}}\left(\frac{1}{2}\left|w_{\omega}(t, \theta)\right|^{2} \tilde{D}(t, \theta)-\tilde{D}(t, \theta) e(t, \tilde{D}(t, \theta))\right) d^{3} \theta d t \\
&+\int_{0}^{T} \tilde{p}\left(\frac{\mathbf{D}^{\nabla^{0},\left(\mathbf{w}_{i}^{1}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}}} g_{\omega}^{1}(t)}{d t}, \frac{\mathbf{D}^{\nabla^{0},\left(\mathbf{w}_{i}^{2}, M_{\omega}^{2, i}\right)_{i=1}^{m_{2}}} g_{\omega}^{2}(t)}{d t}, w_{\omega}(t)\right) d t
\end{aligned}
$$

where $e(t, D)$ is non-random and satisfies $\mathbf{d} e(t)=-p(t) \mathbf{d}\left(\frac{1}{D}\right)$ (with non- random pressure term $p(t))$ and

$$
w_{\omega}(t, \cdot):=T_{g_{\omega}^{1}(t)} R_{g_{\omega}^{1}(t)^{-1}}\left(\frac{\mathscr{D} g_{\omega}^{1}(t)}{d t}\right), \quad \tilde{D}(t, \cdot) d^{3} \theta=\mathbb{E}\left[\left(g_{\omega}^{3}(t, \cdot)^{-1}\right)^{*} \alpha_{0}\right]
$$

Suppose that $u_{\omega}=u$ in (5.2) and (5.22) is non-random and the deformations (5.16) are defined with $v$ non-random. Then $\left(\left(g_{\omega}^{\nu}, H_{i, \nu}, W_{\omega}^{i}\right)_{i=1}^{3},\left(\tilde{g}_{\omega}^{\mu}, H_{i, \mu}, \tilde{W}_{\omega}^{i}\right)_{i=1}^{3}, g_{\omega}^{0}\right)$ is a critical point of $\mathbf{J}$ if and only if (the non-random variables) $(u, \tilde{D})$ satisfy the following (deterministic) classical Navier-Stokes equations for compressible fluid flow

$$
\left\{\begin{array}{l}
d u(t)=-(u(t) \cdot \nabla u(t)) d t+\frac{1}{\tilde{D}(t)}(\nu \Delta u(t) d t+\mu \nabla \operatorname{div} u(t)-\nabla p(t) d t)  \tag{5.24}\\
d \tilde{D}(t)=-\operatorname{div}(u(t) \tilde{D}(t)) d t
\end{array}\right.
$$

Proof. (Theorem 5.5.)

By (5.14), we know that $D_{\omega}(t, \theta)$ satisfies the second equation of (5.23). As explained above, since Theorem 3.11 still holds for $G^{s}$, it suffices to show that the first equation of (5.23) is just the first one in (3.34) for our model.

Relations (5.17)-(5.19), combined with the definition (5.22) of $\tilde{g}_{\omega}^{\nu}$, yield the identities

$$
\begin{align*}
& T_{\tilde{g}_{\omega}^{\nu}(t, \theta)} R_{\tilde{g}_{\omega}^{\nu}(t, \theta)^{-1}}\left(\frac{\mathbf{D}^{\nabla^{0},\left(H_{i, \nu}, \tilde{W}_{\omega}^{i}\right)_{i=1}^{3} \tilde{g}_{\omega}^{\nu}(t)}}{d t}\right)_{i, j}=\nabla_{H_{i, \nu}}^{0} H_{j, \nu} \frac{d \llbracket \tilde{W}_{\omega}^{i}, \tilde{W}_{\omega}^{j} \rrbracket_{t}}{d t}=0 \\
& \left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0} T_{\tilde{g}_{\omega, \varepsilon, v}}(t, \theta)  \tag{5.25}\\
& \quad=\left(\nabla_{\tilde{g}_{\omega}^{\nu}, \varepsilon, v}(t, \theta)^{-1}\right. \\
& \quad 0 \\
& \left.\mathrm{ad}_{v_{\omega}(t)} H_{i, \nu} H_{j, \nu}+\nabla_{H_{i, \nu}}^{0}\left(\operatorname{ad}_{v_{\omega}(t)} H_{j, \nu}\right)\right) \frac{\left.d \llbracket \tilde{W}_{i, \nu,}, \tilde{W}_{\omega}^{i}\right)_{i=1}^{3}}{i} \tilde{g}_{\omega}^{\nu}(t) \\
& d t
\end{align*} \tilde{W}_{\omega, j}^{j} \rrbracket_{t}{ }^{d t}=2 \nu \partial_{i} \partial_{j} v_{\omega}(t, \theta) .
$$

For every $u, \tilde{u} \in \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)$, $\alpha=D(\theta) d^{3} \theta \in U^{*}$, we easily get the following formulas:

$$
\begin{align*}
& \frac{\delta l}{\delta u}(t, u, \alpha)=u D \\
& \operatorname{ad}_{u}^{*}(u D)=(\operatorname{div} u) u D+u \cdot \nabla(D u)+\frac{D}{2} \nabla\left(|u|^{2}\right) \\
& \sum_{i=1}^{3} \operatorname{ad}_{H_{i}}^{*} q d W_{\omega}^{i}(t)=\sqrt{2 \nu} \nabla u \cdot d W_{\omega}(t)  \tag{5.26}\\
& \left(\frac{\delta l_{\omega}}{\delta \alpha}(t, u, \alpha)\right) \diamond \alpha=\frac{D}{2} \nabla\left(|u|^{2}\right)-\nabla p_{\omega}(t)
\end{align*}
$$

The last equality is obtained by repeating the argument in [49, Section 7] (especially (7.4), (7.18), and (7.19)), even though $p_{\omega}(t)$ is random.

On the other hand, for every $A, \tilde{A} \in \mathscr{M}_{n}\left(G^{s}\right)$ and $B, \tilde{B} \in \mathscr{M}_{m}\left(G^{s}\right)$, we have,

$$
\begin{align*}
\left\langle\frac{\delta \tilde{p}}{\delta \xi_{1}}(A, B, u), \tilde{A}\right\rangle & =\frac{1}{2} \int_{\mathbb{T}^{3}} u(\theta) \cdot \operatorname{Tr}(\tilde{A}(\theta)) d^{3} \theta  \tag{5.27}\\
\left\langle\frac{\delta \tilde{p}}{\delta \xi_{2}}(A, B, u), \tilde{B}\right\rangle & =\frac{1}{2} \sum_{i, j=1}^{m} \int_{\mathbb{T}^{3}} \mathbf{P}_{i}(u(\theta)) \mathbf{P}_{j}\left((\tilde{B})_{i, j}(\theta)\right) d^{3} \theta \\
\left\langle\frac{\delta \tilde{p}}{\delta u}(A, B, u), \tilde{u}\right\rangle & =\frac{1}{2} \int_{\mathbb{T}^{3}} \tilde{u}(\theta) \cdot \operatorname{Tr}(A(\theta)) d^{3} \theta+\frac{1}{2} \sum_{i, j=1}^{m} \int_{\mathbb{T}^{3}} \mathbf{P}_{i}(\tilde{u}(\theta)) \mathbf{P}_{j}\left((B)_{i, j}(\theta)\right) d^{3} \theta .
\end{align*}
$$

Hence, using $M_{\omega}^{1}=W_{\omega}$, and (5.17), (5.25), (5.27), we get the formula for the operator $K$ defined by (3.12):

$$
\left\langle K_{\omega}(t, A, B, u), \tilde{u}\right\rangle=-\sum_{i=1}^{3} \int_{\mathbb{T}^{3}} u(\theta) \cdot\left(\nabla_{\left[\tilde{u}, H_{i, \nu}\right]}^{0} H_{i, \nu}+\nabla_{H_{i, \nu}}^{0}\left[\tilde{u}, H_{i, \nu}\right]\right)(\theta) d^{3} \theta
$$

$$
\begin{aligned}
& -\sum_{i, j=1}^{3} \int_{\mathbb{T}^{3}} u_{i}(\theta) \mathbf{P}_{j}\left(\nabla_{\left[\tilde{u}, H_{i, \mu]}\right]}^{0} H_{j, \mu}+\nabla_{H_{i, \mu}}^{0}\left[\tilde{u}, H_{j, \mu}\right]\right)(\theta) d^{3} \theta \\
= & -\nu \int_{\mathbb{T}^{3}} u(\theta) \cdot \Delta \tilde{u}(\theta) d^{3} \theta-\mu \int_{\mathbb{T}^{3}} \tilde{u}(\theta) \cdot \nabla \operatorname{div} u(\theta) d^{3} \theta
\end{aligned}
$$

that is,

$$
\begin{align*}
K_{\omega}(t, A, B, u) & =-\nu \Delta u-\mu \nabla \operatorname{div} u, \quad A, B \in \mathscr{M}_{3}\left(G^{s}\right), u \in \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right), t \in[0, T] \\
\frac{\delta \tilde{p}}{\delta u}(0,0, u) & =0 \tag{5.28}
\end{align*}
$$

Thus, combining the equalities above, the first equation in (3.34) becomes

$$
\begin{align*}
& d\left(u_{\omega}(t) D_{\omega}(t)\right)=-\sqrt{2 \nu} \nabla u_{\omega}(t) \cdot d W_{\omega}(t)-\left(\operatorname{div} u_{\omega}(t)\right) u_{\omega}(t) D_{\omega}(t) d t  \tag{5.29}\\
& -u_{\omega}(t) \cdot \nabla\left(u_{\omega}(t) D_{\omega}(t)\right) d t+\nu \Delta u_{\omega}(t) d t+\mu \nabla\left(\operatorname{div} u_{\omega}\right)(t) d t-\nabla p_{\omega}(t) d t .
\end{align*}
$$

Using the second equation of (5.23), we get

$$
\begin{equation*}
u_{\omega}(t) d D_{\omega}(t)=-\left(\operatorname{div} u_{\omega}(t)\right) u_{\omega}(t) D_{\omega}(t) d t-\left(u_{\omega}(t) \cdot \nabla D_{\omega}(t)\right) u_{\omega}(t) d t \tag{5.30}
\end{equation*}
$$

and together with (5.29), we obtain the first equation of (5.23).
Proof. (Theorem 5.6.) This follows carrying out the same computations as in the previous proof and the one in Theorem 3.12.

Remark 5.7. We emphasize that the usual Navier-Stokes equations for compressible fluids (5.24) were deduced from our stochastic variational principle, without any appeal to thermodynamic considerations in order to get the dissipative terms; these terms appear entirely due to the type of stochastic processes we consider.

Remark 5.8. For the incompressible case, i.e., $D_{\omega}(t, \theta) \equiv 1$, equation (5.23) becomes

$$
d u_{\omega}(t)=-\sqrt{2 \nu} \nabla u_{\omega} \cdot d W_{\omega}(t)-u_{\omega} \cdot \nabla u_{\omega} d t+\nu \Delta u_{\omega} d t-\nabla p_{\omega}(t) d t
$$

which is a stochastic incompressible Navier-Stokes equation.
Remark 5.9. Taking the viscous force $\tilde{p}=0$ in the definition of $\mathbf{J}^{\nu}$ (formula (5.21)) and following the same steps as in Theorem [5.5, it is easy to verify that the associated critical point $\left(u_{\omega}(t), D_{\omega}(t)\right)$ of $\mathbf{J}^{\nu}$ satisfies the following stochastic compressible Euler equation,

$$
\left\{\begin{array}{l}
d u_{\omega}(t)=-u_{\omega}(t) \cdot \nabla u_{\omega}(t) d t-\frac{1}{D_{\omega}(t)}\left(\sqrt{2 \nu} \nabla u_{\omega}(t) \cdot d W_{\omega}(t)+\nabla p_{\omega}(t) d t\right) \\
d D_{\omega}(t)=-\operatorname{div}\left(u_{\omega}(t) D_{\omega}(t)\right) d t
\end{array}\right.
$$

For existence of solutions of stochastic compressible Navier-Stokes equations we refer to [8], [82].

Remark 5.10. We illustrate here how the contraction force $\tilde{p}$, defined by (5.20), gives rise to the term modeling viscosity in the compressible Navier-Stokes equation (and MHD equation later). Other choices for the contraction force $\tilde{p}$ yield different dissipative equations.

For example, let $\tilde{p}: \mathscr{M}\left(G^{s}\right) \times \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right) \times U^{*} \rightarrow \mathbb{R}$ be defined by

$$
\begin{equation*}
\tilde{p}\left(A, u, D(\theta) d^{3} \theta\right)=\frac{1}{2} \int_{\mathbb{T}^{3}} \operatorname{Tr}(A)(\theta) \cdot u(\theta) D(\theta) d^{3} \theta, \tag{5.31}
\end{equation*}
$$

where $A \in \mathscr{M}\left(G^{s}\right), u \in \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right), \quad D(\theta) d^{3} \theta \in U^{*}$. Define the action functional $\mathbf{J}^{\nu}$ in the same way as in Theorem 5.5 with $\tilde{p}$ replaced by the expression in (5.31).

As explained in Remark 3.8, although $\tilde{p}$ depends on $U^{*}$, we can repeat the procedure in Theorem 5.5 to show that $\left(\left(g_{\omega}^{\nu}, H_{i, \nu}, W_{\omega}^{i}\right)_{i=1}^{3},\left(\tilde{g}_{\omega}^{\mu}, H_{i, \mu}, \tilde{W}_{\omega}^{i}\right)_{i=1}^{3}, g_{\omega}^{\nu}\right)$ is a critical point of $\tilde{\mathbf{J}}^{\nu}$ if and only if $\left(u_{\omega}(t), D_{\omega}(t)\right)$ satisfies the following system of equations:

$$
\left\{\begin{aligned}
d u_{\omega}(t)= & -u_{\omega}(t) \cdot \nabla u_{\omega}(t) d t+2 \nu\left\langle\nabla u_{\omega}(t), \nabla \log D_{\omega}(t)\right\rangle d t+\nu \Delta u_{\omega}(t) d t \\
& -\frac{\sqrt{2 \nu} \nabla u_{\omega}(t)}{D_{\omega}(t)} \cdot d W_{\omega}(t)-\frac{\nabla p_{\omega}(t)}{D_{\omega}(t)} d t \\
d D_{\omega}(t)= & -\sqrt{2 \nu \nabla D_{\omega}(t) \cdot d W_{\omega}(t)-\operatorname{div}\left(u_{\omega}(t) D_{\omega}(t)\right) d t+\nu \Delta D_{\omega}(t) d t}
\end{aligned}\right.
$$

The term $\left\langle\nabla u_{\omega}(t), \nabla \log D_{\omega}(t)\right\rangle$ in the equation above is crucial for energy dissipation. This term also appears in Brenner's model; see, e.g., [9, 10, 28].

### 5.3 Compressible MHD equation

Let $\alpha_{0}:=\left(b_{0}(\cdot), \mathbf{B}_{0}(\theta) \cdot \mathbf{d} \mathbf{S}, D_{0}(\theta) d^{3} \theta\right)$, where $b_{0}$ is a $C^{2}$ function on $\mathbb{T}^{3}, \mathbf{B}_{0}(\theta) \cdot \mathbf{d} \mathbf{S}$ is an exact two-form on $\mathbb{T}^{3}$, i.e., there is some one-form $A_{0}(\theta) \cdot \mathbf{d} \theta$ such that

$$
\begin{equation*}
\mathbf{B}_{0}(\theta) \cdot \mathbf{d S}=\mathbf{d}\left(A_{0}(\theta) \cdot \mathbf{d} \theta\right)=\sum_{1 \leq j<k \leq 3, i \neq j, i \neq k}\left(\operatorname{curl} A_{0}(\theta)\right)_{i} \mathbf{d} \theta_{j} \wedge \mathbf{d} \theta_{k} \tag{5.32}
\end{equation*}
$$

and $D_{0}(\theta) d^{3} \theta$ is a density on $\mathbb{T}^{3}$. We let $U^{*}$ denote the vector space of all such triples $\left(b(\cdot), \mathbf{B}(\theta) \cdot \mathbf{d S}, D(\theta) d^{3} \theta\right)$.

As in [49, equation (7.16)], let $l: \Omega \times[0, T] \times \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right) \times U^{*} \rightarrow \mathbb{R}$ be defined by

$$
l_{\omega}(t, u, b, \mathbf{B}, D)=\int_{\mathbb{T}^{3}}\left(\frac{D(\theta)}{2}|u(\theta)|^{2}-D(\theta) e_{\omega}(t, D(\theta), b(\theta))-\frac{1}{2}|\mathbf{B}(\theta)|^{2}\right) d^{3} \theta
$$

where $u \in \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)$ is the Eulerian (spatial) velocity of the fluid, $b \in C^{2}\left(\mathbb{T}^{3}\right)$ is the entropy function, $\mathbf{B}(\theta) \cdot \mathbf{d} \mathbf{S}$ is an exact differential two-form as in (5.32) representing
the magnetic field in the fluid, $D(\theta) d^{3} \theta$ is a density on $\mathbb{T}^{3}$ representing the mass density of the fluid, and the function $e_{\omega}(t, D, b)$ is the fluid's specific internal energy. The pressure $p_{\omega}(t)$ is $\mathcal{P}_{t}$-measurable for all $t$ and the temperature $T_{\omega}(t)$ of the fluid (also $\mathcal{P}_{t}$-measurable for all $t$ ) are given in terms of a thermodynamic equation of state for the specific internal energy $e$, namely $\mathbf{d} e_{\omega}(t)=-p_{\omega}(t) \mathbf{d}\left(\frac{1}{D}\right)+T_{\omega}(t) \mathbf{d} b$. As explained in [49, Section 7] it is assumed that $c_{\omega}^{2}:=\frac{\partial p_{\omega}}{\partial D}>0$, where $c_{\omega}$ is the adiabatic sound speed.

As in subsection 5.2, we work with a contraction force $\tilde{p}: \mathscr{M}\left(G^{s}\right) \times \mathscr{M}\left(G^{s}\right) \times$ $\mathfrak{X}^{s}\left(\mathbb{T}^{3}\right) \rightarrow \mathbb{R}$, defined by (5.20), and a stochastic force $q: \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right) \times U^{*} \rightarrow\left(\mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)\right)^{*}$, defined by

$$
q(u, \alpha):=\langle u, \cdot\rangle, \quad \forall u \in \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)
$$

With $\nabla^{0}, l, \tilde{p}, q, \alpha_{0},\left(H_{i, \nu}, W_{\omega}^{i}\right)_{i=1}^{3}$ as in subsection 5.2, we define the action functional $\mathbf{J}_{1}^{\nu}:=J^{\nabla^{0},\left(H_{i, \nu}, W_{\omega}^{i}\right)_{i=1}^{3}}$ according to (3.29), which in this particular case becomes

$$
\begin{align*}
\mathbf{J}_{1}^{\nu} & \left(\left(g_{\omega}^{1}, \mathbf{w}_{i}^{1}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}},\left(g_{\omega}^{2}, \mathbf{w}_{i}^{1}, M_{\omega}^{2, i}\right)_{i=1}^{m_{1}}, g_{\omega}^{3}, g_{\omega}^{4}, g_{\omega}^{5}\right) \\
:= & \int_{0}^{T} l_{\omega}\left(t, w_{\omega}(t, \cdot), \mathbf{B}_{\omega}(t, \cdot), b_{\omega}(t, \cdot), D_{\omega}(t, \cdot)\right) d t \\
& +\int_{0}^{T} \tilde{p}\left(\frac{\left.\mathbf{D}^{\nabla^{0},\left(\mathbf{w}_{i}^{1}, M_{\omega}^{1, i}\right)}\right)_{i=1}^{m_{1}} g_{\omega}^{1}(t)}{d t}, \frac{\mathbf{D}^{\nabla^{0},\left(\mathbf{w}_{i}^{2}, M_{\omega}^{2, i}\right) i_{i=1}^{m_{2}} g_{\omega}^{2}(t)}}{d t}, w_{\omega}(t)\right) d t  \tag{5.33}\\
& +\int_{0}^{T} \int_{\mathbb{T}^{3}}\left\langle w_{\omega}(t, \theta), d \Xi_{\omega}(t, \theta)\right\rangle d^{3} \theta \\
& -\sum_{i=1}^{3} \sqrt{2 \nu} \int_{0}^{T} \int_{\mathbb{T}^{3}} w_{\omega, i}(t, \theta) d^{3} \theta d W_{\omega}^{i}(t)
\end{align*}
$$

where

$$
\begin{aligned}
w_{\omega}(t, \cdot) & :=T_{g_{\omega}^{1}(t)} R_{g_{\omega}^{1}(t)^{-1}}\left(\frac{\mathscr{D} g_{\omega}^{1}(t)}{d t}\right)=\left(w_{\omega, 1}(t, \cdot), w_{\omega, 2}(t, \cdot), w_{\omega, 3}(t, \cdot)\right) \\
d \Xi_{\omega}(t, \cdot) & :=T_{g_{\omega}^{1}(t)} R_{g_{\omega}^{1}(t)^{-1}}\left(d^{\Delta} g_{\omega}^{1}(t)\right) \\
D_{\omega}(t, \cdot) d^{3} \theta & :=\left(g_{\omega}^{3}(t, \cdot)^{-1}\right)^{*}\left(D_{0}(\theta) d^{3} \theta\right) \\
\mathbf{B}_{\omega}(t, \theta) \cdot \mathbf{d S} & :=\left(g_{\omega}^{4}(t, \cdot)^{-1}\right)^{*}\left(\mathbf{B}_{0}(\theta) \cdot \mathbf{d S}\right) \\
b_{\omega}(t, \theta) & :=\left(g_{\omega}^{5}(t, \cdot)^{-1}\right)^{*} b_{0},
\end{aligned}
$$

and $W_{\omega}(t)$ is a standard $\mathbb{R}^{3}$-valued Brownian motion.
Let $g_{\omega}^{\nu}$ be the solution of (5.2) with $\nu>0$ and $M_{\omega}=W_{\omega}$ the same $\mathbb{R}^{3}$-valued Brownian motion as in the definition of $\mathbf{J}_{1}^{\nu}$ above. Although in the definition of $\mathbf{J}_{1}^{\nu}$, five semimartingales are needed, we can define its critical points in the same way as in (3.7) along deformations (5.16). Moreover, the critical points of $\mathbf{J}_{1}^{\nu}$ are characterized as follows.

Theorem 5.11. (SPDE case) $\left(\left(g_{\omega}^{\nu}, H_{i, \nu}, W_{\omega}^{i}\right)_{i=1}^{3},\left(\tilde{g}_{\omega}^{\mu}, H_{i, \mu}, \tilde{W}_{\omega}^{i}\right)_{i=1}^{3}, g_{\omega}^{0}, g_{\omega}^{\nu_{1}}, g_{\omega}^{\nu_{2}}\right)$ is a critical point of $\mathbf{J}_{1}^{\nu}$ if and only if the following stochastic compressible MHD equations hold for $\left(u_{\omega}(t), b_{\omega}(t), \mathbf{B}_{\omega}(t), D_{\omega}(t)\right)$,

$$
\left\{\begin{array}{l}
d u_{\omega}(t)=-u_{\omega}(t) \cdot \nabla u_{\omega}(t) d t-\frac{1}{D_{\omega}(t)}\left(\sqrt{2 \nu} \nabla u_{\omega}(t) \cdot d W_{\omega}(t)-\nu \Delta u_{\omega}(t) d t-\mu \nabla \operatorname{div} u_{\omega}(t) d t\right.  \tag{5.34}\\
\left.\quad-\mathbf{B}_{\omega}(t) \times \operatorname{curl} \mathbf{B}_{\omega}(t) d t+\nabla p_{\omega}(t) d t\right) \\
d D_{\omega}(t)=-\operatorname{div}\left(u_{\omega}(t) D_{\omega}(t)\right) d t, \\
d \mathbf{B}_{\omega}(t)=-\sqrt{2 \nu_{1}} \nabla \mathbf{B}_{\omega}(t) \cdot d W_{\omega}(t)+\operatorname{curl}\left(u_{\omega}(t) \times \mathbf{B}_{\omega}(t)\right) d t+\nu_{1} \Delta \mathbf{B}_{\omega}(t) d t \\
d b_{\omega}(t)=-\sqrt{2 \nu_{2}} \nabla b_{\omega}(t) \cdot d W_{\omega}(t)-u_{\omega}(t) \cdot \nabla b_{\omega}(t) d t+\nu_{2} \Delta b_{\omega}(t) d t
\end{array}\right.
$$

where $g_{\omega}^{\nu}$ and $\tilde{g}_{\omega}^{\mu}$ are the solution of the $S D E$ (5.2) and (5.22) respectively, $D_{\omega}(t, \theta) d^{3} \theta=$ $\left(g_{\omega}^{0}(t, \cdot)^{-1}\right)^{*}\left(D_{0}(\theta) d^{3} \theta\right), \mathbf{B}_{\omega}(t, \theta) \cdot \mathbf{d S}:=\left(g_{\omega}^{\nu_{1}}(t, \cdot)^{-1}\right)^{*}\left(\mathbf{B}_{0}(\theta) \cdot \mathbf{d S}\right), b_{\omega}(t, \theta):=\left(g_{\omega}^{\nu_{2}}(t, \cdot)^{-1}\right)^{*} b_{0}$.

Theorem 5.12. (PDE case) Set

$$
\begin{aligned}
\mathbf{J}_{1} & \left(\left(g_{\omega}^{1}, \mathbf{w}_{i}^{1}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}},\left(g_{\omega}^{2}, \mathbf{w}_{i}^{1}, M_{\omega}^{2, i}\right)_{i=1}^{m_{1}}, g_{\omega}^{3}, g_{\omega}^{4}, g_{\omega}^{5}\right) \\
:= & \int_{0}^{T} l\left(t, w_{\omega}(t, \cdot), \tilde{\mathbf{B}}_{\omega}(t, \cdot), \tilde{b}_{\omega}(t, \cdot), \tilde{D}_{\omega}(t, \cdot)\right) d t \\
& +\int_{0}^{T} \tilde{p}\left(\frac{\mathbf{D}^{\nabla^{0},\left(\mathbf{w}_{i}^{1}, M_{\omega}^{1, i}\right)_{i=1}^{m_{1}}} g_{\omega}^{1}(t)}{d t}, \frac{\mathbf{D}^{\nabla^{0},\left(\mathbf{w}_{i}^{2}, M_{\omega}^{2, i}\right)_{i=1}^{m_{2}}} g_{\omega}^{2}(t)}{d t}, w_{\omega}(t)\right) d t,
\end{aligned}
$$

where $l$ is non-random (hence the pressure $p(t)$ and temperature $T(t)$ are non-random), $w_{\omega}(t, \cdot):=T_{g_{\omega}^{1}(t)} R_{g_{\omega}^{1}(t)^{-1}}\left(\frac{\mathscr{D} g_{\omega}^{1}(t)}{d t}\right), \tilde{\mathbf{B}}(t, \theta) \cdot \mathbf{d S}:=\mathbb{E}\left[\left(g_{\omega}^{\nu_{1}}(t, \cdot)^{-1}\right)^{*}\left(\mathbf{B}_{0}(\theta) \cdot \mathbf{d S}\right)\right], \tilde{b}(t, \theta):=$ $\mathbb{E}\left[\left(g_{\omega}^{\nu_{2}}(t, \cdot)^{-1}\right)^{*} b_{0}\right], \tilde{D}(t, \cdot) d^{3} \theta=\mathbb{E}\left[\left(g_{\omega}^{0}(t, \cdot)^{-1}\right)^{*}\left(D_{0}(\theta) d^{3} \theta\right)\right]$.

Suppose $u_{\omega}=u$ is non-random in (5.2), (5.22), and that in the deformation $v$ in (5.16) is also non-random. Then $\left(\left(g_{\omega}^{\nu}, H_{i, \nu}, W_{\omega}^{i}\right)_{i=1}^{3},\left(\tilde{g}_{\omega}^{\mu}, H_{i, \mu}, \tilde{W}_{\omega}^{i}\right)_{i=1}^{3}, g_{\omega}^{0}, g_{\omega}^{\nu_{1}}, g_{\omega}^{\nu_{2}}\right)$ is a critical point of $\mathbf{J}_{1}$ if and only if $(u, \tilde{\mathbf{B}}, \tilde{D}, \tilde{b})$ satisfies the following compressible MHD equations

$$
\left\{\begin{array}{l}
d u(t)=-u(t) \cdot \nabla u(t) d t+\frac{1}{\tilde{D}(t)}(\nu \Delta u(t)+\mu \nabla \operatorname{div} u(t)-\tilde{\mathbf{B}}(t) \times \operatorname{curl} \tilde{\mathbf{B}}(t) d t-\nabla p(t)) d t  \tag{5.35}\\
d \tilde{D}(t)=-\operatorname{div}(u(t) \tilde{D}(t)) d t \\
d \tilde{\mathbf{B}}(t)=\operatorname{curl}(u(t) \times \tilde{\mathbf{B}}(t)) d t+\nu_{1} \Delta \tilde{\mathbf{B}}(t) d t \\
d \tilde{b}(t)=-u(t) \cdot \nabla \tilde{b}(t) d t+\nu_{2} \Delta \tilde{b}(t) d t
\end{array}\right.
$$

Proof. (Theorem 5.11.)

Equation (5.14) implies that $D_{\omega}(t, \theta)$ satisfies the second equation of (5.34). Since $\mathbf{B}_{0}(\theta) \cdot \mathbf{d S}=\mathbf{d}\left(A_{0}(\theta) \cdot \mathbf{d} \theta\right)$ for some one-form $A_{0}(\theta) \cdot \mathbf{d} \theta$, it follows that

$$
\begin{aligned}
\mathbf{B}_{\omega}(t, \theta) \cdot \mathbf{d} \mathbf{S} & =\left(g_{\omega}^{\nu_{1}}(t, .)^{-1}\right)^{*}\left(\mathbf{B}_{0}(\theta) \cdot b \mathbf{d} \mathbf{S}\right) \\
& =\left(g_{\omega}^{\nu_{1}}(t, .)^{-1}\right)^{*} \mathbf{d}\left(A_{0}(\theta) \cdot \mathbf{d} \theta\right) \\
& =\mathbf{d}\left(\left(g_{\omega}^{\nu_{1}}(t, .)^{-1}\right)^{*}\left(A_{0}(\theta) \cdot \mathbf{d} \theta\right)\right)(\theta) \\
& =\mathbf{d}\left(A_{\omega}(t, \theta) \cdot \mathbf{d} \theta\right)
\end{aligned}
$$

where

$$
A_{\omega}(t, \theta) \cdot \mathbf{d} \theta:=\left(g^{\nu_{1}}(t, .)^{-1}\right)^{*}\left(A_{0}(\theta) \cdot \mathbf{d} \theta\right), \quad \operatorname{curl} A_{\omega}(t):=\mathbf{B}_{\omega}(t)
$$

By Proposition 5.1, equation (5.4) holds for $A_{\omega}(t)$ with viscosity constant $\nu=\nu_{3}$, and hence $\mathbf{B}_{\omega}(t)=\operatorname{curl} A_{\omega}(t)$ satisfies the third equation of (5.34). We also have $\nabla \cdot \mathbf{B}_{\omega}(t)=\nabla \cdot\left(\operatorname{curl} A_{\omega}(t)\right)=0$.

In the same way, we verify that the fourth equation in (5.34) holds for $b_{\omega}(t)$.
According to Theorem 3.11, (5.19), (5.28) (which implies that $\frac{\delta \tilde{p}}{\delta u}\left(\tilde{H}_{\omega, 1}, \tilde{H}_{\omega, 2}, u_{\omega}(t)\right) \equiv$ 0 here), we conclude that $\left(\left(g_{\omega}^{\nu}, H_{i, \nu}, W_{\omega}^{i}\right)_{i=1}^{3},\left(\tilde{g}_{\omega}^{\mu}, H_{i, \mu}, \tilde{W}_{\omega}^{i}\right)_{i=1}^{3}, g_{\omega}^{0}, g_{\omega}^{\nu_{1}}, g_{\omega}^{\nu_{2}}\right)$ is a critical point of $\mathbf{J}_{1}$ if and only if the following equation holds

$$
\begin{align*}
d\left(\frac{\delta l_{\omega}}{\delta u}\right)(t)= & -\sum_{i=1}^{3} \operatorname{ad}_{H_{i, \nu}}^{*} q\left(t, u_{\omega}, b_{\omega}, \mathbf{B}_{\omega}, D_{\omega}\right) d W_{\omega}^{i}(t)-\operatorname{ad}_{u_{\omega}(t)}^{*} \frac{\delta l_{\omega}}{\delta u} d t+\frac{\delta l_{\omega}}{\delta b} \diamond b_{\omega} d t  \tag{5.36}\\
& +\frac{\delta l_{\omega}}{\delta \mathbf{B}} \diamond \mathbf{B}_{\omega} d t+\frac{\delta l_{\omega}}{\delta D} \diamond D_{\omega} d t-K_{\omega}\left(t, \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t), u_{\omega}(t)\right) d t
\end{align*}
$$

where $K_{\omega}, \tilde{H}_{\omega, 1}(t), \tilde{H}_{\omega, 2}(t)$ are the same terms as in (3.34).
From the computation in [49, Section 7], particularly (7.4), (7.18), and (7.19), we get,

$$
\frac{\delta l_{\omega}}{\delta b} \diamond b+\frac{\delta l_{\omega}}{\delta \mathbf{B}} \diamond \mathbf{B}+\frac{\delta l_{\omega}}{\delta D} \diamond D=\frac{D}{2} \nabla\left(|u|^{2}\right)+\mathbf{B} \times \operatorname{curl} \mathbf{B}-\nabla p_{\omega} .
$$

Combining all of the above with (5.17)-(5.19), (5.26)-(5.28), into (5.36) yields,

$$
\begin{align*}
d\left(u_{\omega}(t) D_{\omega}(t)\right)= & -\sqrt{2 \nu} \nabla u_{\omega}(t) \cdot d W_{\omega}(t)-\left(\operatorname{div} u_{\omega}(t)\right) u_{\omega}(t) D_{\omega}(t) d t \\
& -u_{\omega}(t) \cdot \nabla\left(u_{\omega}(t) D_{\omega}(t)\right) d t+\mathbf{B}_{\omega}(t) \times \operatorname{curl} \mathbf{B}_{\omega}(t) d t  \tag{5.37}\\
& +\nu \Delta u_{\omega}(t) d t+\mu \nabla \operatorname{div} u_{\omega}(t) d t-\nabla p_{\omega}(t) d t
\end{align*}
$$

Putting the second equation of (5.34) into (5.37), we derive the first equation in (5.34).

Proof. (Theorem 5.12.)
The proof of (5.35) follows by repeating the same computations as above and the ones in the proof of Theorem 3.12.

Remark 5.13. The reason for choosing processes $g^{\nu_{i}}$ with different constants $\nu_{i}$ is that the viscosity constants in equation (5.34) are different.

Remark 5.14. In particular, if we take $D(t)=1, b(t)=1$ for every $t \in[0, T]$ in (5.35), we obtain the following incompressible viscous MHD equations (see, e.g., [79]),

$$
\left\{\begin{array}{l}
\partial_{t} u+u \cdot \nabla u+\nabla p+\mathbf{B} \times \operatorname{curl} \mathbf{B}=\nu \Delta u \\
\partial_{t} \mathbf{B}=\operatorname{curl}(u \times \mathbf{B})+\nu_{1} \Delta \mathbf{B} \\
\operatorname{div} u=0
\end{array}\right.
$$

### 5.4 Stochastic Kelvin-Noether Theorem in Continuum Mechanics

We now apply the results on Section 4 in continuum mechanics. Following the formulation in [49, Section 6], we take here $G=G^{s}$ (so $\left.T_{e} G=\mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)\right)$, $U$ a linear space whose formal dual $U^{*}$ represents the advection terms (such as mass density, entropy, the magnetic field viewed as a differential two-form, etc), $\mathscr{C}=\left\{\gamma \in C\left([0,1] ; G^{s}\right) \mid \gamma(0)=\gamma(1)\right\}$ the set of all continuous $G^{s}$-valued loops.

As explained in [49, Section 6], the dual $\left(\mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)\right)^{*}=\Omega^{s}\left(\mathbb{T}^{3}\right) \otimes \operatorname{Den}\left(\mathbb{T}^{3}\right)$, where $\Omega^{s}\left(\mathbb{T}^{3}\right)$ denotes the space of $H^{s}$-differential one-forms on $\mathbb{T}^{3}$, and $\operatorname{Den}\left(\mathbb{T}^{3}\right)$ is the set of all densities on $\mathbb{T}^{3}$. Given a mass density $\rho=\rho(\theta) d^{3} \theta$, we define the circulation map $\mathscr{K}: \mathscr{C} \times U^{*} \rightarrow\left(\mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)\right)^{* *}$ by

$$
\langle\mathscr{K}(\gamma, a), \alpha\rangle=\oint_{\gamma(\cdot)} \frac{\alpha}{\rho}, \quad \gamma \in \mathscr{C}, \quad a \in U^{*}, \quad \alpha \in\left(\mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)\right)^{*}
$$

Since $\alpha \in\left(\mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)\right)^{*}=\Omega^{s}\left(\mathbb{T}^{3}\right) \otimes \operatorname{Den}\left(\mathbb{T}^{3}\right), \rho \in \operatorname{Den}\left(\mathbb{T}^{3}\right)$, and $\frac{\alpha}{\rho} \in \Omega^{s}\left(\mathbb{T}^{3}\right)$, the circulation integral above is well-defined.

Let $\mathscr{L}_{u}$ denote the Lie derivative in the direction $u \in \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)$. As shown in 49, Page 37, formula (6.2)], we have

$$
\operatorname{ad}_{u}^{*} V=\mathscr{L}_{u} V, \quad u \in \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right), \quad V \in\left(\mathfrak{X}^{s}\left(\mathbb{T}^{3}\right)\right)^{*}
$$

Suppose that $g_{\omega}^{\nu}(\cdot)$ is the solution of (5.2) with $M_{\omega}=W_{\omega}$ being a standard $\mathbb{R}^{3}$ valued Brownian motion and $\tilde{g}_{\omega}^{\nu}(\cdot)$ is the solution to (5.22). As illustrated above, Theorem 3.11 still holds for the infinite dimensional group $G^{s}$.

So, for a given Lagrangian functional $l: \Omega \times[0, T] \times \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right) \times U^{*} \rightarrow \mathbb{R}$ such that $\frac{\delta l_{\omega}}{\delta u}$ is non-random, and supposing that $\tilde{p}: \mathscr{M} \times \mathscr{M} \times \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right) \rightarrow \mathbb{R}, q:[0, T] \times$
$\mathfrak{X}^{s}\left(\mathbb{T}^{3}\right) \times U^{*} \rightarrow \mathbb{R}$ are the same terms as in Section 5.2, we define the action functional $J^{\nabla^{0},\left(H_{i, \nu}, W_{\omega}^{i}\right)_{i=1}^{3}}$ by (5.21). Hence by Theorem 3.11 and the computations in Section 5.1 above, we conclude that $\left(\left(g_{\omega}^{\nu}, H_{i, \nu}, W_{\omega}^{i}\right)_{i=1}^{3},\left(\tilde{g}_{\omega}^{\mu}, H_{i, \mu}, \tilde{W}_{\omega}^{i}\right)_{i=1}^{3}, g_{\omega}^{\nu_{1}}\right)$ is a critical point of $J^{\nabla^{0},\left(H_{i, \nu}, W_{\omega}^{i}\right)_{i=1}^{3}}$ if and only if $\left(u_{\omega}(t), \alpha_{\omega}(t)\right)$ satisfy the following equations (note that in the present situation, (5.19) and (5.27) imply $\tilde{H}_{1}(t)=\tilde{H}_{2}(t) \equiv 0$ and $\frac{\delta \tilde{d}}{\delta u}\left(0,0, u_{\omega}(t)\right) \equiv$ 0 ),

$$
\begin{align*}
d \frac{\delta l_{\omega}}{\delta u}(t)= & -\sqrt{2 \nu} \sum_{i=1}^{3} \partial_{i} u_{\omega}(t) d W_{\omega}^{i}(t)-\operatorname{ad}_{u_{\omega}(t)}^{*}\left(\frac{\delta l_{\omega}}{\delta u}(t)\right) d t+\frac{\delta l_{\omega}}{\delta \alpha}(t) \diamond \alpha_{\omega}(t) d t  \tag{5.38}\\
& +\nu \Delta u_{\omega}(t)+\mu \nabla \operatorname{div} u_{\omega}(t) d t \\
d \alpha_{\omega}(t)= & -\sum_{i=1}^{3} \mathscr{L}_{H_{i, \nu_{1}}} \alpha_{\omega}(t) d W_{\omega}^{i}(t)-\mathscr{L}_{u_{\omega}(t)} \alpha_{\omega}(t) d t+\frac{1}{2} \sum_{i=1}^{3} \mathscr{L}_{H_{i, \nu_{1}}} \mathscr{L}_{H_{i, \nu_{1}}} \alpha_{\omega}(t) d t,
\end{align*}
$$

where $u_{\omega}(\cdot)$ denotes the drift in (5.2), $\frac{\delta l_{\omega}}{\delta u}(t):=\frac{\delta l_{\omega}}{\delta u}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right), q_{\omega}(t):=q\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)$, and $\frac{\delta l_{\omega}}{\delta \alpha}(t):=\frac{\delta l_{\omega}}{\delta \alpha}\left(t, u_{\omega}(t), \alpha_{\omega}(t)\right)$. Here, we have also applied the property that $q(t, u, \alpha)=$ $\langle u, \cdot\rangle$.

Given $\gamma_{0} \in \mathscr{C}$, the Kelvin-Noether quantity $I: \mathscr{C} \times \mathfrak{X}^{s}\left(\mathbb{T}^{3}\right) \times U^{*} \rightarrow \mathbb{R}$ is defined by

$$
\begin{equation*}
I_{\omega}(t):=\oint_{\gamma_{\omega}(t)(\cdot)} \frac{1}{\rho_{\omega}(t)} \frac{\delta l_{\omega}}{\delta u}(t) \tag{5.39}
\end{equation*}
$$

where $\gamma_{\omega}(t)(\cdot):=\gamma_{0}\left(g_{\omega}^{\nu}(t, \cdot)\right),\left(u_{\omega}(t), \alpha_{\omega}(t)\right)$ is a solution of equation (5.38), and $\rho_{\omega}(t)=$ $\rho\left(\left(g_{\omega}^{\nu}\right)^{-1}(t, \theta)\right) d^{3} \theta$.

The stochastic Kelvin-Noether Theorem on $G^{s}$ takes the following form.
Proposition 5.15. Let $I_{\omega}(t)$ be defined by (5.39) and suppose that $\left(u_{\omega}(t), \alpha_{\omega}(t)\right)$ satisfies (5.38) with $\nu=\nu_{1}$ and $\frac{\delta l}{\delta u}=q=\langle u, \cdot\rangle$. Then we have

$$
\begin{equation*}
d I_{\omega}(t)=\oint_{\gamma_{\omega}(t)(\cdot)} \frac{1}{\rho_{\omega}(t)}\left(\frac{\delta l_{\omega}}{\delta \alpha}(t) \diamond \alpha_{\omega}(t)+\mu \nabla \operatorname{div} u_{\omega}(t)\right) d t \tag{5.40}
\end{equation*}
$$

Proof. By definition of $\gamma_{\omega}(t)(\cdot)$ and the change of variables formula, we obtain

$$
\begin{equation*}
I_{\omega}(t)=\oint_{\gamma_{\omega}(t)(\cdot)} \frac{1}{\rho_{\omega}(t)} \frac{\delta l_{\omega}}{\delta u}(t)=\oint_{\gamma_{0}(\cdot)} \frac{1}{\rho}\left(g_{\omega}^{\nu}(t)\right)^{*}\left[\frac{\delta l_{\omega}}{\delta u}(t)\right] \tag{5.41}
\end{equation*}
$$

By carefully tracking the proof of Proposition 5.1. we know that the following right
invariant version of (4.6) still holds

$$
\begin{aligned}
d\left(g_{\omega}^{\nu}(t)\right)^{*}\left[\frac{\delta l_{\omega}}{\delta u}(t)\right]= & \left(g_{\omega}^{\nu}(t)\right)^{*}\left[\sum_{i=1}^{3} \operatorname{ad}_{H_{i, \nu}}^{*} \frac{\delta l_{\omega}}{\delta u}(t) d W_{\omega}^{i}(t)+\operatorname{ad}_{u_{\omega}(t)}^{*} \frac{\delta l_{\omega}}{\delta u}(t)+d \frac{\delta l_{\omega}}{\delta u}(t)\right. \\
& \left.+\frac{1}{2} \sum_{i=1}^{3}\left(\operatorname{ad}_{H_{i, \nu}}^{*} \operatorname{ad}_{H_{i, \nu}}^{*} \frac{\delta l_{\omega}}{\delta u}(t)+2 \operatorname{ad}_{H_{i, \nu}}^{*} d \llbracket W_{\omega}^{i}, \frac{\delta l_{\omega}}{\delta u}(t) \rrbracket t\right)\right]
\end{aligned}
$$

Then, replacing here $d \frac{\delta l_{\omega}}{\delta u}(t)$ by its expression given in (5.38) and using the identity $\left\langle u_{\omega}(t), \cdot\right\rangle=q_{\omega}(t)=\frac{\delta l_{\omega}}{\delta u}(t)$, we get

$$
\begin{aligned}
& d\left(g_{\omega}^{\nu}(t)\right)^{*}\left[\frac{\delta l_{\omega}}{\delta u}(t)\right] \\
& =\left(g_{\omega}^{\nu}(t)\right)^{*}\left[\frac{\delta l_{\omega}}{\delta \alpha}(t) \diamond \alpha_{\omega}(t)+\nu \Delta u_{\omega}(t)+\mu \nabla \operatorname{div} u_{\omega}(t)-\frac{1}{2} \sum_{i=1}^{3} \operatorname{ad}_{H_{i, \nu}}^{*} \operatorname{ad}_{H_{i, \nu}}^{*} q_{\omega}(t)\right] d t \\
& =\left(g_{\omega}^{\nu}(t)\right)^{*}\left[\frac{\delta l_{\omega}}{\delta \alpha}(t) \diamond \alpha_{\omega}(t)+\mu \nabla \operatorname{div} u_{\omega}(t)\right] d t .
\end{aligned}
$$

Combining this with (5.41) yields

$$
\begin{aligned}
d I_{\omega}(t) & =\oint_{\gamma_{0}(\cdot)} \frac{1}{\rho} d\left(\left(g_{\omega}^{\nu}(t)\right)^{*}\left[\frac{\delta l_{\omega}}{\delta u}(t)\right]\right) \\
& =\oint_{\gamma_{0}(\cdot)} \frac{1}{\rho}\left(g_{\omega}^{\nu}(t)\right)^{*}\left[\frac{\delta l_{\omega}}{\delta \alpha}(t) \diamond \alpha_{\omega}(t)+\mu \nabla \operatorname{div} u_{\omega}(t)\right] d t \\
& =\oint_{\gamma_{\omega}(t)(\cdot)} \frac{1}{\rho_{\omega}(t)}\left(\frac{\delta l_{\omega}}{\delta \alpha}(t) \diamond \alpha_{\omega}(t)+\mu \nabla \operatorname{div} u_{\omega}(t)\right) d t,
\end{aligned}
$$

which finishes the proof of (5.40).
Remark 5.16. It is worthwhile to note that the second summand in the integrand of (5.40) is the bulk viscosity term appearing in the classical Navier-Stokes equations for compressible fluid flow, except that here, the velocity is random.

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[^1]:    ${ }^{1}$ we delete the constant $1 / 2$ in definition of $K_{\omega}$

