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On the Volume-Ranking of Opportunity Sets in Economic Environments

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Abstract - The domain of polyconvex sets, i.e. finite unions of convex, compact, Euclidean sets, is large enough to encompass most of the opportunity sets typically encountered in economic environments, including non-linear or even non-convex budget sets, and opportunity sets arising from production sets. We provide a characterization of the volume-ranking as defined on the set of all polyconvex sets, relying on a valuation-based volume-characterization theorem due to Klain and Rota (1997).

Keywords - Opportunity sets, valuation, volume total preordering;
JEL classification - D31; D63; I31.

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1. Introduction

In the last decade, a significant amount of work has been devoted to the task of ranking opportunity sets in terms of freedom of choice in various settings including standard economic environments. This paper contributes to this literature by focusing on the volume-ranking of opportunity sets in Euclidean spaces and its characterization.

Indeed, a 'freedom of choice'-based ranking of opportunity sets should arguably only take into account the 'size' of the relevant set, without making any use of information about individual preferences which may be highly unreliable, costly to acquire, or both.[1] In order to cope with standard economic environments we assume that individual options can be represented by points in a n-dimensional Euclidean space, and the problem of ranking opportunity sets in terms of freedom of choice then reduces to comparing their respective sizes. Thus, the total-ordering induced by volume is arguably a natural criterion if Euclidean opportunity sets are to be ranked according to the freedom of choice they allow, regardless of any explicit preferential information. Of course, the volume-ranking embodies an extremely strong threshold effect: a null endowment along any dimension forces indifference to the empty set, which amounts to regarding all dimensions as 'essential' or 'primary' opportunities.[2]

As a matter of fact, the problem of characterizing the volume-ranking of *linear* budget sets has been already addressed by Xu (2004). However, there are a few good reasons which suggest that a larger Euclidean domain should be considered. First, the Lancasterian approach to consumer theory in terms of characteristics of goods requires non-linear (piecewise linear) budgets sets. Moreover, standard consumer choice problems with several income tax rates induce non-convex budget sets (see e.g. Mas-Colell *et alii* (1995)). Also, non-linear convex (or even non-convex) opportunity sets arise whenever basic alternatives are taken to be production, as opposed to consumption, programs. Finally, non-linear, convex, compact opportunity sets are typically met within game-theoretically oriented models such as *abstract economies* and *characteristic-outcome coalitional game forms.* Therefore, we propose to enlarge the domain of opportunity sets to the entire set $polycon\,(n)$ of polyconvex sets (i.e., of finite unions of convex, compact n-dimensional Euclidean opportunity sets).

We provide a characterization of the volume-ranking of polyconvex opportunity sets which takes advantage of the latticial structure of $polycon\,(n)$ and exploits

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[1] Such a 'freedom of choice'-based ranking might arguably also be relevant in order to define a suitable "capability"-ordering of opportunity sets.

[2] See Kolm (2004) for a critical discussion of the volume-ranking and a tentative endorsement of an alternative index-number oriented, distance-based ordering of budget sets.

valuations, i.e. essentially real-valued additive functions on distributive lattices. In particular, our result relies heavily on a basic characterization theorem for volume (see Klain and Rota (1997)), and a classic theorem on extensive measurement (see Krantz *et alii* (1971)). Our work is also closely related to a recent result on ratio-scale representations of rankings of *compact* opportunity sets in the non-negative orthant of an Euclidean space due to Pattanaik and Xu (2000). Our characterization of the volume-ranking requires the introduction of two extra-axioms with respect to the latter work, namely a full dimensionality requirement for non-bottom ranked opportunity sets plus a translation invariance condition.

The organization of the paper is as follows. In section 2, we discuss the motivations for extending to the entire domain of *polycon* (n) the analysis on ranking opportunity sets in economic environments. Section 3 lays down our basic notation and definitions. Section 4 presents our characterization of volume-ranking. We also provide examples which show that each property satisfied by the volume-order is independent from the others: in other words, our volume-characterization is tight.[3] Section 5 contains some concluding remarks, while all the proofs are collected in appendix.

2. Why polyconvex sets? Motivation and overview

As mentioned above, we focus on the entire family of polyconvex sets, i.e. finite unions of convex and compact opportunity sets in Euclidean spaces, an infinite set, that encompasses most of the opportunity sets typically encountered in economic context.

In order to motivate the choice of such a large domain let us provide a few prominent examples which are rather commonly met in the economic literature.

To begin with, let us consider the Lancasterian characteristics-approach to consumption theory (see Lancaster (1968)). Here, competitive budget sets are defined in a characteristics-space, namely the objects of consumer preferences are the properties or characteristics of goods. Therefore, the consumption level of each good corresponds to a point in the characteristics space: the budget set is the convex set having the origin and *efficient* one good bundles as extreme points. The frontier of that budget set is typically piecewise linear (see Figure 1.(a) for a representation

[3] It is worth remarking that we produce an example of a preordered set that violates the Archimedean property, while satisfying all the other properties we have required in our characterization. Such an example also provides a solution to a similar independent problem left open by Pattanaik and Xu (2000, pg. 61).

of vectors of goods in a two-dimensional characteristic space).

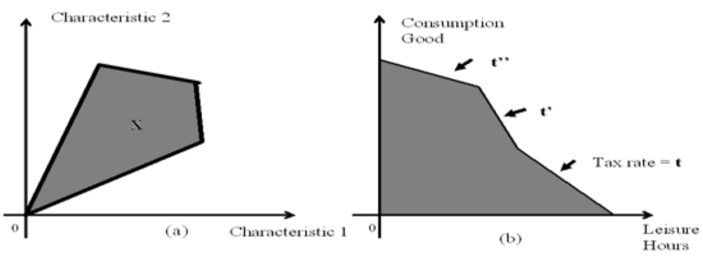

Fig. 1: (a) Lancasterian and (b) Non-Linear Budget Sets

Let us now consider the budget set of a consumer in a standard two-goods competitive economy where the goods are income (consumption goods) and leisure (see Mas-Colell *et alii* (1995)). If three income tax-rates are in place, then the budget sets are a piecewise linear frontier but typically not convex (see Figure 1.(b)).[4]

Next, consider production sets, i.e. the standard models of production technology. Under non-increasing returns to scale the production set is typically a convex but generally non-linear set (see Figure 3.(a)). By contrast, under non-decreasing

[4]It is worth noticing that, in a n-good economy with one *indivisible good*, the standard competitive budget set could still be represented as a polyconvex set. However, such a budget set would typically be $n-1$-dimensional hence by definition equivalent to the null set according to the volume-ranking or for that matter to any ranking induced by a *simple* valuation, as defined below.

Thus, while the Euclidean polyconvex domain also encompasses indivisible opportunities, it should be emphasized that if one insists that indivisibilities are to be considered then the requirement of Simplicity as defined below namely indifference between the empty set and any non-full-dimensional opportunity set must be definitely discarded as entirely inappropriate. In the present paper, we follow the opposite route i.e. we implicitly disregard the 'indivisibility' case and require Simplicity, leaving a proper investigation of the former -and broader- approach as a topic for further research.

returns to scale the production set may well be a non-convex set (see Figure 3.(b)).

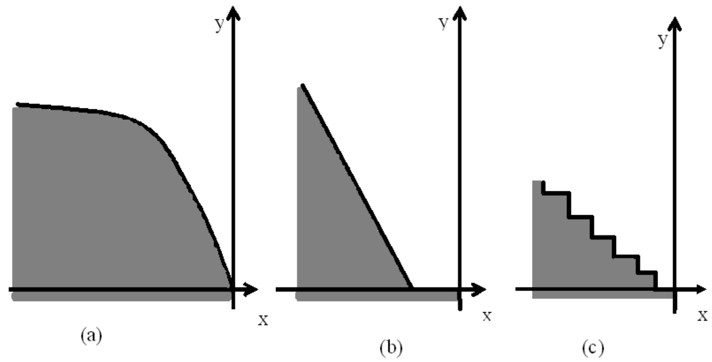

Fig. 2: (a) Nonincreasing and (b) Nondecreasing Returns to Scale; (c) Production Set with Free Entry

Also, additive production sets, which are required in order to ensure *free entry* (see Mas-Colell *et alii* (1995)), are typically non-convex sets (see Figure 3.(c)).

When limits on resources are superimposed on such production sets, non-linear convex, respectively non-convex, polyconvex sets typically obtain (more precisely, the projections on productions sets of the sets of globally feasible allocations, or attainable states, are non-linear convex or non-convex polyconvex sets, respectively: see e.g. Border (1985) Definition 20.2 pg.96).

Finally, one may consider opportunity sets arising in an interactive setting. A first prominent example of that situation occurs whenever the relevant opportunity sets are values of sets of feasible outcomes for coalitions of players in a characteristic-outcome coalitional game form. Coalitional game forms of this type have been typically used to model certain coalition production economies. A characteristic-outcome game form is an array $G = \langle N, X, (X_S)_{S \subseteq N} \rangle$ where N is the finite player set, $X \subseteq \mathbb{R}^{nk}$ with $n = |N|$ and k the number of goods, and for any S, $X_S \subseteq \mathbb{R}^{ns}$, with $s = |S|$, is the feasible outcome set of coalition S. Those feasible sets are taken to be convex and compact (see Border (1985) chapter 23), hence polyconvex.

Another relevant example of interactive-based polyconvex opportunity sets is provided by pseudogames or abstract economies as frequently met in general equilibrium analysis. A pseudogame in strategic form is an array $\Gamma = \left(N, X, (S_i)_{i\in N}, (F_i)_{i\in N}, h, (\succcurlyeq_i)_{i\in N}\right)$ where N is the player set, X is the outcome set, S_i is the strategy set of player i, $F_i : \Pi_{i\in N} S_i \rightarrow\!\!\rightarrow S_i$is the feasibility correspondence of player i (for any $i \in N$), $h : D \subseteq \Pi_{i\in N} S_i \rightarrow X$ is the outcome function (where D denotes the set of all fixed points of the global feasibility correspondence $F : \Pi_{i\in N} S_i \rightarrow\!\!\rightarrow \Pi_{i\in N} S_i$ as defined by the rule $F((s_i)_{i\in N}) = \Pi_{i\in N} F_i((s_i)_{i\in N})$), and $\succcurlyeq_i \subseteq X \times X$ is the preference relation of player i, with $i \in N$. Notice that the values of a feasibility correspondence F_i denote precisely the possible action choices open to player i given the choices made by the other players. In fact, in standard general equilibrium applications the values of consumers' feasibility correspondences are the familiar linear budget sets, while the constant value of the auctioneer's feasibility correspondence is the price simplex, a compact convex set in Euclidean space. However, the individual feasibility correspondences F_i are in general taken to be continuous correspondences with general nonempty compact convex (hence possibly non-linear polyconvex) values in an Euclidean space (see e.g. Border (1985), Theorem 19.8, page 91).

The foregoing list of examples, and, in particular, the inclusion of opportunity sets arising from production sets, also provide a motivation for our interest in opportunity sets comprising points with possibly negative coordinates. Recall that the points of a production set represent production programs where inputs are represented by negative components and outputs are denoted by positive components.

It should be noticed that our characterization theorem is easily adapted to those restricted domains that result from excluding one or more orthants of the n-dimensional real Euclidean space. On the other hand, we can hardly think of a compact but not polyconvex set arising from standard economic or game-theoretic models: that is one of the reasons why we do not allow the entire set of compact Euclidean sets in our domain.[5] Therefore, we shall focus on the full domain of polyconvex sets in $\mathbb{R}^n$ without any further ado, and provide a characterization of the volume-ranking in such a setting.

[5]An example of a non-polyconvex compact body is a standard torus (see fig. 3).

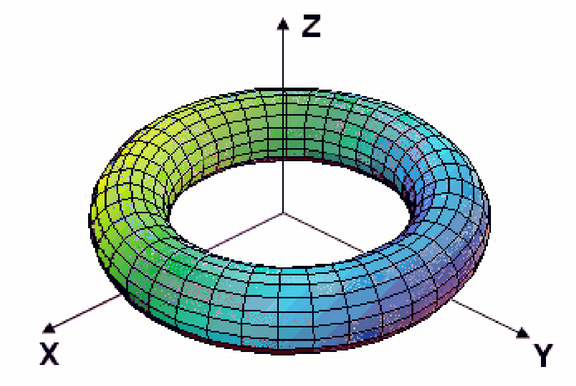

Fig. 3: A Torus

Other possible examples of non-polyconvex compact bodies are polyhedra, with a possibly infinite number of holes, defined as open n-dimensional balls.

Notice that, by contrast, Pattanaik and Xu (2000) consider the set of all compact subsets of the *nonnegative* orthant $\mathbb{R}^n_+$.

3. Notation, definitions and preliminary results

We shall focus on the class of polyconvex sets in an Euclidean finite-dimensional space.

Definition 1. *Let $\mathbb{R}^n$ be a finite dimensional Euclidean space. A* polyconvex set *in $\mathbb{R}^n$ is a finite union of compact and convex subsets of $\mathbb{R}^n$. We denote by $polycon\,(n)$ the set of all polyconvex sets of $\mathbb{R}^n$.*

Moreover, for any bounded *set $A \subseteq \mathbb{R}^n$ the* polyconvex hull *of A is defined as:*

$$k^{pol}(A) = \cap \{B \in polycon(n) : \ A \subseteq B \ \} .$$

The set $polycon\,(n)$ can be endowed with a distributive latticial structure in a very natural way by positing sup $= \cup$ (set-union) and inf $= \cap$ (set-intersection). We shall denote $Polycon\,(n)$ the distributive lattice $(polycon\,(n)\,, \cup, \cap)$.

In particular, we are concerned with the volume-induced preorder $\succcurlyeq_V$ on $polycon\,(n)$.

In order to define the volume on $polycon\,(n)$, we have to introduce first orthogonal parallelotopes. An *orthogonal parallelotope* is a rectilinear box A with sides parallel to the axes of a Cartesian coordinate system in $\mathbb{R}^n$, i.e.:

$$A = \left\{x \in \mathbb{R}^n_+ : x_i \leq k_i \text{ for } i = 1, ..., n \text{ and } k_i > 0 \text{ for any } i = 1, ..., n\right\}$$

hence, an orthogonal parallelotope is described by n inequalities representing straight line parallel to a selected frame. A *parallelotope* is a finite union or intersection of *orthogonal parallelotopes* in an Euclidean space. We denote by $par^{\perp}\,(n)$ the set of all orthogonal parallelotopes and by $par\,(n)$ the set of all finite unions and intersections of orthogonal parallelotopes. It is easily checked that $\langle par\,(n)\,, \cup, \cap\rangle$ is a distributive lattice.

The volume of an orthogonal parallelotope A, as defined above, is:

$$vol\,(A) = k_1 \cdot k_2 \cdot ... \cdot k_n.$$

The volume of a parallelotope $B \in par\,(n)$ can be defined recursively, by positing $vol\,(\emptyset) = 0$ and recalling that $par^{\perp}\,(n)$ is closed under intersections and that for any $A_1, A_2 \in par^{\perp}\,(n)$, $vol\,(A_1 \cup A_2) = vol\,(A_1) + vol\,(A_2) - vol\,(A_1 \cap A_2)$. Finally, the volume of a polyconvex set K is:

$$vol\,(K) = \sup \{vol\,(B) : B \in par\,(n)\,,\ B \subseteq K\} .$$

The volume-induced preorder $\succcurlyeq_V$ on $polycon\,(n)$ is defined by the following rule:

$$A \succcurlyeq_V B \Longleftrightarrow vol\,(A) \geq vol\,(B)\,,$$

for all $A, B \in polycon\,(n)$.

As mentioned above, we shall provide a very simple characterization of $(polycon\,(n)\,, \succcurlyeq_V)$ in terms of latticial valuations. In order to do that, a few definitions are to be introduced:

Definition 2. *Let $\mathbf{L} = (L, \cup, \cap)$ be a lattice of sets. A (real-valued)* valuation *on $\mathbf{L}$ is a function $\mu : L \to \mathbb{R}$ such that, for any $A, B \in L$,*

$$\mu\,(A \cup B) = \mu\,(A) + \mu\,(B) - \mu\,(A \cap B) .$$

- (i)**:** *A valuation μ is* isotone *if, for any $A, B \in L$, if $A \supseteq B$ then $\mu\,(A) \geq \mu\,(B)$;*

(*ii*): *A valuation* μ *on the lattice Polycon* (n) *is* translation invariant *if, for any* $A \in L$ *and for any translation* τ *on* $\mathbb{R}^n$, $\mu(\tau[A]) = \mu(A)$;[6]

(*iii*): *A valuation* μ *on Polycon* (n) *is* simple *if* $\mu(A) = 0$ *for any* $A \in$ *polycon* (n) *such that* $\dim A < n$.

Valuations are a fundamental tool in geometry and are strictly connected to the idea of measuring polygonal regions. Closely related to the concept of valuation is that of dissection:

Definition 3. *Let A be a (polyconvex) subset of* $\mathbb{R}^n$. *A* (polyconvex) dissection *of A is a finite set* $\{B_1, ..., B_k\}$ *of (polyconvex) subsets of* $\mathbb{R}^n$ *such that* $A = B_1 \cup B_2 ... \cup B_k$ *and* $int(B_i \cap B_j) = \varnothing$, *for any* $i, j = 1, ..., k$, *with* $i \neq j$.

A dissection involves the well-known notion of dividing complex geometric regions into figures whose areas are given by more familiar formulas.

In what follows, we provide a characterization of the volume-ranking that results from merging the standard measurement-theoretic treatment of the volume as an instance of a ratio scale arising from a certain extensive structure (as in e.g. Krantz *et alii* (1971)) with the modern geometers' view of the volume as a certain rigid-motion-invariant valuation (see e.g. Klain and Rota (1997)). In view of the shared connection to extensive measurement, the present paper parallels to a large extent Pattanaik and Xu's article (2000) and employs all the axioms used in the latter, namely Total Preordering, Non-Triviality, Independence, Denseness and the Archimedean property, which are the standard requirements for obtaining ratio scales representing ordered extensive structures. Then, in order to provide a characterization of $(polycon\,(n), \succcurlyeq_V)$, the following properties are to be introduced.

Total Preorder (TP): Let D be a non empty subset of $\mathcal{P}(\mathbb{R}^n)$, the power set of X, such that $\varnothing \in D$. A binary relational system $(D, \succcurlyeq)$ is a *total preordered set* if and only if it satisfies:
  i) *Totality*: For any $A, B \in D$, $A \succcurlyeq B$ or $B \succcurlyeq A$ and
  ii) *Transitivity*: For any $A, B, C \in D$, if $A \succcurlyeq B$ and $B \succcurlyeq C$ then $A \succcurlyeq C$.[7]

Non-triviality (NT): Let D be a non empty subset of $\mathcal{P}(\mathbb{R}^n)$ such that $\varnothing \in D$. A binary relational system $(D, \succcurlyeq)$ satisfies *non-triviality* if and only if there exists an $A \in D$ such that $A \succ \varnothing$ and $B \succcurlyeq \varnothing$ for any $B \in D$.

Denseness (D): Let D be a non empty subset of $\mathcal{P}(\mathbb{R}^n)$ such that $\varnothing \in D$. A binary relational system $(D, \succcurlyeq)$ satisfies *denseness* if and only if, for any $A, B \in D \backslash \{\varnothing\}$, such that $A \succcurlyeq B$ there exists an $A' \in D \backslash \{\varnothing\}$ such that $A' \subseteq A$ and $A' \sim B$.

Independence (I): Let D be a non empty subset of $\mathcal{P}(\mathbb{R}^n)$ such that $\varnothing \in D$, and for any $A \in D$, $k^D(A) = \cap \{B \in D : A \subseteq B\}$ if $\{B \in D : A \subseteq B\}$ $\neq \varnothing$ and undefined otherwise. A binary relational system $(D, \succcurlyeq)$ satisfies

[6] Recall that a *translation* on $\mathbb{R}^n$ is a function $\tau : \mathbb{R}^n \to \mathbb{R}^n$ such that there exists a $y \in \mathbb{R}^n$ with $\tau(x) = x + y$ for all $x \in \mathbb{R}^n$.

[7] We respectively denote with $\succ$ and $\sim$ the asymmetric and symmetric components of $\succcurlyeq$.

independence if and only if, for any $A, B, C \subseteq \mathbb{R}^n$, such that $k^D(A \cup C)$ and $k^D(B \cup C)$ are well defined, and $k^D(A \cap C) \sim k^D(B \cap C) \sim \varnothing$,

$$k^D(A) \succcurlyeq k^D(B) \Longleftrightarrow k^D(A \cup C) \succcurlyeq k^D(B \cup C).$$

Archimedean Property (A): Let D be a non empty subset of $\mathcal{P}(\mathbb{R}^n)$ such that $\varnothing \in D$. A transitive binary relational system $(D, \succcurlyeq)$ satisfies the *Archimedean property* if and only if, for any $A, B \in D$, if $A \succ B \succ \varnothing$ then there exists a positive integer m and $B_1, ..., B_m \in D$ such that $B_1 \sim ... \sim B_m \sim B$ and $B \cup B_1 \cup ... \cup B_m \succcurlyeq A$.

For interpretation and discussion of the **TP**, **NT**, **D**, **I**, **A** properties, we refer to the work of Pattanaik and Xu (2000). The new axioms we introduce here are precisely those required for the valuation-theoretic characterization of volume of polyconvex sets, namely Simplicity and Translation Invariance.

Simplicity (S): Let D be a non empty subset of $\mathcal{P}(\mathbb{R}^n)$ such that $\varnothing \in D$. A binary relational system $(D, \succcurlyeq)$ satisfies *simplicity* if and only if $A \sim \varnothing$, for any $A \in D$, such that $\dim A < n$.

Translation Invariance (TI): Let D be a non empty subset of $\mathcal{P}(\mathbb{R}^n)$. A binary relational system $(D, \succcurlyeq)$ satisfies *translation invariance* if and only if, for any $A \in D$ and any translation τ on $\mathbb{R}^n$, if $\tau(A) \in D$ then $A \sim \tau(A)$.

Simplicity is a full dimensionality requirement for non-bottom opportunity sets. As mentioned above, it amounts to an extremely strong minimum threshold for opportunity sets: in order to be valuable, i.e. strictly more valuable than the null set, an opportunity set must include a positive amount of each characteristic. Of course, this is only plausible if each dimension of the opportunity space does indeed represent a basic/primary characteristic.

Translation Invariance requires the ranking to be insensitive to location in space of the opportunity set with respect to a fixed reference frame. This is of course preposterous if the given ranking is meant to reflect some special, fixed preferences but is, we submit, a quite natural requirement when assessing freedom of choice as such.

In order to proceed to our characterization, we shall rely on two basic lemmas, namely:

Lemma 1. *Let $A, B \in polycon(n)$. Then,*

$$\left\{k^{pol}(A \backslash B), k^{pol}(B \backslash A), A \cap B\right\}$$

is a polyconvex dissection of $A \cup B$.

Lemma 2. *i) Let $A, B \in polycon(n)$ such that $B \subseteq A$. If $(polycon(n), \succcurlyeq)$ is a totally preordered set that satisfies **I**, then $A \succ B$ if and only if $k^{pol}(A \backslash B) \succ \varnothing$;*

*ii) let $A, B \in polycon(n)$ such that $B \subseteq A$. If $(polycon(n), \succcurlyeq)$ is a totally preordered set that satisfies **I** and **NT**, then $A \succcurlyeq B$;*

*iii) let $A, B, C, D \in polycon(n)$. If $(polycon(n), \succcurlyeq)$ is a totally preordered set that satisfies **I** and **NT**, $A \sim C$, $B \sim D$ and $(A \cap B) \sim (C \cap D) \sim \varnothing$, then $A \cup B \sim C \cup D$.*

We shall also use a basic theorem on extensive measurement concerning *extensive structures with no essential maximum* as defined below (see Krantz et al.(1971)):

Definition 4. *An* extensive structure with no essential maximum *is a tuple* $(X, \succcurlyeq, \mathcal{B}, \circ)$ *such that:*

(1) *X is a non-empty set,*
(2) *$\varnothing \neq \mathcal{B} \subseteq X \times X$, and*
(3) *$\circ : X \times X \rightharpoonup X$ is a binary partial operation with domain $\mathcal{B}$ such that:*

i): ***(total preorder)****: $(X, \succcurlyeq)$ is a totally preordered set, i.e. $\succcurlyeq$ is a total and transitive binary relation on X;*

ii): ***(local associativity)****: for any $x, y, z \in X$, if $(x, y) \in \mathcal{B}$ and $(x \circ y, z) \in \mathcal{B}$, then $(y, z) \in \mathcal{B}$, $(x, y \circ z) \in \mathcal{B}$, and $(x \circ y) \circ z \succcurlyeq x \circ (y \circ z)$;*

iii): ***(local commutative monotonicity)****: for any $x, y, z \in X$, if $(x, z) \in \mathcal{B}$ and $x \succcurlyeq y$ then $(z, y) \in \mathcal{B}$ and $x \circ z \succcurlyeq z \circ y$;*

iv): ***(solvability)****: for any $x, y \in X$, if $x \succ y$ then there exists $z \in X$ such that $(y, z) \in \mathcal{B}$ and $x \succcurlyeq y \circ z$;*

v): ***(positivity)****: for any $x, y \in X$, if $(x, y) \in \mathcal{B}$, then $x \circ y \succ x$;*

vi): ***(finiteness of strictly bounded standard sequences)****: for any sequence $(x_i)_{i=1}^{k \leqslant \infty}$ in X, if $(x_i)_{i=1}^{k \leqslant \infty}$ is* standard*, (i.e. such that $x_i = x_{i-1} \circ x_1$, $i = 2, ..., k$), and* bounded*, (i.e. there exists $y \in X$ such $y \succ x_i$, $i = 1, ..., k$), then k is a finite integer.*

4. Characterization of the volume-ranking

Let us now proceed to state the main result of this paper, namely:

Theorem 1. *Let $(polycon\,(n), \succcurlyeq)$ be a binary relational system. Then $\succcurlyeq = \succcurlyeq_V$ if and only if $\succcurlyeq$ is a total transitive relation (i.e. a total preorder) and $(polycon\,(n), \succcurlyeq)$ satisfies **NT, D, A, I, S** and **TI**.*

Theorem 1 captures a 'freedom of choice'-based ranking of *polycon* (n), when the 'size' of an opportunity set is assessed by its volume.[8] It is worth noticing that the volume-characterization provided by Theorem 1 is tight. To see this, consider the following examples:

Example 1 ($Totality$): Independence of the totality requirement is immediately shown by considering the binary relational system $(polycon(n), \geqslant_\pi^\circ)$, defined as follows: for any $A, B \in polycon\,(n)$, $A \geqslant_\pi^\circ B$ if and only if either $A >_\pi^\circ B$ or $A \sim_\pi^\circ B$ where $A >_\pi^\circ B$ if and only if there exists $k \in \mathbb{Z}$, $k \geq 1$ such that $vol(A) - vol(B) = k\pi$ and $A \sim_\pi^\circ B$ if and only if $vol(A) = vol(B)$.

Example 2 ($Transitivity$): Independence of the transitivity requirement can be shown by the binary relational system $(polycon\,(n), \geqslant_\pi)$, defined as follows: for any $A, B \in polycon\,(n)$, $A \geqslant_\pi B$ if and only if either $A >_\pi B$ or $A \sim_\pi B$ where $A >_\pi B$ if and only if there exists $k \in \mathbb{Z}$, $k \geq 1$ such that $vol(A) - vol(B) = k\pi$ and $A \sim_\pi B$ if and only if neither $A >_\pi B$ nor $B >_\pi A$.

[8] Incidentally, it should be noticed that virtually all the axioms we are considering are trivially satisfied on the restricted domain of linear budget sets as considered by Xu (2004). It is still to be seen under which amendments, if at all, the axiom-set proposed by Xu himself in the paper mentioned above might still provide (part of) a characterization of the volume-ranking on the polyconvex domain.

Example 3 (NT): To prove independence of the **NT** property from the other conditions, let us consider the binary relational system $\left(polycon\,(n)\,, R^U\right)$, where R^U is the universal binary relation on $polycon\,(n)$, i.e. $R^U = polycon\,(n) \times polycon\,(n)$.

Example 4 (D): To establish independence of the **D** property from the others let us introduce the binary relational system $(polycon\,(n)\,, \geqslant_D)$, defined as follows: for any $A, B \in polycon\,(n)$, $A \geqslant_D B$ if and only if either (i) $vol\,(A) > vol\,(B)$, or (ii) $vol\,(A) = vol\,(B)$ and

$$\max_{i\in\{1,\dots,n\}} \left\{ \left(x_i - x_i^{'}\right) : x, x^{'} \in A \right\} > \max_{i\in\{1,\dots,n\}} \left\{ \left(x_i - x_i^{'}\right) : x, x^{'} \in B \right\},$$

or (iii) $\dim B < n$.

Example 5 (I): To check independence of the **I** property from the other conditions, consider the binary relation system $(polycon\,(n)\,, \succcurlyeq_d)$ with $\succcurlyeq_d$ defined as follows: for any $A, B \in polycon\,(n)$, $A \succcurlyeq_d B$ if and only if $d\,(A) \geqslant d\,(B)$ where $d : polycon\,(n) \to \{0,1\}$ is a function defined by the following rule: $d\,(A) = \begin{cases} 1 & \text{if } \dim A = n \\ 0 & \text{if } \dim A < n \end{cases}$.

A special attention has to be deserved to the following example in which we show that Archimedean property is violated, but **NT**, **D**, **I**, **TI**, **S**, **TP** are satisfied.

Example 6 (A) In order to establish independence of the Archimedean property let us consider the following example. Let $\mathbb{N}$ be the set of natural numbers and $\mathcal{U}$ a *free ultrafilter* in the (boolean) lattice $(\mathcal{P}(\mathbb{N}), \subseteq)$, i.e. a nonempty *proper* subset of $\mathcal{P}(\mathbb{N})$ (the power set of $\mathbb{N}$) such that i) for any $X \subseteq Y \subseteq \mathbb{N}$ if $X \in \mathcal{U}$ then $Y \in \mathcal{U}$, ii) $X \cap Y \in \mathcal{U}$ for any $X, Y \in \mathcal{U}$, iii) for any $X \subseteq \mathbb{N}$ either $X \in \mathcal{U}$ or $\mathbb{N} \backslash X \in \mathcal{U}$, and iv) $\bigcap \{X : X \in \mathcal{U}\} = \emptyset$.[9]

Now, consider the binary relational system $(polycon(n), \succcurlyeq_{\mathcal{U}})$ defined as follows. First, for any $A \in polycon(n)$, take a suitable power series expansion of its (real-valued) volume i.e. $\sum_{i=0}^{\infty} a_i^A z^i = vol(A)$ where $z \in (0,1) \subseteq \mathbb{R}$, and denote $(a_i^A)_{i\in\mathbb{N}}$ the corresponding sequence of coefficients. Then, for any $A, B \in polycon(n)$, define $A \succcurlyeq_{\mathcal{U}} B$ iff $\left\{i \in \mathbb{N} : a_i^A \geqslant a_i^B\right\} \in \mathcal{U}$ (where '$\geqslant$' denotes the natural order of the reals). In particular, we may consider for the sake of simplicity the case $n = 1$ i.e. the real line. It is easily checked that the polyconvex sets on the real line are precisely the finite unions of closed intervals and their volumes obviously reduce to the sums of their lengths.

The binary relational system $(polycon(1), \succcurlyeq_{\mathcal{U}})$ is indeed a *totally preordered set*: $\succcurlyeq_{\mathcal{U}}$ is *total* because $\geqslant$ is total, and for any $X \subseteq \mathbb{N}$ either $X \in \mathcal{U}$ or $\mathbb{N} \backslash X \in \mathcal{U}$, and *transitive* because $X \cap Y \in \mathcal{U}$ for any $X, Y \in \mathcal{U}$, by definition of an ultrafilter.

To see that $(polycon(1), \succcurlyeq_{\mathcal{U}})$ is *non-trivial* just consider any closed interval A such that $a_i^A > 0$ for any $i \in \mathbb{N}$ (e.g. a closed interval A having length $\sum_{i=0}^{\infty} a_i^A z^i$ with $a_i^A = k \in \mathbb{R}_+ \backslash \{0\}$ for each i), while *simplicity* of $(polycon(1), \succcurlyeq_{\mathcal{U}})$ follows from the fact that the volume of a zero-dimensional closed interval i.e. a point is zero hence a point is by definition $\sim_{\mathcal{U}}$-indifferent to $\emptyset$. Also, it is easily checked that $(polycon(1), \succcurlyeq_{\mathcal{U}})$ is *translation-invariant* since lengths *are* translation-invariant. To check *denseness* of $(polycon(1), \succcurlyeq_{\mathcal{U}})$, take any pair A, B of closed intervals such that

[9] See e.g. Skala (1975) for a short introduction to ultrafilters and ultraproducts.

$A \succ_{\mathcal{U}} B$ and define $A' \subseteq A$ as a closed subinterval of A of length $\sum_{i=0}^{\infty} a_i^{A'} z^i$ where for any $i \in \mathbb{N}$, $a_i^{A'} = a_i^B$ if $a_i^A > a_i^B$, and $a_i^{A'} = a_i^A$ otherwise.

Moreover, $(polycon(1), \succcurlyeq_{\mathcal{U}})$ satisfies *independence*: indeed let A, B, C be closed intervals such that $A \cap C \sim B \cap C \sim \emptyset$. If $A \succcurlyeq_{\mathcal{U}} B$, then by definition $\left\{i \in \mathbb{N} : a_i^A \geqslant a_i^B\right\} \in \mathcal{U}$, hence, since $\mathcal{U}$ is $\cap$-closed, $\left\{i \in \mathbb{N} : a_i^A + a_i^C \geqslant a_i^B + a_i^C\right\} \in \mathcal{U}$ as well i.e. $A \cup C \succcurlyeq_{\mathcal{U}} B \cup C$. Conversely, if $A \cup C \succcurlyeq_{\mathcal{U}} B \cup C$ then $\left\{i \in \mathbb{N} : a_i^A + a_i^C \geqslant a_i^B + a_i^C\right\} \in \mathcal{U}$ hence $\left\{i \in \mathbb{N} : a_i^A \geqslant a_i^B\right\} \in \mathcal{U}$ or equivalently $A \succcurlyeq_{\mathcal{U}} B$.

However, $(polycon(1), \succcurlyeq_{\mathcal{U}})$ does *not* satisfy the *Archimedean* property. To check that, consider a pair of closed intervals A, B and a (positive) real number k such that for any $i \in \mathbb{N}$, $a_i^A = i$ and $a_i^B = k$. Then $B \succ_{\mathcal{U}} \emptyset$ since $\left\{i \in \mathbb{N} : a_i^B = k > 0 = a_i^{\emptyset}\right\} = \mathbb{N} \in \mathcal{U}$, while for any $n \in \mathbb{N}$, $\left\{i \in \mathbb{N} : na_i^B = nk \geqslant i = a_i^A\right\} \notin \mathcal{U}$ because it is a finite set (and $\mathcal{U}$ is a *free* ultrafilter hence by property iv) as defined above no finite set can belong to it). It follows that $A \succ_{\mathcal{U}} B \cup B_1 \cup ... \cup B_{n-1}$ for any $n \in \mathbb{N}$ and any t-uple $B_1, .., B_{n-1}$ with $B \sim_{\mathcal{U}} B_j$, $j = 1, .., n-1$.

Notice that, as it is easily checked, our example also holds in the Pattanaik and Xu (2000) setting where opportunity sets are non-empty and *compact* subsets of the non-negative orthant of a n-dimensional Euclidean real space. Hence, the foregoing example also solves an open problem posed by Pattanaik and Xu (2000, pg. 61), who addressed the issue of constructing an example of an ordering satisfying **TP**, **NT**, **D** and **I** but not the Archimedean property (**A**), without solving it.

Example 7 (S): To prove independence of the **S** property from the other assumptions, consider the binary relational system $(polycon\,(n), \geqslant_S)$, defined as follows: for any $A, B \in polycon\,(n)$, $A \geqslant_S B$ if and only if either $\dim A = \dim B$ and $vol\,(A) \geqslant vol\,(B)$, or $\dim A \neq 0$ and $\dim B < n$, or $\dim B = 0$.

Example 8 (TI): Independence of the **TI** requirement can be shown by considering the binary relational system $(polycon(n), \geqslant_\tau)$, defined as follows: for any $A, B \in polycon\,(n)$, $A \geqslant_\tau B$ if and only if either (i) $vol\,(A) \geqslant vol\,(B)$, with $A \neq B \neq A' \in par^{\perp}(n)$ where $\dim A' = n$, or (ii) $volA = vol\,(B) > 0$ with $A = A' \cup \overline{CD}$ where $\overline{CD}$ is a closed line interval and $B = B' \cup \overline{EF}$ where $\overline{EF}$ is a closed line interval and $B' \in par^{\perp}(n)$ with $\dim B' = n$ and $A' \cap \overline{CD} = B' \cap \overline{EF} = \varnothing$, or (iii) $\dim B < n$.

5. Conclusions

We have explored the problem of ranking opportunity sets in terms of freedom in economic contexts, where these sets are typically *non-finite*. In particular, we have focused on opportunity sets that are finite unions of convex and compact Euclidean sets, namely polyconvex sets. On such a large domain, we have provided a characterization of the volume-ranking of polyconvex opportunity sets, which relies on a basic characterization result of volume as a translation-invariant valuation as combined with a classic theorem on extensive measurement.

The results of this paper show that at least one prominent ranking of Euclidean opportunity sets, namely volume-ranking, is amenable to a rather simple characterizations based on the foregoing ideas when applied to the domain of polyconvex sets. When it comes to interpretation, however, there is apparently some tension between such a broad polyconvex domain and the Simplicity axiom, which is in a sense the hallmark of the volume-ranking, asking for indifference between the empty set and any non full-dimensional opportunity set. In fact, as noticed above in the text, while opportunity sets attached to indivisible goods are allowed in the (Euclidean) polyconvex domain, Simplicity forces indifference between any such opportunity set and the empty set. But then, our *valuation-based* characterization of the volume-ranking provides us with an obvious and promising suggestion: look for rankings violating Simplicity, which are induced by a suitable *non-simple* valuation on the polyconvex domain! We leave it as a possible topic for another paper.

Moreover, it should also be recalled that many models of considerable interest rely on infinite horizons hence typically on infinite-dimensional spaces. Now, while distance-based rankings of opportunity sets are at least in principle easily lifted into infinite-dimensional spaces that is not so for the volume-ranking. However, new rankings induced by suitable *valuations other than the volume* might conceivably be identified and singled out for further analysis. Admittedly, this might prove to be not an easy task since apparently very little is known about valuations in infinite-dimensional spaces, but we regard it as a quite interesting topic for further research.

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6. Appendix: Proofs

Proof of Lemma 1. First, notice that

$$k^{pol}(A\backslash B) \cup k^{pol}(B\backslash A) \cup (A \cap B) \supseteq (A\backslash B) \cup (B\backslash A) \cup (A \cap B) = A \cup B,$$

while

$$\begin{aligned} k^{pol}(A\backslash B) \cup k^{pol}(B\backslash A) \cup (A \cap B) &= (k^{pol}(A\backslash B) \cup (A \cap B)) \cup \\ \cup (k^{pol}(B\backslash A) \cup (A \cap B)) &\subseteq k^{pol}((A\backslash B) \cup (A \cap B)) \cup \\ \cup k^{pol}((B\backslash A) \cup (A \cap B)) &= A \cup B, \end{aligned}$$

i.e.

$$k^{pol}(A\backslash B) \cup k^{pol}(B\backslash A) \cup (A \cap B) = A \cup B$$

Let us now consider $k^{pol}(A\backslash B) \cap (A \cap B)$ and suppose that there exists a n-dimensional ball S with a (finite) positive radius such that $S \subseteq k^{pol}(A\backslash B) \cap (A \cap B)$. Clearly, it must be the case that $S \cap (A\backslash B) = \varnothing$, i.e. $S \subseteq k^{pol}(A\backslash B)\backslash(A\backslash B)$.

But then, a convex compact S' can be chosen such that $S\backslash S' \neq \varnothing$ and $A\backslash B \subseteq (k^{pol}(A\backslash B) \cap S') \subset k^{pol}(A\backslash B)$, a contradiction since $k^{pol}(A\backslash B) \cap S'$ is indeed polyconvex. Therefore, $int(k^{pol}(A\backslash B) \cap (A \cap B)) = \varnothing$ and by a similar argument $int(k^{pol}(B\backslash A) \cap (A \cap B)) = \varnothing$.

Now, suppose there exists a n-dimensional ball S with (finite) positive radius such that $S \subseteq k^{pol}(A\backslash B) \cap k^{pol}(B\backslash A)$. If $S \subseteq k^{pol}(A\backslash B)\backslash(A\backslash B)$ or $S \subseteq k^{pol}(B\backslash A)\backslash(B\backslash A)$, a contradiction follows, by the foregoing argument. Thus, it must be the case that both $S \cap k^{pol}(A\backslash B) \cap (A\backslash B) \neq \varnothing$ and $S \cap k^{pol}(B\backslash A) \cap (B\backslash A) \neq \varnothing$, hence $k^{pol}(A\backslash B) \cap (B\backslash A) \neq \varnothing$ and $k^{pol}(B\backslash A) \cap (A\backslash B) \neq \varnothing$. But then e.g. $A\backslash B \subseteq k^{pol}(A\backslash B) \cap A \subset k^{pol}(A\backslash B)$, a contradiction since $k^{pol}(A\backslash B) \cap A$ is a polyconvex set. Therefore, $int(k^{pol}(A\backslash B) \cap k^{pol}(B\backslash A)) = \varnothing$ as well.

It follows that $\{k^{pol}(A\backslash B), k^{pol}(B\backslash A), A \cap B\}$ is indeed a polyconvex dissection of $A \cup B$. $\square$

Proof of Lemma 2. i) Let $A, B \in polycon(n)$, such that $B \subseteq A$ and $A \succ B$, i.e.

$$k^{pol}((A\backslash B) \cup B) = k^{pol}(A) = A \succ B = \varnothing \cup B = k^{pol}(\varnothing \cup B).$$

Now,

$$k^{pol}(B \cap (A\backslash B)) = B \cap (A\backslash B) = \varnothing = B \cap \varnothing$$

hence, by **I**, $k^{pol}(A\backslash B) \succcurlyeq \varnothing$.

If $\varnothing \succcurlyeq k^{pol}(A\backslash B)$ as well then, by **I** again,

$$B = k^{pol}(B) = k^{pol}(\varnothing \cup B) \succcurlyeq k^{pol}((A\backslash B) \cup B) = A,$$

a contradiction.

Therefore, $k^{pol}(A\backslash B) \succ \varnothing$.

Conversely, let $A, B \in polycon(n)$, such that $B \subseteq A$ and $k^{pol}(A\backslash B) \succ \varnothing$.

Then, since $k^{pol}(\varnothing) = \varnothing$ and $B \cap (A\backslash B) = \varnothing = B \cap \varnothing$, **I** entails

$$A = k^{pol}((A\backslash B) \cup B) \succcurlyeq k^{pol}(B) = B.$$

If $B \succcurlyeq A$ as well, then $k^{pol}(\varnothing \cup B) = B \succcurlyeq A = k^{pol}((A\backslash B) \cup B)$, hence by **I** again $\varnothing \succcurlyeq k^{pol}(A\backslash B)$, a contradiction.

Thus, $A \succ B$ as required.

ii) By **NT**, $k^{pol}(A\backslash B) \succcurlyeq \varnothing = k^{pol}(\varnothing)$. Since $B \cap (A\backslash B) = \varnothing = B \cap \varnothing$, **I** entails

$$A = k^{pol}((A\backslash B) \cup B) \succcurlyeq k^{pol}(\varnothing \cup B) = k^{pol}(B) = B.$$

iii) Let $A, B, C, D \in polycon(n)$ such that $(A \cap B) \sim (C \cap D) \sim \varnothing$, $A \sim C$ and $B \sim D$, and define $A' = A\backslash D$ and $D' = D\backslash A$.

Since by definition $C \cap D' \subseteq C \cap D \sim \varnothing$, **NT** and point *ii)* above entail $C \cap D' \sim \varnothing$.

Also, $A \cap D' = \varnothing$ by definition. Hence, by **I**, $k^{pol}(A \cup D') \sim k^{pol}(C \cup D')$, because $k^{pol}(A) = A \sim C = k^{pol}(C)$. But $A \cup k^{pol}(D')$ and $C \cup k^{pol}(D')$ are polyconvex sets, hence by the definition of polyconvex hull, $k^{pol}(A \cup D') = A \cup k^{pol}(D')$ and $k^{pol}(C \cup D') = C \cup k^{pol}(D')$.

It follows that $A \cup k^{pol}(D') \sim C \cup k^{pol}(D')$.

Similarly, by definition, $A' \cap D = \varnothing$ and $A' \cap B \subseteq A \cap B \sim \varnothing$, whence by point *ii)* above, **NT**, and **I**, $k^{pol}(B) = B \sim D = k^{pol}(D)$ entails

$$k^{pol}(A') \cup B = k^{pol}(A' \cup B) \sim k^{pol}(A' \cup D) = k^{pol}(A') \cup D.$$

But notice that by definition $A \cup D' = A \cup D = A' \cup D$, hence

$$A \cup k^{pol}(D') = k^{pol}(A \cup D') = k^{pol}(A' \cup D) = k^{pol}(A') \cup D,$$

and therefore

$$\begin{aligned} k^{pol}(A') \cup B &= k^{pol}(A' \cup B) \sim k^{pol}(A' \cup D) = \\ &= k^{pol}(A \cup D') \sim k^{pol}(C \cup D') = \\ &= C \cup k^{pol}(D'). \end{aligned}$$

Moreover,

$$(A' \cup B) \cap (A \cap D) = (A' \cap (A \cap D)) \cup (B \cap (A \cap D)) \subseteq (A \cap B) \sim \varnothing,$$

and

$$(C \cup D') \cap (A \cap D) = (C \cap (A \cap D)) \cup (D' \cap (A \cap D)) \subseteq (C \cap D) \sim \varnothing.$$

Thus, by **NT** and point *ii*) above, and by the definition of polyconvex hull

$$k^{pol}((A' \cup B) \cap (A \cap D)) \sim \varnothing$$

and

$$k^{pol}((C \cup D') \cap (A \cap D)) \sim \varnothing.$$

Therefore, by **I**

$$k^{pol}((A' \cup B) \cup (A \cap D)) \sim k^{pol}((C \cup D') \cup (A \cap D)).$$

But

$$k^{pol}((A' \cup B) \cup (A \cap D)) = k^{pol}((A' \cup (A \cap D)) \cup B) = k^{pol}(A \cup B) = A \cup B,$$

$$k^{pol}((C \cup D') \cup (A \cap D)) = k^{pol}(C \cup (D' \cup (D \cap A))) = k^{pol}(C \cup D) = C \cup D,$$

whence

$$A \cup B \sim C \cup D.$$

□

Proof of Theorem. ($\Rightarrow$) It is easily checked that $(polycon(n), \succcurlyeq_V)$ is indeed a totally preordered set that satisfies **NT, D, S, A, I** and **TI**;

($\Leftarrow$) Conversely, let us denote by $[A]_\sim$ the $\succcurlyeq$-indifference class of A, for any $A \in polycon(n)$. Also, posit $\mathbf{polycon(n)} = \{[A]_\sim : A \in polycon(n)\}$ and $\mathbf{polycon(n)}^\circ = \mathbf{polycon(n)} \setminus \{[\varnothing]_\sim\}$.

Notice that $\mathbf{polycon(n)}^\circ \neq \varnothing$ by **NT**.

Then, take

$$\mathcal{B} = \left\{ \begin{array}{c} ([A]_\sim, [B]_\sim) : A \succ \varnothing, B \succ \varnothing, \text{ and there exist } A' \in [A]_\sim, B' \in [B]_\sim \\ \text{s.t. } A' \cap B' \sim \varnothing \end{array} \right\}$$

and for any $([A]_\sim, [B]_\sim) \in \mathcal{B}$ define

$$[A]_\sim \circ [B]_\sim = [A \cup B]_\sim if A \cap B \sim \varnothing.$$

It is immediately checked that $\mathcal{B} \neq \varnothing$. By **NT** there exists a $A \in polycon(n)$ such that $A \succ \varnothing$. Thus, take a translation τ such that $A \cap \tau(A) = \varnothing$. By **TI**, $\tau(A) \sim A \succ \varnothing$, hence $(A, \tau(A)) \in \mathcal{B}$.

Notice that $\circ$ is well-defined. To check that, take any $C, D \in polycon(n)$ such that $C \sim A$, $D \sim B$. Then, take any translation τ such $\tau(C) \cap D = \varnothing$. It follows that:

$$[C]_\sim \circ [D]_\sim = [\tau(C) \cup D]_\sim = [\tau(C)]_\sim \circ [D]_\sim = [A]_\sim \circ [B]_\sim.$$

Finally, for any $[A]_\sim, [B]_\sim \in \mathbf{polycon(n)}^\circ$, posit

$$[A]_\sim \succcurlyeq^* [B]_\sim \Longleftrightarrow A \succcurlyeq B.$$

Then $(\mathbf{polycon(n)}^\circ, \mathcal{B}, \succcurlyeq^*, \circ)$ turns out to be an extensive structure with no essential maximum.

Indeed, $(\mathbf{polycon(n)}^\circ, \succcurlyeq^*)$ is a totally preordered set, in particular, it is an antisymmetric totally preordered set, i.e. a totally ordered set, by construction, since $(polycon(n), \succcurlyeq)$ is a totally preordered set.

To check associativity, for any $([A]_\sim, [B]_\sim), ([A]_\sim, [B]_\sim \circ [C]_\sim) \in \mathcal{B}$ just take a suitable pair of translations $\tau_1, \tau_2, \tau_3, \tau_4$ such that

$$\tau_1(B) \cap \tau_2(C) = \tau_3(A) \cap \tau_4(B \cup C)) = \tau_3(A) \cap \tau_1(B) = \tau_3(A) \cap \tau_2(C) = \varnothing.$$

Since clearly, by **TI**, $B \in [\tau_1(B)]_\sim$, $C \in [\tau_2(C)]_\sim$, $A \in [\tau_3(A)]_\sim$, $B \cup C \in [\tau_4(B \cup C)]_\sim$, it follows that both $([B]_\sim, [C]_\sim) \in \mathcal{B}$ and $([A]_\sim, [B \cup C]_\sim) \in \mathcal{B}$. Notice that the foregoing translations do exist by compactness of polyconvex sets. Moreover,

$$\begin{aligned}
([A]_\sim \circ [B]_\sim) \circ [C]_\sim &= [\tau_3(A) \cup \tau_1(B)]_\sim \circ [\tau_2(C)]_\sim = \\
&= [(\tau_3(A) \cup \tau_1(B)) \cup \tau_2(C)]_\sim = \\
&= [\tau_3(A) \cup (\tau_1(B) \cup \tau_2(C))]_\sim = \\
&= [\tau_3(A)]_\sim \circ ([\tau_1(B)]_\sim \circ [\tau_2(C)]_\sim) = \\
&= ([A]_\sim) \circ ([B]_\sim \circ [C]_\sim).
\end{aligned}$$

As for monotonicity, assume $([A]_\sim, [C]_\sim) \in \mathcal{B}$ and $[A]_\sim \succcurlyeq^* [B]_\sim$. Clearly, again, there exist translations τ, τ' such that $\tau(B) \cap \tau'(C) = \varnothing$, whence $([B]_\sim, [C]_\sim) \in \mathcal{B}$.

Also, take a further translation τ'' such that $\tau''(A) \cap \tau'(C) = \varnothing$. Then, by **I** and transitivity of $\succcurlyeq$,

$$\begin{aligned}
[A]_\sim \circ [C]_\sim &= [\tau''(A) \cup \tau'(C)]_\sim \succcurlyeq^* [\tau(B) \cup \tau'(C)]_\sim = \\
&= [\tau(B)]_\sim \circ [\tau'(C)]_\sim = [B]_\sim \circ [C]_\sim.
\end{aligned}$$

To check that 'solvability' holds, let us consider any $[A]_\sim, [B]_\sim \in \mathbf{polycon(n)}^\circ$ such that $[A]_\sim \succ^* [B]_\sim$. Hence $A \succ B$, and thus by **D** there exists a non-empty $A' \subseteq A$ such that $A' \sim B$. Now, there must exist a translation τ such that $\tau(A') \cap k^{pol}(A \backslash A') = \varnothing$. Thus, since of course $B \sim \tau(A')$, $([B]_\sim, [k^{pol}(A \backslash A')]_\sim) \in \mathcal{B}$ and $[A]_\sim = [A' \cup k^{pol}(A \backslash A')]_\sim = [\tau(A')]_\sim \circ [k^{pol}(A \backslash A')]_\sim = [B]_\sim \circ [k^{pol}(A \backslash A')]_\sim$.

Coming to 'positivity', consider any $[A]_\sim, [B]_\sim \in \mathbf{polycon(n)}^\circ$ such that $([A]_\sim, [B]_\sim) \in \mathcal{B}$, and any translation τ such that $A \cap \tau(B) = \varnothing$. Clearly, $[A]_\sim \circ [B]_\sim = [A \cup \tau(B)]_\sim$. Since $\tau(B) \subseteq A \cup \tau(B)$, $A \cup \tau(B) \succ \tau(B)$ if and only if $k^{pol}((A \cup \tau(B)) \backslash \tau(B)) \succ \varnothing$ by Lemma 2.i above.

Now, $k^{pol}((A \cup \tau(B)) \backslash \tau(B)) = k^{pol}(A) = A \succ \varnothing$, by construction.

It follows that $A \cup \tau(B) \succ \tau(B)$, hence

$$[A]_\sim \circ [B]_\sim = [A \cup \tau(B)]_\sim \succ^* [\tau(B)]_\sim = [B]_\sim$$

by **TI**.

It remains to be shown that every strictly bounded standard sequence is finite, which follows immediately from **A.**

Therefore, according to the classic construction of ratio scales for extensive structures with no essential maximum (see Krantz *et alii* (1971), Theorem 3.3, p. 85), there exists a (positive) real-valued function $\varphi : \mathbf{polycon(n)}^{\circ} \to \mathbb{R}_{+}$ such that

$$\varphi([A]_{\sim}) \geqslant \varphi([B]_{\sim}) \text{ if and only if } [A]_{\sim} \succcurlyeq^{*} [B]_{\sim},$$

for any $[A]_{\sim}, [B]_{\sim} \in \mathbf{polycon(n)}^{\circ}$, and

$$\varphi([A]_{\sim} \circ [B]_{\sim}) = \varphi([A]_{\sim}) + \varphi([B]_{\sim}),$$

for any $([A]_{\sim}, [B]_{\sim}) \in \mathcal{B}$.

In particular, φ is uniquely defined up to positive linear transformations.

Then, define $f : polycon(n) \to \mathbb{R}$ by the following rule:

$$\begin{array}{rcll} f(A) & = & \varphi([A]_{\sim}) & \text{if } [A]_{\sim} \neq [\varnothing]_{\sim}, \text{ and} \\ f(A) & = & 0 & \text{if } [A]_{\sim} = [\varnothing]_{\sim}, \end{array}$$

for any $A \in polycon\,(n)$.

Next, we shall prove that f is indeed a valuation on the lattice $Polycon(n)$. To show this, we can either proceed by a direct proof or invoke a general fact involving restrictions of measures. We shall proceed by a direct proof. Thus, consider any $A, B \in polycon(n)$. If $A \succ \varnothing$ and $B \sim \varnothing$ then $(A \cap B) \sim \varnothing$ by **NT** and Lemma 2.ii. Moreover, by Lemma 2.*iii*, $A \cup B \sim A \cup \varnothing = A \succ \varnothing$, thus $f\,(A \cup B) = f\,(A) = f\,(A) + f\,(B) - f\,(A \cap B)$ since by definition $f\,(B) = f\,(A \cap B) = 0$. A similar argument applies to the case $B \succ \varnothing$ and $A \sim \varnothing$. If $A \sim B \sim \varnothing$ then $A \cup B \sim \varnothing$ by Lemma 2.*iii*, hence $f\,(A \cup B) = 0 = f\,(A) + f\,(B) - f\,(A \cap B)$. Therefore, we may assume that both $A \succ \varnothing$ and $B \succ \varnothing$. By a similar argument, we may also assume without loss of generality that $k^{pol}(A \backslash B) \succ \varnothing, k^{pol}(B \backslash A) \succ \varnothing$ and $A \cap B \succ \varnothing$.

Then,

$$\begin{array}{rcl} f(A \cup B) & = & f(k^{pol}(A \backslash B) \cup k^{pol}(B \backslash A) \cup (A \cap B)) = \\ & = & \varphi([k^{pol}(A \backslash B) \cup k^{pol}(B \backslash A) \cup (A \cap B)]_{\sim}). \end{array}$$

Since, by Lemma 1 above,

$$\max \left\{ \begin{array}{c} \dim(k^{pol}(A \backslash B) \cap k^{pol}(B \backslash A)), \\ \dim(k^{pol}(A \backslash B) \cap (A \cap B)), \\ \dim(k^{pol}(B \backslash A) \cap (A \cap B)) \end{array} \right\} < n$$

it follows that

$$\left\{(k^{pol}(A \backslash B), k^{pol}(B \backslash A)), (k^{pol}(A \backslash B), A \cap B), (k^{pol}(B \backslash A), A \cap B)\right\} \subseteq \mathcal{B}$$

and, by **S**,

$$k^{pol}(A \backslash B) \cap k^{pol}(B \backslash A) \sim k^{pol}(A \backslash B) \cap (A \cap B) \sim k^{pol}(B \backslash A) \cap (A \cap B) \sim \varnothing.$$

Of course, by Lemma 2.*ii*,

$$(A \cap B) \cap (k^{pol}(A \backslash B)) \cap ((A \cap B) \cap (k^{pol}(B \backslash A)) \sim \varnothing.$$

Thus, by Lemma 2.*iii* above,

$$\begin{array}{rl} & (A \cap B) \cap (k^{pol}(A \backslash B) \cup k^{pol}(B \backslash A)) = \\ = & ((A \cap B) \cap (k^{pol}(A \backslash B)) \cup ((A \cap B) \cap (k^{pol}(B \backslash A))) \sim \varnothing \end{array}$$

It follows that, by definition of $\circ$, associativity of $\circ$, and additivity of φ over $\mathcal{B}$

$$\begin{aligned} f(A \cup B) &= \varphi([k^{pol}(A \backslash B)_\sim \cup k^{pol}(B \backslash A)_\sim \cup (A \cap B)]_\sim) = \\ &= \varphi(([k^{pol}(A \backslash B) \cup k^{pol}(B \backslash A)]_\sim) \circ [A \cap B]_\sim) = \\ &= \varphi(([k^{pol}(A \backslash B) \cup k^{pol}(B \backslash A)]_\sim)) + \varphi([A \cap B]_\sim) = \\ &= \varphi(([k^{pol}(A \backslash B)]_\sim \circ [k^{pol}(B \backslash A)]_\sim)) + \varphi([A \cap B]_\sim) = \\ &= \varphi([k^{pol}(A \backslash B)]_\sim) + \varphi([k^{pol}(B \backslash A)]_\sim) + \varphi([A \cap B]_\sim), \end{aligned}$$

whence: $f(A \cup B) + f(A \cap B) = (\varphi([k^{pol}(A \backslash B)]_\sim) + \varphi([A \cap B]_\sim)) + (\varphi([k^{pol}(B \backslash A)]_\sim) + \varphi([A \cap B]_\sim)) = (\varphi([k^{pol}(A \backslash B)]_\sim \circ [A \cap B]_\sim)) + (\varphi([k^{pol}(B \backslash A)]_\sim \circ [A \cap B]_\sim)) = (\varphi([k^{pol}(A \backslash B) \cup (A \cap B)]_\sim) + \varphi([k^{pol}(B \backslash A) \cup (A \cap B)]_\sim) = \varphi([k^{pol}(A)]_\sim) + \varphi([k^{pol}(B)]_\sim) = \varphi([A]_\sim) + \varphi([B]_\sim) = f(A) + f(B)$, as claimed.

Moreover, from **S** it follows that f is in particular a *simple* valuation (because for any $A \in polycon(n)$, $\dim A < n$ entails $A \sim \varnothing$, hence $f(A) = 0$), while **TI** entails that it is translation invariant as well (because for any translation τ and any $A \in polycon(n)$, $\tau A \sim A$ entails $f(\tau(A)) = \varphi([\tau(A)]_\sim) = \varphi([A]_\sim) = f(A)$). Thus, $f : polycon\,(n) \to \mathbb{R}$ is an isotone, simple and translation invariant valuation such that, for any $A, B \in polycon\,(n)$, $A \succcurlyeq B$ if and only if $f\,(A) \geqslant f\,(B)$. Then, since f is in particular monotonic, there exists (by Theorem 8.1.1 in Klain and Rota (1997)) a $c \in \mathbb{R}$ such that for any $A \in polycon\,(n)$, $f\,(A) = c \cdot vol(A)$. It follows that $c > 0$ since both f and vol are isotone and non-constant valuations. Hence, for any $A, B \in polycon\,(n)$, $A \succcurlyeq B$ if and only if $c \cdot vol\,(A) \geq c \cdot vol\,(B)$ i.e., if and only if $vol\,(A) \geq vol\,(B)$. It follows that $\succcurlyeq = \succcurlyeq_V$ as required. $\square$

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