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Linear Cost Share Equilibria and the Veto Power of the Grand Coalition

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Linear Cost Share Equilibria and the Veto Power of the Grand Coalition

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Abstract
We consider pure exchange economies with finitely many private goods involving the choice of a public project. We discuss core-equivalence results in the general framework of non-Euclidean representation of the collective goods. We define a contribution scheme to capture the fraction of the total cost of providing the project that each blocking coalition is expected to cover. We show that for each given contribution scheme defined over the wider class of Aubin coalitions, the resulting core is equivalent to the corresponding linear cost share equilibria. We also characterize linear cost share equilibria in terms of the veto power of the grand coalition. It turns out that linear cost share equilibria are exactly those allocations that cannot be blocked by the grand coalition with reference to auxiliary economies with the same space of agents and modified initial endowments and cost functions. Unlike the Aubin-type equivalence, this characterization does not depend on a particular contribution scheme.

Keywords: Public project, cost share equilibrium, core, non-dominated allocation, grand coalition

JEL Classification: D51, D60, H41

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1 Introduction

The main concern of this paper is to analyze the veto power of the grand coalition (namely the coalition made of all agents) in market economies where the choice of a public project is involved and, starting from this analysis, to provide new characterizations of linear cost share equilibria. We consider exchange economies with finitely many private goods and public projects just represented by an abstract set. Budget sets on which agents take individual decisions are influenced by the public good provision. To encompass different situations, throughout the paper the general mathematical framework proposed by [20] is adopted to represent the public sector of the economy. In [20] an economy with only one private commodity (to be interpreted as money), is considered and the choice of a public project takes place over a set where no special structure is imposed. The absence (in contrast with the classical Samuelsonian Euclidean scheme) of a linear structure on the set of public projects allows in particular the treatment of those public goods on which there is no reason to assume a commonly accepted order. This is the case of public goods for which different individuals may have different perceptions and hence different rankings (see the discussion in [4], [5], [6], [9]). Moreover, if public projects are interpreted as public environments, i.e. collections of variables common to all the agents but determined outside the market mechanisms, this general framework incorporates many different economic problems. This is the interpretation of the Mas-Colell approach given by [13], [14], where non-market variables include legal systems (such as the assignment of property rights), tax and benefits systems, but also private goods provided by the public sector.

Finally, according to the original motivation proposed by Mas-Colell in [20]: "*it is not uncommon that a public decision problem be given in terms of a choice among a few (say six or seven) projects*". If this is the case, no structure a priori makes sense on the set of projects. We notice in particular that the absence of an ordered structure on the set of public projects excludes the possibility of a Samuelsonian-like monotonicity assumption on preferences with respect to the public goods provision. Moreover, since the set of public projects has no special structure, it need not be convex. Hence the discussion includes the case of non-convexity in public sector decisions.

In [20] two main results are proved:

Pareto optimal allocations can be decentralized by means of valuation equilibria. In a valuation equilibrium, the notion of valuation function provides non-linear individual prices to have access to the public goods provision. The valuation functions are interpreted as taxes or subsidies that agents have to pay or receive in order the public projects to be realized. Under a valuation equilibrium each agent prefers his consumption plan, defined as a bundle of private commodities and a public project, to any other consumption plan affordable under the valuation function.

As a second main contribution, valuation equilibria with nonnegative valuation functions (referred to as cost share equilibria) are showed to be equivalent to the standard Foley core of the economy ([7]). This core equivalence result does not contradict the traditional failure of core convergence results in public goods models[1]. Indeed, it is worth to observe that the equivalence result in this setting depends crucially on the fact that only one private good is present on the market.

Subsequent papers extend the model of [20] to allow multiple private goods (see [4], [5], [9], [13], [14] and [3], [10], [2] for the general case of infinitely many private commodities). In the corresponding notion of valuation equilibrium, it is assumed that agents are able to maximize their utilities taking into account changes in the price of private commodities deriving form changes in the public project. The assumption that prices may depend on collective projects makes the approach different from the one based on the notion of Lindahl equilibrium and it is motivated by the fact that, since public decisions may exhibit non-convexity, changes in the public project may cause non-negligible changes in the prices of private commodities[2]. In particular, each valuation equilibrium

[1] In [12] the supportability of the Foley core allocations as nonnegative valuation equilibria is proved for economies with one private good and an arbitrary number of public goods assuming separate cost functions. The equivalence of the core and cost share equilibria is shown in [26] focusing on the case of finite economies with one private good and one public good.

[2] In [14] the efficiency and decentralization of valuation equilibria is discussed treating prices of private commodities as fixed. In this case (remaining closer to the Lindahl equilibrium approach) agents compare alternatives needing fewer informational requirements.

yields a core allocation but the (Foley) veto mechanism is not enough to guarantee the equivalence between the core and the set of cost share equilibria even for well-behaved large economies (see [4]). This confirms, also in the case of non-Euclidean approaches, the generally accepted opinion that the Foley core is too large and not useful to produce equivalence and convergence results.

Well known counter-examples show the failure of the Edgeworth conjecture in the public goods context under the veto mechanism adapted from private goods economies. The assumption that the cost for the public good provision has a uniform distribution among the agents implies that the per-capita cost decreases for an increasing number of agents (for example by replicas of the original economy according to Debreu-Scarf procedure), making the small coalitions weaker. According to the Foley veto mechanism, a blocking coalition is expected to produce the public project by itself, covering the whole cost necessary for its provision. Since the blocking power of small coalitions becomes weaker as the number of agents increases, the Foley core becomes larger and the core equivalence fails to be true. These negative results lead to several alternative notions of competitive and core allocations.

In [5] a core equivalence theorem is proved assuming an atomless measure space of agents, an unstructured set of abstract public projects, finitely many private goods. The blocking mechanism is defined requiring that a contribution measure is given to fix a cost sharing among coalitions: a contribution measure is a probability measure that assigns to each coalition the fraction of the total cost of the project that the coalition is expected to cover when blocking an allocation. Cost share equilibria are assumed to be linear, that is a cost share function among traders fix the contribution of each individual to the realization of each project: this cost enters into the individual budget constraint defined under each alternative project. It turns out that, assuming individual cost shares to be Radon-Nikodym densities of contribution measures, the core based on some contribution measure is equivalent to the set of linear cost share equilibria defined at the corresponding cost distribution function. Moreover, [5] shows that, in the case of equal cost share equilibria (i.e. assuming an equal cost share distribution among traders), the core equivalence holds if and only if the contribution fixed for potentially blocking coalitions is proportional to the size of coalitions (for a similar conclusion in the case of Lindahl equilibria see [24]).

Our concern in this paper is twofold. We prove a core equivalence result for the core of a finite economy with an abstract set of public projects and finitely many private goods. To this aim, we consider the approach followed by [1] in the case of finite exchange economies. The veto mechanism introduced in [1] is equivalent to the classical Debreu-Scarf veto system applied to replica economies and leads to a core that coincides with the competitive equilibria (see also [8]). It extends the notion of coalition and the ordinary veto since it is allowed a participation of the agents with a fraction of their endowments when forming a blocking coalition. Focusing on this approach, we are able to extend the idea of a contribution scheme from ordinary coalitions to the wider class of Aubin coalitions. We obtain, as a consequence, a core equivalence theorem for finite economies with an abstract set of public projects. According to the new veto mechanism, the contribution to the realization of the project for a blocking coalition is defined taking into account the share of participation for each agent in the coalition itself. As for atomless economies, the equivalence between the core and linear cost share equilibria for finite economies depends on the given contribution measure and the corresponding cost distribution function. The connection with the Debreu-Scarf approach is also provided. In a second companion result the veto power of the grand coalition is exploited. Precisely, it is proved that, given a cost share function and the corresponding contribution measure, linear cost share equilibria are exactly those allocations that cannot be blocked by a coalition in which each agent participates with a non zero fraction of his initial endowment (see [15] for analogous results in the case of pure exchange economies with asymmetric information). Our results are proved interpreting the public goods economy with a finite number of agents as a continuum economy in which only a finite number of different agents characteristics can be distinguished.

As a second main contribution, we provide a characterization of linear cost share equilibria that again relies only on the grand coalition and is, in addiction, independent of contribution schemes. We generalize to the case of economies with public projects, the veto mechanism recently defined by [16]. To characterize competitive equilibria, the idea proposed in [16] is to replace the veto power of infinitely many coalitions with the veto power of just one coalition (namely the grand coalition) in infinitely many economies. This is possible enlarging the redistribution of endowments by perturbing the original initial endowments. The auxiliary economies in the presence of public projects depend on each alternative public goods provision: We prove that an allocation is a linear cost share equilibrium if and only if, for each project, it is *non-dominated* by the grand coalition in the corresponding auxiliary economy. Since the contribution to the realization of each project for the grand coalition is equal to one under any contribution scheme, we derive a characterization of linear cost share equilibria independent of contribution measures and cost share functions. The intuition underlying this result is that the Foley veto mechanism is enough to obtain a complete characterization of linear cost share equilibria when infinitely many economies are considered. In these economies the space of agents is the same as in the original economy, the initial endowment and the cost functions are modified.

The paper proceeds as follows. In Section 2 we present the model including the notions of contribution measures and cost distribution functions. Section 3 contains the main equilibrium notions and preliminary technical results. In Section 4 we introduce Aubin coalitions for finite economies and extend the idea of contribution measure to this class of coalitions. We obtain the equivalence between the linear cost share equilibria of the finite economy and the Aubin core. Hence we analyze the veto power of the grand coalition under the Aubin veto mechanism. Finally in Sections 5 and 6 we provide the characterization of linear cost share equilibria (of finite and large economies) as non-dominated allocations of a suitable family of associated economies. Unlike the previous ones, this characterization relies on the veto power in the Foley sense.

2 The Economic Model

We present our model in a general measure-theoretical framework that will be specialized to the case of finite economies or continuum economies in the next sections.

An economy with (non-Samuelsonian) public goods is a tuple

$$\mathcal{E} = \{(I, \Sigma, \mu), (\mathcal{Y}, \mathrm{c}), \omega, (u_t)_{t \in I}\}$$

where (I, Σ, μ) is a measure space; $\mathcal{Y}$ is an abstract set; c is a function from $\mathcal{Y}$ into $I\!R^m_+$ (the nonnegative orthant of $I\!R^m$); ω is an integrable function from I into $I\!R^m_+$; for each $t \in I$, u_t is a real valued function defined on $I\!R^m_+ \times \mathcal{Y}$.

According to the standard interpretation, $I\!R^m_+$ is the space of private commodities; I is the set of economic agents; Σ is the Boolean algebra of allowable coalitions; μ describes the size of coalitions; the elements of the set $\mathcal{Y}$ represent public projects; the function c expresses the cost of any public project in terms of private goods; ω_t is the initial endowment density of agent t; u_t is the utility function of t.

The economy $\mathcal{E}$ is finite if $I = \{1, 2, \ldots, n\}$ is a finite set, $\Sigma = \mathcal{P}(I)$ is the power set of I and μ is the counting measure. The economy $\mathcal{E}$ is atomless when (I, Σ, μ) is an atomless measure space (i.e., given any $T \in \Sigma$ with $\mu(T) > 0$, then there exists $S \subseteq T$ such that $\mu(T) > \mu(S) > 0$). In the last case, as a typical example, we shall consider the unit interval $[0, 1]$ with the Lebesgue measure. Throughout the sequel we assume that:

(1) each consumer t has an initial endowment $\omega_t > 0$; the total initial endowment $\omega \equiv \int_I \omega \, d\mu$ satisfies the inequality $\omega \gg c(y)$[3], for each $y \in \mathcal{Y}$. This condition ensures that each private commodity is present on the market regardless to the cost of the project that is going to be realized.

[3]we follow the standard notation according to which for two vectors $x \equiv (x_1, \ldots x_m)$ and $z \equiv (z_1, \ldots z_m)$ of $I\!R^m_+$ $x \gg z$ means that $x_i > z_i$, for each $i = 1, \ldots m$.

(2) For any public project $y \in \mathcal{Y}$ and for each $t \in I$, $u_t(\cdot, y)$ is strictly monotone, continuous, quasi concave and measurable in private goods in the standard sense (see [18]).

An *allocation* for the economy $\mathcal{E}$ is a specification of the amount of private goods assigned to each agent and of the public project chosen to be realized. An allocation is then a pair (f, y), where f is an integrable function from I into $\mathbb{R}_+^m$ and $y \in \mathcal{Y}$ is a public project. It is said to be *feasible* if

$$\int_I f \, d\mu + c(y) \leq \int_I \omega \, d\mu$$

that means that the whole part of initial endowment not used for covering the cost of the realized project is redistributed among the agents.

In order to define the competitive equilibria of our model, we need to introduce cost distribution functions. As we shall see in the next section, they allow to describe how much each economic agent must contribute to the realization of a public project, given a system of prices for private commodities. A *cost distribution* is an integrable function $\varphi : I \to \mathbb{R}_+$ that satisfies $\displaystyle\int_I \varphi \, d\mu = 1$. We denote by Φ the class of all cost distributions for the economy $\mathcal{E}$.

To model the veto mechanism underlying the core notion, we assume that potentially blocking coalitions are responsible for a fixed share of the total cost of the provision of the public goods. This cost is captured by means of contribution measures. A *contribution measure* is a probability measure $\sigma : \Sigma \to [0, 1]$ which is absolutely continuous with respect to μ, i.e. for every $S \in \Sigma$, $\mu(S) = 0$ implies that $\sigma(S) = 0$. We denote by M_μ the collection of all contribution measures of the economy $\mathcal{E}$.
There is a one-to-one relationship between the cost distribution functions and the contribution measures. In fact, if $\varphi \in \Phi$ is a cost distribution function, then the function σ_φ given by

$$\sigma_\varphi(S) = \int_S \varphi \, d\mu, \qquad \text{for all } S \in \Sigma$$

is a contribution measure. Conversely, starting from a contribution measure σ, the function φ_σ given by

$$\varphi_\sigma(t) = \frac{d\sigma}{d\mu}(t), \qquad \text{for all } t \in I$$

where $\dfrac{d\sigma}{d\mu}$ denotes the Radon-Nikodym derivative of σ with respect to μ, is a cost distribution function.
From an Aubin-like perspective, we have the following notion of coalition (see [1], [8], [22]):

Definition 2.1 *Define the set*

$$\mathcal{A} = \{\gamma : I \to [0, 1] : \gamma \textit{ is simple, measurable and } \mu(\{t \in I : \gamma(t) > 0\}) > 0\}.$$

We call any element γ of the set $\mathcal{A}$ an Aubin (or generalized) coalition and the set $\{t \in I : \gamma(t) > 0\}$, denoted by supp$\gamma$, the support of the Aubin coalition γ.

The set $\mathcal{A}$ can be interpreted as a set of generalized coalitions in the following sense: $\gamma(t)$ represents the share of resources employed by agent t in the coalition γ. It is clear that ordinary coalitions of Σ form a subset of $\mathcal{A}$ since each coalition S can be identified with its characteristic function χ_S[4].
To model the veto mechanism on the wider class of Aubin coalitions, we extend each contribution scheme σ from Σ to $\mathcal{A}$ as follows. Once observed that for each coalition $S \in \Sigma$, we have the equality

[4]In the literature, Aubin coalitions are usually referred to as *fuzzy coalitions* in contrast with the term *crisp coalitions* used for ordinary measurable subset of I. The term fuzzy set is used in relation to sets which are sharply defined, so that there is ambiguity in declaring whether an element belongs to the set or to its complement. In the generalized coalitions introduced here, it is intended that agents actually participate in a coalition with a fraction of their initial endowments. Therefore, following [15]we prefer to call them Aubin coalitions.

$\sigma(S) = \int_I \chi_S d\sigma$, the contribution of the Aubin coalition γ to the realization of a public project is defined by

$$\widetilde{\sigma}(\gamma) = \int_I \gamma \, d\sigma.$$

When S is a coalition of Σ, it is true that $\widetilde{\sigma}(\gamma) = \sigma(S)$, hence $\widetilde{\sigma}$ can be seen as a generalization of the measure σ[5] . Let us denote by φ the individual cost distribution corresponding to σ. Clearly, from the relation

$$\widetilde{\sigma}(\gamma) = \int_I \gamma\varphi \, d\mu$$

it follows that the individual cost contribution of agent t when participating in the Aubin coalition γ is equal to $\gamma(t)\varphi(t)$, for each $t \in supp\gamma$. We explicitly remark that in the case of a finite economy $\mathcal{E}$, an Aubin coalition is a vector of real numbers $\gamma \equiv (\gamma_1, \dots, \gamma_n)$ of the interval $[0, 1]$ not all equal to zero and for each contribution measure σ defined over $\Sigma = \mathcal{P}(I)$, the extension of σ to $\mathcal{A}$, when evaluated on γ, is given by $\widetilde{\sigma}(\gamma) = \sum_{i=1}^{n} \gamma_i \sigma(\{i\})$.

3 Equilibrium Notions

In this Section we introduce and discuss the main equilibrium notions that will be analyzed throughout the paper and state the basic relationships among them. We start introducing the notion of dominated allocations. We refer to [15] and [16] for the analogue in the case of pure exchange economies.

Definition 3.1 *Let $z \in \mathcal{Y}$ be a public project. An allocation (f, y) (feasible or not) is z-dominated if there exists a feasible allocation (g, z) such that $u_t(g_t, z) > u_t(f_t, y)$, for almost all $t \in I$.*

The allocation (f, y) is a *non-dominated* allocation if it is not z-dominated, for each $z \in \mathcal{Y}$. A feasible and non-dominated allocation (f, y) is said to be *Pareto optimal*.

Let $\Delta = \left\{ p \in I\!R^m_+ | \sum_{i=1}^{m} p_i = 1 \right\}$ be the $(m-1)$-dimensional price simplex. In the following, the notion of cost share equilibrium introduced by [20] for a model with one private good is adapted to the case of finitely many private goods (compare [4], [5]).

Definition 3.2 *A feasible allocation (f, y) is a linear cost share equilibrium if there exists a price system $p : \mathcal{Y} \to \Delta$ and a cost distribution function φ such that, for almost all $t \in I$, $(f(t), y)$ maximizes the utility u_t on the budget set*

$$\left\{ (h, z) \in I\!R^m_+ \times \mathcal{Y} \,|\, p(z) \cdot h + \varphi(t) p(z) \cdot c(z) \le p(z) \cdot \omega(t) \right\}.$$

Let $\varphi \in \Phi$ be a cost distribution function. The set of linear cost share equilibria whose corresponding cost distribution function is equal to φ will be denoted by $LCE_\varphi(\mathcal{E})$. If $LCE(\mathcal{E})$ is the set of linear cost share equilibria, then

$$LCE(\mathcal{E}) = \bigcup_{\varphi \in \Phi} LCE_\varphi(\mathcal{E}).$$

Notice that the distribution among the agents of the public goods provision costs is not necessarily a constant function. This specification, i.e. the case in which $\varphi(t) = 1$ for almost all $t \in I$, leads to the so called *equal cost share equilibria*.

The utility maximization condition contained in the definition of linear cost share equilibria is the counterpart of similar conditions in usual competitive equilibrium concepts. However, unlike

[5]Interpreting σ as a probability measure on the space (I, Σ), $\widetilde{\sigma}(\gamma)$ can be interpreted, according to [27], as the probability measure of the fuzzy event γ and it coincides with the expectation of the fuzzy event taken with respect to the initial probability measure σ.

the notion of Lindahl equilibrium commonly adopted in a public goods framework, it is important to point out that the price system p for private commodities defining the budget set of each trader, depends on the set of public projects. As a typical request of the model, although only one project will be realized in equilibrium (the project y), the prices for each possible realization of different projects have to be known. The price system $p : \mathcal{Y} \to \Delta$ is usually interpreted as incorporating each possible variation in the private goods sector of the economy due to variations in the public goods choices. The substantial difference with respect to models with Samuelsonian public goods, is due to the fact that preferences are not monotone in the projects since the set $\mathcal{Y}$ is not necessarily Euclidean. It is possible to show that, even assuming the linearity of the set $\mathcal{Y}$, it might be impossible to decentralize optimal allocations without a price system contingent on public projects (see [6]).

The second main task in which the notion of linear cost share equilibrium differs from the usual competitive one is of course the presence of cost distribution functions. The term $\varphi(t)p(z) \cdot c(z)$ contained in the budget constraint, represents the analogue of individual personalized prices for public goods typical of Lindahl equilibrium notions. They are usually interpreted as individual prices that agents have to pay to have access to the public goods consumption. Notice however that, differently by Lindahl prices, they depend on prices for private commodities.

The concept of cost share equilibria defined in [20] for economies with only one private commodity, relies on a more general system of taxes called valuation functions. They fix the individual price that agents pay or receive to have access to the public goods provision. In general, it is possible to show that the set of linear cost share equilibria is a proper subset of the class of cost share equilibria. However, we note that, if there is a unique provision level of public goods, i.e. $|\mathcal{Y}| = 1$, then each cost share equilibrium is a linear cost share equilibrium.

The issue of existence of linear cost share equilibria is discussed in [5] and [9]. It comes out that in our treatment of the public goods sector, differently from the case of Lindahl equilibria, one has not to expect a very general existence theorem. However, the following existence result ensures that the competitive notion introduced in Definition 3.2 is not vacuous.

Theorem 3.3 [5, Theorem 1] *There exists an atomless economy $\mathcal{E}$ with public goods with preferences represented by strictly monotone and quasi-concave utility functions for which the set $LCE(\mathcal{E})$ of linear cost share equilibria is not empty.*

As an immediate optimality property of linear cost share equilibria, we have the following.

Proposition 3.4 *If (f, y) is a linear cost share equilibrium in $\mathcal{E}$, then it is not z-dominated for each $z \in \mathcal{Y}$. In particular, (f, y) is a Pareto optimal allocation.*

PROOF: Assume by contradiction that (f, y) is z-dominated for a project $z \in \mathcal{Y}$. Then there would exists a feasible allocation (g, z) such that $u_t(g(t), z) > u_t(f(t), y)$ for almost all $t \in I$. Let p and φ be, respectively, the price system and the cost distribution function associated to (f, y). Then for almost all $t \in I$, it results

$$p(z) \cdot g(t) + \varphi(t)\, p(z) \cdot c(z) > p(z) \cdot \omega(t)$$

hence

$$p(z) \cdot \int_I g \, d\mu + p(z) \cdot c(z) \int_I \varphi \, d\mu > p(z) \cdot \int_I \omega \, d\mu$$

and then

$$p(z) \cdot \int_I g \, d\mu + p(z) \cdot c(z) > p(z) \cdot \int_I \omega \, d\mu,$$

that contradicts the feasibility of (g, z). □

We introduce in the following the core notion that is most compatible with that of cost sharing. To this aim, we are going to assume, within the framework of contribution measures, a given contribution scheme. According to it, potentially blocking coalitions are not required to cover the whole cost of the new project when dissenting from a given allocation.

Definition 3.5 *Given a contribution measure σ, we say that a coalition $S \in \Sigma$ with $\mu(S) > 0$, σ-blocks an allocation (f, y) if there exist a public good $z \in \mathcal{Y}$ and an integrable assignment of private goods $g : S \to I\!R^m_+$ such that*

$$\int_S g d\mu + \sigma(S)c(z) \leq \int_S \omega d\mu$$

and

$$u_t(g(t), z) > u_t(f(t), y), \qquad \text{for almost all } t \in S.$$

The veto mechanism just defined requires that each member of the blocking coalition is better off under the new assignment and the different project. Moreover, the coalition itself is able to cover the share of the cost of the new public project for which it is responsible, according to the given scheme.

The *σ-core* (or the *σ-budget core*) of the economy $\mathcal{E}$, denoted by $C_\sigma(\mathcal{E})$, is the set of feasible allocations that cannot be σ-blocked by any coalition. When the contribution measure σ is equal to the underlying measure μ, the corresponding core, $C_\mu(\mathcal{E})$, is called the *proportional core.*

Notice that the notion of core that Foley ([7]) originally proposed for economies with public goods, requires that every blocking coalition incurs the entire cost of producing the quantities of the public goods it needs in order to block. This assumption cannot be captured by a contribution measure, but by the contribution function, called *maximal contribution scheme*, which assigns to each non-null coalition a share equal to 1. Let $C(\mathcal{E})$ be the Foley core of the economy $\mathcal{E}$. Clearly, for any contribution measure σ, we have the inclusion $C_\sigma(\mathcal{E}) \subseteq C(\mathcal{E})$, since blocking is hardest under the Foley contribution scheme.

Let us extend now the veto mechanism depending on contribution measures, to the more general case of Aubin coalitions.

Definition 3.6 *Given a contribution measure σ, we say that an Aubin coalition $\gamma \in \mathcal{A}$ σ-blocks an allocation (f, y) if there exist a public good $z \in \mathcal{Y}$ and an integrable assignment $g : I \to I\!R^m_+$ of private goods such that*

$$\int_I \gamma g d\mu + \widetilde{\sigma}(\gamma)c(z) \leq \int_I \gamma\omega d\mu$$

and

$$u_t(g(t), z) > u_t(f(t), y), \qquad \text{for almost all } t \in supp\gamma.$$

A feasible allocation (f, y) is in the *Aubin σ-core* if it cannot be σ-blocked by an Aubin coalition in the previous sense. The Aubin σ-core of $\mathcal{E}$ will be denoted by $C^A_\sigma(\mathcal{E})$. Since ordinary coalitions are a particular case of the Aubin coalitions, then the inclusion $C^A_\sigma(\mathcal{E}) \subseteq C_\sigma(\mathcal{E})$ is obvious. In the Aubin blocking mechanism, the share of the cost of the project that an agent of the dissenting coalition has to cover according to σ, is further weighted by the share of participation in the coalition itself.

Concerning the relation between budget cores and linear cost share equilibria, the first result studies the inclusion of the set $LCE_\varphi(\mathcal{E})$ in a well specified budget core.

Proposition 3.7 *Let (f, y) be a linear cost share equilibrium in $\mathcal{E}$ with cost distribution function φ and let σ_φ be the corresponding contribution measure. Then (f, y) belongs to the Aubin σ_φ-core of $\mathcal{E}$ $C^A_{\sigma_\varphi}(\mathcal{E})$ and, consequently, to the σ_φ-core $C_{\sigma_\varphi}(\mathcal{E})$.*

PROOF: Assume, by contradiction, that (f, y) does not belong to the Aubin σ_φ-core. Then there exist a coalition $\gamma \in \mathcal{A}$, a public project $z \in Y$ and an assignment $g : I \to I\!R^m_+$ of private commodities such that

$$\int_I \gamma g d\mu + \sigma_\varphi(\gamma)c(z) \leq \int_I \gamma\omega d\mu$$

and

$$u_t(g(t), z) > u_t(f(t), y), \qquad \text{for almost all } t \in supp\gamma.$$

Then, denoted by p the system of prices associated to (f, y), by definition of linear cost share equilibrium it follows that, for almost all $t \in S$,

$$p(z) \cdot g(t) + \varphi(t)\, p(z) \cdot c(z) > p(z) \cdot \omega(t).$$

Consequently,

$$p(z) \cdot \gamma(t) g(t) + \gamma(t)\varphi(t)\, p(z) \cdot c(z) > p(z) \cdot \gamma(t)\omega(t).$$

Since $\widetilde{\sigma}_{\varphi}(\gamma) = \int_I \gamma\varphi\, d\mu$, we have that

$$p(z) \cdot \int_I \gamma g d\mu + p(z) \cdot \widetilde{\sigma}_{\varphi}(\gamma) c(z) > p(z) \cdot \int_I \gamma\omega d\mu$$

and a contradiction, given the feasibility of (g, z) on the coalition γ under the contribution measure σ_{φ}. □

In particular, Theorem 3.3 and Proposition 3.7 ensure that the notions of Aubin σ-core and σ-core are not vacuous.

The Foley core is considered to be the core of the economy in [20]. In this paper it is proved that, assuming only one private commodity, this core is equivalent to the set of cost share equilibria (see also [21] and [26] for similar equivalence results with traditional public goods). In the case of finite economies with finitely many private commodities, the failure of this equivalence is proved in [4]. The non-equivalence between the Foley core and the set of cost share equilibria in large atomless economies is confirmed in [5, Theorem 5]. When the economy $\mathcal{E}$ is atomless, the equivalence between well specified budget cores and well chosen linear cost share equilibria, namely the reverse of the inclusion $LCE_{\varphi}(\mathcal{E}) \subseteq C_{\sigma_{\varphi}}(\mathcal{E})$ following from Proposition 3.7, is stated in the next result.

Theorem 3.8 [4, Theorem 4] *Let $\mathcal{E}$ be an atomless economy with public goods. Let $\sigma \in M_{\mu}$ be a contribution measure and let $\frac{d\sigma}{d\mu} \in \Phi$ be the corresponding Radon-Nikodym derivative. Then $C_{\sigma}(\mathcal{E}) = LCE_{\frac{d\sigma}{d\mu}}(\mathcal{E})$.*

The previous equivalence result explicitly depends on the contribution measure and the corresponding cost distribution function. We notice that it contains as a particular case the equivalence between the equal cost share equilibria and the proportional core. Moreover, it implies also that, in the case of atomless economies, for each contribution measure σ, the Aubin σ-core coincides with the set $LCE_{\frac{d\sigma}{d\mu}}(\mathcal{E})$. Hence the Aubin cooperation will provide useful information concerning linear cost share equilibria only in the case of finite economies, namely the situation in which the σ-core is too big.

We close this Section with some technical results that will be useful in the rest of the paper. The first Lemma guarantees, for each Pareto optimal allocation (f, y), the existence, under the project z different from y, of special feasible allocations: They will play, under the project z, the role of f, allowing us to reduce some of the arguments in our proofs to the pure exchange case.

In order to state the Lemma, we have to introduce the following essentiality conditions:

First essentiality condition: For any feasible allocation (f, y), for all z in $\mathcal{Y}$ and for every agent $t \in I$ there exists a bundle g of private commodities such that $u_t(g, z) \geq u_t(f(t), y)$.

Second essentiality condition: For any agent $t \in I$, for all $y, z \in \mathcal{Y}$, for all $f \in I\!R^m_+$, the inequality $u_t(0, z) \leq u_t(f, y)$ holds.

The first essentiality condition ensures that any variation in the public goods provision can be compensated by a suitable quantity of private goods. They are analogous to the essentiality conditions stated in [4][6].

[6] In [13] a condition analogous to the first essentiality condition is introduced restricting the attention to suitable subsets of $\mathcal{Y}$. Under this approach, one excludes a priori those projects which are so bad for some agent t that no choice of a commodity bundle compensates the agents for deviating from the bundle (f_t, y).

The following regularity condition refers to the integrability property of preferences (compare a similar conditions introduced in [5]):

Integrable utilities: For any allocation (f, y) and for all z in $\mathcal{Y}$ there exists a distribution of private commodities $g : I \to I\!R^m_+$ such that (g, z) is feasible and for almost every agent $t \in I$, if for some vector h $u_t(h, z) > u_t(f(t), y)$ then it is also true that $u_t(g(t), z) \geq u_t(f(t), y)$.

Lemma 3.9 *Assume that the utility functions satisfy the essentiality conditions and that are integrable. If (f, y) is a Pareto optimal allocation of the economy $\mathcal{E}$, then there exist, for any public project $z \in \mathcal{Y}$ an integrable function γ_z and a system of prices $p(z)$ such that: (γ_z, z) is a feasible allocation; $u_t(\gamma_z(t), z) \geq u_t(f(t), y)$ for almost all $t \in I$; $u_t(g, z) > u_t(f(t), y) \Rightarrow p(z) \cdot g > p(z) \cdot \gamma_z(t)$, for almost all $t \in I$.*

PROOF: See the Appendix 7.1. □

Notice that in the case of a single project, the content of Lemma 3.9 exactly gives the second welfare Theorem.

The following Proposition deserves interest in itself. It extends to the case of exchange economies with an abstract set of public projects the results of [23] and [**?**]. In the atomless framework, it is enough to consider the veto power of coalitions of a given measure to obtain the σ-core. In particular, when combined with Theorem 3.8, it implies that non-linear cost share equilibria coincide exactly with those allocations that can be σ-blocked by coalitions of arbitrarily small or big measure.

Proposition 3.10 *Assume that the utility functions satisfy the essentiality conditions and that are integrable. Let (f, y) be a Pareto optimal allocation of the atomless economy $\mathcal{E}$ not belonging to the σ-core of $\mathcal{E}$. Then, for any ϵ, with $0 < \epsilon < 1$, there exists a coalition T with $\mu(T) = \epsilon$, and an allocation (g, z) that σ-blocks (f, y) on T.*

PROOF: See the Appendix 7.2. □

4 Equivalence results for finite economies

The aim of this section is to prove a characterization of linear cost share equilibria in terms of σ-core allocations in the case of economies with a finite number of agents. We shall assume that the set of agents is $I = \{1, \dots n\}$, μ is the counting measure over the algebra of coalitions $\Sigma = \mathcal{P}(I)$.

As it is well known, a finite set of coalitions is not enough to obtain equivalence theorems in the finite setting. Indeed, in the case of economies with private commodities, the characterization of competitive allocations is only asymptotic and it is proved, for example, by replicating the original economy. Under a different point of view, as in the approach adopted by [1], the choice of a modified coalition notion implies that the core shrinks to the set of competitive equilibria. According to this, one should not expect, even for finite economies with public projects, an equivalence result without increasing the number of potentially blocking coalitions. Nevertheless, it is well known that the classical convergence results for economies with private goods do not extend to the public goods context under the Foley concept of blocking. This discussion says that the approach to the core equivalence in our setting should require both an increasing number of coalitions and a modification of the veto mechanism. For this reason, the analysis followed in this Section extends the idea of contribution measures from ordinary coalitions of $\mathcal{P}(I)$ to more general Aubin coalitions. The corresponding equivalence theorem will be also stated in terms of replica economies.

In order to prove the converse of Proposition 3.7 in the case of finite economies, we start considering a continuum economy $\mathcal{E}_{\mathcal{C}}$ with n different types of agents canonically associated to $\mathcal{E}$. The construction follows the procedure that is standard in the case of economies with private goods

(see for example [16]), but it takes into account also the relationship between cost distribution functions and contribution measures.
The space of agents is represented by the real interval $I = [0,1]$ with the Lebesgue measure μ. We write $I = \bigcup_{i=1}^{n} I_i$, where $I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right[$ if $i \neq n$ and $I_n = \left]\frac{n-1}{n}, 1\right]$. Each consumer $t \in I_i$ is characterized by the consumption set $\mathbb{R}^m_+$, the utility function $u_t = u_i$ and the initial endowment $\omega(t) = \omega_i$. We will refer to I_i as the set of agents of type i in the atomless economy $\mathcal{E}_{\mathcal{C}}$. Moreover the set $\mathcal{Y}$ represents the set of public projects and the function $\widehat{c} : \mathcal{Y} \to \mathbb{R}^m_+$ defined as $\widehat{c}(y) = \frac{c(y)}{n}$, is the cost function.

Observe that an allocation $(x_1, \ldots, x_n, y)$ in $\mathcal{E}$ can be interpreted as an allocation (f, y) in $\mathcal{E}_{\mathcal{C}}$, where f is the function defined as $f(t) = x_i$, if $t \in I_i$. Reciprocally, an allocation (f, y) in $\mathcal{E}_{\mathcal{C}}$ can be interpreted as an allocation $(x_1, \ldots, x_n, y)$ in $\mathcal{E}$, with $x_i = \frac{1}{\mu(I_i)} \int_{I_i} f d\mu$.

We notice also that

If φ is a cost distribution function for $\mathcal{E}$, then $\widehat{\varphi} : I \to \mathbb{R}^+$ defined as $\widehat{\varphi}(t) = n\varphi(i)$, if $t \in I_i$ is a cost distribution function for $\mathcal{E}_{\mathcal{C}}$.

Reciprocally, if $\widehat{\varphi}$ is a cost distribution function for $\mathcal{E}_{\mathcal{C}}$, then $\varphi : I \to \mathbb{R}^+$ defined as $\varphi(i) = \int_{I_i} \widehat{\varphi} d\mu$ is a cost distribution function for $\mathcal{E}$.

If σ is a contribution measure for $\mathcal{E}$, then $\widehat{\sigma} : \Sigma \to [0,1]$ defined as

$$\widehat{\sigma}(S) = \sum_{i=1}^{n} \sigma(\{i\}) \frac{\mu(S \cap I_i)}{\mu(I_i)}$$

for each $S \in \Sigma$, is a contribution measure for $\mathcal{E}_{\mathcal{C}}$.

Reciprocally, if $\widehat{\sigma}$ is a contribution measure for $\mathcal{E}_{\mathcal{C}}$, then $\sigma : I \to [0,1]$ defined as $\sigma(S) = \sum_{i \in S} \widehat{\sigma}(I_i)$, for each $S \in \mathcal{P}(I)$, is a contribution measure for $\mathcal{E}$.

Proposition 4.1 *If the allocation $(x_1, \ldots, x_n, y)$ is in the Aubin σ-core, then it is a linear cost share equilibrium with cost distribution function φ_σ.*

PROOF: First we prove that the associated allocation (f, y) defined by $f(t) = x_i$, for all $t \in I_i$ is in the $\widehat{\sigma}$-core of the continuum economy $\mathcal{E}_{\mathcal{C}}$, where the contribution measure $\widehat{\sigma}$ is defined by

$$\widehat{\sigma}(E) = \sum_{i=1}^{n} \sigma(\{i\}) \frac{\mu(E \cap I_i)}{\mu(I_i)}.$$

Assume that (f, y) is not in the $\widehat{\sigma}$-core. Then there exist a coalition $S \in \Sigma$ with $\mu(S) > 0$, a public good $z \in \mathcal{Y}$, and an integrable assignment of private commodities $g : S \to \mathbb{R}^m_+$ such that

$$\int_S g d\mu + \widehat{\sigma}(S)\widehat{c}(z) \leq \int_S \omega d\mu$$

and for almost all $t \in S$,

$$u_t(g(t), z) > u_t(f(t), y).$$

The second inequality implies that, for almost all $t \in S \cap I_i$,

$$u_i(g(t), z) > u_i(x_i, y).$$

Set $\gamma_i = \mu(S \cap I_i)$ and, for those i such that $\gamma_i > 0$, define $g_i = \dfrac{1}{\gamma_i} \displaystyle\int_{S \cap I_i} g d\mu$. Then we have

$$\sum_{i=1}^{n} \gamma_i g_i = \int_S g d\mu \leq \int_S \omega d\mu - \widehat{\sigma}(S)\widehat{c}(z) = \sum_{i=1}^{n} \int_{S \cap I_i} \omega d\mu - \sum_{i=1}^{n} \sigma(\{i\}) \frac{\mu(S \cap I_i)}{\mu(I_i)} \frac{c(z)}{n} =$$

$$= \sum_{i=1}^{n} \gamma_i \omega_i - \widetilde{\sigma}(\gamma) c(z)$$

and, by Jensen's integral inequality (see [19]), for all i such that $\gamma_i > 0$,

$$u_i(g_i, z) > u_i(x_i, y).$$

Hence a contradiction that proves our claim.
By Theorem 3.8, (f, y) is a linear cost share equilibrium with respect to the individual cost function $\widehat{\varphi}$ associated to $\widehat{\sigma}$ and defined by $\widehat{\varphi}(t) = n\sigma(\{i\})$, for almost all $t \in I_i$.

Let $p : \mathcal{Y} \to \Delta$ be the price system associated to (f, y).

We claim that $(x_1, \ldots, x_n, y)$ is a linear cost share equilibrium with respect to the contribution function φ and the price system p.

For each $i \in \{1, \ldots, n\}$, $z \in \mathcal{Y}$ and $t \in I_i$, we have that

$$p(z) \cdot x_i + \varphi_i \, p(z) \cdot c(z) = p(z) \cdot f(t) + \widehat{\varphi}(t) \, p(z) \cdot \widehat{c}(z) \leq p(z) \cdot \omega(t),$$

the inequality being an equality when $z = y$. Assume now that for $i \in \{1, \ldots, n\}$, $z \in \mathcal{Y}$ and $g \in I\!R^m_+$ it is

$$u_i(g, z) > u_i(x_i, y).$$

Since (f, y) is a linear cost share equilibrium, we have that, for almost all $t \in I_i$,

$$p(z) \cdot g + \widehat{\varphi}(t) \, p(z) \cdot \widehat{c}(z) > p(z) \cdot \omega(t)$$

hence

$$p(z) \cdot g + \varphi_i \, p(z) \cdot c(z) > p(z) \cdot \omega_i$$

that proves our claim. □

From Propositions 3.7 and 4.1, it follows the analogous of Theorem 3.8 in the case of finite economies with an abstract set of public projects.

Theorem 4.2 *Let $\mathcal{E}$ be a finite economy. Let $\sigma \in M_\mu$ be a contribution measure and let $\varphi_\sigma \in \Phi$ be the corresponding cost distribution function. Then $C^A_\sigma(\mathcal{E}) = LCE_{\varphi_\sigma}(\mathcal{E})$.*

The equivalence expressed by Theorem 4.2 can be interpreted in terms of classical convergence results according to a replica process.

This is possible defining for each positive integer r the *r-fold replica* of the economy $\mathcal{E}$ as the economy $\mathcal{E}_r$ with the following characteristics:

- the economy $\mathcal{E}_r$ has the same commodity-price duality of $\mathcal{E}$; the same set of public projects $\mathcal{Y}$; the cost function defined by $c_r(z) = rc(z)$;
- for each $i = 1, \ldots, n$, there are r agents of type i, each one indexed by (i, j) with $j = 1, \ldots, r$, having the same initial endowment $\omega_{i,j} = \omega_i$ and the same utility functions $u_{i,j}(\cdot, z) = u_i(\cdot, z)$, for any public project $z \in \mathcal{Y}$.

Definition 4.3 *Let $\sigma \in M_\mu$ be a contribution measure of the economy $\mathcal{E}$. Define a contribution measure of the r-fold replica economy $\mathcal{E}_r$ by $\sigma(\{i, j\}) = \frac{\sigma(\{i\})}{r}$, for each $j = 1, \ldots r$. A feasible allocation $(x_1, \ldots, x_n, y)$ is said to be a σ-Edgeworth equilibrium of $\mathcal{E}$ whenever the corresponding equal treatment allocation*

$$(x_{1,1}, \ldots, x_{1,r}, \ldots, x_{n,1}, \ldots, x_{n,r}, y)$$

with $x_{i,h} = x_{i,k}$ for any $h, k = 1, \ldots, r$, for any $i = 1, \ldots, n$, belongs to the σ-core of $\mathcal{E}_r$, for each r.

In the following we interpret Aubin σ-core allocations of the finite economy as σ-Edgeworth equilibria. We state the result by making an additional assumption: This condition implies that under the fixed contribution scheme, each agent owns a positive initial amount of private commodities after paying for the public goods provision.

Proposition 4.4 *Assume that for a contribution measure $\sigma \in M_\mu$ it is true that $(\omega_i - \sigma(\{i\})c(z) \gg 0$, for each agent i. Then the σ-Aubin core of the economy $\mathcal{E}$ coincides with the set of the σ-Edgeworth equilibria.*

PROOF: Let the allocation $(x_1, \ldots, x_n, y)$ be in the σ-Aubin core and assume that it is not a σ-Edgeworth equilibrium. So, there exist an r-replica $\mathcal{E}_r$, a coalition T of $\mathcal{E}_r$, and an allocation $((g_{i,j})_{(i,j)\in T}, z)$ such that $u_{i,j}(g_{i,j}, z) = u_i(g_{i,j}, z) > u_i(x_i, y)$ for all $(i,j) \in T$ and

$$\sum_{(i,j)\in T} g_{i,j} + \sum_{(i,j)\in T} \sigma(\{i,j\})c_r(z) \leq \sum_{(i,j)\in T} \omega_{i,j}.$$

Let us denote by l_i the number of agents of type i belonging to the coalition T, by A the set $A = \{i \ : \ l_i \neq 0\}$ and, for each $i \in A$, by g_i the convex combination $\sum_j \frac{1}{l_i} g_{i,j}$. From the previous inequality we obtain

$$\sum_{i\in A} l_i g_i + \sum_{i\in A} l_i \frac{\sigma(\{i\})}{r} rc(z) \leq \sum_{i\in A} l_i \omega_i$$

so, considering the Aubin coalition γ defined by $\gamma_i = l_i$ for each $i \in A$, by convexity of the utility functions $u_i(\cdot, z)$, we get a contradiction.

Conversely, let $(x_1, \ldots, x_n, y)$ be a σ-Edgeworth equilibrium, and assume that there exists an Aubin coalition $(\gamma_i)_{i=1}^n$ and an allocation $(g_1, \ldots, g_n, z)$ such that $u_i(g_i, z) > u_i(x_i, y)$ for all $i \in supp\gamma$ and

$$\sum_{i\in supp\gamma} \gamma_i g_i + \widetilde{\sigma}(\gamma)c(z) \leq \sum_{i\in supp\gamma} \gamma_i \omega_i.$$

Let $\varepsilon \in (0,1)$ be such that $u_i(\varepsilon g_i, z) > u_i(x_i, y)$, for each $i = 1, \ldots n$. We can rewrite the last inequality in the form

$$\sum_{i\in supp\gamma} \frac{\gamma_i}{\varepsilon} \left[\varepsilon g_i + (1-\varepsilon)(\omega_i - \sigma(\{i\})c(z))\right] + \sum_{i\in supp\gamma} \frac{\gamma_i}{\varepsilon} \sigma(\{i\})c(z) \leq \sum_{i\in supp\gamma} \frac{\gamma_i}{\varepsilon} \omega_i.$$

By monotonicity assumption, we have that $u_i(\varepsilon g_i + (1-\varepsilon)(\omega_i - \sigma(\{i\})c(z)), z) > u_i(x_i, y)$, where the vectors $\varepsilon g_i + (1-\varepsilon)(\omega_i - \sigma(\{i\})c(z))$ are strictly positive. Hence we can assume, without loss of generality, that $g_i \gg 0$ for each i and therefore, again by continuity, that

$$\sum_{i\in supp\gamma} \gamma_i g_i + \sum_{i\in supp\gamma} \gamma_i \sigma(\{i\})c(z) \ll \sum_{i\in supp\gamma} \gamma_i \omega_i.$$

This last inequality ensures that the Aubin coalition γ can be replaced by a rational valued coalition $\gamma^{'}$ in such a way that the inequality still holds.

Let r be an integer such that $l_i = r\gamma_i^{'}$ is integer, for every $i \in supp\gamma$. Define the coalition S in the r-fold replica $\mathcal{E}_r$ of $\mathcal{E}$ as the coalition containing agents (i,j) $j = 1, \ldots l_i$, and for $i \in supp\gamma$. Define $g_{i,j} = g_i$, for $j = 1, \ldots l_i$, for each $i \in supp\gamma$. It follows from the previous inequality that

$$\sum_{i\in supp\gamma} l_i g_{i,j} + \sum_{i\in supp\gamma} l_i \sigma(\{i,j\})c_r(z) \ll \sum_{i\in supp\gamma} l_i \omega_{i,j}$$

contradicting the fact that the allocation $(x_1, \ldots, x_n, y)$ belongs to the σ-core of the economy $\mathcal{E}_r$.
□

The previous result combined with the Aubin core equivalence stated in Theorem 4.2 implies that linear cost share equilibria are exactly Edgeworth equilibria of replica economies.

As a final result of this Section, we make clear the power of the Aubin veto for finite economies. We show that, under the Aubin blocking mechanism, it is enough to consider the Aubin coalitions with full support in order to characterize Aubin σ-core allocations (and, consequently, linear cost share equilibria)[7].

Theorem 4.5 *Assume that the utility functions satisfy the essentiality conditions and that are integrable. Then the Aubin σ-core of E coincides with the set of feasible allocations that cannot be σ-blocked by an Aubin coalition with full support.*

PROOF: The proof of one inclusion is obvious. To show the non trivial one, let us consider a feasible allocation $(x_1, \ldots, x_n, y)$ that cannot be σ-blocked by an Aubin coalition with full support and assume that it does not belong to the Aubin σ-core of $\mathcal{E}$. Notice in particular that the allocation $(x_1, \ldots, x_n, y)$ is Pareto optimal. Then it is easy to show that the allocation (f, y) defined by $f(t) = x_i$ on the subinterval I_i of $[0, 1]$ is Pareto optimal and it does not belong to the σ-core of the associated continuum economy $\mathcal{E}_{\mathcal{C}}$. So there exist an allocation (g, z) and a coalition S with $\mu(S) > 0$ such that

$$u_t(g(t), z) > u_t(f(t), y), \quad \text{for almost all } t \in S$$

and

$$\int_S g d\mu + \widehat{\sigma}(S)\widehat{c}(z) \leq \int_S \omega d\mu.$$

From Proposition 3.10 it follows that the allocation (f, y) can be blocked on a coalition T with $1 > \mu(T) > \frac{n-1}{n}$, so there exists an allocation (h, z) such that

$$u_t(h(t), z) > u_t(f(t), y), \quad \text{for almost all } t \in T$$

and

$$\int_T h d\mu + \widehat{\sigma}(T)\widehat{c}(z) \leq \int_T \omega d\mu$$

Note that $\beta_i = \mu(T \cap I_i) > 0$ for all i and define $h_i = \dfrac{1}{\beta_i} \displaystyle\int_{T \cap I_i} h d\mu$.

We have

$$u_i(h_i, z) > u_i(f_i, y), \quad \text{for all } i = 1, \ldots, n$$

and

$$\sum_{i=1}^{n} \beta_i \frac{1}{\beta_i} \int_{T \cap I_i} h d\mu + \sum_{i=1}^{n} \sigma(\{i\})\beta_i c(z) \leq \sum_{i=1}^{n} \int_{T \cap I_i} \omega_i d\mu$$

that is

$$\sum_{i=1}^{n} \beta_i h_i + \sum_{i=1}^{n} \sigma(\{i\})\beta_i c(z) \leq \sum_{i=1}^{n} \beta_i \omega_i$$

Since each $\beta_i \neq 0$, the allocation $(x_1, \ldots, x_n, y)$ is σ-Aubin blocked by the grand coalition. □

Remark 4.6 We notice that the proof of Theorem 4.5 works in the same way if in the second part we choose a coalition T such that $1 > \mu(T) > \frac{n-\delta}{n}$, with $\delta < 1$. In this case, the coefficient β_i can be replaced by $n\beta_i$ and it results that $n\beta_i > 1 - \delta$. If we choose δ arbitrarily close to zero, then $n\beta_i$ is close to one. Hence the participation of each agent in the grand coalition in the statement of Theorem 4.5 can be actually required to be close to the complete participation.

[7] An interesting application of this equivalence result provides a characterization of Linear cost share equilibria as strong Nash equilibria of a suitable associated two player game (see [17], [11]).

5 Non-dominated allocations in continuum economies

It is the aim of this Section to extend the veto mechanism introduced by [16] to the case of exchange economies with an abstract set of public projects. We shall require that an allocation cannot be blocked in infinitely many economies in which the set of agents does not change, but the initial endowments and the cost functions are suitably modified. In these economies, we shall focus on the veto power of only one coalition, the grand coalition. The interest of this approach in the presence of public projects relies on the fact that the contribution of the grand coalition to the realization of each project is equal to one under any contribution measure.

Given a coalition S, a feasible allocation (g, z) and a real number $\alpha \in [0, 1]$, let us denote by $\mathcal{E}(S, z, \alpha)$ the continuum economy which coincides with $\mathcal{E}$ except for the initial endowment allocation and for the cost function, defined respectively by

$$\omega(S, z, \alpha)(t) = \begin{cases} \omega(t) & \text{if } t \in I \setminus S \\ (1-\alpha)\omega(t) + \alpha g(t) & \text{if } t \in S \end{cases}$$

and

$$c(S, \alpha) = (1 - \alpha\mu(S))c.$$

The continuum perturbed economy $\mathcal{E}(S, z, \alpha)$ represents a path from the initial allocation ω to g. Clearly, $\mathcal{E}(S, z, \alpha)$ coincides with $\mathcal{E}$, when $\alpha = 0$. Note that if the size of the coalition S is either arbitrarily small or arbitrarily big, then the amount of private goods that can be consumed after paying for the realization of the project z, is the same in the economies $\mathcal{E}$ and $\mathcal{E}(S, z, \alpha)$. In fact we have

$$\int_I \omega(S, z, \alpha)d\mu - c(S, \alpha)(z) = \int_{I \setminus S} \omega d\mu + \int_S (1-\alpha)\omega d\mu + \int_S \alpha g d\mu - (1 - \alpha\mu(S))c(z) =$$

$$= \int_I \omega d\mu + \alpha \int_S (g - \omega)d\mu - (1 - \alpha\mu(S))c(z)$$

and then in both cases, by feasibility of (g, z), the difference $\int_I \omega(S, z, \alpha)d\mu - c(S, \alpha)(z)$ is very close to $\int_I \omega d\mu - c(z)$.

Let (f, y) be a Pareto optimal allocation of the economy $\mathcal{E}$. For each $z \in \mathcal{Y}$ define the correspondence

$$t \in I \to F(t, z) \subseteq I\!R^m_+$$

as follows

$$F(t, z) = \left\{ g \in I\!R^m_+ \;\; u_t(g, z) > u_t(f(t), y) \right\}.$$

Let us denote by $F(z)$ the integral of the correspondence $F(t, z) + c(z) - \omega$, i.e.

$$F(z) = \int_I F(t, z)d\mu + c(z) - \omega.$$

The following assumption requires a smoothness property of preferences under each project $z \in \mathcal{Y}$.
Aggregate smoothness condition: For each Pareto optimal allocation (f, y) and for each public project $z \in \mathcal{Y}$, there exists at most one price $p(z)$ supporting the set $F(z)$ at the point 0, i.e. at most one price $p(z)$ s.t.

$$p(z) \cdot F(z) \geq 0.$$

Clearly the uniqueness of $p(z)$ is referred to its direction.

Proposition 5.1 *Assume that the utility functions satisfy the essentiality conditions, that are integrable and verify the aggregate smoothness condition. Let (f, y) be a linear cost share equilibrium of $\mathcal{E}$. Then, for any public project $z \in \mathcal{Y}$, there exists a feasible allocation (γ_z, z) such that (f, y) is not z-dominated in the corresponding economy $\mathcal{E}(S, z, \alpha)$, for any $\alpha \in [0, 1]$ and for any coalition S.*

PROOF: First observe that (f, y) is Pareto optimal. Let (γ_z, z) be the feasible allocation defined by Lemma 3.9. Assume that there exist $\alpha \in [0, 1]$ and a coalition S such that (f, y) is z-blocked by the grand coalition in the corresponding economy $\mathcal{E}(S, z, \alpha)$. Then there exists a feasible allocation (g, z) such that

$$\int_I g d\mu + c(S, \alpha)(z) \leq \int_I \omega(S, z, \alpha) d\mu$$

$$u_t(g(t), z) > u_t(f(t), y) \text{ for almost all } t \in I.$$

If p and φ are, respectively, the price system and the cost distribution function associated to (f, y), for almost all $t \in I$, it results

$$p(z) \cdot g(t) + p(z) \cdot \varphi(t) c(z) > p(z) \cdot \omega(t).$$

Moreover, by smoothness assumption and the properties of the vectors γ_z defined by Lemma 3.9, it follows that

$$p(z) \cdot g(t) > p(z) \cdot \gamma_z(t).$$

So,

$$p(z) \cdot (1 - \alpha) g(t) + p(z) \cdot (1 - \alpha) \varphi(t) c(z) > p(z) \cdot (1 - \alpha) \omega(t)$$

and

$$p(z) \cdot \alpha g(t) > p(z) \cdot \alpha \gamma_z(t)$$

and adding

$$p(z) \cdot g(t) > p(z) \cdot (1 - \alpha) \omega(t) + p(z) \cdot \alpha \gamma_z(t) - p(z) \cdot (1 - \alpha) \varphi(t) c(z) \geq$$

$$\geq p(z) \cdot (1 - \alpha) \omega(t) + p(z) \cdot \alpha \; \gamma_z(t) - p(z) \cdot (1 - \alpha) c(z).$$

Then

$$\int_I p(z) \cdot g d\mu = \int_{I \setminus S} p(z) \cdot g d\mu + \int_S p(z) \cdot g d\mu >$$

$$\int_{I \setminus S} p(z) \cdot g d\mu + \int_S p(z) \cdot (1 - \alpha) \omega d\mu + \int_S p(z) \cdot \alpha \; \gamma_z(t) d\mu - p(z) \cdot (1 - \alpha) \mu(S) c(z) >$$

$$\int_{I \setminus S} p(z) \cdot \omega d\mu - \int_{I \setminus S} p(z) \cdot c(z) d\mu + \int_S p(z) \cdot (1 - \alpha) \omega d\mu + \int_S p(z) \cdot \alpha \; \gamma_z(t) d\mu - p(z) \cdot (1 - \alpha) \mu(S) c(z) =$$

$$= \int_I p(z) \cdot \omega d\mu + \int_S p(z) \cdot \alpha \left(\gamma_z(t) - \omega\right) d\mu - p(z) \cdot \left(\mu(I \setminus S) + \mu(S) - \alpha \mu(S)\right) c(z)$$

that, given the definition of $\omega(S, z, \alpha)$, contradicts the feasibility of (g, z). □

Proposition 5.2 *Assume that the utility functions satisfy the essentiality conditions and that are integrable. Let (f, y) be a Pareto optimal allocation of the economy $\mathcal{E}$. If (f, y) is not a linear cost share equilibrium, then there exist $\alpha \in [0, 1]$, a coalition S, a public project $z \in \mathcal{Y}$ and a feasible allocation (γ_z, z) such that (f, y) is z-dominated in the economy $\mathcal{E}(S, f, \alpha)$.*

PROOF: In light of Theorem 3.8, the allocation (f, y) does not belong to the σ-core of $\mathcal{E}$ for any contribution measure σ. In particular for $\sigma = \mu$. So it is possible to find a coalition S with $\mu(S) > 0$, a public good $z \in \mathcal{Y}$, and an integrable function $g : S \to I\!R^m_+$ such that

$$\int_S g d\mu + \mu(S) c(z) \leq \int_S \omega d\mu$$

and

$$u_t(g(t), z) > u_t(f(t), y), \quad \text{for almost all } t \in S.$$

Since utility functions are continuous and monotone and since $\omega - c(z) \gg 0$, we can assume that

$$\int_S g d\mu + \mu(S)c(z) - \int_S \omega d\mu = -\delta < 0.$$

For each public project $z \in \mathcal{Y}$, let us denote by (γ_z, z) the feasible allocation defined by Lemma 3.9. As in the proof of Proposition 3.10, for any $\alpha \in [0,1]$, there exists an integrable function h such that, for almost all $t \in S$,

$$\int_S h d\mu = \int_S (\alpha g + (1-\alpha)\gamma_z) d\mu$$

and

$$u_t(h(t), z) > u_t(f(t), y).$$

Let us choose α and define the function

$$\widetilde{g}(t) = \begin{cases} h(t) & \text{if } t \in S \\ \gamma_z(t) + \frac{\delta}{\mu(I \setminus S)} & \text{if } t \in I \setminus S \end{cases}$$

Then

$$\int_I (\widetilde{g} - \omega(I \setminus S, z, \alpha)) d\mu + c(I \setminus S, \alpha)(z) =$$

$$\int_S h d\mu + \int_{I \setminus S} \gamma_z d\mu + \alpha\delta - \int_S \omega d\mu - \int_{I \setminus S} (1-\alpha)\omega d\mu - \int_{I \setminus S} \alpha\gamma_z d\mu + (1 - \alpha\mu(I \setminus S))c(z) =$$

$$\int_S (\alpha g + (1-\alpha)\gamma_z) d\mu + \int_{I \setminus S} \gamma_z d\mu + \alpha\delta - \int_S \omega d\mu - \int_{I \setminus S} (1-\alpha)\omega d\mu - \int_{I \setminus S} \alpha\gamma_z d\mu + (1 - \alpha\mu(I \setminus S))c(z) =$$

$$(1-\alpha)\left(\int_I (\gamma_z - \omega) d\mu + c(z)\right) = 0.$$

So (f, z) is z-blocked by the grand coalition in the economy $\mathcal{E}(I \setminus S, f, \alpha)$. $\square$

As consequence of Propositions 5.1 and 5.2, we obtain the following characterization of linear cost share equilibria in terms of the veto power of the grand coalition considered in infinitely many economies.

Theorem 5.3 *Assume that the utility functions satisfy the essentiality conditions, that are integrable and verify the aggregate smoothness condition. Let (f, y) be a Pareto optimal allocation of the economy $\mathcal{E}$. Then (f, y) is a linear cost share equilibrium if and only if there exists a family of feasible allocations $\{(\gamma_z, z)\}_{z \in \mathcal{Y}}$ such that (f, y) is not z-dominated in the corresponding economies $\mathcal{E}(S, z, \alpha)$, for any $\alpha \in [0,1]$ and for any coalition S.*

Remark 5.4 Let (f, y) be an allocation that is z-dominated in the economy $\mathcal{E}(S, z, \alpha)$, with $\alpha < 1$, defined starting by a certain feasible allocation (g, z). Hence there exists (h, z) such that $u_t(h(t), z) > u_t(f(t), y)$ for almost all $t \in I$ and

$$\int_I h d\mu + (1 - \alpha\mu(S))c(z) \leq \int_I \omega d\mu - \alpha \int_S (\omega - g)\, d\mu.$$

Consider the measure ν defined as $\nu(A) = \left(\mu(A), \int_A (\omega - g)\, d\mu\right)$, for every $A \subset S$. Applying the Lyapunov's convexity theorem to the vector measure ν restricted to S, we obtain that there exists $S' \subset S$ with $\mu(S') = \alpha\mu(S)$ and

$$\int_{S'} (\omega - \gamma_z)\, d\mu = \alpha \int_S (\omega - \gamma_z)\, d\mu.$$

So

$$\int_I hd\mu + (1-\mu(S'))c(z) \leq \int_I \omega d\mu - \int_{S'} (\omega - g)\, d\mu = \int_{I\setminus S'} \omega d\mu + \int_{S'} gd\mu$$

that is (f, y) is also dominated in the economy $\mathcal{E}_{\mathcal{C}}(S', z, 1)$.

Remark 5.5 We notice that, as it is clear from the proof of Propositions 5.1 and 5.2, the feasible allocations $\{(\gamma_z, z)\}_{z\in\mathcal{Y}}$ in the statement of Theorem 5.3 are exactly the same determined by Lemma 3.9.

If the continuum economy $\mathcal{E}$ coincides with the economy $\mathcal{E}_C$ associated to a finite economy (see the construction in Section 4), and the allocation (f, y) is constant over agents of the same type (i.e. on each interval I_i), then we can assume without loss of generality that, for each project $z \in \mathcal{Y}$, $\gamma_z(t)$ is constant on I_i, for $i = 1 \ldots n$. Indeed, the properties stated in Lemma 3.9 remain valid replacing the allocation (γ_z, z) with a new allocation $(\widetilde{\gamma}_z, z)$ where $\widetilde{\gamma}_z(t) = \dfrac{1}{\mu(I_i)} \displaystyle\int_{I_i} \gamma d\mu$, for each $t \in I_i$.

6 Non-dominated allocations in finite economies

The aim of this Section is to apply results obtained in Section 5 to the case of economies with a finite number of agents. The analogous of Theorem 5.3 will be derived using the correspondence of the finite economy $\mathcal{E}$ with the continuum economy $\mathcal{E}_{\mathcal{C}}$ constructed in Section 4.

Given a feasible allocation $(g_1, \ldots, g_n, z)$ of the finite economy $\mathcal{E}$, a public project z and the vector of real numbers $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $0 \leq \alpha_i \leq 1$, let $\mathcal{E}(z, \alpha)$ be a finite economy which coincides with $\mathcal{E}$ except for the initial endowment allocation and for the cost function given respectively by

$$\omega_i(z, \alpha) = \alpha_i \omega_i + (1 - \alpha_i) g_i$$

and

$$c_\alpha = \sum_{i=1}^{n} \frac{\alpha_i}{n} c.$$

The main result of this Section is the characterization of linear cost share equilibria in terms of the veto power of the grand coalition in infinitely many economies.

Proposition 6.1 *Assume that the utility functions satisfy the essentiality conditions, that are integrable and verify the aggregate smoothness condition.*

Let $(x_1, \ldots, x_n, y)$ be a linear cost share equilibrium of $\mathcal{E}$. Then, for any public project $z \in \mathcal{Y}$, there exists a feasible allocation $(\gamma_{1,z}, \ldots, \gamma_{n,z}, z)$ such that for any $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $0 \leq \alpha_i \leq 1$, the allocation $(x_1, \ldots, x_n, y)$ is not z-dominated in the corresponding economy $\mathcal{E}(z, \alpha)$.

PROOF: Let φ be the cost distribution function associated to $(x_1, \ldots, x_n, y)$. It is easy to verify that the allocation (f, y) defined by $f(t) = x_i$ if $t \in I_i$, is a linear cost share equilibrium of the associated continuum economy $\mathcal{E}_{\mathcal{C}}$ with individual cost function defined by $\widehat{\varphi}(t) = n\varphi(i)$ for almost all $t \in I_i$. In particular (f, y) is Pareto optimal. For any $z \in \mathcal{Y}$, let (γ_z, z) be the feasible allocation defined by Theorem 5.3. In view of Remark 5.5, we can assume that γ_z is constant among agents of the same type. Denote by $(\gamma_{1,z}, \ldots, \gamma_{n,z}, z)$ the corresponding feasible allocation in $\mathcal{E}$.

If there exists $z \in \mathcal{Y}$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $0 \leq \alpha_i \leq 1$, such that $(x_1, \ldots, x_n, y)$ is z-blocked by the grand coalition in the economy $\mathcal{E}(z, \alpha)$, then

$$\sum_{i=1}^{n} g_i + c_\alpha(z) \leq \sum_{i=1}^{n} \alpha_i \omega_i + \sum_{i=1}^{n} (1 - \alpha_i) \gamma_{i,z} \tag{1}$$

and $u_i(g_i, z) > u_i(x_i, y)$, for all $i \in I$.

Let us denote by S the coalition $S = \cup_{i=1}^{n} A_i$, where A_i is a subset of I_i with $\mu(A_i) = \dfrac{\alpha_i}{n}$. Then

$\mu(S) = \sum_{i=1}^{n} \frac{\alpha_i}{n}$ and $\mu(I \setminus S) = 1 - \sum_{i=1}^{n} \frac{\alpha_i}{n}$. Let us define $\widetilde{g}(t) = g_i$ if $t \in I_i$, then $u_t(g(t), z) > u_t(f(t), y)$ for almost all $t \in I$ and, dividing (1) by n, we obtain

$$\int_I \widetilde{g} d\mu + (1 - \mu(I \setminus S))\widehat{c}(z) \leq \int_S \omega + \int_{I \setminus S} \gamma_z d\mu.$$

So (f, y) is z-blocked by the grand coalition in the economy $\mathcal{E}_{\mathcal{C}}(I \setminus S, z, \alpha)$ with $\alpha = 1$, hence a contradiction. □

Proposition 6.2 *Assume that the utility functions satisfy the essentiality conditions and that are integrable. Let $(x_1, \ldots, x_n, y)$ be a Pareto optimal allocation of the economy $\mathcal{E}$. If $(x_1, \ldots, x_n, y)$ is not a linear cost share equilibrium, then there exist $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $0 \leq \alpha_i \leq 1$, a public project $z \in \mathcal{Y}$ and a feasible allocation $(\gamma_{1,z}, \ldots, \gamma_{n,z}, z)$ such that $(x_1, \ldots, x_n, y)$ is z-blocked by the grand coalition in the economy $\mathcal{E}(z, \alpha)$.*

PROOF: It is easy to verify that the allocation (f, y) defined by $f(t) = x_i$ if $t \in I_i$, is a Pareto optimal allocation of the associated continuum economy $\mathcal{E}_{\mathcal{C}}$ and that it is not a linear cost share equilibrium. For any $z \in \mathcal{Y}$, let (γ_z, z) be the feasible allocation defined by Theorem 5.3. In view of Remark 5.5, we can assume that γ_z is constant over agents of the same type. Denote by $(\gamma_{1,z}, \ldots, \gamma_{n,z}, z)$ the corresponding feasible allocation in $\mathcal{E}$. It follows from Theorem 5.3 that there exists a public project $z \in \mathcal{Y}$, a coalition S and a number $\alpha \in [0, 1]$ such that (f, y) is z-dominated in $\mathcal{E}_{\mathcal{C}}(S, z, \alpha)$, that is there exists an integrable function g such that $u_t(g(t), z) > u_t(f(t), y)$ for almost all $t \in I$ and

$$\int_I g d\mu + (1 - \alpha\mu(S))\widehat{c}(z) \leq \int_{I \setminus S} \omega d\mu + \int_S [(1 - \alpha)\omega + \alpha\gamma_z] \, d\mu \tag{2}$$

Let us define $g_i = \frac{1}{\mu(I_i)} \int_{I_i} g d\mu$, then $u_i(g_i, z) > u_i(x_i, y)$ for any $i \in I$, and, from (2),

$$\sum_{i=1}^{n} \frac{1}{n} g_i + (1 - \alpha \sum_{i=1}^{n} \beta_i) \frac{c(z)}{n} \leq \sum_{i=1}^{n} \alpha_i \omega_i + \sum_{i=1}^{n} \beta_i (1 - \alpha)\omega_i + \sum_{i=1}^{n} \beta_i \alpha \gamma_{i,z}$$

with $\alpha_i = \mu\left((I \setminus S) \cap I_i\right)$, $\beta_i = \mu(S \cap I_i)$ and $\alpha_i + \beta_i = \frac{1}{n}$.

So
$$\sum_{i=1}^{n} g_i + (1 - \sum_{i=1}^{n} \beta_i \alpha) c(z) = \sum_{i=1}^{n} n\alpha_i \omega_i + \sum_{i=1}^{n} n\beta_i (1 - \alpha)\omega_i + \sum_{i=1}^{n} n\beta_i \alpha \gamma_{i,z} =$$

$$\sum_{i=1}^{n} n(\alpha_i + \beta_i - \alpha_i \beta_i)\omega_i + \sum_{i=1}^{n} n\beta_i \alpha \gamma_{i,z} =$$

$$\sum_{i=1}^{n} (1 - n\alpha\beta_i)\omega_i + \alpha\beta_i \gamma_{i,z}$$

and $(x_1, \ldots, x_n, y)$ is z-blocked by the grand coalition in the economy $\mathcal{E}(x, \widehat{\alpha})$ with $\widehat{\alpha}_i = n\alpha\beta_i$. □

From the previous results we derive the following characterization of linear cost share equilibria independently of any given cost distribution function.

Theorem 6.3 *Assume that the utility functions satisfy the essentiality conditions, that are integrable and verify the aggregate smoothness condition. Let $(x_1, \ldots, x_n, y)$ be a Pareto optimal allocation of the finite economy $\mathcal{E}$. Then $(x_1, \ldots, x_n, y)$ is a linear cost share equilibrium if and only if there exists a family of feasible allocations $\{(\gamma_{1,z}, \ldots, \gamma_{n,z}, z)\}_{z \in \mathcal{Y}}$ such that for each $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $0 \leq \alpha_i \leq 1$, $(x_1, \ldots, x_n, y)$ is not z-dominated in the economy $\mathcal{E}(z, \alpha)$.*

7 Appendix

7.1 Proof of Lemma 3.9.

For any public project $z \in \mathcal{Y}$, let us define the sets

$$F(t,z) = \left\{ g \in I\!R^m_+ \;\; u_t(g,z) > u_t(f(t),y) \right\}$$

and

$$\overline{F}(t,z) = \left\{ g \in I\!R^m_+ \;\; u_t(g,z) \geq u_t(f(t),y) \right\} .$$

The first essentiality condition and strict monotonicity ensure that $F(t,z)$ is non-empty. Moreover, since $u_t(\cdot,z)$ is continuous and quasi-concave, it results that it is open and convex and $clF(t,z) = \overline{F}(t,z)$. Define

$$F(z) = \int_I F(t,z) d\mu + c(z) - \int_I \omega d\mu$$

and

$$\overline{F}(z) = \int_I \overline{F}(t,z) d\mu + c(z) - \int_I \omega d\mu.$$

The assumption of integrable utilities ensures that these definitions are indeed proper. If $x \in \overline{F}(z)$, then

$$x = \int_I g(t) d\mu + c(z) - \int_I \omega d\mu$$

with $u_t(g(t),z) \geq u_t(f(t),y)$, for almost all $t \in I$. Let us choose $v \gg 0$ and define

$$x_n = \int_I g_n(t,z) d\mu + c(z) - \int_I \omega d\mu$$

with $g_n = g + \frac{1}{n} v$.

Since $x_n \in F(z)$ and $x_n \to x$, then $x \in clF(z)$ and $\overline{F}(z) \subseteq clF(z)$. Moreover, in light of the essentiality and monotonicity assumptions and by Pareto optimality of the allocation (f,y), we have that $0 \in \overline{F}(z) \backslash F(z)$, so for any $z \in \mathcal{Y}$ we can find an integrable function γ_z, such that

$$\int_I \gamma_z d\mu + c(z) - \int_I \omega d\mu = 0$$

and $u_t(\gamma_z(t),z) \geq u_t(f(t),y)$, for almost all $t \in I$. Let $p(z)$ be the price system separating $F(z)$ from $-I\!R^m_+$ (notice that $F(z)$ is convex by Lyapunov's Theorem, hence the Minkowski's separating hyperplane Theorem can be applied, see e.g. [18, page 38]). Then $p(z) \geq 0$ and $p(z) \cdot \overline{F}(z) \geq 0$, since $\overline{F}(z) \subseteq clF(z)$.

We claim that, for any integrable selection $h(t)$ of the correspondence $F(t,z)$, it results $p(z) \cdot \gamma_z(t) \leq p(z) \cdot h(t)$, for almost all $t \in I$. Assume on the contrary that for a selection $h(t)$ of $F(t,z)$ and for a coalition S of positive measure we would have $p(z) \cdot \gamma_z(t) > p(z) \cdot h(t)$, for almost all $t \in S$. Then we could define

$$\widehat{h}(t) = \begin{cases} h(t) & \text{if } t \in S \\ \gamma_z(t) & \text{if } t \in I \setminus S \end{cases}$$

having that $u_t(\widehat{h}(t),z) \geq u_t(f(t),y)$, $v = \int_I \widehat{h} d\mu + c(z) - \int_I \omega d\mu \in \overline{F}(z)$ and

$$p(z) \cdot v = p(z) \cdot \int_I \widehat{h} d\mu + p(z) \cdot c(z) - p(z) \cdot \int_I \omega d\mu =$$

$$p(z) \cdot \int_{I \backslash S} \gamma_z d\mu + p(z) \cdot \int_S h d\mu + p(z) \cdot c(z) - p(z) \cdot \int_I \omega d\mu = p(z) \cdot \int_S (h - \gamma_z)\, d\mu < 0$$

hence a contradiction.

From the assumption $c(z) \ll \omega$, it follows that $p(z) \cdot \int_I \gamma_z \, d\mu > 0$. Hence there exists a coalition A of positive measure such that $p(z) \cdot \gamma_z(t) > 0$, for μ-almost all $t \in A$. Let us assume that for a vector $x > 0$ we have $p(z) \cdot x = 0$. Define $g_z(t) = \gamma_z(t) + x$, for μ-almost all $t \in A$, and observe that $u_t(g_z(t), z) > u_t(f(t), y)$ and $p(z) \cdot g_z(t) > 0$. Let $A^{'} \subseteq A$ be a coalition of positive measure and $\varepsilon \in (0,1)$ be such that $u_t(\varepsilon g_z(t), z) > u_t(f(t), y)$, for μ-almost all $t \in A^{'}$. Then $p(z) \cdot \varepsilon g_z(t) < p(z) \cdot g_z(t) = p(z) \cdot \gamma_z(t)$ on $A^{'}$ and, consequently, we can easily construct an integrable selection of $\overline{F}(z)$ contradicting the previous claim. Hence $p(z) \cdot x > 0$ and, since x is arbitrary, we derive that $p(z) \gg 0$.

Now we want to prove that for almost all $t \in I$, if $u_t(g, z) > u_t(f(t), y)$, then $p(z) \cdot g(t) > p(z) \cdot \gamma_z(t)$. Assume not, then there would exist a coalition S with positive measure such that the correspondence

$$\Gamma : t \in S \rightarrow \{g : u_t(g, z) > u_t(f(t), y)\} \cap \{g : p(z) \cdot g \leq p(z) \cdot \gamma_z(t)\}$$

has nonempty values. By measurability assumption, there exists an integrable selection $g(t)$ of $\Gamma(t)$ defined over S. For $g(t)$ it is true that

$$u_t(g(t), z) > u_t(f(t), y)$$

and moreover, by previous claim,

$$p(z) \cdot g(t) = p(z) \cdot \gamma_z(t).$$

Let $C \subseteq S$ be a coalition of positive measure and $\varepsilon > 0$ be such that, for almost all $t \in C$

$$u_t(\varepsilon g(t), z) > u_t(f(t), y).$$

By strict positivity of the price $p(z)$ we easily get a contradiction. $\square$

7.2 Proof of Proposition 3.10.

Since (f, y) does not belong to the σ-core, by continuity and measurability assumption, there exists an allocation (g, z), such that $u_t(g(t), z) > u_t(f(t), y)$, for almost all $t \in S$, with $\mu(S) > 0$ and

$$\int_S g d\mu + \sigma(S) c(z) - \int_S \omega d\mu = -\delta < 0.$$

First we prove that (f, y) can be blocked by a coalition with arbitrarily big measure.
Denote by (γ_z, z) the feasible allocation defined according to Lemma 3.9.

We claim that there exists an assignment (h, z) such that $u_t(h(t), z) > u_t(f(t), y)$, for almost all $t \in S$, and

$$\int_S h d\mu = \int_S (\varepsilon g + (1 - \varepsilon)\gamma_z) d\mu.$$

To show our claim, consider the vector measure ν defined over measurable subsets A of S by

$$\left(\mu(A), \int_A (g - \gamma_z) d\mu\right).$$

By Lyapunov Theorem, there exists a measurable subset A of S such that $\nu(A) = \varepsilon \nu(S)$. Let $\bar{g}(t)$ be an assignment for A and $\gamma > 0$ be such that

$$\int_A \bar{g} d\mu = \int_A (g - \gamma) d\mu$$

and $u_t(\bar{g}(t), z) > u_t(f(t), y)$, for almost all $t \in A$. Define a new assignment for the allocation S by the following law

$$h(t) = \begin{cases} \bar{g}(t) & \text{if } t \in A \\ \gamma_z(t) + \frac{\gamma \mu(A)}{\mu(S \setminus A)} & \text{if } t \in S \setminus A \end{cases}$$

Then

$$\int_S h(t)d\mu = \int_A \bar{g}d\mu + \int_{S\setminus A} \gamma_z d\mu + \gamma = \int_A gd\mu + \int_S \gamma_z d\mu - \int_A \gamma_z d\mu =$$

$$\int_A (g-\gamma_z)d\mu + \int_S \gamma_z d\mu = \varepsilon \int_S (g-\gamma_z)d\mu + \int_S \gamma_z d\mu = \int_S (\varepsilon g + (1-\varepsilon)\gamma_z)d\mu$$

and by monotonicity assumption, $u_t(h(t),z) > u_t(f(t),y)$, for almost all $t \in S$, proving our claim. Let $\varepsilon \in (0,1)$ and denote by S^c the coalition $S^c = I \setminus S$. Consider the vector measure ν defined over measurable subsets A of S^c by

$$\left(\mu(A), \sigma(A), \int_A (\gamma_z - \omega)d\mu\right).$$

Again Lyapunov Theorem ensures the existence of a subset S' of S^c such that $\mu(S') = (1-\epsilon)\mu(S^c)$, $\sigma(S') = (1-\epsilon)\widehat{\sigma}(S^c)$ and $\int_{S'} (\gamma_z - \omega)\, d\mu = (1-\epsilon)\int_{S^c} (\gamma_z - \omega)\, d\mu$.

Let us define

$$\gamma(t) = \begin{cases} h(t) & \text{if } t \in S \\ \gamma_z(t) + \frac{\epsilon\delta}{\mu(S')} & \text{if } t \in S' \end{cases}$$

then, $u_t(\gamma(t),z) > u_t(f(t),y)$ for almost all $t \in S \cup S'$ and

$$\int_{S\cup S'} \gamma d\mu + \sigma(S\cup S')c(z) - \int_{S\cup S'} \omega d\mu =$$

$$= \int_S (\epsilon g + (1-\epsilon)\gamma_z)\, d\mu + \int_{S'} \gamma_z d\mu + \epsilon\delta + \sigma(S\cup S')c(z) - \int_{S\cup S'} \omega d\mu =$$

$$= \epsilon \int_S (g-\omega)\, d\mu + \epsilon \int_S \omega d\mu + (1-\epsilon)\int_S \gamma_z d\mu + \int_{S'} \gamma_z d\mu + \epsilon\delta + \sigma(S\cup S')c(z) - \int_{S\cup S'} \omega d\mu =$$

$$= \epsilon \int_S \omega d\mu + (1-\epsilon)\int_S \gamma_z d\mu + (1-\epsilon)\int_{S^c} (\gamma_z - \omega)\, d\mu + (1-\epsilon)\left(\sigma(S) + \sigma(S^c)\right) c(z) - \int_S \omega d\mu =$$

$$= (1-\epsilon)\left[\int_I (\gamma_z - \omega)\, d\mu + c(z)\right] \leq 0.$$

So, being $\mu(S\cup S') = 1 - \epsilon\mu(S^c)$, if $\epsilon \to 0$ the measure of the coalition blocking (f,y) can be chosen close to one.

Now we prove that (f,y) can be blocked by a coalition with arbitrarily small measure.

Let $\alpha \in (0,1)$ with $\alpha < \mu(S)$ and assume that $\alpha = \beta\mu(S)$. As before, we can chose a coalition $S' \subseteq S$ such that $\mu(S') = \alpha = \beta\mu(S)$, $\sigma(S') = \beta\sigma(S)$ and

$$\int_{S'} (g-\omega)d\mu = \beta \int_S (g-\omega)d\mu.$$

Hence we have the inequality

$$\int_{S'} gd\mu + \sigma(S')c(z) - \int_{S'} \omega d\mu = \beta\left[\int_S gd\mu + \sigma(S)c(z) - \int_S \omega d\mu\right] \leq 0$$

that completes the proof. □

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