SEQUENTIAL CONTESTS WITH SYNERGY AND BUDGET CONSTRAINTS

Reut Megidish and Aner Sela

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Monaster Center for
Economic Research
Ben-Gurion University of the Negev
P.O. Box 653
Beer Sheva, Israel

Fax: 972-8-6472941
Tel: 972-8-6472286

Sequential Contests with Synergy and Budget Constraints

Reut Megidish and Aner Sela*

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Abstract

We study a sequential Tullock contest with two stages and two identical prizes. The players compete for one prize in each stage and each player may win either one or two prizes. The players have either decreasing or increasing marginal values for the prizes, which are commonly known, and there is a constraint on the total effort that each player can exert in both stages. We analyze the players' allocations of efforts along both stages when the budget constraints (effort constraints) are either restrictive, non-restrictive or partially restrictive. In particular, we show that when the players are either symmetric or asymmetric and the budget constraints are restrictive, independent of the players' values for the prizes, each player allocates his effort equally along both stages of the contest.

Keywords: Sequential contests, Tullock contests, budget constrains.

JEL classification: D44, O31, O32

1 Introduction

In real life contests contestants usually face budget constraints, which implies that there will be constraints on the total effort that the contestants are able to exert. A budget constraint completely changes the contestants' equilibrium behavior compared to the same contests without budget constraints. This was shown, among others, by Che and Gale (1997, 1998) and Gavious, Moldovanu and Sela (2003) in a single-stage contest.[1] In sequential multi-stage contests, however, the effect of the budget constraints on the players' strategies is even more complex than in single-stage contests since the choice of efforts in the early stages of the contest influences the choice of efforts in the later stages.[2] In this regard, Amegashie, Cadsby and Song (2007) as

*Department of Economics, Ben-Gurion University, Israel. Email: anersela@bgu.ac.il

[1] Che and Gale (1998) and Gavious, Moldovanu and Sela (2003) deal with all-pay auctions with bid caps. The bid cap is a budget constraint that the contest designer imposes on the contestants.

[2] Several papers in the literature (see, for example, Leininger (1993), Morgan (2003), Konrad (2004) and Klumpp and Polborn (2006)) compare simultaneous (one-stage) and sequential (multi-stage) contests.

well as Matros (2006) showed that if players have budget constraints they exert more effort in the initial rounds than in the following ones, and Harbaugh and Klumpp (2005) showed in a two-stage contest that weak players exert more effort in the first stage whereas strong players save more effort for the second stage.

In this paper we analyze the model of a two-stage Tullock contest which is similar to the two-stage all-pay auction model studied by Sela (2009). In contrast to that paper, we consider a multi-stage contest with budget-constrained players and, furthermore, unlike most of the literature on multi-stage contests with budget-constrained players, we assume that a synergy exists between the players' values for the prizes in both stages of the contest. These two factors combined makes the analysis of our sequential contest complicated but also more interesting and realistic. In fact even without any budget constraints more complex strategies are involved since each player may win more than one prize and therefore players may face many options that depend on the identity of the winner in each stage, and each of these options may have a different effect on the chance of each player to win the other prizes in the later stages. In particular, in sequential multi-prize contests, each player has to decide in which stages he will compete to win and in which stages he will quit and reserve his effort for the other rounds. Moreover, the players' decisions become more complicated when we add a constraint on the total effort that each player can exert in both stages.

Formally, our model considers a sequential Tullock contest with two stages and two identical prizes. The players compete for one prize in each stage and each player may win either one or two prizes. We first assume that the players are symmetric and have the same marginal values (decreasing or increasing) for the prizes, which are commonly known, and we also assume that there is a constraint on the total effort that each player can exert in both stages. We show that when the budget constraint is nonrestrictive and the players' marginal values for the prizes are decreasing the total effort in the first stage of the contest is always lower than the total effort in the second stage. On the other hand, when the players' marginal values for the prizes are increasing and the budget constraint is nonrestrictive the total effort in the first stage of the contest is always higher than the total effort in the second stage. Then, we let the players be either symmetric or asymmetric and we show the main result of this paper, namely, if the budget constraint is restrictive, each player allocates his budget constraint equally along the contest's stages independent of the players' values for both the prizes and the budget constraints. In particular, the players' total effort in the first stage of the contest is always equal to the total effort in the second stage. We conclude that in sequential Tullock contests with synergy if the players have sufficiently low budget constraints, the players' values for the prizes in both stages do not have any effect on their allocation of efforts.

The paper most related to our work is that of Benoit and Krishna (2001) who analyzed sequential first and second price auctions with synergy between the stages and a budget constraint.[3] They found that in

[3] Several papers deal with sequential auctions. These include, Pitchik and Schoter (1998) who analyzed sequential first and second price auctions with a budget constraint and two different prizes; Pitchik (2009) who analyzed a sequential auction with a budget constraint under incomplete information, and Brusco and Lopomo (2008, 2009) who considered sequential auctions

a sequential auction with a budget constraint it is optimal to sell the more valuable object first. They also showed that if the discrepancy in the values is large, the sequential auction yields more revenue than the simultaneous auction, but if it is small the simultaneous auction is superior. Furthermore, in Benoit and Krishna's model it might be advantageous for a bidder to bid aggressively for one object even when he does not plan to win since by increasing the price he depletes his opponent's budget such that the other objects may then be obtained at a lower price. In our sequential contest, the players incur costs as a result of their efforts in any case, and therefore a player does not have an incentive to increase his effort in a stage at which he does not want to win since then he depletes his budget and his options in the following stages. Other papers that are related to our paper in which the focus is on the dependence between the effort decisions along the different stages in the contest as a result of the budget constraint include Robson (2005) and Klumpp and Polborn (2006). These authors consider the Colonel Blotto game, where in each battlefield a Tullock contest takes place. In these models, the dependence between the stages is caused only by the budget constraint, while in our model the dependence is caused by the budget constraint and also by the synergy between the players' values for the prizes in each stage.

The rest of the paper is organized as follows: Section 2 presents our sequential two-stage Tullock contest with budget-constrained players. Section 3 analyzes this contest when the players are symmetric, and Section 4 presents several examples of the contest with different values of winning. Section 5 analyzes the contest with asymmetric players, Section 6 concludes.

2 The model

We consider a sequential Tullock contest with two symmetric players $i, j \in \{1, 2\}$ and two stages. In each of the stages a single (identical) prize is awarded. The values for the prizes are given by the vector $(v_1, v_2) \in \{v^a, v^b\}, v^a \geq v^b > 0$, where v_1 denotes the players' marginal value for winning the first prize and v_2 denotes the players' marginal value for winning the second prize. That is, if a player wins only one prize his value is v_1 and if he wins two prizes his value is $v_1 + v_2$. We assume that the players' marginal values are either decreasing $(v_1, v_2) = (v^a, v^b)$ or increasing $(v_1, v_2) = (v^b, v^a)$ and that they are common knowledge.

The players have a budget constraint denoted by w such that in both stages a player cannot exert a total effort which is higher than w. We assume that a money unit is identical to an effort unit. The players simultaneously exert efforts x_i, x_j in the first stage, then the players' probabilities of winning are $\frac{x_i}{x_i+x_j}$ and $\frac{x_j}{x_i+x_j}$ respectively, and all the players bear the costs of their efforts. The players know the identity of the winner in the first stage before the beginning of the second stage, which means that the players' values in the second stage are common knowledge. Like in the first stage, the players simultaneously exert efforts

with a budget constraint and with and without a synergy between the values of the prizes.

$\widetilde{x}_i, \widetilde{x}_j$, then the players' probabilities of winning are $\frac{\widetilde{x}_i}{\widetilde{x}_i+\widetilde{x}_j}$ and $\frac{\widetilde{x}_j}{\widetilde{x}_i+\widetilde{x}_j}$ respectively, and all the players bear the costs of their efforts.

3 Symmetric players

Consider a sequential Tullock contest with two symmetric players $i, j \in \{1, 2\}$. We denote by x_i^k, $k = a, b$ player i's effort in the first stage of the contest in which he competes to win a prize that is equal to v^k. We also denote by $\widetilde{x}_i^k$, $k = a, b$ player i's effort in the second stage of the contest when he competes to win a prize that is equal to v^k. We consider below two different scenarios: the first is when players have decreasing marginal values and the second one is when the players have increasing marginal values.

1) If the players' marginal values for the prizes $(v_1, v_2) = (v^a, v^b)$ are decreasing, $v^a \geq v^b > 0$, then, in the second stage we have a standard Tullock contest where one of the players has the value v^a and the other has the value v^b. Thus, if player i wins in the first stage his maximization problem in the second stage is

$$\max_{\widetilde{x}_i^b} v^b \frac{\widetilde{x}_i^b}{\widetilde{x}_i^b + \widetilde{x}_j^a} - \widetilde{x}_i^b \tag{1}$$

and if player i loses in the first stage his maximization problem in the second stage is

$$\max_{\widetilde{x}_i^a} v^a \frac{\widetilde{x}_i^a}{\widetilde{x}_i^a + \widetilde{x}_j^b} - \widetilde{x}_i^a \tag{2}$$

Thus, if the budget constraint w is nonrestrictive the players' effort in the second stage will be identical to the equilibrium efforts in the one-stage Tullock contest with values of $(v_i, v_j) \in \{v^a, v^b\}$, otherwise if the budget constraint w is restrictive, player i's effort in the second stage will be equal to $w - x_i^a$ where x_i^a is player i's effort in the first stage.

Given the players' strategies in the second stage, player i's maximization problem in the first stage is

$$\max_{x_i^a}(v^a + v^b \frac{\widetilde{x}_i^b}{\widetilde{x}_i^b + \widetilde{x}_j^a} - \widetilde{x}_i^b) \frac{x_i^a}{x_i^a + x_j^a} + (v^a \frac{\widetilde{x}_i^a}{\widetilde{x}_i^a + \widetilde{x}_j^b} - \widetilde{x}_i^a) \frac{x_j^a}{x_i^a + x_j^a} - x_i^a \tag{3}$$

$$s.t.$$

$$\begin{aligned} x_i^a + \widetilde{x}_i^b &\leq w \\ x_i^a + \widetilde{x}_i^a &\leq w \end{aligned}$$

where $\frac{x_i^a}{x_i^a+x_j^a}$ is player i's winning probability in the first stage of the contest; $\frac{\widetilde{x}_i^b}{\widetilde{x}_i^b+\widetilde{x}_j^a}$ is player i's winning probability in the second stage of the contest if he wins in the first stage; and $\frac{\widetilde{x}_i^a}{\widetilde{x}_i^a+\widetilde{x}_j^b}$ is player i's winning probability in the second stage of the contest if he loses in the first stage. Note that the value of winning the first stage is $v^a + v^b \frac{\widetilde{x}_i^b}{\widetilde{x}_i^b+\widetilde{x}_j^a} - \widetilde{x}_i^b$, while the value of losing the first stage is $v^a \frac{\widetilde{x}_i^a}{\widetilde{x}_i^a+\widetilde{x}_j^b} - \widetilde{x}_i^a$. By our assumption of symmetry, player j's maximization problems are identical to those of player i.

2) If the players' marginal values for the prizes $(v_1, v_2) = (v^b, v^a)$ are increasing, $v^a \geq v^b > 0$, then, if player i wins in the first stage his maximization problem in the second stage is

$$\max_{\widetilde{x}_i^a} v^a \frac{\widetilde{x}_i^a}{\widetilde{x}_i^a + \widetilde{x}_j^b} - \widetilde{x}_i^a \tag{4}$$

and if player i loses in the first stage his maximization problem in the second stage is

$$\max_{\widetilde{x}_i^b} v^b \frac{\widetilde{x}_i^b}{\widetilde{x}_i^b + \widetilde{x}_j^a} - \widetilde{x}_i^b \tag{5}$$

Given the players' strategies in the second stage, player i's maximization problem in the first stage is:

$$\max_{x_i^b} (v^b + v^a \frac{\widetilde{x}_i^a}{\widetilde{x}_i^a + \widetilde{x}_j^b} - \widetilde{x}_i^a) \frac{x_i^b}{x_i^b + x_j^b} + (v^b \frac{\widetilde{x}_i^b}{\widetilde{x}_i^b + \widetilde{x}_j^a} - \widetilde{x}_i^b) \frac{x_j^b}{x_i^b + x_j^b} - x_i^b \tag{6}$$

$$s.t.$$

$$\begin{aligned} x_i^b + \widetilde{x}_i^a &\leq w \\ x_i^b + \widetilde{x}_i^b &\leq w \end{aligned}$$

where $\frac{x_i^b}{x_i^b + x_j^b}$ is player i's winning probability in the first stage of the contest; $\frac{\widetilde{x}_i^a}{\widetilde{x}_i^a + \widetilde{x}_j^b}$ is player i's winning probability in the second stage of the contest if he wins in the first stage; and $\frac{\widetilde{x}_i^b}{\widetilde{x}_i^b + \widetilde{x}_j^a}$ is player i's winning probability in the second stage of the contest if he loses in the first stage. The value of winning the first stage is $v^b + v^a \frac{\widetilde{x}_i^a}{\widetilde{x}_i^a + \widetilde{x}_j^b} - \widetilde{x}_i^a$, while the value of losing the first stage is $v^b \frac{\widetilde{x}_i^b}{\widetilde{x}_i^b + \widetilde{x}_j^a} - \widetilde{x}_i^b$. By our assumption of symmetry, player j's maximization problems are identical to those of player i.

For each scenario, either decreasing marginal values or increasing marginal values, we divide our analysis of the players' allocation of effort along the contest's stages into three cases:

1. Case A: the budget constraint is nonrestrictive (both of the restrictions in the above maximization problems ((3) and (6)) are nonrestrictive).
2. Case B: the budget constraint is restrictive (both of the restrictions in the above maximization problems ((3) and (6)) are restrictive).
3. Case C: the budget constraint is partially restrictive (only one of the restrictions in the above maximization problems ((3) and (6)) is restrictive).

3.1 Case A: Nonrestrictive budget constraints

We assume first that the players have decreasing marginal values $v = (v^a, v^b)$,$v^a = 1 > v^b$, and also that both of the restrictions in the maximization problem (3) are nonrestrictive.[4] Then, the following proposition

[4] We assume tat the players' values are in the interval $[0, 1]$. This is only a normalization. Considering higher values wold not qualitatively affect the results..

defines the range of the budget constraint's values for which the budget constraint is nonrestrictive and characterizes the players' effort allocation in both stages of the contest.

Proposition 1 *In the sequential Tullock contest with symmetric players and decreasing marginal values* $(v^a = 1, v^b)$*, the budget constraint is nonrestrictive iff*

$$w > \frac{(v^b)^3 + (v^b)^2 + 6v^b}{4(v^b+1)^2}$$

Then, the subgame perfect equilibrium is given by the following strategies:[5]

The equilibrium effort in the first stage is

$$x^a = \frac{(v^b)^3 + (v^b)^2 + 2v^b}{4(1+v^b)^2}$$

The equilibrium efforts in the second stage are

$$\begin{aligned} \widetilde{x}^a &= \frac{v^b}{(1+v^b)^2} \\ \widetilde{x}^b &= \frac{(v^b)^2}{(1+v^b)^2} \end{aligned}$$

The above equilibrium strategies satisfy:

1) $x^a > \widetilde{x}^b$*; that is, if a player wins in the first stage his effort in that stage is always higher than his effort in the second stage.*

2) $x^a < \widetilde{x}^a$*; that is, if a player loses in the first stage his effort in that stage is always lower than his effort in the second stage.*

Proof. See Appendix. ■

In order to explain the players' effort allocations over both stages of the contest we examine their 'real values'. In the first stage, a player's induced value ('real value') is the difference between his expected payoff in the entire contest if he wins in the first stage and his expected payoff if he loses in the first stage. Thus, a player's induced value in the first stage is

$$v^a + \frac{(v^b)^3}{(v^a+v^b)^2} - \frac{(v^a)^3}{(v^a+v^b)^2}$$

where $v^a + \frac{(v^b)^3}{(v^a+v^b)^2}$ is a player's expected payoff in the entire contest if he wins in the first stage, and $\frac{(v^a)^3}{(v^a+v^b)^2}$ if he loses in the first stage.

The sum of the induced values in the first stage is

$$2v^a + \frac{2(v^b)^3 - 2(v^a)^3}{(v^a+v^b)^2}$$

[5] The uniqueness of the subgame perfect equilibrium is obtained by the uniqueness of the equilibrium in the one-stage Tullock contest with two players.

while the sum of the values in the second stage is

$$v^a + v^b$$

By comparing the sum of the induced values in the first stage and the sum of the values in the second stage we obtain that the sum of the induced values in the first stage is lower than the sum of the values in the second stage. On the other hand, the variance of the players' induced values in the first stage is lower than the variance of the players' values in the second stage. Thus, it is not clear whether the total effort in the second stage would be higher or lower than the total effort in the first stage. Nonetheless, we can show that

Proposition 2 *In the subgame perfect equilibrium of the sequential Tullock contest with symmetric players and decreasing marginal values* $(v^a = 1, v^b)$*, if the budget constraint is nonrestrictive, the total effort in the first stage of the contest is always lower than the total effort in the second stage.*

Proof. See Appendix. ■

We assume now that players have increasing marginal values $v = (v^b, v^a), v^b < v^a = 1$, and also that both of the restrictions in the maximization problem (6) are nonrestrictive. The following proposition defines the range of the budget constraint's values for which the budget constraint is nonrestrictive and characterizes the players' effort allocation in both stages of the contest.

Proposition 3 *In the sequential Tullock contest with symmetric players and increasing marginal values* $(v^b, v^a = 1)$*, the budget constraint is nonrestrictive iff*

$$w > \frac{2(v^b)^2 + 5v^b + 1}{4(v^b+1)^2}$$

Then, the subgame perfect equilibrium is given by the following strategies:[6]

The equilibrium effort in the first stage is

$$x^b = \frac{2(v^b)^2 + v^b + 1}{4(1+v^b)^2}$$

The equilibrium efforts in the second stage are

$$\begin{aligned} \widetilde{x}^a &= \frac{v^b}{(1+v^b)^2} \\ \widetilde{x}^b &= \frac{(v^b)^2}{(1+v^b)^2} \end{aligned}$$

The above equilibrium strategies satisfy:

[6] The uniqueness of the subgame perfect equilibrium is obtained by the uniqueness of the equilibrium in the one-stage Tullock contest with two players.

1) $x^b \geq \widetilde{x}^a > \widetilde{x}^b$ *if* $0 < v^b \leq 0.5$*; that is, the player's effort in the first stage is larger than his effort in the second stage given that he wins in the first stage. In addition, the effort in the second stage given that he wins in the first stage is larger than his effort in that stage given that he loses in the first one.*

2) $\widetilde{x}^a \geq x^b > \widetilde{x}^b$ *if* $0.5 \leq v^b < 1$*; that is, the player's effort in the first stage is smaller than his effort in the second stage given that he wins in the first stage, but it is larger than his effort in the second stage given that he loses in the first one.*

Proof. See Appendix. ■

In this scenario, the sum of the players' induced values in the first stage is

$$2v^b + \frac{2(v^a)^3 - 2(v^b)^3}{(v^a + v^b)^2}$$

while the sum of the values in the second stage is

$$v^a + v^b$$

By comparing the sum of the induced values in the first stage with the sum of the values in the second stage we obtain that the sum of the induced values in the first stage is higher than the sum of the values in the second stage. Furthermore, since the variance of the players' induced values in the first stage is lower than the variance of the players' values in the second stage, we can conclude that the total effort in the first stage is higher than in the second stage.

Proposition 4 *In the subgame perfect equilibrium of the sequential Tullock contest with symmetric players and increasing marginal values* $(v^b, v^a = 1)$*, if the budget constraint is nonrestrictive, the total effort in the first stage of the contest is always higher than the total effort in the second stage.*

Proof. See Appendix. ■

3.2 Case B: Restrictive budget constraints

We assume now that the players have either increasing or decreasing marginal values and that both of the restrictions in the maximization problems (3) and (6) are restrictive. The following proposition defines the range of the budget constraint's values for which the budget constraint is restrictive and characterizes the players' effort allocation in both stages of the contest.

Proposition 5 *In the sequential Tullock contest with symmetric players and either decreasing* $(v^a = 1, v^b)$ *or increasing* $(v^b, v^a = 1)$ *marginal values, the budget constraint is restrictive iff*

$$w < \frac{v^b}{2}$$

in the subgame perfect equilibrium each player allocates his budget constraint equally along the contest's stages; that is,

$$x^a = \widetilde{x}^a = \widetilde{x}^b = x^b$$

In particular, the total effort in the first stage of the contest is always equal to the total effort in the second stage.

Proof. See Appendix. ■

According to Proposition 5, independently of the players' values in both stages, they allocate their effort equally along both of the stages. This essentially means that when the budget constraint is relatively low such that the players are restricted in both stages of the contest, the players' values do not affect their effort allocation along the contest. Later we will show that this result does not depend on the assumption of symmetry between the players.

3.3 Case C: Partially restrictive budget constraints

Here we assume that the players have decreasing marginal values and that only one of the restrictions in the maximization problem (3) is restrictive and the other is not. The former assumption implies that it is not possible that the budget constraint would be restrictive if a player wins in the first stage of the contest but would not be restrictive if he loses in the first stage of the contest. In other words, if $x_i^a + \widetilde{x}_i^b \leq w$ is restrictive then $x_i^a + \widetilde{x}_i^a \leq w$ is restrictive as well. Therefore, we consider here the situation where only the second restriction in the maximization problem (3) is restrictive. The following proposition defines the range of the budget constraint's values for which the budget constraint is partially restrictive and presents the implicit equation that characterizes the players' allocation of effort.

Proposition 6 *In the sequential Tullock contest with symmetric players and decreasing marginal values ($v^a = 1, v^b$), the budget constraint is partially restrictive iff*

$$\frac{v^b}{2} < w < \frac{(v^b)^3 + (v^b)^2 + 6v^b}{4(v^b + 1)^2}$$

Then in the subgame perfect equilibrium each player's effort in the first stage (x^a) is determined by the following equation

$$\begin{aligned} & [1 + v^b + 2w - 2x^a - \sqrt{w - x^a}(2\sqrt{v^b} + \frac{1}{\sqrt{v^b}})]v^b(w - x^a) \\ = \quad & [\sqrt{v^b(w - x^a)} - (w - x^a)(v^b + 1)]2x^a + v^b(w - x^a)4x^a \end{aligned} \tag{7}$$

The efforts in the second stage are given by

$$\begin{aligned}\widetilde{x}^a &= w - x^a \\ \widetilde{x}^b &= \sqrt{v^b(w - x^a)} - w + x^a\end{aligned}$$

Proof. See Appendix. ■

Next we assume that the players have increasing marginal values and that only one of the restrictions in the maximization problem (6) is restrictive and the other is not. The former assumption implies that it is not possible that the budget constraint would be restrictive if a player loses in the first stage of the contest and would not be restrictive if he wins in the first stage of the contest. Therefore, we consider here the situation where only the first restriction in the maximization problem (6) is restrictive. The following proposition defines the range of the budget constraint's values for which the budget constraint is partially restrictive and presents the implicit equation that characterizes the player's allocation of effort.

Proposition 7 *In the sequential Tullock contest with symmetric players and increasing marginal values* $(v^b, v^a = 1)$*, the budget constraint is partially restrictive iff*

$$\frac{v^b}{2} < w < \frac{2(v^b)^2 + 5v^b + 1}{4(v^b + 1)^2}$$

Then in the subgame perfect equilibrium each player's effort in the first stage (x^b) *is determined by the following equation*

$$\frac{\sqrt{w - x^b}}{\sqrt{v^b}} + 2\sqrt{v^b(w - x^b)} - 2w - 2x^b = \frac{[\sqrt{v^b} - (v^b + 1)\sqrt{w - x^b}]2x^b}{v^b\sqrt{w - x^b}} \tag{8}$$

Thus the efforts in the second stage are given by

$$\begin{aligned}\widetilde{x}^a &= w - x^b \\ \widetilde{x}^b &= \sqrt{v^b(w - x^b)} - w + x^b\end{aligned}$$

Proof. See Appendix. ■

In the following section we present some examples which describe the players' allocations of effort for all the ranges of the budget constraint.

4 Examples

In the following we consider two different situations:

- We assume that the players are symmetric and have decreasing marginal values. Figure 1 presents a player's effort in each stage of the contest as a function of the budget constraint w where $v^a = 1, v^b = 0.5$.

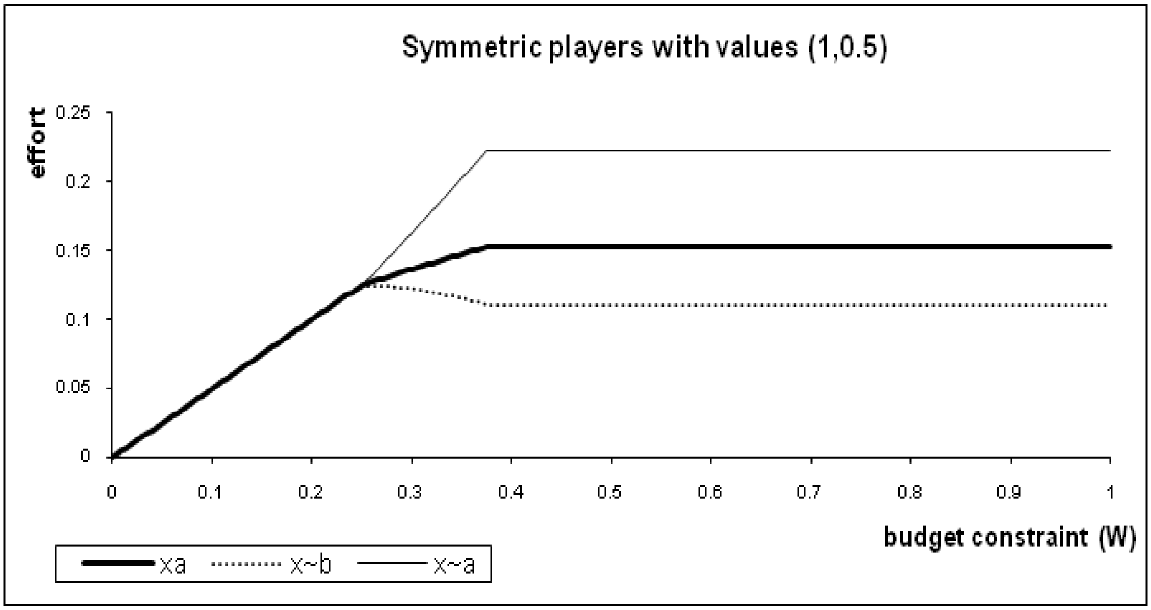

Figure 1: Players with decreasing marginal values $(v^a, v^b) = (1, 0.5)$.

Here if $w \leq 0.25$ the budget constraint is restrictive for both players. If $0.25 < w \leq 0.375$ the budget constraint is restrictive only for the player who loses in the first stage; and if $0.375 < w$ the budget constraint is not restrictive for both players. We can see from Figure 1 that for every budget constraint w, $\widetilde{x}^a \geq x^a \geq \widetilde{x}^b$. Furthermore, the total effort in the second stage of the contest, $\widetilde{x}^a + \widetilde{x}^b$ is higher than or equal to the total effort in its first stage, $2x^a$, namely, $TE_1 \leq TE_2$.

- We assume that the players are symmetric and have increasing marginal values. Figure 2 presents a player's effort in each stage of the contest as a function of the budget constraint w when $v^b = 0.2, v^a = 1$.

We can observe that when the budget constraint is partially restrictive the player's effort in the second stage if he wins in the first stage ($\widetilde{x}^a$) increases in the value of the budget constraint. The player's effort in the first stage (x^b) increases more strongly in the value of the budget constraint. However, the player's effort in the second stage if he loses in the first stage ($\widetilde{x}^b$) decreases in the value of the budget constraint. This phenomenon holds for all $0 < v^b \leq 0.5$.

5 Asymmetric players

We consider now the general case with asymmetric players who have different values and different budget constraints. Player 1 has a budget constraint w_1 and his marginal values for the prizes are (v^a, v^b) while

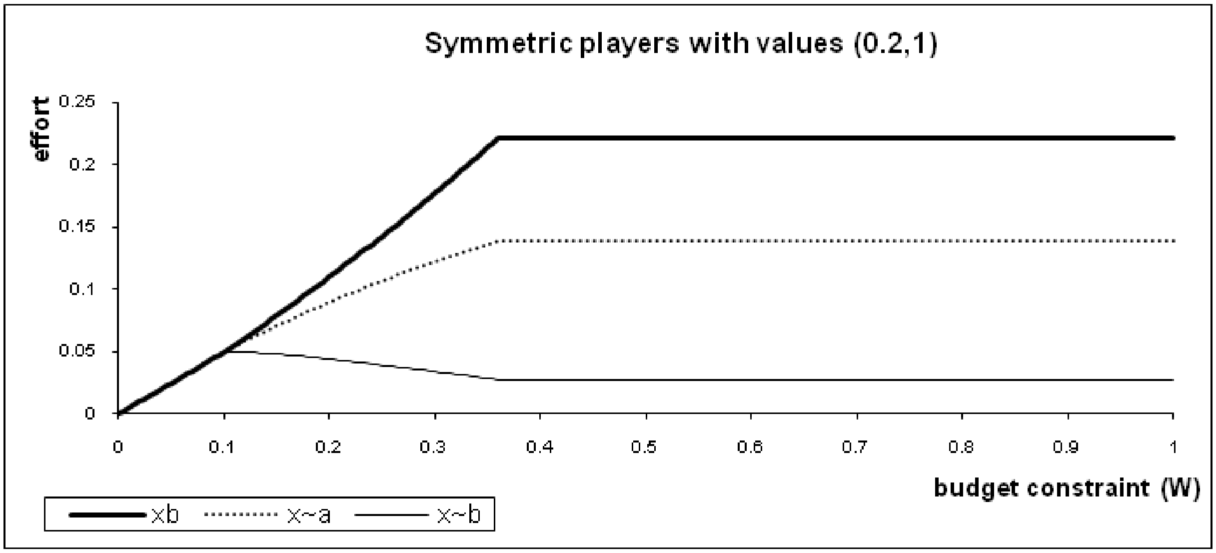

Figure 2: Players with increasing marginal values $(v^b, v^a) = (0.2, 1)$.

player 2 has a budget constraint w_2 and his marginal values for the prizes are (v^c, v^d). Moreover, we also consider the generalized Tullock model in which if players exert efforts of x_i,x_j their probabilities of winning are $\frac{(x_i)^r}{(x_i)^r+(x_j)^r}$, $\frac{(x_j)^r}{(x_i)^r+(x_j)^r}$ respectively, where r is a constant that satisfies $0 < r < 2$ (so far we assumed that $r = 1$). We focus here on the situation where both of the players have a restrictive budget constraint, which, it turns out, has somewhat unexpected results.

Denote by x^a player 1's effort in the first stage of the contest; by $\widetilde{x}^b$ player 1's effort in the second stage of the contest if he wins in the first stage; and by $\widetilde{x}^a$ player 1's effort in the second stage of the contest if he loses in the first stage. Similarly, denote by y^c player 2's effort in the first stage of the contest; by $\widetilde{y}^d$ player 2's effort in the second stage of the contest if he wins in the first stage; and by $\widetilde{y}^c$ player 2's effort in the second stage of the contest if he loses in the first stage. Thus, in the second stage we have a standard Tullock contest where players 1 and 2's efforts are either $(\widetilde{x}^b, \widetilde{y}^c)$ or $(\widetilde{x}^a, \widetilde{y}^d)$. Therefore, if player 1 wins in the first stage, the players' maximization problems in the second stage are

$$\max_{\widetilde{x}^b} v^b \frac{(\widetilde{x}^b)^r}{(\widetilde{x}^b)^r + (\widetilde{y}^c)^r} - \widetilde{x}^b \tag{9}$$

and

$$\max_{\widetilde{y}^c} v^c \frac{(\widetilde{y}^c)^r}{(\widetilde{x}^b)^r + (\widetilde{y}^c)^r} - \widetilde{y}^c \tag{10}$$

If, on the other hand, player 2 wins in the first stage, the players' maximization problems are

$$\max_{\widetilde{x}^a} v^a \frac{(\widetilde{x}^a)^r}{(\widetilde{x}^a)^r + (\widetilde{y}^d)^r} - \widetilde{x}^a \tag{11}$$

and

$$\max_{\widetilde{y}^d} v^d \frac{(\widetilde{y}^d)^r}{(\widetilde{x}^a)^r + (\widetilde{y}^d)^r} - \widetilde{y}^d \tag{12}$$

Given the players' strategies in the second stage, player 1's maximization problem in the first stage is

$$\begin{aligned}
&\max_{x^a} (v^a + v^b \frac{(\widetilde{x}^b)^r}{(\widetilde{x}^b)^r + (\widetilde{y}^c)^r} - \widetilde{x}^b) \frac{(x^a)^r}{(x^a)^r + (y^c)^r} \\
&+ (v^a \frac{(\widetilde{x}^a)^r}{(\widetilde{x}^a)^r + (\widetilde{y}^d)^r} - \widetilde{x}^a) \frac{(y^c)^r}{(x^a)^r + (y^c)^r} - x^a \\
&s.t.
\end{aligned} \tag{13}$$

$$\begin{aligned}
x^a + \widetilde{x}^b &\leq w_1 \\
x^a + \widetilde{x}^a &\leq w_1
\end{aligned}$$

Likewise, player 2's maximization problem in the first stage is

$$\begin{aligned}
&\max_{y^c} (v^c + v^d \frac{(\widetilde{y}^d)^r}{(\widetilde{x}^a)^r + (\widetilde{y}^d)^r} - \widetilde{y}^d) \frac{(y^c)^r}{(x^a)^r + (y^c)^r} \\
&+ (v^c \frac{(\widetilde{y}^c)^r}{(\widetilde{x}^b)^r + (\widetilde{y}^c)^r} - \widetilde{y}^c) \frac{(x^a)^r}{(x^a)^r + (y^c)^r} - y^c \\
&s.t.
\end{aligned} \tag{14}$$

$$\begin{aligned}
y^c + \widetilde{y}^d &\leq w_2 \\
y^c + \widetilde{y}^c &\leq w_2
\end{aligned}$$

Since we assume that both players have a restrictive budget constraint we have

$$\begin{aligned}
\widetilde{x}^b &= \widetilde{x}^a = w_1 - x^a \\
\widetilde{y}^d &= \widetilde{y}^c = w_2 - y^c
\end{aligned}$$

Thus, player 1's maximization problem in the first stage (equation (13)) is then

$$\begin{aligned}
&\max_{x^a} (v^a + v^b \frac{(w_1 - x^a)^r}{(w_1 - x^a)^r + (w_2 - y^c)^r} - (w_1 - x^a)) \frac{(x^a)^r}{(x^a)^r + (y^c)^r} + \\
&(v^a \frac{(w_1 - x^a)^r}{(w_1 - x^a)^r + (w_2 - y^c)^r} - (w_1 - x^a)) \frac{(y^c)^r}{(x^a)^r + (y^c)^r} - x^a
\end{aligned} \tag{15}$$

and player 2's maximization problem in the first stage (equation (14)) is then

$$\begin{aligned}
&\max_{y^c} (v^c + v^d \frac{(w_2 - y^c)^r}{(w_1 - x^a)^r + (w_2 - y^c)^r} - (w_2 - y^c)) \frac{(y^c)^r}{(x^a)^r + (y^c)^r} + \\
&(v^c \frac{(w_2 - y^c)^r}{(w_1 - x^a)^r + (w_2 - y^c)^r} - (w_2 - y^c)) \frac{(x^a)^r}{(x^a)^r + (y^c)^r} - y^c
\end{aligned} \tag{16}$$

The following theorem characterizes the players' allocation of effort.

Theorem 1 *In the subgame perfect equilibrium of the sequential generalized Tullock contest with asymmetric players, independent of the players' values for the prizes in each stage, if the budget constraint is restrictive then each player allocates his budget constraint equally along both stages of the contest. In particular, the total effort in the first stage of the contest is always equal to the total effort in the second stage.*

Proof. The proof is in the Appendix. In the following we provide another mathematical explanation for the above result.

If the budget constraints are restrictive for both players, in both stages, then all of the four restrictions in the maximization problems (13) and (14) are restrictive such that

$$\begin{aligned} x^a + \widetilde{x}^b &= w_1 \\ x^a + \widetilde{x}^a &= w_1 \\ y^c + \widetilde{y}^d &= w_2 \\ y^c + \widetilde{y}^c &= w_2 \end{aligned}$$

Thus we denote

$$\begin{aligned} \widetilde{x} &= \widetilde{x}^b = \widetilde{x}^a \\ \widetilde{y} &= \widetilde{y}^d = \widetilde{y}^c \\ x &= x^a \\ y &= y^c \end{aligned}$$

Then the three first-order conditions of player 1's maximization problems ((13), (9), (11)) are

$$\frac{d}{dx} : [v^a + v^b \frac{(\widetilde{x})^r}{(\widetilde{x})^r + (\widetilde{y})^r} - \widetilde{x} - v^a \frac{(\widetilde{x})^r}{(\widetilde{x})^r + (\widetilde{y})^r} + \widetilde{x}] \frac{(y)^r r (x)^{r-1}}{((y)^r + (x)^r)^2} - 1 = \lambda_1 + \lambda_2 \tag{17}$$

$$\frac{d}{d\widetilde{x}^a} : [v^a \frac{(\widetilde{y})^r r (\widetilde{x})^{r-1}}{((\widetilde{x})^r + (\widetilde{y})^r)^2} - 1] \frac{(y)^r}{(x)^r + (y)^r} = \lambda_1 \tag{18}$$

$$\frac{d}{d\widetilde{x}^b} : [v^b \frac{(\widetilde{y})^r r (\widetilde{x})^{r-1}}{((\widetilde{x})^r + (\widetilde{y})^r)^2} - 1] \frac{(x)^r}{(x)^r + (y)^r} = \lambda_2 \tag{19}$$

where λ_1 and λ_2 are the Lagrangian multipliers. The first-order conditions (18) and (19) can be unified as follows

$$\frac{d}{d\widetilde{x}} : [v^a \frac{(\widetilde{y})^r r (\widetilde{x})^{r-1}}{((\widetilde{x})^r + (\widetilde{y})^r)^2}] \frac{(y)^r}{(x)^r + (y)^r} + [v^b \frac{(\widetilde{y})^r r (\widetilde{x})^{r-1}}{((\widetilde{x})^r + (\widetilde{y})^r)^2}] \frac{(x)^r}{(x)^r + (y)^r} - 1 = \lambda_1 + \lambda_2 \tag{20}$$

Note that both first-order conditions of player 1's maximization problem (17) and (20) are exactly the same if $x = \widetilde{x}$ and $y = \widetilde{y}$. In other words, if player 2's allocation of effort is symmetric, $y = \widetilde{y}$, then player 1's maximization problems in both stages are actually symmetric and therefore his allocation of effort between both stages is the same, namely, $x = \widetilde{x}$. Similarly it can be shown that if player 1's allocation of effort is symmetric $x = \widetilde{x}$, then player 2's maximization problems in both stages are symmetric, and therefore his allocation of effort between both stages is the same, namely, $y = \widetilde{y}$. Thus, we obtain an equilibrium according to which $x = \widetilde{x}$ and $y = \widetilde{y}$. ■

Theorem 1 generalizes Proposition 5 to show that each player allocates his effort equally along both of the contest's stages independently of the relation between his values and the relation between his values and those of his opponent. Furthermore, each player allocates his effort equally along the contest's stages independently of the players' budget constraints as long as these budget constraints are restrictive. To state this somewhat differently, Theorem 1 establishes that when players have sufficiently low budget constraints, the players' values as well as their budget constraints do not have any effect on their allocations of efforts in the sequential contest.

6 Concluding remarks

This paper studied a sequential Tullock contest with budget-constrained players and synergy between the players' values for the prizes in both stages of the contest. We showed that when the players are symmetric with the same values over the contest's stages and their budget constraints are not restrictive, then the total effort in the first stage of the contest is always higher than the total effort in the second stage if the players' marginal values are increasing, and the opposite holds when the marginal values are decreasing. On the other hand, when the players' budget constraints are restrictive the total effort in the first stage of the contest is always equal to the total effort in the second stage. We prove that this result holds even when the players are asymmetric regarding their values for the prizes and the budget constraints.

Our results have an interesting implication. Let us suppose that the sum of the players' marginal values is fixed but the designer of the contest controls the allocation of the players' values along both stages of the contest. As such, he can determine whether the players' marginal values for the prizes are increasing or decreasing. A question that naturally arises is what should then be the optimal allocation of prizes for a designer who wishes to maximize the players' expected total effort? Should the prizes' values be increasing or decreasing over both stages of the contest? Based on the analysis in the paper, if the budget constraints are restrictive it does not matter whether the prizes' value are increasing or decreasing since the allocation of prizes does not affect the players' allocation of effort. However, when there is a nonrestrictive budget constraint, our analysis indicates that, independent of whether the marginal values are increasing or decreasing, the total effort in the second stage of the contest is identical. On the other hand, the total effort in the first stage of the contest is always higher when the players' marginal values for the prizes are increasing. Hence, if the players' budget constraints are nonrestrictive the contest designer who wishes to maximize the expected total effort will prefer a contest with increasing marginal values. However, if the players' budget constraints are restrictive the contest designer cannot influence the players' allocations of effort in the sequential contest.

7 Appendix

7.1 The Proof of Proposition 1

If the budget constraint is nonrestrictive both of the restrictions in the maximization problem (3) are non-restrictive such that

$$\begin{aligned} x_i^a + \widetilde{x}_i^b &< w \\ x_i^a + \widetilde{x}_i^a &< w \end{aligned}$$

Then the first-order conditions of the maximization problems in the second stage (1) and (2) are

$$\begin{aligned} v^b \frac{\widetilde{x}_j^a}{(\widetilde{x}_i^b + \widetilde{x}_j^a)^2} - 1 &= 0 \\ v^a \frac{\widetilde{x}_j^b}{(\widetilde{x}_i^a + \widetilde{x}_j^b)^2} - 1 &= 0 \end{aligned}$$

Because of the symmetry we denote

$$\begin{aligned} \widetilde{x}_i^a &= \widetilde{x}_j^a = \widetilde{x}^a \\ \widetilde{x}_i^b &= \widetilde{x}_j^b = \widetilde{x}^b \end{aligned}$$

The solution of the above two first-order conditions is:

$$\begin{aligned} \widetilde{x}^a &= \frac{v^b (v^a)^2}{(v^a + v^b)^2} \\ \widetilde{x}^b &= \frac{v^a (v^b)^2}{(v^a + v^b)^2} \end{aligned} \tag{21}$$

The first-order condition of the maximization problem in the first stage (equation (3)) is

$$[v^a + v^b \frac{\widetilde{x}_i^b}{\widetilde{x}_i^b + \widetilde{x}_j^a} - \widetilde{x}_i^b - v^a \frac{\widetilde{x}_i^a}{\widetilde{x}_i^a + \widetilde{x}_j^b} + \widetilde{x}_i^a] \frac{x_j^a}{(x_i^a + x_j^a)^2} = 1$$

where $\widetilde{x}_i^a, \widetilde{x}_j^a, \widetilde{x}_i^b, \widetilde{x}_j^b$ are given by (21). Because of the symmetry we denote

$$x_i^a = x_j^a = x^a$$

Then the solution of the above first-order condition is

$$x^a = \frac{(v^b)^3 + v^a(v^b)^2 + 2v^b(v^a)^2}{4(v^a + v^b)^2}$$

By normalizing ($v^a = 1$) we obtain

$$\widetilde{x}^a - x^a = \frac{-v^b[(v^b)^2 + v^b - 2]}{4(v^b + 1)^2}$$

Since the expression $(v^b)^2 + v^b - 2$ is negative for all $0 < v^b < 1$, the difference $\widetilde{x}^a - x^a$ is always positive. Furthermore,

$$x^a - \widetilde{x}^b = \frac{v^b[(v^b)^2 - 3v^b + 2]}{4(v^b + 1)^2}$$

Since the expression $(v^b)^2 - 3v^b + 2$ is positive for all $0 < v^b < 1$, the difference $x^a - \widetilde{x}^b$ is always positive.

Now we examine the conditions under which the budget constraint is nonrestrictive. If the restrictions are nonrestrictive we have

$$\begin{aligned} x^a + \widetilde{x}^b &= \frac{(v^b)^3 + 5(v^b)^2 + 2v^b}{4(v^b + 1)^2} < w \\ x^a + \widetilde{x}^a &= \frac{(v^b)^3 + (v^b)^2 + 6v^b}{4(v^b + 1)^2} < w \end{aligned}$$

Since $x^a + \widetilde{x}^a > x^a + \widetilde{x}^b$ we obtain that the constraints are nonrestrictive iff $w > x^a + \widetilde{x}^a$. Thus, the condition that implies nonrestrictive budget constraints is

$$w > \frac{(v^b)^3 + (v^b)^2 + 6v^b}{4(v^b + 1)^2}$$

Q.E.D.

7.2 The Proof of Proposition 2

We proved in Proposition 1 that if v^a is normalized to be 1, the budget constraint is nonrestrictive if

$$w > \frac{(v^b)^3 + (v^b)^2 + 6v^b}{4(v^b + 1)^2}$$

In this case the total effort in the first stage of the contest is

$$TE_1 = 2x^a = \frac{(v^b)^3 + (v^b)^2 + 2v^b}{2(1 + v^b)^2}$$

and the total effort in the second stage of the contest is

$$TE_2 = \widetilde{x}^a + \widetilde{x}^b = \frac{v^b}{1 + v^b}$$

The difference between the total efforts in both stages when the budget constraint is nonrestrictive is

$$TE_1 - TE_2 = \frac{(v^b)^2(v^b-1)}{2(1+v^b)^2}$$

Since $v^b < v^a = 1$ (decreasing marginal values) this difference is negative and therefore $TE_1 < TE_2$. *Q.E.D.*

7.3 The Proof of Proposition 3

If the budget constraint is nonrestrictive both of the restrictions in the maximization problem (6) are non-restrictive such that

$$\begin{aligned} x_i^b + \widetilde{x}_i^a &< w \\ x_i^b + \widetilde{x}_i^b &< w \end{aligned}$$

The first-order conditions of the maximization problems in the second stage (4) and (5) are

$$\begin{aligned} v^a \frac{\widetilde{x}_j^b}{(\widetilde{x}_i^a + \widetilde{x}_j^b)^2} - 1 &= 0 \\ v^b \frac{\widetilde{x}_j^a}{(\widetilde{x}_i^b + \widetilde{x}_j^a)^2} - 1 &= 0 \end{aligned}$$

Because of the symmetry we denote

$$\begin{aligned} \widetilde{x}_i^a &= \widetilde{x}_j^a = \widetilde{x}^a \\ \widetilde{x}_i^b &= \widetilde{x}_j^b = \widetilde{x}^b \end{aligned}$$

The solution of the above two first-order conditions is

$$\begin{aligned} \widetilde{x}^a &= \frac{v^b(v^a)^2}{(v^a+v^b)^2} \\ \widetilde{x}^b &= \frac{v^a(v^b)^2}{(v^a+v^b)^2} \end{aligned} \tag{22}$$

The first-order condition of the maximization problem in the first stage (equation (6)) is

$$[v^b + v^a \frac{\widetilde{x}_i^a}{\widetilde{x}_i^a + \widetilde{x}_j^b} - \widetilde{x}_i^a - v^b \frac{\widetilde{x}_i^b}{\widetilde{x}_i^b + \widetilde{x}_j^a} + \widetilde{x}_i^b] \frac{x_j^b}{(x_i^b + x_j^b)^2} = 1$$

where $\widetilde{x}_i^a, \widetilde{x}_j^a, \widetilde{x}_i^b, \widetilde{x}_j^b$ are given by (22). Because of the symmetry we denote

$$x_i^b = x_j^b = x^b$$

Then the solution of the above first-order condition is

$$x^b = \frac{2(v^b)^2 v^a + v^b(v^a)^2 + (v^a)^3}{4(v^a+v^b)^2}$$

By using the normalization ($v^a = 1$) we obtain

$$\widetilde{x}^a - \widetilde{x}^b = \frac{v^b(1-v^b)}{(v^b+1)^2}$$

Since $0 < v^b < 1$, the difference $\widetilde{x}^a - \widetilde{x}^b$ is always positive. Furthermore,

$$x^b - \widetilde{x}^b = \frac{-2(v^b)^2 + v^b + 1}{4(v^b+1)^2}$$

Since the expression $-2(v^b)^2 + v^b + 1$ is positive for all $0 < v^b < 1$, the difference $x^b - \widetilde{x}^b$ is always positive. We also have

$$x^b - \widetilde{x}^a = \frac{2(v^b)^2 - 3v^b + 1}{4(v^b+1)^2}$$

Since the expression $2(v^b)^2 - 3v^b + 1$ is positive for all $0 < v^b < 0.5$ and negative for all $0.5 < v^b < 1$ we obtain that the difference $x^b - \widetilde{x}^a$ is positive for all $0 < v^b < 0.5$ and is negative for all $0.5 < v^b < 1$. The relations between a player's allocations of effort is therefore

$$\begin{aligned} x^b &\geq \widetilde{x}^a > \widetilde{x}^b \text{ if } 0 < v^b \leq 0.5 \\ \widetilde{x}^a &\geq x^b > \widetilde{x}^b \text{ if } 0.5 < v^b < 1 \end{aligned}$$

Now we examine the conditions under which the budget constraint is nonrestrictive. If the restrictions are nonrestrictive we have

$$\begin{aligned} x^b + \widetilde{x}^a &= \frac{2(v^b)^2 + 5v^b + 1}{4(v^b+1)^2} < w \\ x^b + \widetilde{x}^b &= \frac{6(v^b)^2 + v^b + 1}{4(v^b+1)^2} < w \end{aligned}$$

Since $x^b + \widetilde{x}^a > x^b + \widetilde{x}^b$ we obtain that the constraints are nonrestrictive iff $w > x^b + \widetilde{x}^a$. Thus, the condition that implies nonrestrictive budget constraints is

$$w > \frac{2(v^b)^2 + 5v^b + 1}{4(v^b+1)^2}$$

Q.E.D.

7.4 The Proof of Proposition 4

We proved in Proposition 3 that if v^a is normalized to be 1, the budget constraint is nonrestrictive if

$$w > \frac{2(v^b)^2 + 5v^b + 1}{4(v^b+1)^2}$$

In this case the total effort in the first stage of the contest is

$$TE_1 = 2x^b = \frac{2(v^b)^2 + v^b + 1}{2(1+v^b)^2}$$

and the total effort in the second stage of the contest is

$$TE_2 = \widetilde{x}^a + \widetilde{x}^b = \frac{v^b}{1+v^b}$$

The difference between the total efforts in both stages when the budget constraint is nonrestrictive is

$$TE_1 - TE_2 = \frac{1-v^b}{2(1+v^b)^2}$$

Since $v^b < v^a = 1$ (increasing marginal values) this difference is positive and therefore $TE_1 > TE_2$. *Q.E.D.*

7.5 The Proof of Proposition 5

1) Assume first that the players have decreasing marginal values. If the budget constraint is restrictive both of the restrictions in the maximization problem (3) are restrictive such that

$$\begin{aligned} x_i^a + \widetilde{x}_i^b &= w \\ x_i^a + \widetilde{x}_i^a &= w \end{aligned}$$

Player i's maximization problem in the first stage is then

$$\max_{x_i^a}(v^a + v^b \frac{w - x_i^a}{w - x_i^a + w - x_j^a} - (w - x_i^a))\frac{x_i^a}{x_i^a + x_j^a} + (v^a \frac{w - x_i^a}{w - x_i^a + w - x_j^a} - (w - x_i^a))(1 - \frac{x_i^a}{x_i^a + x_j^a}) - x_i^a$$

Therefore the first-order condition is

$$\begin{aligned} & [v^a + v^b \frac{w - x_i^a}{w - x_i^a + w - x_j^a} - (w - x_i^a) - v^a \frac{w - x_i^a}{w - x_i^a + w - x_j^a} + (w - x_i^a)]\frac{x_j^a}{(x_i^a + x_j^a)^2} \\ & +(v^b - v^a)\frac{-(w - x_j^a)}{(w - x_i^a + w - x_j^a)^2}\frac{x_i^a}{x_i^a + x_j^a} + v^a \frac{-(w - x_j^a)}{(w - x_i^a + w - x_j^a)^2} \\ = \; & 0 \end{aligned}$$

Because of the symmetry we denote

$$\begin{aligned} x_i^a &= x_j^a = x^a \\ \widetilde{x}_i^a &= \widetilde{x}_j^a = \widetilde{x}^a \\ \widetilde{x}_i^b &= \widetilde{x}_j^b = \widetilde{x}^b \end{aligned}$$

Then, the solution of the above first-order condition is:

$$x^a = \frac{w}{2}$$

and then by our assumption

$$\widetilde{x}^a = \widetilde{x}^b = w - x^a = \frac{w}{2}$$

In the second stage, player i's maximization problems are given by (1) and (2). The first-order conditions of these maximization problems are

$$v^b \frac{\widetilde{x}_j^a}{(\widetilde{x}_i^b + \widetilde{x}_j^a)^2} - 1$$
$$v^a \frac{\widetilde{x}_j^b}{(\widetilde{x}_i^a + \widetilde{x}_j^b)^2} - 1$$

In order that both constraints will be restrictive these first-order conditions of the maximization problems in the second stage should be positive. Thus, both constraints are restrictive iff

$$\begin{aligned} \frac{v^b}{2w} - 1 &> 0 \\ &\Rightarrow w < \frac{v^b}{2} \end{aligned}$$

In this case the total effort in the first stage of the contest is

$$TE_1 = 2x^a = w$$

and the total effort in the second stage is

$$TE_2 = \widetilde{x}^a + \widetilde{x}^b = w$$

Therefore

$$TE_1 = TE_2$$

2) Assume now that the players have increasing marginal values. When the budget constraint is restrictive both of the restrictions in the maximization problem (6) are restrictive such that

$$\begin{aligned} x_i^b + \widetilde{x}_i^a &= w \\ x_i^b + \widetilde{x}_i^b &= w \end{aligned}$$

Player i's maximization problem in the first stage is then

$$\max_{x_i^b} (v^b + v^a \frac{w - x_i^b}{w - x_i^b + w - x_j^b} - (w - x_i^b)) \frac{x_i^b}{x_i^b + x_j^b} + (v^b \frac{w - x_i^b}{w - x_i^b + w - x_j^b} - (w - x_i^b))(1 - \frac{x_i^b}{x_i^b + x_j^b}) - x_i^b$$

The first-order condition is

$$
\begin{aligned}
& [v^b + v^a \frac{w - x_i^b}{w - x_i^b + w - x_j^b} - (w - x_i^b) - v^b \frac{w - x_i^b}{w - x_i^b + w - x_j^b} + (w - x_i^b)] \frac{x_j^b}{(x_i^b + x_j^b)^2} \\
& + (v^a - v^b) \frac{-(w - x_j^b)}{(w - x_i^b + w - x_j^b)^2} \frac{x_i^b}{x_i^b + x_j^b} + v^b \frac{-(w - x_j^b)}{(w - x_i^b + w - x_j^b)^2} \\
= \; & 0
\end{aligned}
$$

Because of the symmetry we denote

$$
\begin{aligned}
x_i^b &= x_j^b = x^b \\
\widetilde{x}_i^b &= \widetilde{x}_j^b = \widetilde{x}^b \\
\widetilde{x}_i^a &= \widetilde{x}_j^a = \widetilde{x}^a
\end{aligned}
$$

The solution of the above first-order condition is

$$x^b = \frac{w}{2}$$

and then by our assumption

$$\widetilde{x}^a = \widetilde{x}^b = w - x^b = \frac{w}{2}$$

In the second stage, player i's maximization problems are given by (4) and (5). The first-order conditions of these maximization problems are

$$
\begin{aligned}
& v^a \frac{\widetilde{x}_j^b}{(\widetilde{x}_i^a + \widetilde{x}_j^b)^2} - 1 \\
& v^b \frac{\widetilde{x}_j^a}{(\widetilde{x}_i^b + \widetilde{x}_j^a)^2} - 1
\end{aligned}
$$

In order that both constraints will be restrictive these first order conditions of the maximization problems in the second stage should be positive. Thus, both constraints are restrictive iff

$$
\begin{aligned}
\frac{v^b}{2w} - 1 &> 0 \\
&\Rightarrow w < \frac{v^b}{2}
\end{aligned}
$$

In this case the total effort in the first stage of the contest is

$$TE_1 = 2x^b = w$$

and the total effort in the second stage is

$$TE_2 = \widetilde{x}^a + \widetilde{x}^b = w$$

Therefore

$$TE_1 = TE_2$$

Q.E.D.

7.6 The Proof of Proposition 6

If the budget constraint is partially restrictive only the second restriction in the maximization problem (3) is restrictive such that

$$\begin{aligned} x_i^a + \widetilde{x}_i^b &< w \\ x_i^a + \widetilde{x}_i^a &= w \end{aligned}$$

Thus, if player i does not win in the first stage his effort in the second stage is $\widetilde{x}_i^a = w - x_i^a$. If, on the other hand, he wins in the first stage his maximization problem in the second stage is

$$\max_{\widetilde{x}_i^b} v^b \frac{\widetilde{x}_i^b}{\widetilde{x}_i^b + \widetilde{x}_j^a} - \widetilde{x}_i^b$$

The first-order condition of this maximization problem is

$$v^b \frac{\widetilde{x}_j^a}{(\widetilde{x}_i^b + \widetilde{x}_j^a)^2} - 1 = 0 \tag{23}$$

Player i's maximization problem in the first stage is then

$$\max_{x_i^a}(v^a + v^b \frac{\widetilde{x}_i^b}{\widetilde{x}_i^b + w - x_j^a} - \widetilde{x}_i^b)\frac{x_i^a}{x_i^a + x_j^a} + (v^a \frac{w - x_i^a}{w - x_i^a + \widetilde{x}_j^b} - (w - x_i^a))(1 - \frac{x_i^a}{x_i^a + x_j^a}) - x_i^a$$

Therefore the first-order condition is

$$\begin{aligned} & [v^a + v^b \frac{\widetilde{x}_i^b}{\widetilde{x}_i^b + w - x_j^a} - \widetilde{x}_i^b - v^a \frac{w - x_i^a}{w - x_i^a + \widetilde{x}_j^b} + (w - x_i^a)] \frac{x_j^a}{(x_i^a + x_j^a)^2} \\ & + (v^a \frac{-\widetilde{x}_j^b}{(w - x_i^a + \widetilde{x}_j^b)^2} + 1)(1 - \frac{x_i^a}{x_i^a + x_j^a}) - 1 \\ = & \ 0 \end{aligned} \tag{24}$$

Because of the symmetry we denote

$$\begin{aligned} x_i^a &= x_j^a = x^a \\ \widetilde{x}_i^a &= \widetilde{x}_j^a = \widetilde{x}^a \\ \widetilde{x}_i^b &= \widetilde{x}_j^b = \widetilde{x}^b \end{aligned}$$

The solution of the first-order conditions (when $v^a = 1$) from both stages (24) and (23) implies that the equilibrium effort in the first stage x^a is determined by the following equation

$$\begin{aligned} & [1 + v^b + 2w - 2x^a - \sqrt{w - x^a}(2\sqrt{v^b} + \frac{1}{\sqrt{v^b}})]v^b(w - x^a) \\ = \; & [\sqrt{v^b(w - x^a)} - (w - x^a)(v^b + 1)]2x^a + v^b(w - x^a)4x^a \end{aligned}$$

where

$$\begin{aligned} \widetilde{x}^a &= w - x^a \\ \widetilde{x}^b &= \sqrt{v^b(w - x^a)} - w + x^a \end{aligned}$$

According to Propositions 1 and 5, the budget constraint is partially restrictive iff

$$\frac{v^b}{2} < w < \frac{(v^b)^3 + (v^b)^2 + 6v^b}{4(v^b + 1)^2}$$

Q.E.D.

7.7 The Proof of Proposition 7

If the budget constraint is partially restrictive only the first restriction in the maximization problem (6) is restrictive such that

$$\begin{aligned} x_i^b + \widetilde{x}_i^a &= w \\ x_i^b + \widetilde{x}_i^b &< w \end{aligned}$$

Thus if player i wins in the first stage his effort in the second stage is $\widetilde{x}_i^a = w - x_i^b$. If, on the other hand, he does not win in the first stage his maximization problem in the second stage is

$$\max_{\widetilde{x}_i^b} v^b \frac{\widetilde{x}_i^b}{\widetilde{x}_i^b + \widetilde{x}_j^a} - \widetilde{x}_i^b$$

The first order of this maximization problem is

$$v^b \frac{\widetilde{x}_j^a}{(\widetilde{x}_i^b + \widetilde{x}_j^a)^2} - 1 = 0 \tag{25}$$

Player i's maximization problem in the first stage is then

$$\max_{x_i^b}(v^b + v^a \frac{w - x_i^b}{w - x_i^b + \widetilde{x}_j^b} - (w - x_i^b))\frac{x_i^b}{x_i^b + x_j^b} + (v^b \frac{\widetilde{x}_i^b}{\widetilde{x}_i^b + w - x_j^b} - \widetilde{x}_i^b)(1 - \frac{x_i^b}{x_i^b + x_j^b}) - x_i^b$$

The first-order-condition is

$$\begin{aligned} & [v^b + v^a \frac{w - x_i^b}{w - x_i^b + \widetilde{x}_j^b} - (w - x_i^b) - v^b \frac{\widetilde{x}_i^b}{\widetilde{x}_i^b + w - x_j^b} + \widetilde{x}_i^b] \frac{x_j^b}{(x_i^b + x_j^b)^2} \\ & +(v^a \frac{-\widetilde{x}_j^b}{(w - x_i^b + \widetilde{x}_j^b)^2} + 1) \frac{x_i^b}{x_i^b + x_j^b} - 1 \\ = \; & 0 \end{aligned} \tag{26}$$

Because of the symmetry we denote

$$\begin{aligned} x_i^b &= x_j^b = x^b \\ \widetilde{x}_i^a &= \widetilde{x}_j^a = \widetilde{x}^a \\ \widetilde{x}_i^b &= \widetilde{x}_j^b = \widetilde{x}^b \end{aligned}$$

The solution of the first-order conditions (when $v^a = 1$) from both stages (26) and (25) implies that the equilibrium effort in the first stage x^b is determined by the following equation

$$\begin{aligned} & \frac{\sqrt{w - x^b}}{\sqrt{v^b}} + 2\sqrt{v^b(w - x^b)} - 2w - 2x^b \\ = \; & \frac{[\sqrt{v^b} - (v^b + 1)\sqrt{w - x^b}]2x^b}{v^b\sqrt{w - x^b}} \end{aligned}$$

where

$$\begin{aligned} \widetilde{x}^a &= w - x^b \\ \widetilde{x}^b &= \sqrt{v^b(w - x^b)} - w + x^b \end{aligned}$$

According to Propositions 3 and 5 the budget constraint is partially restrictive iff

$$\frac{v^b}{2} < w < \frac{2(v^b)^2 + 5v^b + 1}{4(v^b + 1)^2}$$

Q.E.D.

7.8 The Proof of Theorem 1

If the budget constraint is restrictive all of the four restrictions in the maximization problems (13) and (14) are restrictive such that

$$\begin{aligned} x^a + \widetilde{x}^b &= w_1 \\ x^a + \widetilde{x}^a &= w_1 \\ y^c + \widetilde{y}^d &= w_2 \\ y^c + \widetilde{y}^c &= w_2 \end{aligned}$$

Thus we denote

$$\begin{aligned}
\widetilde{x} &= \widetilde{x}^b = \widetilde{x}^a \\
\widetilde{y} &= \widetilde{y}^d = \widetilde{y}^c \\
x &= x^a \\
y &= y^c
\end{aligned}$$

Then the first-order condition of player 1's maximization problem (15) is

$$\begin{aligned}
& ((v^a - v^b)\frac{-r(w_1 - x)^{r-1}(w_2 - y)^r}{((w_1 - x)^r + (w_2 - y)^r)^2})\frac{(y)^r}{(x)^r + (y)^r} + \\
& ((v^a - v^b)\frac{(w_1 - x)^r}{(w_1 - x)^r + (w_2 - y)^r} - v^a)\frac{-r(x)^{r-1}(y)^r}{((x)^r + (y)^r)^2} + \\
& v^b\frac{-r(w_1 - x)^{r-1}(w_2 - y)^r}{((w_1 - x)^r + (w_2 - y)^r)^2} \\
= \ & 0
\end{aligned} \tag{27}$$

Similarly, the first-order condition of player 2's maximization problem (16) is

$$\begin{aligned}
& ((v^c - v^d)\frac{-r(w_2 - y)^{r-1}(w_1 - x)^r}{((w_1 - x)^r + (w_2 - y)^r)^2})\frac{(x)^r}{(x)^r + (y)^r} + \\
& ((v^c - v^d)\frac{(w_2 - y)^r}{(w_1 - x)^r + (w_2 - y)^r} - v^c)\frac{-r(y)^{r-1}(x)^r}{((x)^r + (y)^r)^2} + \\
& v^d\frac{-r(w_2 - y)^{r-1}(w_1 - x)^r}{((w_1 - x)^r + (w_2 - y)^r)^2} \\
= \ & 0
\end{aligned} \tag{28}$$

Thus, it can be verified that the solution of the above first-order conditions (27) and (28) is

$$\begin{aligned}
x &= \widetilde{x} = \frac{w_1}{2} \\
y &= \widetilde{y} = \frac{w_2}{2}
\end{aligned}$$

The budget constraints are restrictive if the first-order conditions of the maximization problems in the second stage (9),(10),(11) and (12) are positive. Thus,

$$
\begin{aligned}
v^a \frac{(\widetilde{y})^r r(\widetilde{x})^{r-1}}{((\widetilde{x})^r + (\widetilde{y})^r)^2} - 1 &> 0 \\
v^b \frac{(\widetilde{y})^r r(\widetilde{x})^{r-1}}{((\widetilde{x})^r + (\widetilde{y})^r)^2} - 1 &> 0 \\
v^c \frac{(\widetilde{x})^r r(\widetilde{y})^{r-1}}{((\widetilde{x})^r + (\widetilde{y})^r)^2} - 1 &> 0 \\
v^d \frac{(\widetilde{x})^r r(\widetilde{y})^{r-1}}{((\widetilde{x})^r + (\widetilde{y})^r)^2} - 1 &> 0
\end{aligned}
$$

This happens when

$$
\begin{aligned}
\frac{2v^a (w_2)^r r(w_1)^{r-1}}{((w_1)^r + (w_2)^r)^2} &> 1 \\
\frac{2v^b (w_2)^r r(w_1)^{r-1}}{((w_1)^r + (w_2)^r)^2} &> 1 \\
\frac{2v^c (w_1)^r r(w_2)^{r-1}}{((w_1)^r + (w_2)^r)^2} &> 1 \\
\frac{2v^d (w_1)^r r(w_2)^{r-1}}{((w_1)^r + (w_2)^r)^2} &> 1
\end{aligned}
$$

In this case the total effort in the first stage of the contest is

$$
TE_1 = x^a + y^c = \frac{w_1 + w_2}{2}
$$

The total effort in the second stage of the contest is

$$
TE_2 = \widetilde{x}^a + \widetilde{y}^d = \widetilde{x}^b + \widetilde{y}^c = \frac{w_1 + w_2}{2}
$$

Therefore

$$
TE_1 = TE_2
$$

Q.E.D.

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