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M E X I C O

Simple Plurality versus Plurality Runoff with Privately Informed Voters

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Abstract: This paper compares two voting methods commonly used in presidential elections: simple plurality voting and plurality runoff. In a situation in which a group of voters have common interests but do not agree on which candidate to support due to private information, information aggregation requires them to split their support between their favorite candidates. However, if a group of voters split their support between their favorite candidates, they increase the probability that the winner of the election is not one of them. In a model with three candidates, due to this tension between information aggregation and the need for coordination, plurality runoff leads to higher expected utility for the majority than simple plurality voting if the information held by voters about the candidates is not very accurate.

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1 Introduction

Consider a group of voters who must decide between three candidates, say, A, B, and C, for a single public office. Some of the voters clearly prefer candidate C to the other two. The rest of the voters, presumably the majority, must decide whether to support candidate A or B. Under certain circumstances, it would be better for the majority to elect A, while under other circumstances it would be better to elect B. Voters do not know with certainty which of the two cases holds. There is, however, some information dispersed among voters. An election might be an opportunity to aggregate the information about the desirability of electing A or B. The ability to aggregate information rests on the possibility of majority voters splitting their support between A and B, according to the private information they may have. Nevertheless, if voters in the majority split their support between two candidates, they run the risk of losing the election to the candidate supported by the minority. Thus, there is a potential conflict between information aggregation and voter coordination. This paper explores this conflict within the context of a game-theoretic model where voters behave as rational, strategic agents. Attention is restricted to the two common voting methods used in presidential elections: simple plurality voting and plurality runoff.

The model shows that there are at least three equilibria under simple plurality. In one of them, majority voters vote according to their private information, and in the other two, majority voters coordinate in supporting only A or only B. If the expected voting share of candidate C is low enough, the first equilibrium is likely to lead to successful information aggregation in the sense that the most desirable alternative for the majority is elected. However, if the expected voting share of candidate C is high enough, successful information aggregation is possible only if the information held by majority voters about the candidates is very accurate. Otherwise, it is better for majority voters to disregard their private information and to coordinate in supporting only A or only B.

Under plurality runoff, on the other hand, there is (under certain conditions) only one equilibrium,[1] in which voters vote according to their private

[1]To avoid dealing with abstention, the analysis of plurality runoff focuses on the case in which candidate C is high enough, a condition which will be elaborated on further in

information. In this equilibrium, information is successfully aggregated in the sense that the most desirable alternative for the majority either wins an outright victory in the first round or makes it into the runoff election.

This model can help illuminate some common concerns about the effects of plurality runoff in presidential regimes, particularly in Latin America, where five out of 18 countries employ plurality voting and twelve employ plurality runoff in presidential elections (Nohlen 1997). Opinion in Latin America turned towards plurality runoff in part as a result of the victory of Allende in a disputed three-way race in Chile and the subsequent demise of democracy by the military intervention of 1973.[2] More recently, plurality runoff has been criticized as leading to a proliferation of candidates, due to low costs of entry, which may lead in the long run to a lack of consolidation of a bipartisan system (Shugart and Carey 1992). In the model proposed, equilibrium under plurality runoff does lead potentially to a larger number of serious candidates than plurality voting. However, the model casts a positive light on this apparent weakness of plurality runoff as it is a condition for successful information aggregation in a single election.

Moreover, in some episodes such as Fujimori's victory in Peru's 1990 presidential election, the runoff voting method has been charged with leading to seemingly erratic behavior by voters.[3] The evolution of voters' intentions before the first round of Peru's 2000 presidential elections was again consider erratic by the media. This time Fujimori was running for reelection, and opinion polls showed swift variations in the support for the different opposition candidates.[4] Our model suggests that voters may appear "fickle" before the first round of a plurality runoff election precisely because equilibrium behavior requires them to react to all sources of information about

the paper.

[2]Chile's 1925 constitution provided for the Chamber of Deputies to decide between the two candidates with the most votes, assuming neither one had the majority. However, until the military coup, presidential elections operated de facto under plurality voting, since whenever the chamber was required to decide, it picked the candidate with the most votes.

[3]Schmidt (1996) emphasizes the importance of electoral rules to explain Fujimori's victory over Vargas Llosa. Vargas Llosa (1993) contains a very readable account of that election.

[4]See e.g. "Cholo Challenge," The Economist, March 25, 2000. Eventually, Fujimori won a runoff election amidst widespread allegations of fraud.

the desirability of the alternatives with less regard for possible coordination problems than under simple plurality. The apparent fickleness of voters may be a rational reaction to poor information about the candidates. [5]

It has been noted that plurality runoff has the potential for leading to the election of a president with little congressional support. This might be the case if a completely different method such as proportional representation with large electoral districts were used to elect the congress. The model proposed here does not take into account the possible incompatibility between the methods used for electing the president and those used for the congress (an issue dealt with by Shugart and Carey in their 1992 analysis of electoral institutions and also by Mainwaring and Shugart, 1997). This important topic is left for future research. Also, the analysis of a lower threshold than 50% for victory in the first round is left for future work.

On a more technical note, the analysis of large voting games within a framework of rational strategic voters carried out here is complicated because it requires voters to compare the probabilities of the different situations in which a vote may be decisive, even if all these probabilities are nearly zero for a large electorate. The task has been made relatively easier by the introduction of Poisson games by Myerson (1997, 1998a, 1998b), which allow to compute the ratios between limit probabilities of different events with large electorates. The analysis in this paper is carried out within a framework of Poisson games. Other applications of this framework include Feddersen and Pesendorfer's (1999) analysis of abstention in two-way races and Myerson's (1998c) analysis of scoring rules (a class of voting rules that excludes runoff) in three-way races with complete information.

2 Basics

A group of voters must choose one out of three alternatives. Alternatives are denoted by $X \in \{A, B, C\}$. There are three types of voters, denoted by $t \in \{t_1, t_2, t_3\}$. Each voter's preferences over the alternatives depend on the voter type and the state of the world, $\omega \in \{\omega_1, \omega_2\}$. Denote by $U(X, t, \omega)$ the

[5] A formal analysis of opinion polls would require to introduce another stage in the game, before the elections, and is beyond the scope of this paper. Some early and interesting references are Simon (1954) and McKelvey and Ordeshook (1985).

utility payoff for a type t voter if the state is ω and alternative X is chosen. It is assumed that for every alternative X, $U(X, t_1, \omega) = U(X, t_2, \omega)$, and for $t = t_1, t_2$,

$$\begin{aligned} U(C, t, \omega_1) &= U(C, t, \omega_2) = U(B, t, \omega_1) \\ &< U(A, t, \omega_1) = U(A, t, \omega_2) < U(B, t, \omega_2). \end{aligned}$$

That is, voters of types t_1 and t_2 have common preferences, they would like alternative A to be chosen in state ω_1 and alternative B to be chosen in state ω_2, and they are at best indifferent between alternative C and any other alternative. It might be useful to think of alternative B as an "entrant" about which there is imperfect information disseminated among majority (i.e., t_1 and t_2) voters.

Define

$$u_{AB} = \frac{U(B, t_1, \omega_2) - U(A, t_1, \omega_2)}{U(A, t_1, \omega_2) - U(C, t_1, \omega_2)} \text{ and } u_{CB} = \frac{U(B, t_1, \omega_2) - U(C, t_1, \omega_2)}{U(A, t_1, \omega_2) - U(C, t_1, \omega_2)}$$

to be the relative gain for t_1 and t_2 voters of choosing alternative B instead of A or C in state ω_2 . Note that $u_{AB}, u_{CB} > 0$.

It is also assumed that for $\omega = \omega_1, \omega_2$,

$$U(A, t_3, \omega) = U(B, t_3, \omega) < U(C, t_3, \omega).$$

That is, voters of type t_3 would like alternative C to be chosen, regardless of the state of the world, and they are indifferent between the other two alternatives.

Prior beliefs about the state of the world are denoted by $q(\omega)$, with $q(\omega_1) + q(\omega_2) = 1$. The number of voters is a random variable that has a Poisson distribution with mean n. That is, the probability that there are k voters is

$$P(k; n) = e^{-n} n^k / k!$$

Each voter type is drawn from a distribution function that depends on the state of the world. Denoting by $r(t|\omega)$ the probability that a random voter is of type t given state ω, it is assumed that

$$\begin{aligned} r(t_1|\omega_1) &= r(t_2|\omega_2) = r_{11} > 0, \\ r(t_1|\omega_2) &= r(t_2|\omega_1) = r_{12} > 0, \\ r(t_3|\omega_1) &= r(t_3|\omega_2) = r_3 > 0, \end{aligned}$$

with

$$r_{11} + r_{12} + r_3 = 1 \quad \text{and} \quad r_{11} > r_{12}.$$

That is, a voter is more likely to be of type t_1 than of type t_2 if the state is ω_1, and the opposite if the state is ω_2. It is also assumed that $r_3 < 1/2$, so that voters of type t_3 are likely to be a minority with respect to the combined population of types t_1 and t_2. Given a state ω, the random number of type t voters has a conditional probability distribution that is Poisson with mean $nr(t|\omega)$, as discussed by Myerson (1997). The fraction

$$\frac{r_{11}}{r_{11} + r_{12}}$$

represents how accurate is the information held by majority voters about candidates A and B.

Let $q(\omega|t)$ denote the posterior beliefs of a type $t = t_1, t_2$ voter about state ω. Then

$$q(\omega_1|t_1) = \frac{r_{11}q(\omega_1)}{r_{11}q(\omega_1) + r_{12}q(\omega_2)}, \qquad q(\omega_2|t_1) = 1 - q(\omega_1|t_1),$$

$$q(\omega_1|t_2) = \frac{r_{12}q(\omega_1)}{r_{12}q(\omega_1) + r_{11}q(\omega_2)}, \qquad q(\omega_2|t_2) = 1 - q(\omega_1|t_2).$$

Note that $r_{11} > r_{12}$ implies

$$q(\omega_1|t_1)/q(\omega_2|t_1) \; > \; q(\omega_1|t_2)/q(\omega_2|t_2).$$

In what follows, two different voting methods are considered in this setup: simple plurality voting and plurality runoff. These methods presents voters with different games, so that equilibrium predictions are potentially different. Since it has been assumed that the combined population of t_1 and t_2 voters are likely to be a majority (arbitrarily likely, for large n), a reasonable criterion to judge a voting method is to look at the expected utility of t_1 and t_2 voters in equilibrium. Note that the expected utility of t_1 and t_2 voters in equilibrium is bounded from above by $U(A, t_1, \omega_1)q(\omega_1) + U(B, t_1, \omega_2)q(\omega_2)$. It will be of particular interest to determine under which conditions this upper bound is attainable under each voting method.

3 Plurality Voting

Under *plurality voting*, each voter must cast simultaneously a vote for either A, B or C, and the alternative that receives most votes is chosen. Ties are broken alphabetically. This choice of a tie breaking rule is made only for expositional convenience and other tie breaking rules such as a fair coin toss will lead to the same results.[6]

A *strategy function* is a mapping $\sigma : \{t_1, t_2, t_3\} \rightarrow \Delta(\{A, B, C\})$, where $\sigma(X|t)$ is the probability that a voter will vote for alternative X if his type is t. Note that $\sigma(X|t) \geq 0$ for every X and every t, and $\sum_X \sigma(X|t) = 1$ for every t. Each voter's behavior as predicted by the strategy function depends only on his type because, as discussed by Myerson (1997), two voters of the same type have no commonly known attributes by which others can distinguish them. A strategy function σ is an *equilibrium* under plurality rule if it maximizes the expected utility of every given voter when other voters use the strategy σ.

For a given strategy function, the probability that a random voter votes for alternative X in state ω is denoted by $\tau(X|\omega)$ where

$$\tau(X|\omega) \quad = \quad \sum_t \sigma(X|t) r(t|\omega).$$

Given a strategy function, the number of voters who choose X if the state is ω is a Poisson random variable with mean $\tau(X|\omega)n$, for $X = A, B, C$. From the discussion in Myerson (1997a), these are independent random variables. Moreover, from the viewpoint of any given voter, the number of other voters who vote for X in state ω is described by the same random variable that the total number of voters who vote for X. Thus, the probability that k, l, m other voters vote for A,B and C, respectively, is just

$$P(k; \tau(A|\omega)n) P(l; \tau(B|\omega)n) P(m; \tau(C|\omega)n).$$

This probability will be denoted $P^\sigma(k, l, m|\omega)$.

[6]A similar choice is made by Palfrey (1989) and Fey (1997), for identical reasons. A random tie-breaking rule would increase the number of events to be considered in Lemma 2 below. The events disregarded, however, correspond to near three-way ties, so their probability converges very fast to zero.

Note that, for a t_1 or t_2 voter, voting for C is a strictly dominated strategy if there is some positive probability that voting for C actually leads to C being chosen, while voting for A leads to A being chosen. But this probability is positive independently of the strategy chosen by other voters because the event that the total number of voters is one has no zero probability. For a t_3 voter, voting for C is a strictly dominant strategy for the same reason. Thus:

Lemma 1 *Given any strategy σ followed by every other voter, if a voter is playing a best response then he will vote for C with probability 0 if his type is t_1 or t_2 and with probability 1 if his type is t_3. Thus, if σ is an equilibrium strategy, $\sigma(C|t_1) = \sigma(C|t_2) = 0$ and $\sigma(C|t_3) = 1$.* □

It remains to determine conditions under which t_1 and t_2 voters will support alternative A or B. In deciding how to vote, a strategic voter takes into account exclusively the events in which his vote is pivotal – i.e., it actually makes a difference with respect to the outcome of the election. Those events are in this model {voting for A yields A but voting for B yields B}, {voting for A yields A but voting for B yields C}, and {voting for A yields C but voting for B yields B}. The probabilities of these events depend on the state of the world and on the strategy followed by other voters, and are given below:

$$
\begin{aligned}
p^\sigma_{AB}(\omega) &= \sum_{k=0}^{\infty}\sum_{m=0}^{k+1} P^\sigma(k,k,m|\omega) + P^\sigma(k,k+1,m|\omega),\\
p^\sigma_{AC}(\omega) &= \sum_{k=0}^{\infty}\sum_{m=0}^{k} P^\sigma(k,m,k+1|\omega),\\
p^\sigma_{CB}(\omega) &= \sum_{k=0}^{\infty}\sum_{m=0}^{k} P^\sigma(m,k+1,k+2|\omega).
\end{aligned}
$$

These three probabilities involve near two-way ties, and converge to zero as the expected number of voters n goes to infinity. To determine the best response of a t_1 or t_2 voter in a large election, it is useful to know how fast these probabilities converge to 0. The following lemma answers that question.

Define

$$T(\omega) \;=\; (\tau(A|\omega)\tau(B|\omega)\tau(C|\omega))^{1/3}.$$

Then

Lemma 2 *For* $(X,Y,Z) = (A,B,C),(A,C,B),(C,B,A)$ *and* $\omega = \omega_1, \omega_2$,

$$\lim_{n\to\infty} \frac{\log p^{\sigma}_{XY}(\omega)}{n} \;=\; \begin{cases} -(\tau(X|\omega)^{1/2} - \tau(Y|\omega)^{1/2})^2 & \textit{if } T(\omega) \geq \tau(Z|\omega) \\ -1 + 3T(\omega) & \textit{if } T(\omega) < \tau(Z|\omega). \end{cases}$$

□

Note that

$$-(\tau(X|\omega)^{1/2} - \tau(Y|\omega)^{1/2})^2 \geq -1 + 3T(\omega)$$

with strict equality if and only if $T(\omega) = \tau(Z|\omega) > 0$. The proof of the lemma is in the appendix and is an application of the “magnitude theorem” in Myerson (1998b). The lemma states that the probability of a near tie between two alternatives goes faster to zero if the probability that a random voter votes for the excluded alternative is higher than the geometric mean of the probabilities of voting for each of the three alternatives. Otherwise, the probability of a near tie between two alternatives goes faster to zero the larger is the difference between the probabilities that a random voter will vote for each of the two alternatives. Note also that if

$$\lim_{n\to\infty} \frac{\log p^{\sigma}_{AB}(\omega)}{n} \;>\; \lim_{n\to\infty} \frac{\log p^{\sigma}_{AC}(\omega)}{n}$$

then

$$\lim_{n\to\infty} \frac{p^{\sigma}_{AC}(\omega)}{p^{\sigma}_{AB}(\omega)} \;=\; 0,$$

and similarly for the other pairs of probabilities. This fact will be used throughout the paper.

Going back to the problem of t_1 and t_2 voters, let

$$\begin{aligned} G^{\sigma}(t) \;=\; & q(\omega_1|t)\left(p^{\sigma}_{AB}(\omega_1) + p^{\sigma}_{AC}(\omega_1)\right) \\ & -q(\omega_2|t)\left(u_{AB}p^{\sigma}_{AB}(\omega_2) + u_{CB}p^{\sigma}_{CB}(\omega_2) - p^{\sigma}_{AC}(\omega_2)\right). \end{aligned}$$

$G^\sigma(t)$ is the expected (normalized) gain of voting for A instead of for B for a t_1 or t_2 voter, conditional on the information possessed by the voter. Since $q(\omega_1|t_1)/q(\omega_2|t_1) > q(\omega_1|t_2)/q(\omega_2|t_2)$, this gain is strictly larger for a t_1 voter than for a t_2 voter. Thus

Lemma 3 *Given a strategy σ followed by every other voter, if a voter is playing a best response and his type is t_1 or t_2, he will vote for B with probability 0 if $G^\sigma(t) > 0$ and with probability 1 if $G^\sigma(t) < 0$. Moreover, $G^\sigma(t_1) > G^\sigma(t_2)$. If σ is an equilibrium strategy, then*

$$\sigma(B|t_2) < 1 \Rightarrow \sigma(B|t_1) = 0 \quad \textit{and} \quad \sigma(B|t_1) > 0 \Rightarrow \sigma(B|t_2) = 1.$$

□

This lemma simplifies the search for equilibrium strategies. In particular, define

$$h \;=\; \sigma(B|t_2) + \sigma(B|t_1).$$

By an abuse of notation, denote by h any strategy σ that satisfies the restrictions imposed by Lemmas 1 and 3, and such that $\sigma(B|t_2)+\sigma(B|t_1) = h$. Note that $h = 0$ denotes an equilibrium in which all t_1 and t_2 voters vote for alternative A, $h = 2$ denotes an equilibrium in which all t_1 and t_2 voters vote for alternative B, and $h = 1$ denotes an equilibrium in which t_2 voters vote for B and t_1 voters vote for A.

We have:

Theorem 1 *For large enough n, there is an equilibrium with $h = 0$, an equilibrium with $h = 2$, and at least one equilibrium with $0 < h < 2$. The value of h corresponding to a sequence of these intermediate equilibria converges to 1 as n goes to infinity.*

□

Proof To prove that there is an equilibrium with $h = 0$ and an equilibrium with $h = 2$, we can use Lemma 2 and $p^0_{AC}(\omega_1) = p^0_{AC}(\omega_2)$ to show that

$$\lim_{n\to\infty} \frac{G^0(t)}{p^0_{AC}(\omega_1)} = q(\omega_1|t) + q(\omega_2|t) = 1$$

$$\lim_{n\to\infty} \frac{G^2(t)}{p^2_{CB}(\omega_2)} = -q(\omega_2|t)u_{CB}.$$

Thus, for large enough n, $G^0(t_1), G^0(t_2) > 0$ and $G^2(t_1), G^2(t_2) < 0$.

Note that, from Lemma 3, $0 < h < 1$ is an equilibrium iff $G^h(t_2) = 0$; $1 < h < 2$ is an equilibrium iff $G^h(t_1) = 0$; and $h = 1$ is an equilibrium iff $G^1(t_2) \leq 0$ and $G^1(t_1) \geq 0$. The proof that there is an equilibrium with an intermediate value of h consists of three cases. Let

$$R = (r_{11}r_{12}r_3)^{1/3}.$$

Consider first the case $r_{12} > r_3$. This implies that $r_{11} > R > r_3$. By Lemma 2,

$$\lim_{n\to\infty} \frac{\log p^1_{AB}(\omega_1)}{n} = \lim_{n\to\infty} \frac{\log p^1_{AB}(\omega_2)}{n} = -(r_{11}^{1/2} - r_{12}^{1/2})^2$$

$$\lim_{n\to\infty} \frac{\log p^1_{AC}(\omega_1)}{n} = \lim_{n\to\infty} \frac{\log p^1_{CB}(\omega_2)}{n} = \max\left\{-(r_{11}^{1/2} - r_3^{1/2})^2, -1 + 3R\right\}$$

$$\lim_{n\to\infty} \frac{\log p^1_{AC}(\omega_2)}{n} = -1 + 3R.$$

Recall that $-(r_{11}^{1/2} - r_{12}^{1/2})^2 > -1 + 3R$ (see the discussion after Lemma 2). This implies that

$$\begin{aligned}\lim_{n\to\infty} \frac{\log p^1_{AB}(\omega_1)}{n} &= \lim_{n\to\infty} \frac{\log p^1_{AB}(\omega_2)}{n} \\ &> \lim_{n\to\infty} \frac{\log p^1_{AC}(\omega_1)}{n} = \lim_{n\to\infty} \frac{\log p^1_{CB}(\omega_2)}{n} \geq \lim_{n\to\infty} \frac{\log p^1_{AC}(\omega_2)}{n}.\end{aligned}$$

Since $p^1_{AB}(\omega_1) = p^1_{AB}(\omega_2)$, we have

$$\lim_{n\to\infty} \frac{G^1(t)}{p^1_{AB}(\omega_1)} = q(\omega_1|t) - u_{AB}q(\omega_2|t).$$

Note that

$$q(\omega_1|t_2) - u_{AB}q(\omega_2|t_2) < q(\omega_1|t_1) - u_{AB}q(\omega_2|t_1).$$

We have five subcases. Suppose first that $q(\omega_1|t_1) - u_{AB}q(\omega_2|t_1) < 0$. Then $\lim_{n\to\infty} G^1(t_1), G^1(t_2) < 0$. Consider a strategy $h = 1 + \epsilon$, with $\epsilon > 0$ and such that $r_{12}(1-\epsilon) > r_3$ and $r_{11}(1-\epsilon) > r_{12} + \epsilon r_{11}$. These two conditions imply that $\tau(A|\omega_1) > \tau(B|\omega_1) > \tau(C|\omega_1)$ and $\tau(B|\omega_2) > \tau(A|\omega_2) > \tau(C|\omega_2)$. Note also that

$$-((r_{11}(1-\epsilon))^{1/2} - (r_{12} + \epsilon r_{11})^{1/2})^2 > -((r_{11} + \epsilon r_{12})^{1/2} - (r_{12}(1-\epsilon))^{1/2})^2$$

or, equivalently, for the strategy $1+\epsilon$,

$$-(\tau(A|\omega_1)^{1/2} - (\tau(B|\omega_1)^{1/2})^2 > -(\tau(A|\omega_2)^{1/2} - (\tau(B|\omega_2)^{1/2})^2.$$

Using Lemma 2, we can show that

$$\lim_{n\to\infty} \frac{G^{1+\epsilon}(t_1)}{p_{AB}^{1+\epsilon}(\omega_1)} = q(\omega_1|t_1).$$

Thus, $\lim_{n\to\infty} G^{1+\epsilon}(t_1) > 0$. Since $G^h(t_1)$ is continuous in h, this implies that for large enough n there exists some $\mu(n) \in (0,\epsilon)$ such that $G^{1+\mu(n)}(t_1) = 0$. But then $h = 1 + \mu(n)$ is an equilibrium. Since ϵ can be chosen to be arbitrarily small, for large n we can find an equilibrium arbitrarily close from above to $h = 1$.

Suppose that $q(\omega_1|t_2) - u_{AB}q(\omega_2|t_2) > 0$. Then $\lim_{n\to\infty} G^1(t_1), G^1(t_2) > 0$. For $\epsilon > 0$ such that $r_{12}(1-\epsilon) > r_3$ and $r_{11}(1-\epsilon) > r_{12} + \epsilon r_{11}$, we can show that

$$\lim_{n\to\infty} \frac{G^{1-\epsilon}(t_2)}{p_{AB}^{1-\epsilon}(\omega_2)} = -q(\omega_2|t_2).$$

Thus, $\lim_{n\to\infty} G^{1-\epsilon}(t_2) < 0$. Since $G^h(t_2)$ is continuous in h, this implies that for large enough n there exists some $\mu(n) \in (0,\epsilon)$ such that $G^{1-\mu(n)}(t_2) = 0$. But then $h = 1 - \mu(n)$ is an equilibrium. Since ϵ can be chosen to be arbitrarily small, for large n we can find an equilibrium arbitrarily close from below to $h = 1$.

Suppose that $q(\omega_1|t_2) - u_{AB}q(\omega_2|t_2) < 0 < q(\omega_1|t_1) - u_{AB}q(\omega_2|t_1)$. In this subcase, for large enough n, $h = 1$ is an equilibrium.

Suppose that $q(\omega_1|t_1) - u_{AB}q(\omega_2|t_1) = 0$ and $\lim_{n\to\infty} G^1(t_1) = 0$. For any subsequence such that $G^1(t_1)$ converges from above, we are back in the previous subcase, while for any subsequence such that $G^1(t_1)$ converges from below we are in situation similar to the first subcase considered. The analysis is similar in the last subcase, $q(\omega_1|t_2) - u_{AB}q(\omega_2|t_2) = 0$.

Now consider the case $r_3 > r_{12}$. This implies that $R > r_{12}$. By Lemma 2,

$$\lim_{n\to\infty} \frac{\log p^1_{AC}(\omega_1)}{n} = \lim_{n\to\infty} \frac{\log p^1_{CB}(\omega_2)}{n} = -(r_{11}^{1/2} - r_3^{1/2})^2$$

$$\lim_{n\to\infty} \frac{\log p^1_{AB}(\omega_1)}{n} = \lim_{n\to\infty} \frac{\log p^1_{AB}(\omega_2)}{n} = \max\left\{-(r_{11}^{1/2} - r_{12}^{1/2})^2, -1+3R\right\}$$

$$\lim_{n\to\infty} \frac{\log p^1_{AC}(\omega_2)}{n} = \max\left\{-(r_3^{1/2} - r_{12}^{1/2})^2, -1+3R\right\}.$$

Recall that $-(r_{11}^{1/2} - r_3^{1/2})^2 > -1 + 3R$. Also, $\lim_{n\to\infty} \log p^1_{AC}(\omega_2)/n = -(r_3^{1/2} - r_{12}^{1/2})^2$ iff $r_{11} < R$ (from Lemma 2). But $r_{11} < R$ implies $r_3 > r_{11}$ and $-(r_{11}^{1/2} - r_3^{1/2})^2 > -(r_3^{1/2} - r_{12}^{1/2})^2$. Thus,

$$\lim_{n\to\infty} \frac{\log p^1_{AC}(\omega_1)}{n} = \lim_{n\to\infty} \frac{\log p^1_{CB}(\omega_2)}{n}$$
$$> \max\left\{\lim_{n\to\infty} \frac{\log p^1_{AB}(\omega_1)}{n}, \lim_{n\to\infty} \frac{\log p^1_{AB}(\omega_2)}{n}, \lim_{n\to\infty} \frac{\log p^1_{AC}(\omega_2)}{n}\right\}.$$

Since $p^1_{AC}(\omega_1) = p^1_{CB}(\omega_2)$, we have

$$\lim_{n\to\infty} \frac{G^1(t)}{p^1_{AC}(\omega_1)} = q(\omega_1|t) - u_{CB}q(\omega_2|t).$$

The rest of the analysis is similar to the previous case. In particular, if $r_{11} > r_3$, for small enough ϵ,

$$-((r_{11}(1-\epsilon))^{1/2} - r_3^{1/2})^2 > -((r_{11} + \epsilon r_{12})^{1/2} - r_3^{1/2})^2.$$

If $q(\omega_1|t_1) - u_{CB}q(\omega_2|t_1) < 0$, $\lim_{n\to\infty} G^1(t_1) < 0$ and $\lim_{n\to\infty} G^{1+\epsilon}(t_1) > 0$. Thus, for large enough n there is an equilibrium arbitrarily close from above to $h = 1$, while if $q(\omega_1|t_2) - u_{CB}q(\omega_2|t_2) > 0$ there is an equilibrium arbitrarily close from below. If $r_{11} \le r_3$, the position of the intermediate equilibrium with respect to $h = 1$ in those subcases is reversed.

Finally, consider the case $r_{12} = r_3$. This implies that $r_{11} > R > r_{12}$. By

Lemma 2,

$$\lim_{n\to\infty} \frac{\log p^1_{AB}(\omega_1)}{n} = \lim_{n\to\infty} \frac{\log p^1_{AB}(\omega_2)}{n} = -(r_{11}^{1/2} - r_{12}^{1/2})^2$$

$$\lim_{n\to\infty} \frac{\log p^1_{AC}(\omega_1)}{n} = \lim_{n\to\infty} \frac{\log p^1_{CB}(\omega_2)}{n} = -(r_{11}^{1/2} - r_{3}^{1/2})^2$$

$$\lim_{n\to\infty} \frac{\log p^1_{AC}(\omega_2)}{n} = -1 + 3R.$$

Moreover, $p^1_{AB}(\omega_1) = p^1_{AB}(\omega_2) = p^1_{AC}(\omega_1) = p^1_{CB}(\omega_2)$. Then

$$\lim_{n\to\infty} \frac{G^1(t)}{p^1_{AB}(\omega_1)} = 2q(\omega_1|t) - (1 + u_{CB})q(\omega_2|t).$$

The rest of the analysis is similar to previous cases. In particular, if $2q(\omega_1|t_1) - (1 + u_{CB})q(\omega_2|t_1) < 0$, there is an equilibrium arbitrarily close from above to $h = 1$, while if $2q(\omega_1|t) - (1 + u_{CB})q(\omega_2|t) > 0$, there is an equilibrium arbitrarily close from below. ■

The equilibria with $h = 0$ and $h = 2$ are sometimes referred as "Duvergerian equilibria" in the political science literature (see, e.g., Riker 1986, Cox 1997 and the references therein). In the Duvergerian equilibria of the model, only two of the three alternatives receive a positive fraction of the votes cast in the election. The idea is that, if a voter perceives that no one else will vote for a given alternative, voting for that alternative would be equivalent to "wasting a vote" because the probability that this alternative will be nearly tied for the first place is negligible compared to the probability that the other two alternatives will be nearly tied. The next result establishes that, if $r_{11} \leq r_3$, a non-Duvergerian equilibrium would entail a utility loss for t_1 and t_2 voters. Since t_1 and t_2 voters are in the majority, we may think about a non-Duvergerian equilibrium as a sort of coordination failure. For the case $r_{11} > r_3$, however, a Duvergerian equilibrium would constitute a coordination failure:

Corollary 1 *If $r_{11} \leq r_3$, the maximal equilibrium expected payoff for t_1 and t_2 voters converges to $\max\{U(A, t_1, \omega_1), q(\omega_1)U(B, t_1, \omega_1) + q(\omega_2)U(B, t_1, \omega_2)\}$ as n goes to infinity. If $r_{11} > r_3$, the maximal equilibrium expected payoff for*

t_1 and t_2 voters converges to $q(\omega_1)U(A, t_1, \omega_1) + q(\omega_2)U(B, t_1, \omega_2)$ as n goes to infinity. □

Proof For the first part, note that an equilibrium with $h = 0$ has expected payoff of $U(A, t_1, \omega_1)$, while an equilibrium with $h = 2$ has an expected payoff of $q(\omega_1)U(B, t_1, \omega_1)+q(\omega_2)U(B, t_1, \omega_2)$ for t_1 and t_2 voters. From Theorem 1, both equilibria exist. An equilibrium with $0 < h < 2$ can improve upon both $h = 0$ and $h = 2$ only if in this equilibrium $\tau(A|\omega_1) \geq r_3$ and $\tau(B|\omega_2) \geq r_3$. Thus we need

$$\begin{aligned} \sigma(A|t_1)r_{11} + \sigma(A|t_2)r_{12} &\geq r_3, \\ \sigma(B|t_1)r_{12} + \sigma(B|t_2)r_{11} &\geq r_3. \end{aligned}$$

Since $r_{11} > r_{12}$, we have $(2 - h)r_{11} > r_3$ and $hr_{11} > r_3$. Adding up these two inequalities we obtain $r_{11} > r_3$.

For the second part, from Theorem 1, there is a sequence of equilibria with $0 < h < 2$ such that h converges to 1. But if $r_{11} > r_3$ and h is close enough to 1, alternative A will win the election with probability arbitrarily close to 1 in state ω_1, while alternative B will win the election with probability arbitrarily close to 1 in state ω_2. ■

4 Plurality Runoff

Under *plurality runoff*, each voter must cast simultaneously a vote for either A, B or C, and the alternative that receives most votes is chosen if the number of votes it receives exceeds half the total number of votes cast in the election. Otherwise, the two alternatives that receive more votes go into a runoff election, and the alternative with most votes in the runoff is chosen. Ties are broken alphabetically.

With respect to the runoff, suppose the electorate is drawn again according to the process described in section 2. For a large n, this means that the probability of A or B defeating C in a runoff is arbitrarily close to 1. Also, from Theorem 2 in Myerson (1998a), there is a sequence of equilibria in the continuation game corresponding to a runoff between A and B such that the probability of A being chosen in state ω_1 and the probability of B being chosen in state ω_2 converge to 1 as n goes to infinity.[7] We do not model explicitly the continuation game corresponding to a runoff and instead assume that A or B defeat C with probability 1 in a runoff, and that A defeats B with probability 1 if and only if the state is ω_1.

A *strategy function* $\tilde{\sigma}$ in a plurality runoff game is a mapping $\tilde{\sigma}$: $\{t_1, t_2, t_3\} \rightarrow \Delta(\{A, B, C\})$, where $\tilde{\sigma}(X|t)$ is the probability that a voter will vote in the first round for alternative X if his type is t. We must have $\tilde{\sigma}(X|t) \geq 0$ for every X and every t, and $\sum_X \tilde{\sigma}(X|t) = 1$ for every t. A strategy function $\tilde{\sigma}$ is an *equilibrium* if it maximizes the expected utility of every voter when other voters use the strategy $\tilde{\sigma}$.

As before, let $\tau(X|\omega)$ be the probability that a random voter votes in the first round for alternative X in state ω for a given strategy $\tilde{\sigma}$. The number of voters who choose alternative X if the state is ω is, then, a Poisson random variable with mean $\tau(X|\omega)n$. We will denote by $\tilde{P}^{\tilde{\sigma}}(k, l, m|\omega)$ the probability that k, l, m voters vote for alternatives A, B and C respectively in the first round, in state ω given the strategy $\tilde{\sigma}$.

Note that a vote for C in the first round increases the probability of C being chosen in the first round or disputing a runoff, while leaving unaffected the outcome of a runoff, if every voter uses runoff pooling strategies. On the other hand, voting for A or B reduces the probability of C being chosen in the first round or disputing a runoff, while leaving unaffected the expected outcome in case of a runoff. Thus, voting for C is a strictly dominant strategy for t_3 voters. However, as opposed to the case of plurality rule, voting for C is not a strictly dominated strategy for t_1 and t_2 voters. The reason is that we do not allow for abstention, and for some strategy profiles t_1 or t_2 voter may prefer to avoid being decisive for a victory of A or B in the first round, if the risk of C winning in the first round or getting into the runoff is small enough.[8] To keep matters simple, rather than allowing for abstention, we restrict our attention to situations in which C is a "serious contender."

[7]The same result is shown by Feddersen and Pesendorfer (1997) in a game without population uncertainty: if the size and preferences of the electorate are common knowledge, elections fully aggregate information in a two-way race in the sense that the chosen alternative would not change if all private information would be common knowledge.

[8]See Feddersen and Pesendorfer (1996) and (1999) for an analysis of two-way races in which voters may be better off abstaining.

Formally, C is a *serious contender* if

$$-1+2(1-r_3)^{1/2}r_3^{1/2} \;>\; -(r_{11}^{1/2}-r_{12}^{1/2})^2$$

and $r_3 > 1/3$. This condition requires that r_3 is closer to $1/2$ the closer is r_{12} to r_{11}. The intuition for this requirement is that a voter will be less tempted to abstain to vote for either A or B the larger is the risk of C winning the election and the better is the information contained in the voter's type. The following result is proved in the Appendix:

Lemma 4 *If a voter is playing a best response, then in the first round he will vote for C with probability 1 if his type is t_3. Moreover, if C is a serious contender and the strategy followed by other voters satisfies $\tilde{\sigma}(C|t_1) = \tilde{\sigma}(C|t_2) = 0$ and $\tilde{\sigma}(C|t_3) = 1$, for large enough n if a voter is playing a best response then in the first round he will vote for C with probability 0 if his type is t_1 or t_2.* □

A t_1 or t_2 voter deciding whether to support A or B must be concerned about the events in which his vote is pivotal. Those events are: {voting for A in the first round ultimately yields A but voting for B yields B}, {voting for A in the first round ultimately yields A but voting for B yields C}, and {voting for A in the first round ultimately yields C but voting for B yields B}. The probabilities of the last two events are 0, regardless of the state of the world. This is because a near tie between C and any other alternative in the first round already implies that a runoff will take place once one more vote is cast for either A or B, and C will ultimately be defeated. The probability of the first event given state ω_1, if other voters are following the strategy $\tilde{\sigma}$, is given by

$$\begin{aligned} p_{AB}^{\tilde{\sigma}}(\omega_1) &= \sum_{k=0}^{\infty}\left(\sum_{m=k+1}^{2k+1}\tilde{P}^{\tilde{\sigma}}(k,k,m|\omega)+\sum_{m=\max\{1,k\}}^{2k+1}\tilde{P}^{\tilde{\sigma}}(k,k+1,m+1|\omega)\right) \\ &+ \sum_{k=0}^{\infty}\sum_{m=0}^{k+1}\left(\tilde{P}^{\tilde{\sigma}}(k,k+m,m|\omega)+\tilde{P}^{\tilde{\sigma}}(k,k+m+1,m|\omega)\right). \end{aligned}$$

The first term represents situations in which a vote for either A or B determines which of the two alternatives disputes a runoff with C. The second term represent situations in which a vote for A leads to a runoff between A and B while a vote for B leads to a victory of B in the first round. Similarly, the probability of the first event given state ω_2, if other voters are following the strategy $\tilde{\sigma}$, is given by

$$\begin{aligned} p_{AB}^{\tilde{\sigma}}(\omega_2) &= \sum_{k=0}^{\infty}\left(\sum_{m=k+1}^{2k+1}\tilde{P}^{\tilde{\sigma}}(k,k,m|\omega)+\sum_{m=\max\{1,k\}}^{2k+1}\tilde{P}^{\tilde{\sigma}}(k,k+1,m+1|\omega)\right) \\ &+ \sum_{k=0}^{\infty}\sum_{m=0}^{k+1}\left(\tilde{P}^{\tilde{\sigma}}(k+m,k,m|\omega)+\tilde{P}^{\tilde{\sigma}}(k+m+1,k,m|\omega)\right). \end{aligned}$$

The two probabilities just described converge to 0 as the expected number of voters n goes to infinity. The following lemma tells us how fast they converge. Let

$$\begin{aligned} S(\omega_1) &= (\tau(A|\omega_1)+\tau(C|\omega_1))^{1/2}\tau(B|\omega_1)^{1/2}, \\ S(\omega_2) &= \tau(A|\omega_2)^{1/2}(\tau(B|\omega_2)+\tau(C|\omega_2))^{1/2}. \end{aligned}$$

We have

Lemma 5 *If $\tau(A|\omega)^{1/2}\tau(B|\omega)^{1/2} < \tau(C|\omega) < 1/2$ for $\omega = \omega_1, \omega_2$, then*

$$\lim_{n\to\infty}\frac{\log p_{AB}^{\tilde{\sigma}}(\omega)}{n} = \max\{-(\tau(A|\omega)^{1/2}-\tau(B|\omega)^{1/2})^2, -1+2S(\omega)\}.$$

□

The proof of Lemma 5 is in the appendix. Lemma 5 reflects the fact that the more likely event in which a vote for A or B is decisive is either a near tie between A and B for the second place in the first round or a situation in which alternative B in state ω_1 (or alternative A in state ω_2) is close to winning the election in the first round (and avoiding losing the runoff).

Let

$$\tilde{G}^{\tilde{\sigma}}(t) = q(\omega_1|t)p_{AB}^{\tilde{\sigma}}(\omega_1) - q(\omega_2|t)u_{AB}p_{AB}^{\tilde{\sigma}}(\omega_2).$$

$\tilde{G}^{\tilde{\sigma}}(t)$ is the expected (normalized) gain of voting for A instead of for B for a t_1 or t_2 voter, conditional on the information possessed by the voter. As in the case of plurality, this gain is larger for a t_1 voter than for a t_2 voter. Thus

Lemma 6 *Given a strategy $\tilde{\sigma}$ followed by every other voter, if a voter is playing a best response and his type is t_1 or t_2, he will vote for B in the first round with probability 0 if $\tilde{G}^{\tilde{\sigma}}(t) > 0$ and he will vote for A in the first round with probability 0 if $\tilde{G}^{\tilde{\sigma}}(t) < 0$. Moreover, $\tilde{G}^{\tilde{\sigma}}(t_1) > \tilde{G}^{\tilde{\sigma}}(t_2)$. If $\tilde{\sigma}$ is an equilibrium strategy, then*

$$\tilde{\sigma}(B|t_2) < 1 \Rightarrow \tilde{\sigma}(B|t_1) = 0 \quad \text{and} \quad \tilde{\sigma}(B|t_1) > 0 \Rightarrow \tilde{\sigma}(A|t_2) = 0.$$

□

This lemma simplifies the search for equilibrium strategies. In particular, define

$$\tilde{h} \;=\; \tilde{\sigma}(B|t_2) + \tilde{\sigma}(B|t_1).$$

By an abuse of notation, we denote by $\tilde{h}$ any strategy $\tilde{\sigma}$ that satisfies the restrictions imposed by lemmas 4 and 6, with $\tilde{\sigma}(B|t_2) + \tilde{\sigma}(B|t_1) = \tilde{h}$. As before, $\tilde{h} = 1$ denotes an equilibrium in which t_1 voters vote for A and t_2 voters vote for B in the first round. We have

Theorem 2 *If C is a serious contender, for large enough n there is at least one equilibrium with $0 < \tilde{h} < 2$. The value of $\tilde{h}$ corresponding to any sequence of equilibria converges to 1 as n goes to infinity.* □

Remark This theorem implies in particular that in a large election there is no equilibrium in which t_1 and t_2 voters "coordinate" in voting only for A or only for B.

Proof From Lemmas 4 and 6, for large n, $\tilde{h} = 0$ is an equilibrium iff $\tilde{G}^{\tilde{h}}(t_2) \geq 0$, $0 < \tilde{h} < 1$ is an equilibrium iff $\tilde{G}^{\tilde{h}}(t_2) = 0$; $\tilde{h} = 1$ is an equilibrium iff $\tilde{G}^1(t_2) \leq 0$ and $\tilde{G}^1(t_1) \geq 0$; $1 < \tilde{h} < 2$ is an equilibrium iff $\tilde{G}^{\tilde{h}}(t_1) = 0$; and $\tilde{h} = 2$ is an equilibrium iff $\tilde{G}^{\tilde{h}}(t_1) \leq 0$.

If C is a serious contender, $r_3 > \tau(A|\omega)^{1/2}\tau(B|\omega)^{1/2}$ for any strategy $\tilde{h}$. Thus, we can use Lemma 5 for any such strategy. Now, consider any strategy $\tilde{h} = 1 - \epsilon$, with $0 < \epsilon \leq 1$. Note that

$$-((r_{11}(1-\epsilon))^{1/2} - (r_{12} + \epsilon r_{11})^{1/2})^2 > -((r_{11} + \epsilon r_{12})^{1/2} - ((1-\epsilon)r_{12})^{1/2})^2$$

and

$$\begin{aligned} & -1 + 2((1-\epsilon)r_{11} + r_3)^{1/2}((r_{12} + \epsilon r_{11})^{1/2} \\ & \qquad\qquad > -1 + 2(r_{11} + \epsilon r_{12} + r_3)^{1/2}((1-\epsilon)r_{12})^{1/2}. \end{aligned}$$

It follows from Lemma 5 that $\lim_{n\to\infty} \tilde{G}^{\tilde{h}}(t_1), \tilde{G}^{\tilde{h}}(t_2) < 0$ so that there cannot be a sequence of equilibria converging to some $\tilde{h} < 1$. Now consider any strategy $\tilde{h} = 1 + \epsilon$, with $0 < \epsilon \leq 1$. By an argument similar to the one put forward above, we have $\lim_{n\to\infty} \tilde{G}^{\tilde{h}}(t_1), \tilde{G}^{\tilde{h}}(t_2) > 0$. Thus, there cannot be an equilibrium sequence converging to some $\tilde{h} > 1$.

The proof that there is in fact a sequence of equilibria converging to $\tilde{h} = 1$ has five cases. Since $p^1_{AB}(\omega_1) = p^1_{AB}(\omega_2)$, we have

$$\frac{\tilde{G}^1(t)}{p^1_{AB}(\omega_1)} = q(\omega_1|t) - u_{AB}q(\omega_2|t).$$

Recall that $q(\omega_1|t_1) - u_{AB}q(\omega_2|t_1) > q(\omega_1|t_2) - u_{AB}q(\omega_2|t_2)$. Suppose first that $q(\omega_1|t_1) - u_{AB}q(\omega_2|t_1) < 0$. Then $\lim_{n\to\infty} \tilde{G}^1(t_1), \tilde{G}^1(t_2) < 0$. By the previous argument, $\lim_{n\to\infty} \tilde{G}^{1+\epsilon}(t_1), \tilde{G}^{1+\epsilon}(t_2) > 0$ for every $\epsilon \in (0,1]$. Since $\tilde{G}^{\tilde{h}}(t_1)$ is continuous in $\tilde{h}$ for every ϵ for large enough n there exists some $\mu(n) \in (0,\epsilon)$ such that $\tilde{G}^{1+\mu(n)}(t_1) = 0$. But then $\tilde{h} = 1 + \mu(n)$ is an equilibrium. Since ϵ can be chosen to be arbitrarily small, for large enough n we can find an equilibrium arbitrarily close from above to $\tilde{h} = 1$.

Similarly, if $q(\omega_1|t_2) - u_{AB}q(\omega_2|t_2) > 0$, there is an equilibrium arbitrarily close from below to $\tilde{h} = 1$, and if

$$q(\omega_1|t_2) - u_{AB}q(\omega_2|t_2) < 0 < q(\omega_1|t_1) - u_{AB}q(\omega_2|t_1),$$

for large enough n, $\tilde{h} = 1$ is an equilibrium. For the other two subcases, $q(\omega_1|t_1) - u_{AB}q(\omega_2|t_1) = 0$ and $q(\omega_1|t_2) - u_{AB}q(\omega_2|t_2) = 0$, see the proof of Theorem 1. ■

Recall that $r_{11} > r_{12}$ and, if C is a serious contender, $1/2 > r_3 > r_{12}$. Thus, from Theorem 2, A is arbitrarily likely to win the election in the first round or to dispute a runoff with C in state ω_1, and B is arbitrarily likely to win the election in the first round or to dispute a runoff with C in state ω_2. Thus

Corollary 2 *If C is a serious contender, the equilibrium expected payoff for t_1 and t_2 voters converges to $U(A, t_1, \omega_1)q(\omega_1) + U(B, t_1, \omega_2)q(\omega_2)$ as n goes to infinity.* □

5 Conclusion

This paper compares simple plurality voting and plurality runoff from the point of view of a group of voters with common preferences that is likely to be the majority but which have divided opinions about which candidate to support due to private information. A situation with three candidates is modeled. The analysis gets complicated as it is conducted in a framework of rational, strategic voters which are able to compare near zero probabilities. The results of the analysis are clear, however. Simple plurality allows for successful information aggregation among majority voters only if the candidate they like the least is not supported by a large minority. If the candidate majority voters like the least is in fact supported by a large minority, plurality runoff gives majority voters a higher expected payoff. The advantage of plurality runoff over simple plurality in terms of information aggregation seems likely to hold in more complex situations (with more candidates and more heterogeneity of preferences), even if completely successful information aggregation of the sort obtained in this paper is not likely to hold. Ignoring the possibility of coordination failures under plurality voting, the advantage of the runoff system consists in introducing an stage in the election game in which voters can express their opinions without risking completely the final result of the election. The advantage of the runoff system disappears if primaries take place or if a sequence of preelection polls allow voters to successfully pool their information about candidates.

Appendix

Preliminaries We state here a result from Myerson (1998b) that will be useful in the proofs that follow. Let $\lambda_n(\omega) = (\lambda_n(A|\omega), \lambda_n(B|\omega), \lambda_n(C|\omega))$ be any sequence of voting profiles. Define

$$\Psi(\theta) \; = \; \theta(1 - \log \theta).$$

By Lemma 1 in Myerson (1988b), if either limit exists,

$$\lim_{n\to\infty} \frac{\log \Pr(\lambda_n(\omega))}{n} \quad = \quad \lim_{n\to\infty} \sum_{X\in\{A,B,C\}} \tau(X|\omega)\Psi\left(\frac{\lambda_n(X|\omega)}{n\tau(X|\omega)}\right).$$

Moreover, if $\Lambda_n(\omega)$ is a sequence of events, from Theorem 1 in Myerson (1988b),

$$\lim_{n\to\infty} \frac{\log \Pr(\Lambda_n(\omega))}{n} \quad = \quad \lim_{n\to\infty} \max_{\lambda_n(\omega)\in\Lambda_n(\omega)} \sum_{X\in\{A,B,C\}} \tau(X|\omega)\Psi\left(\frac{\lambda_n(X|\omega)}{n\tau(X|\omega)}\right).$$

That is, the probability of a sequence of events is concentrated in the limit in the voting profiles in that event with maximum probability.

Proof of Lemma 2 Consider the sequence of events

$$L_n(\omega) \quad = \quad \bigcup_{\substack{k\geq 0 \\ 0\leq m\leq k+1}} \{(k,k,m),(k,k+1,m)\},$$

with probability $p^{\sigma}_{AB}(\omega)$. If $\tau(C|\omega) < (\tau(A|\omega)\tau(B|\omega))^{1/2}$ or, equivalently,

$$\tau(C|\omega) < (\tau(A|\omega)\tau(B|\omega)\tau(C|\omega))^{1/3},$$

then, for large n, $\sum_{X\in\{A,B,C\}} \tau(X|\omega)\Psi(\lambda_n(X|\omega)/(n\tau(X|\omega)))$ is maximized on $\lambda_n(\omega) \in L_n(\omega)$ by

$$\lambda_n(\omega) \quad = \quad \left(n(\tau(A|\omega)\tau(B|\omega))^{1/2}, n(\tau(A|\omega)\tau(B|\omega))^{1/2}, n\tau(C|\omega)\right)$$

(ignoring integer constraints). Thus,

$$\lim_{n\to\infty} \frac{\log p^{\sigma}_{AB}(\omega)}{n}$$

$$= \tau(A|\omega)\Psi\left(\frac{(\tau(B|\omega))^{1/2}}{(\tau(A|\omega))^{1/2}}\right) + \tau(B|\omega)\Psi\left(\frac{(\tau(A|\omega))^{1/2}}{(\tau(B|\omega))^{1/2}}\right) + \tau(C|\omega)\Psi(1)$$

$$= -(\tau(A|\omega)^{1/2} - \tau(B|\omega)^{1/2})^2.$$

On the other hand, if

$$\tau(C|\omega) > (\tau(A|\omega)\tau(B|\omega)\tau(C|\omega))^{1/3},$$

then, for large n, $\sum_{X\in\{A,B,C\}} \tau(X|\omega)\Psi(\lambda_n(X|\omega)/(n\tau(X|\omega)))$ is maximized on $\lambda_n(\omega) \in L_n(\omega)$ by

$$\lambda_n(X|\omega) \;=\; n(\tau(A|\omega)\tau(B|\omega)\tau(C|\omega))^{1/3}$$

for $X = A, B, C$ (ignoring integer constraints). Thus,

$$\lim_{n\to\infty} \frac{\log p^{\sigma}_{AB}(\omega)}{n} \;=\; -1 + 3(\tau(A|\omega)\tau(B|\omega)\tau(C|\omega))^{1/3}.$$

We can proceed similarly with respect to the other sequences of events in Lemma 2.

Proof of Lemma 4 It is argued in the text that a t_3 voter will vote for C for any strategy of the other voters. Thus, in equilibrium, $\tilde{\sigma}(C|t_3) = 1$. In what follows we show that under the conditions of the lemma a t_1 or t_2 voter obtains a larger payoff by voting for A or B rather than for C if n is large enough. Let $p^{\tilde{\sigma}}_{XY'}, p^{\tilde{\sigma}}_{XY''}$ denote, respectively, the probability that voting for A in the first round ultimately yields X but voting for C yields Y, and the probability that voting for B in the first round ultimately yields X but voting for C yields Y, with $X, Y = A, B, C$. Then, the statement of the lemma obtains if, for n large enough and $t = t_1, t_2$,

$$q(\omega_1|t)(p^{\tilde{\sigma}}_{AB'}(\omega_1) + p^{\tilde{\sigma}}_{AC'}(\omega_1))$$
$$> q(\omega_2|t)(u_{AB}p^{\tilde{\sigma}}_{AB'}(\omega_2) - u_{BC}p^{\tilde{\sigma}}_{BC'}(\omega_2) - p^{\tilde{\sigma}}_{AC'}(\omega_2))$$

or

$$q(\omega_1|t)(p^{\tilde{\sigma}}_{AC''}(\omega_1) - p^{\tilde{\sigma}}_{BA''}(\omega_1))$$
$$> q(\omega_2|t)(-u_{AB}p^{\tilde{\sigma}}_{BA''}(\omega_2) - u_{BC}p^{\tilde{\sigma}}_{BC''}(\omega_2) - p^{\tilde{\sigma}}_{AC''}(\omega_2)).$$

A sufficient condition for this is

$$u_{BC}p^{\tilde{\sigma}}_{BC'}(\omega_2) > u_{AB}p^{\tilde{\sigma}}_{AB'}(\omega_2) \quad \text{or} \quad p^{\tilde{\sigma}}_{AC''}(\omega_1) > p^{\tilde{\sigma}}_{BA''}(\omega_1).$$

Using the same techniques as in the proof of Lemmas 2 and 5 (below) we can show that:

$$\begin{aligned}
\lim_{n\to\infty}(\log p^{\tilde{\sigma}}_{BC'}(\omega_2))/n &= \lim_{n\to\infty}(\log p^{\tilde{\sigma}}_{AC''}(\omega_2))/n = -1 + 2(1-r_3)^{1/2}r_3^{1/2}, \\
\lim_{n\to\infty}(\log p^{\tilde{\sigma}}_{AB'}(\omega_2))/n &= -(\tau(A|\omega_2)^{1/2} - \tau(B|\omega_2)^{1/2})^2, \\
\lim_{n\to\infty}(\log p^{\tilde{\sigma}}_{BA''}(\omega_1)/n &= -(\tau(A|\omega_1)^{1/2} - \tau(B|\omega_1)^{1/2})^2.
\end{aligned}$$

Since $\min\{-(\tau(A|\omega_1)^{1/2} - \tau(B|\omega_1)^{1/2})^2, -(\tau(A|\omega_2)^{1/2} - \tau(B|\omega_2)^{1/2})^2\}$ is less than or equal to $-(r_{11}^{1/2} - r_{12}^{1/2})^2$, we have that $u_{BC}p^{\tilde{\sigma}}_{BC'}(\omega_2) > u_{AB}p^{\tilde{\sigma}}_{AB'}(\omega_2)$ or $p^{\tilde{\sigma}}_{AC''}(\omega_1) > p^{\tilde{\sigma}}_{BA''}(\omega_1)$ if

$$-1 + 2(1-r_3)^{1/2}r_3^{1/2} > -(r_{11}^{1/2} - r_{12}^{1/2})^2.$$

which is satisfied if C is a serious contender.

Proof of Lemma 5 Consider the sequence of events

$$M_n(\omega) = \left(\bigcup_{k=0}^{\infty}\bigcup_{m=k+1}^{2k+1}(k,k,m)\right)\bigcup\left(\bigcup_{k=0}^{\infty}\bigcup_{m=\max\{1,k\}}^{2k+1}(k,k+1,m+1)\right),$$

corresponding to the first term in $p^{\tilde{\sigma}}_{AB}(\omega)$. Recall that, by assumption of the lemma, $\tau(C|\omega) < 1/2$. Following the steps of the proof of Lemma 2, we can show that

$$\lim_{n\to\infty}\frac{\log \Pr(M_n(\omega))}{n} = -(\tau(A|\omega)^{1/2} - \tau(B|\omega)^{1/2})^2.$$

Now consider the sequence of events

$$N_n(\omega_1) = \bigcup_{\substack{k\geq 0\\ 0\leq m\leq k+1}} \{(k,k+m,m),(k,k+m+1,m)\},$$

corresponding to the second term in $p_{AB}^{\tilde{\sigma}}(\omega_1)$. For $\lambda_n(\omega_1) \in N_n(\omega_1)$ and large n, $\sum_{X \in \{A,B,C\}} \tau(X|\omega_1)\Psi(\lambda_n(X|\omega_1)/(n\tau(X|\omega_1)))$ is maximized by

$$\lambda_n(A|\omega_1) = \frac{n\tau(A|\omega_1)\tau(B|\omega_1)^{1/2}}{(\tau(A|\omega_1) + \tau(C|\omega_1))^{1/2}},$$

$$\lambda_n(B|\omega_1) = n(\tau(A|\omega_1) + \tau(C|\omega_1))^{1/2}\tau(B|\omega_1)^{1/2},$$

$$\lambda_n(C|\omega_1) = \frac{n\tau(C|\omega_1)\tau(B|\omega_1)^{1/2}}{(\tau(A|\omega_1) + \tau(C|\omega_1))^{1/2}}.$$

(ignoring integer constraints). Thus,

$$\lim_{n\to\infty}(\log \Pr(N_n(\omega_1)))/n = -1 + 2(\tau(A|\omega_1) + \tau(C|\omega_1))^{1/2}\tau(B|\omega_1)^{1/2}.$$

The claim about state ω_2 obtains similarly.

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