# Treelike families of multiweights 

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#### Abstract

Let $\mathcal{T}=(T, w)$ be a weighted finite tree with leaves $1, \ldots, n$. For any $I:=\left\{i_{1}, \ldots, i_{k}\right\} \subset$ $\{1, \ldots, n\}$, let $D_{I}(\mathcal{T})$ be the weight of the minimal subtree of $T$ connecting $i_{1}, \ldots, i_{k}$; the $D_{I}(\mathcal{T})$ are called $k$-weights of $\mathcal{T}$. Given a family of real numbers parametrized by the $k$-subsets of $\{1, \ldots, n\},\left\{D_{I}\right\}_{I \in(\{1, \ldots, n\}}$, we say that a weighted tree $\mathcal{T}=(T, w)$ with leaves $1, \ldots, n$ realizes the family if $D_{I}(\mathcal{T})=D_{I}$ for any $I$. We give a characterization of the families of real numbers that are realized by some weighted tree.


## 1 Introduction

For any graph $G$, let $E(G), V(G)$ and $L(G)$ be respectively the set of the edges, the set of the vertices and the set of the leaves of $G$. A weighted graph $\mathcal{G}=(G, w)$ is a graph $G$ endowed with a function $w: E(G) \rightarrow \mathbb{R}$. For any edge $e$, the real number $w(e)$ is called the weight of the edge. If all the weights are nonnegative (respectively positive), we say that the graph is nonnegative-weighted (respectively positive-weighted); if the weights of the internal edges are nonzero (respectively positive), we say that the graph is internal-nonzero-weighted (respectively internal-positive-weighted). For any finite subgraph $G^{\prime}$ of $G$, we define $w\left(G^{\prime}\right)$ to be the sum of the weights of the edges of $G^{\prime}$. In this paper we will deal only with weighted finite trees.

Definition 1. Let $\mathcal{T}=(T, w)$ be a weighted tree. For any distinct $i_{1}, \ldots ., i_{k} \in V(T)$, we define $D_{\left\{i_{1}, \ldots, i_{k}\right\}}(\mathcal{T})$ to be the weight of the minimal subtree containing $i_{1}, \ldots ., i_{k}$. We call this subtree "the subtree realizing $D_{\left\{i_{1}, \ldots, i_{k}\right\}}(\mathcal{T})$ ". More simply, we denote $D_{\left\{i_{1}, \ldots, i_{k}\right\}}(\mathcal{T})$ by $D_{i_{1}, \ldots, i_{k}}(\mathcal{T})$ for any order of $i_{1}, \ldots, i_{k}$. We call the $D_{i_{1}, \ldots, i_{k}}(\mathcal{T})$ the $k$-weights of $\mathcal{T}$ and we call a $k$-weight of $\mathcal{T}$ for some $k$ a multiweight of $\mathcal{T}$.

For any $S \subset V(T)$, we call the family $\left\{D_{I}\right\}_{I \in\binom{S}{k}}$ the family of the $k$-weights of $(\mathcal{T}, S)$ or the $k$-dissimilarity family of $(\mathcal{T}, S)$.
We can wonder when a family of real numbers is the family of the $k$-weights of some weighted tree and of some subset of the set of its vertices. If $S$ is a finite set, $k \in \mathbb{N}$ and $k<\# S$,

[^0]we say that a family of real numbers $\left\{D_{I}\right\}_{I \in\binom{S}{k}}$ is treelike (respectively p-treelike, nn-treelike, inz-treelike, ip-treelike, nn-ip-treelike) if there exist a weighted (respectively positive-weighted, nonnegative-weighted, internal-nonzero-weighted, internal-positive-weighted, nonegative-weighted and internal-positive weighted) tree $\mathcal{T}=(T, w)$ and a subset $S$ of the set of its vertices such that $D_{I}(\mathcal{T})=D_{I}$ for any $k$-subset $I$ of $S$. In this case, we say that $\mathcal{T}=(T, w)$ realizes the family $\left\{D_{I}\right\}_{I \in\binom{S}{k}}$.
If in addition $S \subset L(T)$, we say that the family is l-treelike (respectively p-l-treelike, nn-l-treelike, inz-l-treelike, ip-l-treelike, nn-ip-l-treelike).
A criterion for a metric on a finite set to be nn-l-treelike was established in [4], [11], [12]:
Theorem 2. Let $\left\{D_{I}\right\}_{I \in(\{1, \ldots, n\})}$ be a set of positive real numbers satisfying the triangle inequalities. It is $p$-treelike (or nn-l-treelike) if and only if, for all distinct $i, j, k, h \in\{1, \ldots, n\}$, the maximum of
$$
\left\{D_{i, j}+D_{k, h}, D_{i, k}+D_{j, h}, D_{i, h}+D_{k, j}\right\}
$$
is attained at least twice.
In [3], Bandelt and Steel proved a result, analogous to Theorem[2, for general weighted trees; more precisely, they proved that, for any set of real numbers $\left\{D_{I}\right\}_{I \in(\{1, \ldots, n\})}$, there exists a weighted tree $\mathcal{T}$ with leaves $1, \ldots, n$ such that $D_{I}(\mathcal{T})=D_{I}$ for any $I \in\left(\frac{\{1, \ldots, n\}}{2}\right)$ if and only if, for any $a, b, c, d \in\{1, \ldots, n\}$, at least two among $D_{a, b}+D_{c, d}, \quad D_{a, c}+D_{b, d}, \quad D_{a, d}+D_{b, c}$ are equal. For higher $k$ the literature is more recent. Some of the most important results are the following.

Theorem 3. (Pachter-Speyer, [8]). Let $k, n \in \mathbb{N}$ with $3 \leq k \leq(n+1) / 2$. A positive-weighted tree $\mathcal{T}$ with leaves $1, \ldots, n$ and no vertices of degree 2 is determined by the values $D_{I}(\mathcal{T})$, where $I$ varies in $(\underset{k}{\{1, \ldots, n\}})$.

Theorem 4. (Herrmann, Huber, Moulton, Spillner, [5]). If $n \geq 2 k$, a family of positive real numbers $\left\{D_{I}\right\}_{I \in\binom{\{1, \ldots, n\}}{k}}$ is nn-ip-l-treelike if and only if the restriction to every $2 k$-subset of $\{1, \ldots, n\}$ is nn-ip-l-treelike.

Theorem 5. (Levy-Yoshida-Pachter) Let $\mathcal{T}=(T, w)$ be a positive-weighted tree with $L(T)=$ $\{1, \ldots, n\}$. For any $i, j \in\{1, \ldots, n\}$, define

$$
S(i, j)=\sum_{Y \in\binom{\{1, \ldots, n\}-\{i, j\}}{k-2}} D_{i, j, Y}(\mathcal{T}) .
$$

Then there exists a positive-weighted tree $\mathcal{T}^{\prime}=\left(T^{\prime}, w^{\prime}\right)$ such that $D_{i, j}\left(\mathcal{T}^{\prime}\right)=S(i, j)$ for all $i, j \in$ $\{1, \ldots, n\}$, the quartet system of $T^{\prime}$ contains the quartet system of $T$ and, defined $T_{\leq s}$ the subforest of $T$ whose edge set consists of edges whose removal results in one of the components having size at most $s$, we have $T_{\leq n-k} \cong T_{\leq n-k}^{\prime}$.

In [9] and [10], the author gave an inductive characterization of the families of real numbers that are indexed by the subsets of $\{1, \ldots, n\}$ of cardinality greater than or equal to 2 and are the families of the multiweights of a tree with $n$ leaves.
Let $n, k \in \mathbb{N}$ with $n>k$. In [1] we studied the problem of the characterization of the families of positive real numbers, indexed by the $k$-subsets of an $n$-set, that are p-treelike in the "border" case $k=n-1$. Moreover we studied the analogous problem for graphs. See [6] for other results on graphs and see the introduction of 11 for a survey.
Here we examine the case of trees for general $k$. To illustrate the result, we need the following definition.

Definition 6. Let $k \in \mathbb{N}-\{0\}$. We say that a tree $P$ is a pseudostar of kind $(n, k)$ if $\# L(P)=n$ and any edge of $P$ divides $L(P)$ into two sets such that at least one of them has cardinality greater than or equal to $k$.


Figure 1: A pseudostar of kind $(10,8)$

In [2] we proved that, if $3 \leq k \leq n-1$, given a l-treelike family of real numbers, $\left\{D_{I}\right\}_{I \in\binom{\{1, \ldots, n\}}{k}}$, there exists exactly one internal-nonzero-weighted pseudostar $\mathcal{P}$ of kind $(n, k)$ with leaves $1, \ldots, n$ and no vertices of degree 2 such that $D_{I}(\mathcal{P})=D_{I}$ for any $I$. Here we associate to any pseudostar of kind $(n, k)$ with leaves $1, \ldots, n$ a hierarchy on $\{1, \ldots, n\}$ with clusters of cardinality between 2 and $n-k$ and, by using this association and by pushing forward the ideas in [9] and [10], we get a theorem (Theorem 18) characterizing l-treelike dissimilarity families; consequently, we obtain also a characterization of p-l-treelike dissimilarity families (see Remark 19).

## 2 Notation and recalls

Notation 7. - We use the symbols $\subset$ and $\subsetneq$ respectively for the inclusion and the strict inclusion. - For any $n \in \mathbb{N}$ with $n \geq 1$, let $[n]=\{1, \ldots, n\}$.

- For any set $S$ and $k \in \mathbb{N}$, let $\binom{S}{k}$ be the set of the $k$-subsets of $S$ and let $\binom{S}{\geq k}$ be the set of the $t$-subsets of $S$ with $t \geq k$.
- For any $A, B \subset[n]$, we will write $A B$ instead of $A \cup B$. Moreover, we will write $a, B$, or even $a B$, instead of $\{a\} \cup B$.
- Throughout the paper, the word "tree" will denote a finite tree.
- We say that a vertex of a tree is a node if its degree is greater than 2 .
- Let $F$ be a leaf of a tree $T$. Let $N$ be the node such that the path $p$ between $N$ and $F$ does not contain any node apart from $N$. We say that $p$ is the twig associated to $F$. We say that an edge is internal if it is not an edge of a twig.
- We say that a tree is essential if it has no vertices of degree 2 .
- Let $T$ be a tree and let $S$ be a subset of $L(T)$. We denote by $\left.T\right|_{S}$ the minimal subtree of $T$ whose set of vertices contains $S$. If $\mathcal{T}=(T, w)$ is a weighted tree, we denote by $\left.\mathcal{T}\right|_{S}$ the tree $\left.T\right|_{S}$ with the weight induced by $w$.

Definition 8. Let $T$ be a tree.
We say that two leaves $i$ and $j$ of $T$ are neighbours if in the path from $i$ to $j$ there is only one node; furthermore, we say that $C \subset L(T)$ is a cherry if any $i, j \in C$ are neighbours.
We say that a cherry is complete if it is not strictly contained in another cherry.
The stalk of a cherry is the unique node in the path with endpoints any two elements of the cherry. Let $C$ be a cherry in $T$. We say that a tree $T^{\prime}$ is obtained from $T$ by pruning $C$ if it is obtained from $T$ by "deleting" all the twigs associated to leaves of $C$ (more precisely, by contracting all the edges of the twigs associated to leaves of $C$ ).
We say that a cherry $C$ in $T$ is good if it is complete and, if $T^{\prime}$ is the tree obtained from $T$ by pruning $C$, the stalk of $C$ is a leaf of $T^{\prime}$. We say that a cherry is bad if it is not good.
Let $i, j, l, m \in L(T)$. We say that $\langle i, j \mid l, m\rangle$ holds if in $\left.T\right|_{\{i, j, l, m\}}$ we have that $i$ and $j$ are neighbours, $l$ and $m$ are neighbours, and $i$ and $l$ are not neighbours; in this case we denote by $\gamma_{i, j, l, m}$ the path between the stalk of $\{i, j\}$ and the stalk of $\{l, m\}$ in $\left.T\right|_{\{i, j, l, m\}}$. The symbol $\langle i, j \mid l, m\rangle$ is called Buneman's index of $i, j, l, m$.

Example. In the tree in Figure 2 the only good cherries are $\{1,2,3\}$ and $\{6,7\}$.


Figure 2: Good cherries and bad cherries

Remark 9. (i) A pseudostar of kind $(n, n-1)$ is a star, that is, a tree with only one node.
(ii) Let $k, n \in \mathbb{N}-\{0\}$. If $\frac{n}{2} \geq k$, then every tree with $n$ leaves is a pseudostar of kind $(n, k)$, in fact if we divide a set with $n$ elements into two parts, at least one of them has cardinality greater than or equal to $\frac{n}{2}$, which is greater than or equal to $k$.

Definition 10. Let $S$ be a set. We say that a set system $\mathcal{H}$ of $S$ is a hierarchy over $S$ if, for any $H, H^{\prime} \in \mathcal{H}$, we have that $H \cap H^{\prime}$ is one among $\emptyset, H, H^{\prime}$. We say that $\mathcal{H}$ covers $S$ if $S=\cup_{H \in \mathcal{H}} H$.
Definition 11. Let $k, n \in \mathbb{N}$ with $2 \leq k \leq n-2$. Let $P$ be an essential pseudostar of kind $(n, k)$ with $L(P)=[n]$.
We define inductively $P^{s}$ for any $s \geq 0$ as follows: let $P^{0}=P$ and let $P^{s}$ be the tree obtained from $P^{s-1}$ by pruning all the good cherries of cardinality $\leq n-k$.
We say that $x \in[n]$ descends from $y \in L\left(P^{s}\right)$ for some $s$ if the path between $x$ and $y$ in $P$ contains no leaf of $P^{s}$ apart from $y$. For any $Y \subset L\left(P^{s}\right)$, let $\partial Y$ denote the subset of the elements of $[n]$ descending from any element of $Y$.
We define the hierarchy $\mathcal{H}$ over $[n]$ associated to the pseudostar $P$ (depending on $k$ ) as follows:
we say that a cherry $C$ of $P$ is in $\mathcal{H}$ if and only if $C$ is good and $\# C \leq n-k$;
if $C$ is a cherry of $P^{s}$ for some $s$, we say that $\partial C$ is in $\mathcal{H}$ if and only if $C$ is good and $\# \partial C \leq n-k$; if, for some s, we have that $L\left(P^{s}\right)$ is the union of two complete cherries, $C_{1}$ and $C_{2}$, and both have cardinality less than or equal to $n-k$, we put in $\mathcal{H}$ only $\partial C_{i}$ for $i$ such that $\partial C_{i}$ contains the minimum of $\partial C_{1} \cup \partial C_{2}$.
The elements of $\mathcal{H}$ are only the ones above. We call the elements of $\mathcal{H}$ " $\mathcal{H}$-clusters".
For any $H \in \mathcal{H}$, we define $e_{H}$ as follows: let $H=\partial C$ for some $C$ cherry of $P^{s}$; we call $e_{H}$ the twig of $P^{s+1}$ associated to the stalk of $C$. For any $i \in[n]$, we call $e_{i}$ the twig associated to $i$.

Observe that the set of the leaves of a star is a bad cherry; so, according to our definition of $\mathcal{H}$, if for some $s \in \mathbb{N}$ we have that $P^{s}$ is a star, we do not consider $\partial L\left(P^{s}\right)$, which is [ $n$ ], a cluster of $\mathcal{H}$. So, for instance, the hierarchy of a star is empty.

Examples. Let $P$ be the pseudostar of kind $(12,6)$ in Figure 3 (a). The associated hierarchy over [12] (with $k=6$ ) is

$$
\mathcal{H}=\{\{4,5,6\},\{7,8,9\},\{1,2,3,4,5,6\}\}
$$

Let $Q$ be the pseudostar of kind $(10,5)$ in Figure 3 (b). The associated hierarchy over [10] (with $k=5$ ) is

$$
\mathcal{H}=\{\{1,2\},\{3,4\},\{1,2,3,4\}\} .
$$

Let $R$ be the pseudostar of kind $(12,5)$ in Figure 3 (c). The associated hierarchy over [12] is

$$
\mathcal{H}=\{\{3,4,5\},\{6,7\},\{8,9\},\{1,2,3,4,5\}\} .
$$

Remark 12. It is easy to reconstruct the pseudostar $P$ from the hierarchy $\mathcal{H}$ :
Let $\mathcal{H}$ be a hierarchy on $[n]$ such that its clusters have cardinality between 2 and $n-k$. Let us consider a star $B$ with $L(B)=[n]-\cup_{H \in \mathcal{H}} H$ and call $O$ its stalk. For any $M$ maximal element of $\mathcal{H}$, we add an edge $e_{M}$ with endpoint $O$; let $V_{M}$ be the other endpoint of $e_{M}$. Then we add a cherry with stalk $V_{M}$ and leaves $M-\cup_{H \in \mathcal{H}, H \subsetneq M} H$; for every element $M^{\prime}$ of $\mathcal{H}$ strictly contained in $M$ which is maximal among the elements of $\mathcal{H}$ strictly contained in $M$, we add an edge with endpoint $V_{M}$ and we call $V_{M^{\prime}}$ the other endpoint and so on. When we arrive at a minimal element $N$ of $\mathcal{H}$, we add a cherry with stalk $V_{N}$ and set of leaves $N$.


Figure 3: Pseudostars and hierarchies

Example. Let $r=6$. Consider the following hierarchy over [12]:

$$
\mathcal{H}=\{\{1,2,3,4,5,6\},\{4,5,6\},\{7,8,9,10\},\{7,8\},\{9,10\}\} .
$$

The associated 6-pseudostar is the one in Figure[4, in fact: $L(B)=\{11,12\}$, the maximal elements of $\mathcal{H}$ are $\{1,2,3,4,5,6\}$ and $\{7,8,9,10\}$; for $M=\{1,2,3,4,5,6\}$, the set $M-\cup_{H \in \mathcal{H}, H \subsetneq M} H$ is $\{1,2,3\}$ and the only element of $\mathcal{H}$ strictly contained in $M$ is $\{4,5,6\}$, which is minimal in $\mathcal{H}$; for $M=\{7,8,9,10\}$ the set $M-\cup_{H \in \mathcal{H}, H \subsetneq M} H$ is empty and the only elements of $\mathcal{H}$ strictly contained in $M$ are $\{7,8\}$ and $\{9,10\}$, which are minimal in $\mathcal{H}$.


Figure 4: How to recover the pseudostar from the hierarchy

We report now two results (Proposition 13 and Theorem (14) we proved in [2], because we need them in the proof of our main result (Theorem 18).

Proposition 13. Let $k, n \in \mathbb{N}$ with $2 \leq k \leq n-2$. Let $\mathcal{P}=(P, w)$ be a weighted tree with $L(P)=[n]$.

1) Let $i, j \in[n]$.
(1.1) If $i, j$ are neighbours, then $D_{i, X}(\mathcal{P})-D_{j, X}(\mathcal{P})$ does not depend on $X \in\binom{[n]-\{i, j\}}{k-1}$.
(1.2) If $\mathcal{P}$ is an internal-nonzero-weighted essential pseudostar of $\operatorname{kind}(n, k)$, then also the converse is true.
2) Let $i, j, x, y \in[n]$. Let $k \geq 4$ and $\mathcal{P}$ be an internal-nonzero-weighted essential pseudostar of kind $(n, k)$. We have that $\langle i, j \mid x, y\rangle$ holds if and only if at least one of the following conditions holds:
(a) $\{i, j\}$ and $\{x, y\}$ are complete cherries in $P$,
(b) there exist $S, R \in\binom{[n]-\{i, j, x, y\}}{k-2}$ such that

$$
\begin{align*}
& D_{i, j, S}(\mathcal{P})+D_{x, y, S}(\mathcal{P}) \neq D_{i, x, S}(\mathcal{P})+D_{j, y, S}(\mathcal{P})  \tag{1}\\
& D_{i, j, R}(\mathcal{P})+D_{x, y, R}(\mathcal{P}) \neq D_{i, y, R}(\mathcal{P})+D_{j, x, R}(\mathcal{P}) \tag{2}
\end{align*}
$$

Theorem 14. Let $n, k \in \mathbb{N}$ with $3 \leq k \leq n-1$. Let $\left\{D_{I}\right\}_{I \in\binom{[n]}{k}}$ be a family of real numbers. If it is l-treelike, then there exists exactly one internal-nonzero-weighted essential pseudostar $\mathcal{P}$ of kind $(n, k)$ realizing the family. If the family $\left\{D_{I}\right\}_{I \in\binom{[n]}{k}}$ is $p$-l-treelike, then $\mathcal{P}$ is positive-weighted.

## 3 Characterization of treelike families

In $\S 2$ we established a relation between pseudostars and hierarchies. In this section, firstly we associate to any family of real numbers $\left\{D_{I}\right\}_{I \in\binom{[n]}{k}}$ a hierarchy (see Definition [16) in such a way that, if the family is the family of $k$-weights of a pseudostar, the hierachy associated to the pseudostar and the one associated to the family coincide. Then, in the main theorem (Theorem 18), we give necessary and sufficient conditions for a family $\left\{D_{I}\right\}_{I \in\binom{[n]}{k}}$ to be l-treelike and these conditions involve the hierarchy associated to the family.

Remark 15. Let $T$ be a tree with $L(T)=[n]$. Let $C \subset[n]$.
a By definition, we have that $C$ is a cherry if and only if, for any $i, j \in C, i$ and $j$ are neighbours, and this is true if and only if, for any $i, j \in C$, there do not exist $x, y \in[n]-\{i, j\}$ such that $\langle i, x \mid j, y\rangle$ holds;
$b$ Let $C$ be a complete cherry. Let $i, j \in C$. Then $C$ is good if and only if, for any $x, y \in[n]-C$, we have that $\langle i, j \mid x, y\rangle$ holds.

Let $r \in \mathbb{N}$ and let us define $T^{0}=T$ and $T^{s}$ to be the tree obtained from $T^{s-1}$ by pruning the good cherries of cardinality less or equal than r. If $J$ is a good cherry of $T^{s}$, we denote the stalk of $J$, which is a leaf of $T^{s+1}$, by $\lceil\min (J)\rceil$. Let $C \subset L\left(T^{s}\right)$.
c By definition, we have that $C$ is a cherry of $T^{s}$ if and only if, for any $\lceil i\rceil,\lceil j\rceil \in C$, $\lceil i\rceil$ and $\lceil j\rceil$ are neighbours, and this is true if and only if, for any $\lceil i\rceil,\lceil j\rceil \in C$, there do not exist $\lceil x\rceil,\lceil y\rceil \in L\left(T^{s}\right)-\{\lceil i\rceil,\lceil j\rceil\}$ such that $\langle\lceil i\rceil,\lceil x\rceil \mid\lceil j\rceil,\lceil y\rceil\rangle$ holds. This is true if and only if, for any $\lceil i\rceil,\lceil j\rceil \in C$, there do not exist $\lceil x\rceil,\lceil y\rceil \in L\left(T^{s}\right)-\{\lceil i\rceil,\lceil j\rceil\}$ such that $\langle i, x \mid j, y\rangle$ holds.
$d$ Let $C$ be a complete cherry of $T^{s}$. Let $\lceil i\rceil,\lceil j\rceil \in C$. Then $C$ is good if and only if, for any $\lceil x\rceil,\lceil y\rceil \in L\left(T^{s}\right)-C$, we have that $\langle\lceil i\rceil,\lceil j\rceil \mid\lceil x\rceil,\lceil y\rceil\rangle$ holds. This is true if and only if for any $\lceil x\rceil,\lceil y\rceil \in L\left(T^{s}\right)-C$, we have that $\langle i, j \mid x, y\rangle$ holds.

Resuming,
a) Let $C \subset[n]$;
$C$ is a cherry $\Longleftrightarrow \forall i, j \in C, \nexists x, y \in[n]-\{i, j\}$ such that $\langle i, x \mid j, y\rangle$ holds.
b) Let $C$ be a complete cherry;
$C$ is good $\Longleftrightarrow \forall i, j \in C, \forall x, y \in[n]-C$, we have that $\langle i, j \mid x, y\rangle$ holds.
c) Let $C \subset L\left(T^{s}\right)$;
$C$ is a cherry of $T^{s} \Longleftrightarrow \forall\lceil i\rceil,\lceil j\rceil \in C, \nexists\lceil x\rceil,\lceil y\rceil \in L\left(T^{s}\right)-\{\lceil i\rceil,\lceil j\rceil\}$ such that $\langle i, x \mid j, y\rangle$ holds.
d) Let $C$ be a complete cherry of $T^{s}$;
$C$ is good $\Longleftrightarrow \forall\lceil i\rceil,\lceil j\rceil \in C, \forall\lceil x\rceil,\lceil y\rceil \in L\left(T^{s}\right)-C$, we have that $\langle i, j \mid x, y\rangle$ holds.
Definition 16. Let $n, k \in \mathbb{N}$ with $5 \leq k \leq n-1$. Let $\left\{D_{I}\right\}_{I \in\binom{[n]}{k}}$ be a family in $\mathbb{R}$.
Let

$$
\mathcal{C}^{0}=\left\{\left.Z \in\binom{[n]}{\geq 2} \right\rvert\, D_{i, X}-D_{j, X} \text { does not depend on } X \in\binom{[n]-\{i, j\}}{k-1} \forall i, j \in Z\right\}
$$

let $\underline{\mathcal{C}}^{0}$ be the set of the maximal elements of $\mathcal{C}^{0}$ and

$$
\mathcal{G}^{0}=\left\{\begin{array}{ll}
Z \in \underline{\mathcal{C}}^{0} \mid & \# Z \leq n-k \text { and } \forall i, j \in Z, \forall x, y \in[n]-Z \text { one of the following holds: } \\
& \left(\text { a) }\{i, j\},\{x, y\} \in \underline{\mathcal{C}}^{0}\right.
\end{array}\right\}
$$

Let $[n]^{0}=[n]$. We define inductively $[n]^{s}, \mathcal{C}^{s}, \mathcal{\mathcal { C }}^{s}, \mathcal{G}^{s}$ as follows: for $s \geq 1$, we define $[n]^{s}$ to be the set obtained from $[n]^{s-1}$ by eliminating for every $Z \in \mathcal{G}^{s-1}$ all the elements of $Z$ apart from the minimum

$$
\mathcal{C}^{s}= \begin{cases}\left.Z \in\binom{[n]^{s}}{\geq 2} \right\rvert\, \quad & \forall i, j \in Z, \forall x, y \in[n]^{s}-\{i, j\} \text { both the following do not hold: } \\
& (a)\{i, x\},\{j, y\} \in \underline{\mathcal{C}}^{0} \\
& (b) \exists R, S \in\binom{[n]-\{i, j, x, y\}}{k-2} \text { s.t. }\left\{\begin{array}{l}
D_{i, x, R}+D_{j, y, R} \neq D_{i, j, R}+D_{x, y, R} \\
D_{i, x, S}+D_{j, y, S} \neq D_{i, y, S}+D_{j, x, S}
\end{array}\right\}, ~\end{cases}
$$

let $\underline{\mathcal{C}}^{s}$ be the set of the maximal elements of $\mathcal{C}^{s}$ and

$$
\mathcal{G}^{s}=\left\{\begin{aligned}
Z \in \underline{\mathcal{C}}^{s} \mid & \# \partial Z \leq n-k \text { and } \forall i, j \in Z, \forall x, y \in[n]-Z \text { one of the following holds: } \\
& \text { (a) }\{i, j\},\{x, y\} \in \mathcal{C}^{0} \\
& \text { (b) } \exists R, S \in\binom{[n]-\{i, j, x, y\}}{k-2} \text { s.t. }\left\{\begin{array}{l}
D_{i, j, R}+D_{x, y, R} \neq D_{i, x, R}+D_{j, y, R} \\
D_{i, j, S}+D_{x, y, S} \neq D_{i, y, S}+D_{j, x, S}
\end{array}\right.
\end{aligned}\right\},
$$

where: we say $y_{0} \in[n]$ descends from $y_{s} \in[n]^{s}$ if and only if there exist (not necessarily distinct) $y_{1}, \ldots, y_{s-1} \in[n]$ and, for any $t=0, \ldots, s-1$, an element of $\mathcal{G}^{s}$ containing both $y_{t}$ and $y_{t+1}$; for any $Z \in G^{s}$, let $\partial Z$ be the set of the $y \in[n]$ descending from some element of $Z$ and let $\partial \mathcal{G}^{s}=\left\{\partial Z \mid Z \in \mathcal{G}^{s}\right\}$

Finally, we define

$$
\mathcal{H}=\cup_{s \geq 0} \partial \mathcal{G}^{s}
$$

and we call $\mathcal{H}$ the hierarchy associated to the family $\left\{D_{I}\right\}_{I \in\binom{[n]}{k}}$.
Let $n, k \in \mathbb{N}$ with $3 \leq k \leq n-2$. Let $\mathcal{P}=(P, w)$ be an internal-nonzero-weighted essential pseudostar of kind $(n, k)$ with $L(P)=[n]$ and let us denote $D_{I}(\mathcal{P})$ by $D_{I}$ for any $I \in\binom{[n]}{k}$. Observe that, by Remark 15 and Proposition [13, the hierarchy $\mathcal{H}$ over [ $n$ ] defined by $P$ as in Definition 11 is equal to the hierarchy associated to the family $\left\{D_{I}\right\}_{I}$; precisely $\mathcal{C}^{s}$ is the set of the cherries of the tree $P^{s}$ in Definition [11, $\underline{\mathcal{C}}^{s}$ is the set of the complete cherries of $P^{s}$, and $\mathcal{G}^{s}$ is the set of the good cherries of $P^{s}$.

Remark 17. Let $n, k \in \mathbb{N}$ with $2 \leq k \leq n-2$. Let $\mathcal{P}=(P, w)$ be a weighted pseudostar of kind $(n, k)$ with $L(P)=[n]$. Let $\mathcal{H}$ be a hierarchy over $[n]$ associated to Pas in Definition 11. Observe that, for any $J \in \mathcal{H}$ and any $I \in\binom{[n]}{k}$, the subtree realizing $D_{I}(\mathcal{P})$ contains $e_{J}$ if and only if $I \cap J \neq \emptyset$ and $I \not \subset J$.

We are ready now to state the characterization of treelike families. In the proof, it will be necessary to use two technical lemmas; we postpone them to the appendix.
Theorem 18. Let $n, k \in \mathbb{N}$ with $5 \leq k \leq n-1$. Let $\left\{D_{I}\right\}_{I \in\binom{[n]}{k}}$ be a family of real numbers. If $k \leq n-2$, the family $\left\{D_{I}\right\}_{I}$ is l-treelike if and only if the hierarchy $\mathcal{H}$ over $[n]$ associated to the family $\left\{D_{I}\right\}_{I \in\binom{[n]}{k}}$ is such that:
(i) if $\mathcal{H}$ covers $[n]$, then the number of the maximal clusters of $\mathcal{H}$ is not 2 ,
(ii) for any $q \in\{1, \ldots, n-1\}, s \in\{1, k-1\}$ and for any $W, W^{\prime} \in\binom{[n]}{s}$

$$
\sum_{i=1, \ldots, q} D_{W, Z_{i}}-D_{W^{\prime}, Z_{i}}
$$

does not depend on $Z_{i} \in\binom{[n]-W-W^{\prime}}{k-s}$ under the condition that, in the free $\mathbb{Z}$-module $\oplus_{H \in \mathcal{H}} \mathbb{Z} H$, the sum

$$
\sum_{i=1, \ldots, q}\left[\sum_{H \in \mathcal{H}, H \cap\left(W Z_{i}\right) \neq \emptyset, H \not \supset\left(W Z_{i}\right)} H-\sum_{H \in \mathcal{H}, H \cap\left(W^{\prime} Z_{i}\right) \neq \emptyset, H \not \supset\left(W^{\prime} Z_{i}\right)} H\right]
$$

does not change.
If $k=n-1$, the family $\left\{D_{I}\right\}_{I}$ is always l-treelike.
Proof. If $k=n-1$, it is easy to show that there exists a weighted star realizing the family $\left\{D_{I}\right\}_{I}$. Suppose $k \leq n-2$. If the family $\left\{D_{I}\right\}_{I}$ is l-treelike, then there exists a weighted pseudostar of kind $(n, k)$ realizing it by Theorem [14, it induces a hierarchy over $[n]$ as in Definition 11 and it is easy to see that conditions (i) and (ii) hold; by Remark 17, we have also that condition (iii) holds. Suppose now that the hierarchy $\mathcal{H}$ over $[n]$ associated to the family $\left\{D_{I}\right\}_{I \in\binom{[n]}{k}}$ satisfies (i) and (ii). Let $P$ be the essential pseudostar of kind $(n, k)$ determined by $\mathcal{H}$ (see Remark 12); observe that it is essential by condition (i). For any $J \in \mathcal{H}$, let $e_{J}$ be defined as in Remark 12, we define

$$
\begin{equation*}
w\left(e_{J}\right):=D_{a, X}-D_{a^{\prime}, X}-D_{a, X^{\prime}}+D_{a^{\prime} X^{\prime}} \tag{3}
\end{equation*}
$$

for any $a, a^{\prime} \in[n], X, X^{\prime} \subset[n]$ such that $a, a^{\prime} \notin X, X^{\prime}$ and

is equal to $J$. Let us check that the definition of $w\left(e_{J}\right)$ is a good definition:

- to see that it does not depend on $X$, it is sufficient to see that $D_{a, X}-D_{a^{\prime}, X}$ does not depend on $X$ under the condition that the sum in (4) does not depend on $X$; obviously this is equivalent to the fact that $\sum_{H \in \mathcal{H}, H \cap(a X) \neq \emptyset, H \not \supset(a X)} H-\sum_{H \in \mathcal{H}, H \cap\left(a^{\prime} X\right) \neq \emptyset, H \not \supset\left(a^{\prime} X\right)} H$ does not depend on $X$; so our assertion follows from condition (ii) by taking $q=1, s=1, W=\{a\}$, $W^{\prime}=\left\{a^{\prime}\right\}$ and $Z_{1}=X$; in an analogous way we can see that it does not depend on $X^{\prime}$;
- to see that it does not depend on $a$, it is sufficient to see that $D_{a, X}-D_{a, X^{\prime}}$ does not depend on $a$ under the condition that the sum in (4) does not depend on $a$; obviously this is equivalent to the fact that $\sum_{H \in \mathcal{H}, H \cap(a X) \neq \emptyset, H \not \supset(a X)} H-\sum_{H \in \mathcal{H}, H \cap\left(a X^{\prime}\right) \neq \emptyset, H \not \supset\left(a X^{\prime}\right)} H$ does not depend on $a$; so our assertion follows from condition (ii) by taking $q=1, s=k-1, W=X, W^{\prime}=X^{\prime}$, and $Z_{1}=\{a\}$; in an analogous way we can see that it does not depend on $a^{\prime}$.

Moreover, observe that, by Lemma 21, it is possible to find $a, a^{\prime}, X, X^{\prime}$ as required.
For any $i \in[n]$, we define the weight of the twig $e_{i}$ as follows:

$$
\begin{equation*}
w\left(e_{i}\right):=\frac{1}{k}\left[D_{I}+\sum_{l \in I}\left(D_{i, X(i, l)}-D_{l, X(i, l)}\right)-\sum_{J \in \mathcal{H}, J \cap I \neq \emptyset, J \not \supset I} w\left(e_{J}\right)\right] \tag{5}
\end{equation*}
$$

for any $I \in\binom{[n]}{k}, X(i, l) \in\binom{[n]-\{i, l\}}{k-1}$ such that

$$
\sum_{H \in \mathcal{H}, H \cap(i, X(i, l)) \neq \emptyset,} H-\sum_{H \not \supset(i, X(i, l))} \sum_{H \in \mathcal{H}, H \cap(l, X(i, l)) \neq \emptyset,} H=0 .
$$

Observe that, by Lemma 20, it is possible to find $X(i, l)$ as required. The definition of $w\left(e_{i}\right)$ does not depend on the choice of $X(i, l)$ by condition (ii); we have to show that it does not depend on $I$. Let $I=(a, Y)$ and $I^{\prime}=\left(a^{\prime}, Y\right)$ for some distinct $a, a^{\prime} \in[n], Y \in\binom{[n]-\left\{a, a^{\prime}\right\}}{k-1}$. We have to show that

$$
\begin{aligned}
& D_{a, Y}+\sum_{l \in(a Y)}\left(D_{i, X(i, l)}-D_{l, X(i, l)}\right)-\sum_{J \in \mathcal{H}, J \cap(a Y) \neq \emptyset, J \not \supset(a Y)} w\left(e_{J}\right)= \\
= & D_{a^{\prime}, Y}+\sum_{l \in\left(a^{\prime} Y\right)}\left(D_{i, X(i, l)}-D_{l, X(i, l)}\right)-\sum_{J \in \mathcal{H}, J \cap\left(a^{\prime} Y\right) \neq \emptyset, J \not \supset\left(a^{\prime} Y\right)} w\left(e_{J}\right),
\end{aligned}
$$

that is

$$
D_{a, Y}+D_{i, X(i, a)}-D_{a, X(i, a)}-\sum_{\substack{J \in \mathcal{H}, J \cap(a Y) \neq \emptyset, J \not \supset(a Y)}} w\left(e_{J}\right)=D_{a^{\prime}, Y}+D_{i, X\left(i, a^{\prime}\right)}-D_{a^{\prime}, X\left(i, a^{\prime}\right)}-\sum_{\substack{J \in \mathcal{H}, J \cap\left(a^{\prime} Y\right) \neq \emptyset, J \not \supset\left(a^{\prime} Y\right)}} w\left(e_{J}\right) .
$$

Observe that $\{J \in \mathcal{H} \mid J \cap(a Y) \neq \emptyset, J \not \supset(a Y)\}$ can be written as disjoint union of the following sets:

$$
\begin{aligned}
& \left\{J \in \mathcal{H} \mid J \ni a, J \not \supset a^{\prime}, J \cap(a Y) \neq \emptyset, J \not \supset(a Y)\right\},\left\{J \in \mathcal{H} \mid J \not \supset a, J \ni a^{\prime}, J \cap(a Y) \neq \emptyset, J \not \supset(a Y)\right\}, \\
& \left\{J \in \mathcal{H} \mid J \ni a, J \ni a^{\prime}, J \cap(a Y) \neq \emptyset, J \not \supset(a Y)\right\},\left\{J \in \mathcal{H} \mid J \not \supset a, J \not \supset a^{\prime}, J \cap(a Y) \neq \emptyset, J \not \supset(a Y)\right\},
\end{aligned}
$$

that is, as disjoint union of

$$
\begin{array}{ll}
\left\{J \in \mathcal{H} \mid J \ni a, J \not \supset a^{\prime}, J \not \supset Y\right\}, & \left\{J \in \mathcal{H} \mid J \not \supset a, J \ni a^{\prime}, J \cap Y \neq \emptyset\right\}, \\
\left\{J \in \mathcal{H} \mid J \ni a, J \ni a^{\prime}, J \not \supset Y\right\}, & \left\{J \in \mathcal{H} \mid J \not \supset a, J \not \supset a^{\prime}, J \cap Y \neq \emptyset\right\},
\end{array}
$$

and then as disjoint union of

$$
\begin{array}{ll}
\left\{J \in \mathcal{H} \mid J \ni a, J \not \supset a^{\prime}, J \cap Y \neq \emptyset, J \not \supset Y\right\}, & \left\{J \in \mathcal{H} \mid J \not \supset a, J \ni a^{\prime}, J \cap Y \neq \emptyset, J \not \supset Y\right\}, \\
\left\{J \in \mathcal{H} \mid J \ni a, J \not \supset a^{\prime}, J \cap Y=\emptyset\right\}, & \left\{J \in \mathcal{H} \mid J \not \supset a, J \ni a^{\prime}, J \supset Y\right\}, \\
\left\{J \in \mathcal{H} \mid J \ni a, J \ni a^{\prime}, J \not \supset Y\right\}, & \left\{J \in \mathcal{H} \mid J \not \supset a, J \not \supset a^{\prime}, J \cap Y \neq \emptyset\right\},
\end{array}
$$

Analogously we can write $\left\{J \in \mathcal{H} \mid J \cap\left(a^{\prime} Y\right) \neq \emptyset, J \not \supset\left(a^{\prime} Y\right)\right\}$.
Let us take both $X(i, a)$ and $X\left(i, a^{\prime}\right)$ equal to a set $X$ satisfying the conditions of Lemma 20 for $i, a$, for $i, a^{\prime}$ and for $a, a^{\prime}$ (there exists since $k \geq 5$ ). By simplifying, the assertion becomes

$$
D_{a, Y}-D_{a, X}-\sum_{\substack{J \in \mathcal{H} \text { and } \\ \text { either } J \ni a, \not \supset a^{\prime}, J \cap Y=\emptyset \\ \text { or } J \ni a^{\prime}, J \nexists a, Y \subset J}} w\left(e_{J}\right)=D_{a^{\prime}, Y}-D_{a^{\prime}, X}-\sum_{\substack{J \in \mathcal{H} \text { and } \\ \text { either } J \ni a^{\prime}, J \neq a, J \cap Y=\emptyset \\ \text { or } J \ni a, J \nexists a^{\prime}, Y \subset J \subset}} w\left(e_{J}\right) .
$$

For any $J \in \mathcal{H}$ such that $J \ni a^{\prime}, J \not \supset a$, and $J \cap Y=\emptyset$ or $Y \subset J$, let $Z_{J}, Z_{J}^{\prime}$ be such that the sum

$$
\sum_{\substack{H \in \mathcal{H}, H \cap\left(a^{\prime} Z_{J}\right) \neq \emptyset \\ H \not \supset\left(a^{\prime} Z_{J}\right)}} H-\sum_{\substack{H \in \mathcal{H}, H \cap\left(a Z_{J}\right) \neq \emptyset \\ H \not \supset\left(a Z_{J}\right)}} H-\sum_{\substack{H \in \mathcal{H}, H \cap\left(a^{\prime} Z_{J}^{\prime}\right) \neq \emptyset \\ H \not \supset\left(a^{\prime} Z_{J}^{\prime}\right)}} H+\sum_{\substack{H \in \mathcal{H}, H \cap\left(a Z_{J}^{\prime}\right) \neq \emptyset \\ H \not \supset\left(a Z_{J}^{\prime}\right)}} H
$$

is equal to $J$. By the definition in (3), we have that

$$
w\left(e_{J}\right)=D_{a^{\prime}, Z_{J}}-D_{a, Z_{J}}-D_{a^{\prime}, Z_{J}^{\prime}}+D_{a, Z_{J}^{\prime}}
$$

For any $J \in \mathcal{H}$ such that $J \ni a, J \nexists a^{\prime}$, and $J \cap Y=\emptyset$ or $Y \subset J$, let $R_{J}, R_{J}^{\prime}$ be such that the sum

$$
\sum_{\substack{H \in \mathcal{H}, H \cap\left(a R_{J}\right) \neq \emptyset \\ H \not \supset\left(a R_{J}\right)}} H-\sum_{\substack{H \in \mathcal{H}, H \cap\left(a^{\prime} R_{J}\right) \neq \emptyset \\ H \not \supset\left(a^{\prime} R_{J}\right)}} H \sum_{\substack{H \in \mathcal{H}, H \cap\left(a R_{J}^{\prime}\right) \neq \emptyset \\ H \not \supset\left(a R_{J}^{\prime}\right)}} H+\sum_{\substack{H \in \mathcal{H}, H \cap\left(a^{\prime} R_{J}^{\prime}\right) \neq \emptyset \\ H \not \supset\left(a^{\prime} R_{J}^{\prime}\right)}} H
$$

is equal to $J$; by the definition in (3), we have that

$$
w\left(e_{J}\right)=D_{a, R_{J}}-D_{a^{\prime}, R_{J}}-D_{a, R_{J}^{\prime}}+D_{a^{\prime}, R_{J}^{\prime}} .
$$

So our assertion becomes

$$
\begin{align*}
D_{a, Y}-D_{a^{\prime}, Y}= & D_{a, X}-D_{a^{\prime}, X} \\
& -\sum_{J \in \mathcal{H}, J \cap Y=\emptyset, J \ni a^{\prime}, J \not \supset a}\left(D_{a^{\prime}, Z_{J}}-D_{a, Z_{J}}-D_{a^{\prime}, Z_{J}^{\prime}}+D_{a, Z_{J}^{\prime}}\right) \\
& -\sum_{J \in \mathcal{H}, Y \subset J, J \ni a, J \not \supset a^{\prime}}\left(D_{a, R_{J}}-D_{a^{\prime}, R_{J}}-D_{a, R_{J}^{\prime}}+D_{a^{\prime}, R_{J}^{\prime}}\right)  \tag{6}\\
& +\sum_{J \in \mathcal{H}, Y \subset J, J \ni a^{\prime}, J \not \supset a}\left(D_{a^{\prime}, Z_{J}}-D_{a, Z_{J}}-D_{a^{\prime}, Z_{J}^{\prime}}+D_{a, Z_{J}^{\prime}}\right) \\
& +\sum_{J \in \mathcal{H}, J \cap Y=\emptyset, J \ni a, J \not \supset a^{\prime}}\left(D_{a, R_{J}}-D_{a^{\prime}, R_{J}}-D_{a, R_{J}^{\prime}}+D_{a^{\prime}, R_{J}^{\prime}}\right),
\end{align*}
$$

that is

$$
\left(\begin{array}{r}
D_{a, Y}-D_{a^{\prime}, Y}  \tag{7}\\
+\sum_{J \in \mathcal{H}, J \cap Y=\emptyset, J \ni a^{\prime}, J \not \not a a}\left(D_{a, Z_{J}^{\prime}}-D_{a^{\prime}, Z_{J}^{\prime}}\right) \\
+\sum_{J \in \mathcal{H}, Y \subset J, J \ni a, J \not \supset a^{\prime}}\left(D_{a, R_{J}}-D_{a^{\prime}, R_{J}}\right) \\
+\sum_{J \in \mathcal{H}, Y \subset J, J \ni a^{\prime}, J \not \supset a}\left(D_{a, Z_{J}}-D_{a^{\prime}, Z_{J}}\right) \\
+\sum_{J \in \mathcal{H}, J \cap Y=\emptyset, J \ni a, J \not \supset a^{\prime}}\left(D_{a, R_{J}^{\prime}}-D_{a^{\prime}, R_{J}^{\prime}}\right)
\end{array}\right)=\left(\begin{array}{l}
D_{a, X}-D_{a^{\prime}, X} \\
+\sum_{J \in \mathcal{H}, J \cap Y=\emptyset, J \ni a^{\prime}, J \not \supset a}\left(D_{a, Z_{J}}-D_{a^{\prime}, Z_{J}}\right) \\
+\sum_{J \in \mathcal{H}, Y \subset J, J \ni a, J \not \supset a^{\prime}}\left(D_{a, R_{J}^{\prime}}-D_{a^{\prime}, R_{J}^{\prime}}\right) \\
+\sum_{J \in \mathcal{H}, Y \subset J, J \ni a^{\prime}, J \not \supset a}\left(D_{a, Z_{J}^{\prime}}-D_{a^{\prime}, Z_{J}^{\prime}}\right) \\
+\sum_{J \in \mathcal{H}, J \cap Y=\emptyset, J \ni a, J \not \supset a^{\prime}}\left(D_{a, R_{J}}-D_{a^{\prime}, R_{J}}\right) .
\end{array}\right)
$$

Observe that

$$
\#\left(\left\{J \in \mathcal{H} \mid J \ni a^{\prime}, J \not \supset a\right\} \cup\left\{J \in \mathcal{H} \mid J \ni a, J \not \supset a^{\prime}\right\}\right) \leq n-2,
$$

in fact: let

$$
x:=\#\left\{J \in \mathcal{H} \mid J \ni a^{\prime}, J \not \supset a\right\}, \quad y:=\#\left\{J \in \mathcal{H} \mid J \ni a, J \not \supset a^{\prime}\right\} ;
$$

the set $\left\{J \in \mathcal{H} \mid J \ni a^{\prime}, J \not \supset a\right\}$ is a chain, so in its largest $\mathcal{H}$-cluster, call it $A$, there are at least $x+1$ elements; analogously in the largest $\mathcal{H}$-cluster contained in $\left\{J \in \mathcal{H} \mid J \ni a^{\prime}, J \not \supset a\right\}$, call it $B$, there are at least $y+1$ elements; since $A$ and $B$ are disjoint, we have that

$$
(x+1)+(y+1) \leq n,
$$

thus $x+y \leq n-2$, as we wanted to prove. Hence the number of the terms at each member of (7) is at most $n-1$. Therefore it is easy to see that our assertion (7) follows from condition (ii): write it as (6) and observe that the sum

$$
\sum_{H \in \mathcal{H}, H \cap(a X) \neq \emptyset, H \not \supset(a X)} H-\sum_{H \in \mathcal{H}, H \cap\left(a^{\prime} X\right) \neq \emptyset, H \not \supset\left(a^{\prime} X\right)} H
$$

is 0 for the definition of $X$.
So we have defined the weight of $e_{i}$ for every $i \in[n]$ and the weight of $e_{J}$ for every $J \in \mathcal{H}$. Let $\mathcal{P}=(P, w)$, where $w$ is the weight we have just defined. We have to show that $D_{I}(\mathcal{P})=D_{I}$ for any $I \in\binom{[n]}{k}$. First we show that, for any $i, j \in[n]$,

$$
\begin{equation*}
w\left(e_{i}\right)-w\left(e_{j}\right)=D_{i, X(j, i)}-D_{j, X(j, i)} \tag{8}
\end{equation*}
$$

for any $X(i, j)$ such that

$$
\sum_{\substack{H \in \mathcal{H}, H \cap(j, X(j, i)) \neq \emptyset \\ H \not \supset(j, X(j, i))}} H-\sum_{\substack{H \in \mathcal{H}, H \cap(i, X(j, i)) \neq \emptyset \\ H \not \supset(i, X(j, i))}} H=0 .
$$

Let us choose the same $I$ in the definition of $w\left(e_{i}\right)$ and $w\left(e_{j}\right)$ (see (5)) and let us choose it containing neither $i$ nor $j$; so we get

$$
\begin{gathered}
w\left(e_{i}\right)-w\left(e_{j}\right)=\frac{1}{k}\left[\sum_{t \in I}\left(D_{i, X(t, i)}-D_{t, X(t, i)}\right)-\sum_{t \in I}\left(D_{j, X(t, j)}-D_{t, X(t, j)}\right)\right]= \\
=\frac{1}{k}\left[\sum_{t \in I}\left(D_{i, X(t, i)}-D_{t, X(t, i)}-D_{j, X(t, j)}+D_{t, X(t, j)}\right)\right] .
\end{gathered}
$$

For any $t \in I$, take $X(t, i)$ and $X(t, j)$ equal to a set $X_{t}$ satisfying the conditions of Lemma 20 for the couple $t, i$, for the couple $t, j$ and for the couple $i, j$ (there exists since $k \geq 5$ ). So we get

$$
w\left(e_{i}\right)-w\left(e_{j}\right)=\frac{1}{k}\left[\sum_{t \in I}\left(D_{i, X_{t}}-D_{j, X_{t}}\right)\right] .
$$

Moreover, by condition (ii), we have that $D_{j, X_{t}}-D_{i, X_{t}}=D_{j, X(j, i)}-D_{i, X(j, i)}$ for any $t \in I$, since

$$
\sum_{\substack{\left.H \in \mathcal{H}, H \cap\left(j, X_{t}\right)\right) \neq \emptyset \\ H \not \supset\left(j, X_{t}\right)}} H-\sum_{\substack{H \in \mathcal{H}, H \cap\left(i, X_{t}\right) \neq \emptyset \\ H \not \supset\left(i, X_{t}\right)}} H=0 .
$$

Hence we get (8).
Obviously, for any $I \in\binom{[n]}{k}$, we have that

$$
D_{I}(\mathcal{P})=\sum_{l \in I} w\left(e_{l}\right)+\sum_{J \in \mathcal{H}, J \cap I \neq \emptyset, J \not \supset I} w\left(e_{J}\right) .
$$

So, for any $i \in I$,

$$
w\left(e_{i}\right)=\frac{1}{k}\left[D_{I}(\mathcal{P})+\sum_{l \in I}\left(w\left(e_{i}\right)-w\left(e_{l}\right)\right)-\sum_{J \in \mathcal{H}, J \cap I \neq \emptyset, J \not \supset I} w\left(e_{J}\right)\right],
$$

which, by (8), is equal to

$$
w\left(e_{i}\right)=\frac{1}{k}\left[D_{I}(\mathcal{P})+\sum_{l \in I}\left(D_{i, X(l, i)}-D_{l, X(l, i)}\right)-\sum_{J \in \mathcal{H}, J \cap I \neq \emptyset, J \not \supset I} w\left(e_{J}\right)\right] .
$$

On the other side we have defined $w\left(e_{i}\right)$ to be

$$
\frac{1}{k}\left[D_{I}+\sum_{l \in I}\left(D_{i, X(l, i)}-D_{l, X(l, i)}\right)-\sum_{J \in \mathcal{H}, J \cap I \neq \emptyset, J \not \supset I} w\left(e_{J}\right)\right],
$$

so we get $D_{I}(\mathcal{P})=D_{I}$ for any $I$.
Remark 19. Let $n, k \in \mathbb{N}$ with $2 \leq k \leq n-2$. Let $\left\{D_{I}\right\}_{I \in\binom{[n]}{k}}$ be a family of positive real numbers. Obviously the family $\left\{D_{I}\right\}_{I}$ is p-l-treelike if and only if there exists a hierarchy $\mathcal{H}$ over $[n]$ such that the conditions (i) and (ii) of Theorem 18 hold, and, in addition, the numbers in (3) and (5) are positive for any $i \in[n], J \in \mathcal{H}$.

## 4 Appendix

Lemma 20. Let $k, n \in \mathbb{N}$ with $4 \leq k \leq n-2$. Let $\mathcal{H}$ be a hierarchy on $[n]$ such that its clusters have cardinality less than or equal to $n-k$ and greater than or equal to 2. For any $t \in \cup_{H \in \mathcal{H}} H$, denote by $m_{t}$ the minimal $\mathcal{H}$-cluster containing $t$ and by $M_{t}$ the maximal $\mathcal{H}$-cluster containing $t$. Let $i, l \in[n]$ and $X \in\binom{[n]-\{i, l\}}{k-1}$ satisfy the following conditions:

- if i,l $\in \cup_{H \in \mathcal{H}} H$ and $M_{i} \cap M_{l}=\emptyset$, then
$X$ contains an element $\bar{i} \in m_{i}$ different from $i$,
$X$ contains an element $\bar{l} \in m_{l}$ different from $l$;
- if $i, l \in \cup_{H \in \mathcal{H}} H$ and $M_{i} \cap M_{l} \neq \emptyset$ then
- if $m_{i} \subset m_{l}$ :
$X$ contains an element $\bar{i} \in m_{i}$ different from $i$,
$X$ contains an element $\hat{i} \in[n]-M_{i} ;$
- if $m_{l} \subset m_{i}$ :
$X$ contains an element $\bar{l} \in m_{l}$ different from $l$,
$X$ contains an element $\hat{l} \in[n]-M_{l} ;$
- if $m_{i} \cap m_{l}=\emptyset$ :
$X$ contains an element $\bar{i} \in m_{i}$ different from $i$,
$X$ contains an element $\bar{l} \in m_{l}$ different from $l$, $X$ contains an element $\hat{i} \in[n]-M_{i} ;$
- if $i \in \cup_{H \in \mathcal{H}} H$ and $l \notin \cup_{H \in \mathcal{H}} H$, then
$X$ contains an element $\bar{i} \in m_{i}$ different from $i$, $X$ contains an element $\hat{i} \in[n]-M_{i}$;
- if $l \in \cup_{H \in \mathcal{H}} H$ and $i \notin \cup_{H \in \mathcal{H}} H$, then
$X$ contains an element $\bar{l} \in m_{l}$ different from $l$, $X$ contains an element $\hat{l} \in[n]-M_{l}$.
Then, in the free $\mathbb{Z}$-module $\oplus_{H \in \mathcal{H}} \mathbb{Z} H$,

$$
\sum_{H \in \mathcal{H}, H \cap(i X) \neq \emptyset, H \not \supset(i X)} H-\sum_{H \in \mathcal{H},} \sum_{H \cap(l X) \neq \emptyset, H \not \supset(l X)} H=0
$$

Proof. We have to show that, for every $V \in \mathcal{H}$, we have that $V \cap(i X) \neq \emptyset$ and $V \not \supset(i X)$ if and only if $V \cap(l X) \neq \emptyset$ and $V \not \supset(l X)$. We have five possible cases.

- $V \cap X=\emptyset$.

We want to prove that, in this case, we have that $V \cap(i X)=\emptyset$. Suppose on the contrary that $V \cap(i X) \neq \emptyset$; hence $i \in V$ and then, obviously, $i \in \cup_{H \in \mathcal{H}} H$. If $l \in \cup_{H \in \mathcal{H}} H, M_{i} \cap M_{l} \neq \emptyset$ and
$m_{l} \subset m_{i}$, then $\bar{l} \in X$ by assumption; by definition, we have that $\bar{l} \in m_{l}$ and, since $m_{l} \subset m_{i} \subset V$, we have $\bar{l} \in V$ and thus $X \cap V \neq \emptyset$, which is absurd. In the other cases, by assumption we have that $X \ni \bar{i}$; moreover $\bar{i} \in V$, since $m_{i}$ contains $\bar{i}$ and is contained in $V$; so we get that $V \cap X \neq \emptyset$, which is absurd. Analogously, we can show that $V \cap(l X)=\emptyset$ and then we can conclude.

- $V \cap X \neq \emptyset, V \not \supset i, l$.

In this case, we have obviously that $V \cap(i X) \neq \emptyset, V \cap(l X) \neq \emptyset, V \not \supset(i X), V \not \supset(l X)$ and we can conclude.

- $V \cap X \neq \emptyset, V \ni i, l$.

In this case, we have obviously that $V \cap(i X) \neq \emptyset$ and $V \cap(l X) \neq \emptyset$. Furthermore, $V \supset(i X)$ if and only if $V \supset X$ and this holds if and only if $V \supset(l X)$, so we can conclude.

- $V \cap X \neq \emptyset, V \ni i, V \not \supset l$.

In this case, we have obviously that $i \in \cup_{H \in \mathcal{H}} H$; moreover $V \cap(i X) \neq \emptyset$ and $V \cap(l X) \neq \emptyset$. Furthermore, $V \not \supset(l X)$ since $V \not \supset l$. So we have to prove that $V \not \supset(i X)$. Suppose on the contrary that $V \supset(i X)$; thus $V \supset X$.
If $l \notin \cup_{H \in \mathcal{H}} H$, then, by assumption, $X \ni \hat{i}$; since $V \supset X$, we have that $V \ni \hat{i}$, and thus $\hat{i} \in M_{i}$, which is absurd. We can argue analogously in case $l \in \cup_{H \in \mathcal{H}} H, M_{i} \cap M_{l} \neq \emptyset$, and $m_{i} \subset m_{l}$ or $m_{i} \cap m_{l}=\emptyset$.
If $l \in \cup_{H \in \mathcal{H}} H, M_{i} \cap M_{l} \neq \emptyset$, and $m_{l} \subset m_{i}$, then, by assumption, $X \ni \hat{l}$; since $V \supset X$ and $M_{i} \supset V$ (because $V$ contains $i$, we have that $M_{i} \ni \hat{l}$; furthermore observe that $M_{i}=M_{l}$, because if two $\mathcal{H}$-clusters have a nonempty intersection and are maximal, then they are equal; so $M_{l} \ni \hat{l}$, which is absurd.
If $l \in \cup_{H \in \mathcal{H}} H$ and $M_{i} \cap M_{l}=\emptyset$, then by assumption $X \ni \bar{l}$; since $X \subset V$, we have that $\bar{l} \in V$; since $V \subset M_{i}$ (because $V$ contains $i$, we get that $\bar{l} \in M_{i}$ and thus $\bar{l} \in M_{i} \cap M_{l}$, which is absurd. - $V \cap X \neq \emptyset, V \ni l, V \not \supset i$.

Analogous to the previous case.
Lemma 21. Let $k, n \in \mathbb{N}$ with $4 \leq k \leq n-2$. Let $\mathcal{H}$ be a hierarchy on $[n]$ such that its clusters have cardinality less than or equal to $n-k$ and greater than or equal to 2. Let $a, a^{\prime} \in[n]$, $J \in \mathcal{H}$ with $a \in J, a^{\prime} \notin J$. Let denote the maximal cluster containing $a^{\prime}$ and the minimal cluster containing $a^{\prime}$ respectively by $M_{a^{\prime}}$ and $m_{a^{\prime}}$. Let $X, X^{\prime} \in\binom{[n]-\left\{a, a^{\prime}\right\}}{k-1}$ satisfy the following conditions:

1. if $a^{\prime} \in \cup_{H \in \mathcal{H}} H$, then
$1.1 X$ and $X^{\prime}$ contain an element $b$ of $m_{a^{\prime}}$ with $b \neq a^{\prime}$;
1.2 $X$ contains an element $c$ which is not in $M_{a^{\prime}}$ and $X^{\prime}$ contains an element $c^{\prime}$ which is not in $M_{a^{\prime}}$;
2. if $a^{\prime} \notin \cup_{H \in \mathcal{H}} H$, then $X$ and $X^{\prime}$ contain an element $d$ which is not in the maximal cluster containing $J$;
3. if there exists $\bar{J}$ in $\mathcal{H}$ with $a \in \bar{J} \subsetneq J$, suppose that $\bar{J}$ is maximal among the $\mathcal{H}$-clusters with these characteristics; then $X^{\prime}$ contains an element of $J-\bar{J}$ and $X^{\prime} \cap \bar{J}=\emptyset$; if there does not exist $\bar{J}$ in $\mathcal{H}$ with $a \in \bar{J} \subsetneq J$, then $X^{\prime} \cap J \neq \emptyset$;
4. $X \cap J=\emptyset$; moreover, if there exists $\tilde{J}$ in $\mathcal{H}$ with $J \subsetneq \tilde{J}$, suppose that $\tilde{J}$ is minimal among the $\mathcal{H}$-clusters with these characteristics; then $X$ contains an element of $\tilde{J}-J$;

Then, in the free $\mathbb{Z}$-module $\oplus_{H \in \mathcal{H}} \mathbb{Z} H$,

$$
\begin{equation*}
J=\sum_{\substack{H \in \mathcal{H}, H \cap(a X) \neq \emptyset, H \not \supset(a X)}} H-\sum_{\substack{H \in \mathcal{H}, H \cap\left(a^{\prime} X\right) \neq \emptyset, H \not \supset\left(a^{\prime} X\right)}} H-\sum_{\substack{H \in \mathcal{H}, H \cap\left(a X^{\prime}\right) \neq \emptyset, H \not \supset\left(a X^{\prime}\right)}} H+\sum_{\substack{H \in \mathcal{H}, H \cap\left(a^{\prime} X^{\prime}\right) \neq \emptyset, H \not \supset\left(a^{\prime} X^{\prime}\right)}} I \tag{9}
\end{equation*}
$$

Proof. In order to prove (9), we have to show that every $\mathcal{H}$-cluster $V$ different from $J$ does not appear in the second member of (9) and that $J$ appears with coefficient 1 . Let $V \in \mathcal{H}$.

- Suppose $V \not \supset a, a^{\prime}($ so $V \neq J)$.

In this case $V$ does not contain any of $a X, a^{\prime} X, a X^{\prime}, a^{\prime} X^{\prime}$ and we can conclude easily by considering the four possible cases:
$-V \cap X \neq \emptyset, V \cap X^{\prime} \neq \emptyset$,
$-V \cap X=\emptyset, V \cap X^{\prime} \neq \emptyset$,

- $V \cap X \neq \emptyset, V \cap X^{\prime}=\emptyset$,
- $V \cap X=\emptyset, V \cap X^{\prime}=\emptyset$.
- Suppose $V \ni a^{\prime}($ so $V \neq J)$.

Then $a^{\prime} \in \cup_{H \in \mathcal{H}} H$, therefore, by assumption (1), $b \in X, X^{\prime}, c \in X, c^{\prime} \in X^{\prime}$. Moreover $V \ni a^{\prime}$, thus $V \ni b$, so $V \cap X \neq \emptyset$ and $V \cap X^{\prime} \neq \emptyset$. Since $c \in X, c^{\prime} \in X^{\prime}$ and $c, c^{\prime} \notin V$, we have that $X \not \subset V$ and $X^{\prime} \not \subset V$, so we can conclude.

- Suppose $V \ni a, V \not \supset a^{\prime}$.

There are at most three possible cases: $V \subset \bar{J}, V \supset \tilde{J}, V=J$.

- If $V \subset \bar{J}$, then $V \cap X=\emptyset$ and $V \cap X^{\prime}=\emptyset$ by assumptions (3) and (4), thus $V \not \supset\left(a^{\prime} X^{\prime}\right)$, $V \not \supset\left(a^{\prime} X\right), V \not \supset\left(a X^{\prime}\right), V \not \supset(a X), V \cap\left(a^{\prime} X^{\prime}\right)=\emptyset$ and $V \cap\left(a^{\prime} X\right)=\emptyset$. Moreover, since $V \ni a$, $V \cap\left(a X^{\prime}\right) \neq \emptyset$ and $V \cap(a X) \neq \emptyset$ and we conclude easily.
- If $V \supset \tilde{J}$, then $V \cap X \neq \emptyset$ and $V \cap X^{\prime} \neq \emptyset$ since $\tilde{J} \cap X \neq \emptyset$ and $\tilde{J} \cap X^{\prime} \neq \emptyset$ by assumptions (3) and (4).
Suppose $a^{\prime} \in \cup_{H \in \mathcal{H}} H$. Then, if $V$ contained $X$, then it would contain $b$ and thus $V \cap m_{a^{\prime}}$ would be nonempty; thus either $m_{a^{\prime}} \subset V$ or $V \subset m_{a^{\prime}}$; if $m_{a^{\prime}} \subset V$, we would have $a^{\prime} \in V$, which is absurd; if $V \subset m_{a^{\prime}}$, we would have $c \in X \subset V \subset m_{a^{\prime}} \subset M_{a^{\prime}}$, so $c \in M_{a^{\prime}}$, which is absurd; so $V$ does not contain $X$. Analogously $V$ does not contain $X^{\prime}$. So $V \not \supset\left(a^{\prime} X^{\prime}\right)$, $V \not \supset\left(a^{\prime} X\right), V \not \supset\left(a X^{\prime}\right)$, $V \not \supset(a X)$, and we conclude.
Suppose $a^{\prime} \notin \cup_{H \in \mathcal{H}} H$. Hence $X$ and $X^{\prime}$ contain $d$ by assumption (2). Then, if $V$ contained $X$, then it would contain $d$, which is absurd since $d$ is not in the maximal cluster containing $J$; thus $V$ does not contain $X$. Analogously $V$ does not contain $X^{\prime}$. So $V \not \supset\left(a^{\prime} X^{\prime}\right), V \not \supset\left(a^{\prime} X\right), V \not \supset\left(a X^{\prime}\right)$, $V \not \supset(a X)$, and we conclude.
- Finally consider the cluster $J$. We have that $J \cap X^{\prime} \neq \emptyset$ by assumption (3) and $J \ni a$, so $J \cap(a X) \neq \emptyset, J \cap\left(a X^{\prime}\right) \neq \emptyset, J \cap\left(a^{\prime} X^{\prime}\right) \neq \emptyset$. Since $a^{\prime} \notin J$ and $J \cap X=\emptyset$ by assumption (4), we have that $J \cap\left(a^{\prime} X\right)=\emptyset$. Moreover $J \not \supset(a X)$, since $J \cap X=\emptyset$, and $J \not \supset\left(a^{\prime} X\right)$ and $J \not \supset\left(a^{\prime} X^{\prime}\right)$, since $J \not \supset a^{\prime}$. Finally $J \not \supset X^{\prime}$, in fact: if $a^{\prime} \in \cup_{H \in \mathcal{H}} H$, then $b \in X^{\prime}$ by assumption (1), so, if $J$ contained $X^{\prime}$, it would contain $b$, thus $J \cap m_{a^{\prime}}$ would be nonempty, hence either
$J \subset m_{a^{\prime}}$ or $m_{a^{\prime}} \subset J$; if $m_{a^{\prime}} \subset J$, we would have $a^{\prime} \in J$, which is absurd; if $J \subset m_{a^{\prime}}$, we would have $c^{\prime} \in X^{\prime} \subset J \subset m_{a^{\prime}} \subset M_{a^{\prime}}$, thus $c^{\prime} \in M_{a^{\prime}}$, which is absurd; if $a^{\prime} \notin \cup_{H \in \mathcal{H}} H$, then $d \in X^{\prime}$ by assumption (2), so, if $J$ contained $X^{\prime}$, it would contain $d$, which is absurd. So $J \not \supset X^{\prime}$, thus $J \not \supset\left(a X^{\prime}\right)$ and we can conclude.


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[^0]:    2010 Mathematical Subject Classification: 05C05, 05C12, 05C22
    Key words: weighted trees, dissimilarity families

