Treelike families of multiweights

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Abstract

Let $\mathcal{T} = (T, w)$ be a weighted finite tree with leaves 1, ..., n. For any $I := \{i_1, ..., i_k\} \subset \{1, ..., n\}$, let $D_I(\mathcal{T})$ be the weight of the minimal subtree of T connecting $i_1, ..., i_k$; the $D_I(\mathcal{T})$ are called k-weights of \mathcal{T} . Given a family of real numbers parametrized by the k-subsets of $\{1, ..., n\}, \{D_I\}_{I \in \binom{\{1, ..., n\}}{k}}$, we say that a weighted tree $\mathcal{T} = (T, w)$ with leaves 1, ..., n realizes the family if $D_I(\mathcal{T}) = D_I$ for any I. We give a characterization of the families of real numbers that are realized by some weighted tree.

1 Introduction

For any graph G, let E(G), V(G) and L(G) be respectively the set of the edges, the set of the vertices and the set of the leaves of G. A weighted graph $\mathcal{G} = (G, w)$ is a graph Gendowed with a function $w : E(G) \to \mathbb{R}$. For any edge e, the real number w(e) is called the weight of the edge. If all the weights are nonnegative (respectively positive), we say that the graph is **nonnegative-weighted** (respectively **positive-weighted**); if the weights of the internal edges are nonzero (respectively positive), we say that the graph is **internal-nonzero-weighted** (respectively **internal-positive-weighted**). For any finite subgraph G' of G, we define w(G') to be the sum of the weights of the edges of G'. In this paper we will deal only with weighted finite trees.

Definition 1. Let $\mathcal{T} = (T, w)$ be a weighted tree. For any distinct $i_1, \ldots, i_k \in V(T)$, we define $D_{\{i_1,\ldots,i_k\}}(\mathcal{T})$ to be the weight of the minimal subtree containing i_1, \ldots, i_k . We call this subtree "the subtree realizing $D_{\{i_1,\ldots,i_k\}}(\mathcal{T})$ ". More simply, we denote $D_{\{i_1,\ldots,i_k\}}(\mathcal{T})$ by $D_{i_1,\ldots,i_k}(\mathcal{T})$ for any order of i_1, \ldots, i_k . We call the $D_{i_1,\ldots,i_k}(\mathcal{T})$ the k-weights of \mathcal{T} and we call a k-weight of \mathcal{T} for some k a multiweight of \mathcal{T} .

For any $S \subset V(T)$, we call the family $\{D_I\}_{I \in \binom{S}{k}}$ the **family of the** *k*-weights of (\mathcal{T}, S) or the *k*-dissimilarity family of (\mathcal{T}, S) .

We can wonder when a family of real numbers is the family of the k-weights of some weighted tree and of some subset of the set of its vertices. If S is a finite set, $k \in \mathbb{N}$ and k < #S,

²⁰¹⁰ Mathematical Subject Classification: 05C05, 05C12, 05C22

Key words: weighted trees, dissimilarity families

we say that a family of real numbers $\{D_I\}_{I \in \binom{S}{k}}$ is **treelike** (respectively p-treelike, nn-treelike, inz-treelike, ip-treelike, nn-ip-treelike) if there exist a weighted (respectively positive-weighted, nonnegative-weighted, internal-nonzero-weighted, internal-positive-weighted, nonegative-weighted and internal-positive weighted) tree $\mathcal{T} = (T, w)$ and a subset S of the set of its vertices such that $D_I(\mathcal{T}) = D_I$ for any k-subset I of S. In this case, we say that $\mathcal{T} = (T, w)$ **realizes the family** $\{D_I\}_{I \in \binom{S}{k}}$.

If in addition $S \subset L(T)$, we say that the family is **l-treelike** (respectively p-l-treelike, nn-l-treelike, inz-l-treelike, ip-l-treelike).

A criterion for a metric on a finite set to be nn-l-treelike was established in [4], [11], [12]:

Theorem 2. Let $\{D_I\}_{I \in \binom{\{1,...,n\}}{2}}$ be a set of positive real numbers satisfying the triangle inequalities. It is p-treelike (or nn-l-treelike) if and only if, for all distinct $i, j, k, h \in \{1, ..., n\}$, the maximum of

$$\{D_{i,j} + D_{k,h}, D_{i,k} + D_{j,h}, D_{i,h} + D_{k,j}\}$$

is attained at least twice.

In [3], Bandelt and Steel proved a result, analogous to Theorem 2, for general weighted trees; more precisely, they proved that, for any set of real numbers $\{D_I\}_{I \in \binom{\{1,\dots,n\}}{2}}$, there exists a weighted tree \mathcal{T} with leaves $1, \dots, n$ such that $D_I(\mathcal{T}) = D_I$ for any $I \in \binom{\{1,\dots,n\}}{2}$ if and only if, for any $a, b, c, d \in \{1, \dots, n\}$, at least two among $D_{a,b} + D_{c,d}$, $D_{a,c} + D_{b,d}$, $D_{a,d} + D_{b,c}$ are equal.

For higher k the literature is more recent. Some of the most important results are the following.

Theorem 3. (Pachter-Speyer, [8]). Let $k, n \in \mathbb{N}$ with $3 \leq k \leq (n+1)/2$. A positive-weighted tree \mathcal{T} with leaves 1, ..., n and no vertices of degree 2 is determined by the values $D_I(\mathcal{T})$, where I varies in $\binom{\{1,...,n\}}{k}$.

Theorem 4. (Herrmann, Huber, Moulton, Spillner, [5]). If $n \ge 2k$, a family of positive real numbers $\{D_I\}_{I \in \binom{\{1,...,n\}}{k}}$ is nn-ip-l-treelike if and only if the restriction to every 2k-subset of $\{1,...,n\}$ is nn-ip-l-treelike.

Theorem 5. (Levy-Yoshida-Pachter) Let $\mathcal{T} = (T, w)$ be a positive-weighted tree with $L(T) = \{1, \ldots, n\}$. For any $i, j \in \{1, \ldots, n\}$, define

$$S(i,j) = \sum_{\substack{Y \in \binom{\{1,\dots,n\} - \{i,j\}}{k-2}}} D_{i,j,Y}(\mathcal{T}).$$

Then there exists a positive-weighted tree $\mathcal{T}' = (T', w')$ such that $D_{i,j}(\mathcal{T}') = S(i, j)$ for all $i, j \in \{1, \ldots, n\}$, the quartet system of T' contains the quartet system of T and, defined $T_{\leq s}$ the subforest of T whose edge set consists of edges whose removal results in one of the components having size at most s, we have $T_{\leq n-k} \cong T'_{< n-k}$.

In [9] and [10], the author gave an inductive characterization of the families of real numbers that are indexed by the subsets of $\{1, ..., n\}$ of cardinality greater than or equal to 2 and are the families of the multiweights of a tree with n leaves.

Let $n, k \in \mathbb{N}$ with n > k. In [1] we studied the problem of the characterization of the families of positive real numbers, indexed by the k-subsets of an n-set, that are p-treelike in the "border" case k = n - 1. Moreover we studied the analogous problem for graphs. See [6] for other results on graphs and see the introduction of [1] for a survey.

Here we examine the case of trees for general k. To illustrate the result, we need the following definition.

Definition 6. Let $k \in \mathbb{N} - \{0\}$. We say that a tree P is a **pseudostar** of kind (n, k) if #L(P) = n and any edge of P divides L(P) into two sets such that at least one of them has cardinality greater than or equal to k.



Figure 1: A pseudostar of kind (10, 8)

In [2] we proved that, if $3 \leq k \leq n-1$, given a l-treelike family of real numbers, $\{D_I\}_{I \in \binom{\{1,...,n\}}{k}}$, there exists exactly one internal-nonzero-weighted pseudostar \mathcal{P} of kind (n, k) with leaves 1, ..., nand no vertices of degree 2 such that $D_I(\mathcal{P}) = D_I$ for any I. Here we associate to any pseudostar of kind (n, k) with leaves 1, ..., n a hierarchy on $\{1, ..., n\}$ with clusters of cardinality between 2 and n-k and, by using this association and by pushing forward the ideas in [9] and [10], we get a theorem (Theorem 18) characterizing l-treelike dissimilarity families; consequently, we obtain also a characterization of p-l-treelike dissimilarity families (see Remark 19).

2 Notation and recalls

Notation 7. • We use the symbols \subset and \subsetneq respectively for the inclusion and the strict inclusion.

• For any $n \in \mathbb{N}$ with $n \ge 1$, let $[n] = \{1, ..., n\}$.

• For any set S and $k \in \mathbb{N}$, let $\binom{S}{k}$ be the set of the k-subsets of S and let $\binom{S}{\geq k}$ be the set of the t-subsets of S with $t \geq k$.

• For any $A, B \subset [n]$, we will write AB instead of $A \cup B$. Moreover, we will write a, B, or even aB, instead of $\{a\} \cup B$.

• Throughout the paper, the word "tree" will denote a finite tree.

• We say that a vertex of a tree is a **node** if its degree is greater than 2.

• Let F be a leaf of a tree T. Let N be the node such that the path p between N and F does not contain any node apart from N. We say that p is the **twig** associated to F. We say that an edge is **internal** if it is not an edge of a twig.

• We say that a tree is **essential** if it has no vertices of degree 2.

• Let T be a tree and let S be a subset of L(T). We denote by $T|_S$ the minimal subtree of T whose set of vertices contains S. If $\mathcal{T} = (T, w)$ is a weighted tree, we denote by $\mathcal{T}|_S$ the tree $T|_S$ with the weight induced by w.

Definition 8. Let T be a tree.

We say that two leaves i and j of T are **neighbours** if in the path from i to j there is only one node; furthermore, we say that $C \subset L(T)$ is a **cherry** if any $i, j \in C$ are neighbours.

We say that a cherry is **complete** if it is not strictly contained in another cherry.

The **stalk** of a cherry is the unique node in the path with endpoints any two elements of the cherry. Let C be a cherry in T. We say that a tree T' is obtained from T by **pruning** C if it is obtained from T by "deleting" all the twigs associated to leaves of C (more precisely, by contracting all the edges of the twigs associated to leaves of C).

We say that a cherry C in T is **good** if it is complete and, if T' is the tree obtained from T by pruning C, the stalk of C is a leaf of T'. We say that a cherry is **bad** if it is not good.

Let $i, j, l, m \in L(T)$. We say that $\langle i, j | l, m \rangle$ holds if $in T|_{\{i,j,l,m\}}$ we have that i and j are neighbours, l and m are neighbours, and i and l are not neighbours; in this case we denote by $\gamma_{i,j,l,m}$ the path between the stalk of $\{i, j\}$ and the stalk of $\{l, m\}$ in $T|_{\{i,j,l,m\}}$. The symbol $\langle i, j | l, m \rangle$ is called **Buneman's index** of i, j, l, m.

Example. In the tree in Figure 2 the only good cherries are $\{1, 2, 3\}$ and $\{6, 7\}$.



Figure 2: Good cherries and bad cherries

Remark 9. (i) A pseudostar of kind (n, n - 1) is a star, that is, a tree with only one node. (ii) Let $k, n \in \mathbb{N} - \{0\}$. If $\frac{n}{2} \ge k$, then every tree with n leaves is a pseudostar of kind (n, k), in fact if we divide a set with n elements into two parts, at least one of them has cardinality greater than or equal to $\frac{n}{2}$, which is greater than or equal to k. **Definition 10.** Let S be a set. We say that a set system \mathcal{H} of S is a **hierarchy** over S if, for any $H, H' \in \mathcal{H}$, we have that $H \cap H'$ is one among \emptyset, H, H' . We say that \mathcal{H} covers S if $S = \bigcup_{H \in \mathcal{H}} H$.

Definition 11. Let $k, n \in \mathbb{N}$ with $2 \le k \le n-2$. Let P be an essential pseudostar of kind (n, k) with L(P) = [n].

We define inductively P^s for any $s \ge 0$ as follows: let $P^0 = P$ and let P^s be the tree obtained from P^{s-1} by pruning all the good cherries of cardinality $\le n - k$.

We say that $x \in [n]$ descends from $y \in L(P^s)$ for some s if the path between x and y in P contains no leaf of P^s apart from y. For any $Y \subset L(P^s)$, let ∂Y denote the subset of the elements of [n]descending from any element of Y.

We define the hierarchy \mathcal{H} over [n] associated to the pseudostar P (depending on k) as follows:

we say that a cherry C of P is in \mathcal{H} if and only if C is good and $\#C \leq n-k$;

if C is a cherry of P^s for some s, we say that ∂C is in \mathcal{H} if and only if C is good and $\#\partial C \leq n-k$; if, for some s, we have that $L(P^s)$ is the union of two complete cherries, C_1 and C_2 , and both have cardinality less than or equal to n-k, we put in \mathcal{H} only ∂C_i for i such that ∂C_i contains the minimum of $\partial C_1 \cup \partial C_2$.

The elements of \mathcal{H} are only the ones above. We call the elements of \mathcal{H} " \mathcal{H} -clusters".

For any $H \in \mathcal{H}$, we define e_H as follows: let $H = \partial C$ for some C cherry of P^s ; we call e_H the twig of P^{s+1} associated to the stalk of C. For any $i \in [n]$, we call e_i the twig associated to i.

Observe that the set of the leaves of a star is a bad cherry; so, according to our definition of \mathcal{H} , if for some $s \in \mathbb{N}$ we have that P^s is a star, we do not consider $\partial L(P^s)$, which is [n], a cluster of \mathcal{H} . So, for instance, the hierarchy of a star is empty.

Examples. Let P be the pseudostar of kind (12, 6) in Figure 3 (a). The associated hierarchy over [12] (with k = 6) is

$$\mathcal{H} = \{\{4, 5, 6\}, \{7, 8, 9\}, \{1, 2, 3, 4, 5, 6\}\}.$$

Let Q be the pseudostar of kind (10, 5) in Figure 3 (b). The associated hierarchy over [10] (with k = 5) is

$$\mathcal{H} = \{\{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}.$$

Let R be the pseudostar of kind (12, 5) in Figure 3 (c). The associated hierarchy over [12] is

 $\mathcal{H} = \{\{3, 4, 5\}, \{6, 7\}, \{8, 9\}, \{1, 2, 3, 4, 5\}\}.$

Remark 12. It is easy to reconstruct the pseudostar P from the hierarchy \mathcal{H} :

Let \mathcal{H} be a hierarchy on [n] such that its clusters have cardinality between 2 and n - k. Let us consider a star B with $L(B) = [n] - \bigcup_{H \in \mathcal{H}} H$ and call O its stalk. For any M maximal element of \mathcal{H} , we add an edge e_M with endpoint O; let V_M be the other endpoint of e_M . Then we add a cherry with stalk V_M and leaves $M - \bigcup_{H \in \mathcal{H}, H \subsetneq M} H$; for every element M' of \mathcal{H} strictly contained in M which is maximal among the elements of \mathcal{H} strictly contained in M, we add an edge with endpoint V_M and we call $V_{M'}$ the other endpoint and so on. When we arrive at a minimal element N of \mathcal{H} , we add a cherry with stalk V_N and set of leaves N.



Figure 3: Pseudostars and hierarchies

Example. Let r = 6. Consider the following hierarchy over [12]:

 $\mathcal{H} = \{\{1, 2, 3, 4, 5, 6\}, \{4, 5, 6\}, \{7, 8, 9, 10\}, \{7, 8\}, \{9, 10\}\}.$

The associated 6-pseudostar is the one in Figure 4, in fact: $L(B) = \{11, 12\}$, the maximal elements of \mathcal{H} are $\{1, 2, 3, 4, 5, 6\}$ and $\{7, 8, 9, 10\}$; for $M = \{1, 2, 3, 4, 5, 6\}$, the set $M - \bigcup_{H \in \mathcal{H}, H \subseteq M} H$ is $\{1, 2, 3\}$ and the only element of \mathcal{H} strictly contained in M is $\{4, 5, 6\}$, which is minimal in \mathcal{H} ; for $M = \{7, 8, 9, 10\}$ the set $M - \bigcup_{H \in \mathcal{H}, H \subseteq M} H$ is empty and the only elements of \mathcal{H} strictly contained in M are $\{7, 8\}$ and $\{9, 10\}$, which are minimal in \mathcal{H} .



Figure 4: How to recover the pseudostar from the hierarchy

We report now two results (Proposition 13 and Theorem 14) we proved in [2], because we need them in the proof of our main result (Theorem 18).

Proposition 13. Let $k, n \in \mathbb{N}$ with $2 \leq k \leq n-2$. Let $\mathcal{P} = (P, w)$ be a weighted tree with L(P) = [n]. 1) Let $i, j \in [n]$. (1.1) If i, j are neighbours, then $D_{i,X}(\mathcal{P}) - D_{j,X}(\mathcal{P})$ does not depend on $X \in {[n]-\{i,j\} \choose k-1}$. (1.2) If \mathcal{P} is an internal-nonzero-weighted essential pseudostar of kind (n, k), then also the converse is true. 2) Let $i, j, x, y \in [n]$. Let $k \ge 4$ and \mathcal{P} be an internal-nonzero-weighted essential pseudostar of kind (n, k). We have that $\langle i, j | x, y \rangle$ holds if and only if at least one of the following conditions holds:

(a) $\{i, j\}$ and $\{x, y\}$ are complete cherries in P, (b) there exist $S, R \in {\binom{[n]-\{i,j,x,y\}}{k-2}}$ such that

$$D_{i,j,S}(\mathcal{P}) + D_{x,y,S}(\mathcal{P}) \neq D_{i,x,S}(\mathcal{P}) + D_{j,y,S}(\mathcal{P}).$$
(1)

$$D_{i,j,R}(\mathcal{P}) + D_{x,y,R}(\mathcal{P}) \neq D_{i,y,R}(\mathcal{P}) + D_{j,x,R}(\mathcal{P}).$$
(2)

Theorem 14. Let $n, k \in \mathbb{N}$ with $3 \leq k \leq n-1$. Let $\{D_I\}_{I \in \binom{[n]}{k}}$ be a family of real numbers. If it is *l*-treelike, then there exists exactly one internal-nonzero-weighted essential pseudostar \mathcal{P} of kind (n,k) realizing the family. If the family $\{D_I\}_{I \in \binom{[n]}{k}}$ is *p*-*l*-treelike, then \mathcal{P} is positive-weighted.

3 Characterization of treelike families

In §2 we established a relation between pseudostars and hierarchies. In this section, firstly we associate to any family of real numbers $\{D_I\}_{I \in \binom{[n]}{k}}$ a hierarchy (see Definition 16) in such a way that, if the family is the family of k-weights of a pseudostar, the hierachy associated to the pseudostar and the one associated to the family coincide. Then, in the main theorem (Theorem 18), we give necessary and sufficient conditions for a family $\{D_I\}_{I \in \binom{[n]}{k}}$ to be l-treelike and these conditions involve the hierarchy associated to the family.

Remark 15. Let T be a tree with L(T) = [n]. Let $C \subset [n]$.

- a By definition, we have that C is a cherry if and only if, for any $i, j \in C$, i and j are neighbours, and this is true if and only if, for any $i, j \in C$, there do not exist $x, y \in [n] - \{i, j\}$ such that $\langle i, x | j, y \rangle$ holds;
- b Let C be a complete cherry. Let $i, j \in C$. Then C is good if and only if, for any $x, y \in [n] C$, we have that $\langle i, j | x, y \rangle$ holds.

Let $r \in \mathbb{N}$ and let us define $T^0 = T$ and T^s to be the tree obtained from T^{s-1} by pruning the good cherries of cardinality less or equal than r. If J is a good cherry of T^s , we denote the stalk of J, which is a leaf of T^{s+1} , by $\lceil \min(J) \rceil$. Let $C \subset L(T^s)$.

- c By definition, we have that C is a cherry of T^s if and only if, for any $\lceil i \rceil, \lceil j \rceil \in C$, $\lceil i \rceil$ and $\lceil j \rceil$ are neighbours, and this is true if and only if, for any $\lceil i \rceil, \lceil j \rceil \in C$, there do not exist $\lceil x \rceil, \lceil y \rceil \in L(T^s) \{\lceil i \rceil, \lceil j \rceil\}$ such that $\langle \lceil i \rceil, \lceil x \rceil | \lceil j \rceil, \lceil y \rceil \rangle$ holds. This is true if and only if, for any $\lceil i \rceil, \lceil j \rceil \in C$, there do not exist $\lceil x \rceil, \lceil y \rceil \in L(T^s) \{\lceil i \rceil, \lceil j \rceil\}$ such that $\langle i \rangle, \lceil y \rceil \in L(T^s) \{\lceil i \rceil, \lceil j \rceil\}$ such that $\langle i, x | j, y \rangle$ holds.
- d Let C be a complete cherry of T^s . Let $\lceil i \rceil, \lceil j \rceil \in C$. Then C is good if and only if, for any $\lceil x \rceil, \lceil y \rceil \in L(T^s) C$, we have that $\langle \lceil i \rceil, \lceil j \rceil | \lceil x \rceil, \lceil y \rceil \rangle$ holds. This is true if and only if for any $\lceil x \rceil, \lceil y \rceil \in L(T^s) C$, we have that $\langle i, j | x, y \rangle$ holds.

Resuming, a) Let $C \subset [n]$; C is a cherry $\iff \forall i, j \in C, \not\exists x, y \in [n] - \{i, j\}$ such that $\langle i, x | j, y \rangle$ holds. b) Let C be a complete cherry; C is good $\iff \forall i, j \in C, \forall x, y \in [n] - C$, we have that $\langle i, j | x, y \rangle$ holds. c) Let $C \subset L(T^s)$; C is a cherry of $T^s \iff \forall \lceil i \rceil, \lceil j \rceil \in C, \not\exists \lceil x \rceil, \lceil y \rceil \in L(T^s) - \{\lceil i \rceil, \lceil j \rceil\}$ such that $\langle i, x | j, y \rangle$ holds. d) Let C be a complete cherry of T^s ; C is good $\iff \forall \lceil i \rceil, \lceil j \rceil \in C, \forall \lceil x \rceil, \lceil y \rceil \in L(T^s) - C$, we have that $\langle i, j | x, y \rangle$ holds.

Definition 16. Let $n, k \in \mathbb{N}$ with $5 \leq k \leq n-1$. Let $\{D_I\}_{I \in \binom{[n]}{k}}$ be a family in \mathbb{R} . Let

$$\mathcal{C}^{0} = \left\{ Z \in \binom{[n]}{\geq 2} \mid D_{i,X} - D_{j,X} \text{ does not depend on } X \in \binom{[n] - \{i, j\}}{k - 1} \forall i, j \in Z \right\},\$$

let \underline{C}^0 be the set of the maximal elements of C^0 and

$$\mathcal{G}^{0} = \left\{ \begin{array}{ccc} Z \in \underline{\mathcal{C}}^{0} \mid & \#Z \leq n-k \text{ and } \forall i, j \in Z, \ \forall x, y \in [n]-Z \text{ one of the following holds:} \\ & (a) \ \{i, j\}, \{x, y\} \in \underline{\mathcal{C}}^{0} \\ & (b) \ \exists R, S \in \binom{[n]-\{i,j,x,y\}}{k-2} \text{ s.t. } \left\{ \begin{array}{c} D_{i,j,R} + D_{x,y,R} \neq D_{i,x,R} + D_{j,y,R} \\ D_{i,j,S} + D_{x,y,S} \neq D_{i,y,S} + D_{j,x,S} \end{array} \right\}.$$

Let $[n]^0 = [n]$. We define inductively $[n]^s$, \mathcal{C}^s , $\underline{\mathcal{C}}^s$, \mathcal{G}^s as follows: for $s \ge 1$, we define $[n]^s$ to be the set obtained from $[n]^{s-1}$ by eliminating for every $Z \in \mathcal{G}^{s-1}$ all the elements of Z apart from the minimum

$$\mathcal{C}^{s} = \left\{ \begin{array}{ccc} Z \in {\binom{[n]^{s}}{\geq 2}} & | & \forall i, j \in Z, \ \forall x, y \in [n]^{s} - \{i, j\} \text{ both the following do not hold:} \\ & (a) \ \{i, x\}, \{j, y\} \in \underline{\mathcal{C}}^{0} \\ & (b) \ \exists R, S \in {\binom{[n] - \{i, j, x, y\}}{k-2}} \text{ s.t. } \left\{ \begin{array}{c} D_{i, x, R} + D_{j, y, R} \neq D_{i, j, R} + D_{x, y, R} \\ D_{i, x, S} + D_{j, y, S} \neq D_{i, y, S} + D_{j, x, S} \end{array} \right\}, \end{array} \right\},$$

let \underline{C}^s be the set of the maximal elements of C^s and

$$\mathcal{G}^{s} = \left\{ \begin{array}{ccc} Z \in \underline{\mathcal{C}}^{s} \mid & \# \partial Z \leq n-k \text{ and } \forall i, j \in Z, \ \forall x, y \in [n]-Z \text{ one of the following holds:} \\ & (a) \ \{i, j\}, \{x, y\} \in \underline{\mathcal{C}}^{0} \\ & (b) \ \exists R, S \in \binom{[n]-\{i,j,x,y\}}{k-2} \text{ s.t. } \left\{ \begin{array}{c} D_{i,j,R} + D_{x,y,R} \neq D_{i,x,R} + D_{j,y,R} \\ D_{i,j,S} + D_{x,y,S} \neq D_{i,y,S} + D_{j,x,S} \end{array} \right\}, \end{array} \right\}$$

where: we say $y_0 \in [n]$ descends from $y_s \in [n]^s$ if and only if there exist (not necessarily distinct) $y_1, \ldots, y_{s-1} \in [n]$ and, for any $t = 0, \ldots, s - 1$, an element of \mathcal{G}^s containing both y_t and y_{t+1} ; for any $Z \in \mathcal{G}^s$, let ∂Z be the set of the $y \in [n]$ descending from some element of Z and let $\partial \mathcal{G}^s = \{\partial Z \mid Z \in \mathcal{G}^s\}$

Finally, we define

$$\mathcal{H} = \cup_{s \ge 0} \partial \mathcal{G}^s$$

and we call \mathcal{H} the hierarchy associated to the family $\{D_I\}_{I \in \binom{[n]}{2}}$.

Let $n, k \in \mathbb{N}$ with $3 \leq k \leq n-2$. Let $\mathcal{P} = (P, w)$ be an internal-nonzero-weighted essential pseudostar of kind (n, k) with L(P) = [n] and let us denote $D_I(\mathcal{P})$ by D_I for any $I \in {[n] \choose k}$. Observe that, by Remark 15 and Proposition 13, the hierarchy \mathcal{H} over [n] defined by P as in Definition 11 is equal to the hierarchy associated to the family $\{D_I\}_I$; precisely \mathcal{C}^s is the set of the cherries of the tree P^s in Definition 11, $\underline{\mathcal{C}}^s$ is the set of the complete cherries of P^s , and \mathcal{G}^s is the set of the good cherries of P^s .

Remark 17. Let $n, k \in \mathbb{N}$ with $2 \leq k \leq n-2$. Let $\mathcal{P} = (P, w)$ be a weighted pseudostar of kind (n, k) with L(P) = [n]. Let \mathcal{H} be a hierarchy over [n] associated to Pas in Definition 11. Observe that, for any $J \in \mathcal{H}$ and any $I \in {[n] \choose k}$, the subtree realizing $D_I(\mathcal{P})$ contains e_J if and only if $I \cap J \neq \emptyset$ and $I \not\subset J$.

We are ready now to state the characterization of treelike families. In the proof, it will be necessary to use two technical lemmas; we postpone them to the appendix.

Theorem 18. Let $n, k \in \mathbb{N}$ with $5 \leq k \leq n-1$. Let $\{D_I\}_{I \in \binom{[n]}{k}}$ be a family of real numbers. If $k \leq n-2$, the family $\{D_I\}_I$ is l-treelike if and only if the hierarchy \mathcal{H} over [n] associated to the family $\{D_I\}_{I \in \binom{[n]}{k}}$ is such that: (i) if \mathcal{H} covers [n], then the number of the maximal clusters of \mathcal{H} is not 2, (ii) for any $q \in \{1, ..., n-1\}$, $s \in \{1, k-1\}$ and for any $W, W' \in \binom{[n]}{s}$

$$\sum_{i=1,\dots,q} D_{W,Z_i} - D_{W',Z_i}$$

does not depend on $Z_i \in {\binom{[n]-W-W'}{k-s}}$ under the condition that, in the free \mathbb{Z} -module $\bigoplus_{H \in \mathcal{H}} \mathbb{Z}H$, the sum

$$\sum_{i=1,\dots,q} \left[\sum_{H \in \mathcal{H}, H \cap (WZ_i) \neq \emptyset, \ H \not\supset (WZ_i)} H - \sum_{H \in \mathcal{H}, \ H \cap (W'Z_i) \neq \emptyset, \ H \not\supset (W'Z_i)} H \right]$$

does not change. If k = n - 1, the family $\{D_I\}_I$ is always l-treelike.

Proof. If k = n - 1, it is easy to show that there exists a weighted star realizing the family $\{D_I\}_I$. Suppose $k \leq n - 2$. If the family $\{D_I\}_I$ is l-treelike, then there exists a weighted pseudostar of kind (n, k) realizing it by Theorem 14; it induces a hierarchy over [n] as in Definition 11 and it is easy to see that conditions (i) and (ii) hold; by Remark 17, we have also that condition (iii) holds. Suppose now that the hierarchy \mathcal{H} over [n] associated to the family $\{D_I\}_{I \in {[n] \choose k}}$ satisfies (i) and (ii). Let P be the essential pseudostar of kind (n, k) determined by \mathcal{H} (see Remark 12); observe that it is essential by condition (i). For any $J \in \mathcal{H}$, let e_J be defined as in Remark 12; we define

$$w(e_J) := D_{a,X} - D_{a',X} - D_{a,X'} + D_{a'X'}, \tag{3}$$

for any $a, a' \in [n], X, X' \subset [n]$ such that $a, a' \notin X, X'$ and

$$\sum_{\substack{H \in \mathcal{H}, \\ H \cap (aX) \neq \emptyset, \ H \not\supseteq (aX)}} H - \sum_{\substack{H \in \mathcal{H}, \\ H \cap (a'X) \neq \emptyset, \ H \not\supseteq (a'X)}} H - \sum_{\substack{H \in \mathcal{H}, \\ H \cap (aX') \neq \emptyset, \ H \not\supseteq (aX')}} H + \sum_{\substack{H \in \mathcal{H}, \\ H \cap (a'X') \neq \emptyset, \ H \not\supseteq (a'X')}} H$$
(4)

is equal to J. Let us check that the definition of $w(e_J)$ is a good definition:

- to see that it does not depend on X, it is sufficient to see that $D_{a,X} D_{a',X}$ does not depend on X under the condition that the sum in (4) does not depend on X; obviously this is equivalent to the fact that $\sum_{H \in \mathcal{H}, H \cap (aX) \neq \emptyset, H \not\supseteq (aX)} H - \sum_{H \in \mathcal{H}, H \cap (a'X) \neq \emptyset, H \not\supseteq (a'X)} H$ does not depend on X; so our assertion follows from condition (ii) by taking $q = 1, s = 1, W = \{a\}$, $W' = \{a'\}$ and $Z_1 = X$; in an analogous way we can see that it does not depend on X';
- to see that it does not depend on a, it is sufficient to see that $D_{a,X} D_{a,X'}$ does not depend on a under the condition that the sum in (4) does not depend on a; obviously this is equivalent to the fact that $\sum_{H \in \mathcal{H}, H \cap (aX) \neq \emptyset, H \not\supseteq (aX)} H \sum_{H \in \mathcal{H}, H \cap (aX') \neq \emptyset, H \not\supseteq (aX')} H$ does not depend on a; so our assertion follows from condition (ii) by taking q = 1, s = k 1, W = X, W' = X', and $Z_1 = \{a\}$; in an analogous way we can see that it does not depend on a'.

Moreover, observe that, by Lemma 21, it is possible to find a, a', X, X' as required. For any $i \in [n]$, we define the weight of the twig e_i as follows:

$$w(e_i) := \frac{1}{k} \left[D_I + \sum_{l \in I} \left(D_{i,X(i,l)} - D_{l,X(i,l)} \right) - \sum_{J \in \mathcal{H}, \ J \cap I \neq \emptyset, \ J \not\supseteq I} w(e_J) \right]$$
(5)

for any $I \in {\binom{[n]}{k}}, X(i,l) \in {\binom{[n]-\{i,l\}}{k-1}}$ such that

$$\sum_{H \in \mathcal{H}, \ H \cap (i, X(i, l)) \neq \emptyset, \ H \not\supseteq (i, X(i, l))} H - \sum_{H \in \mathcal{H}, \ H \cap (l, X(i, l)) \neq \emptyset, \ H \not\supseteq (l, X(i, l))} H = 0.$$

Observe that, by Lemma 20, it is possible to find X(i, l) as required. The definition of $w(e_i)$ does not depend on the choice of X(i, l) by condition (ii); we have to show that it does not depend on I. Let I = (a, Y) and I' = (a', Y) for some distinct $a, a' \in [n], Y \in {\binom{[n]-\{a,a'\}}{k-1}}$. We have to show that

$$D_{a,Y} + \sum_{l \in (aY)} \left(D_{i,X(i,l)} - D_{l,X(i,l)} \right) - \sum_{J \in \mathcal{H}, J \cap (aY) \neq \emptyset, J \not\supseteq (aY)} w(e_J) =$$
$$= D_{a',Y} + \sum_{l \in (a'Y)} \left(D_{i,X(i,l)} - D_{l,X(i,l)} \right) - \sum_{J \in \mathcal{H}, J \cap (a'Y) \neq \emptyset, J \not\supseteq (a'Y)} w(e_J),$$

that is

$$D_{a,Y} + D_{i,X(i,a)} - D_{a,X(i,a)} - \sum_{\substack{J \in \mathcal{H}, \\ J \cap (aY) \neq \emptyset, J \not\supset (aY)}} w(e_J) = D_{a',Y} + D_{i,X(i,a')} - D_{a',X(i,a')} - \sum_{\substack{J \in \mathcal{H}, \\ J \cap (a'Y) \neq \emptyset, J \not\supset (a'Y)}} w(e_J).$$

Observe that $\{J \in \mathcal{H} | J \cap (aY) \neq \emptyset, J \not\supseteq (aY)\}$ can be written as disjoint union of the following sets:

$$\{J \in \mathcal{H} | J \ni a, J \not\ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a, J \ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \ni a, J \ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a, J \not\ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a, J \not\ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a, J \not\ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a, J \not\ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a, J \not\ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a, J \not\ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a, J \not\ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a, J \not\ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a, J \not\ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a, J \not\ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a, J \not\ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a, J \not\ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a, J \not\ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a, J \not\ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a, J \not\ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a, J \not\ni a', J \cap (aY) \neq \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a', J \cap (aY) \not= \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a', J \cap (aY) \not= \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a', J \cap (aY) \not= \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a', J \cap (aY) \not= \emptyset, J \not\supseteq (aY) \}, \{J \in \mathcal{H} | J \not\ni a', J \cap (aY) \not= \emptyset, J \not= \emptyset$$

that is, as disjoint union of

$$\{J \in \mathcal{H} | J \ni a, J \not\supseteq a', J \not\supseteq Y\}, \ \{J \in \mathcal{H} | J \not\supseteq a, J \ni a', J \cap Y \neq \emptyset\}, \\ \{J \in \mathcal{H} | J \ni a, J \ni a', J \supseteq Y\}, \ \{J \in \mathcal{H} | J \not\supseteq a, J \not\supseteq a', J \cap Y \neq \emptyset\},$$

and then as disjoint union of

$$\begin{split} \{J \in \mathcal{H} | \ J \ni a, \ J \not\supseteq a', \ J \cap Y \neq \emptyset, \ J \not\supseteq Y \}, & \{J \in \mathcal{H} | \ J \not\supseteq a, \ J \ni a', \ J \cap Y \neq \emptyset, \ J \not\supseteq Y \}, \\ \{J \in \mathcal{H} | \ J \ni a, \ J \not\supseteq a', \ J \cap Y = \emptyset \}, & \{J \in \mathcal{H} | \ J \not\supseteq a, \ J \ni a', \ J \supset Y \}, \\ \{J \in \mathcal{H} | \ J \ni a, \ J \ni a', \ J \not\supseteq Y \}, & \{J \in \mathcal{H} | \ J \not\supseteq a, \ J \not\supseteq a', \ J \cap Y \neq \emptyset \}, \end{split}$$

Analogously we can write $\{J \in \mathcal{H} | J \cap (a'Y) \neq \emptyset, J \not\supseteq (a'Y) \}$.

Let us take both X(i, a) and X(i, a') equal to a set X satisfying the conditions of Lemma 20 for i, a, for i, a' and for a, a' (there exists since $k \ge 5$). By simplifying, the assertion becomes

$$D_{a,Y} - D_{a,X} - \sum_{\substack{J \in \mathcal{H} \text{ and} \\ \text{either } J \ni a, J \not\ni a', J \cap Y = \emptyset \\ \text{ or } J \ni a', J \not\ni a, Y \subset J}} w(e_J) = D_{a',Y} - D_{a',X} - \sum_{\substack{J \in \mathcal{H} \text{ and} \\ \text{either } J \ni a', J \not\ni a, J \cap Y = \emptyset \\ \text{ or } J \ni a', J \not\ni a, Y \subset J}} w(e_J).$$

For any $J \in \mathcal{H}$ such that $J \ni a', J \not\supseteq a$, and $J \cap Y = \emptyset$ or $Y \subset J$, let Z_J, Z'_J be such that the sum

$$\sum_{\substack{H \in \mathcal{H}, H \cap (a'Z_J) \neq \emptyset \\ H \not\supseteq (a'Z_J)}} H - \sum_{\substack{H \in \mathcal{H}, H \cap (aZ_J) \neq \emptyset \\ H \not\supseteq (aZ_J)}} H - \sum_{\substack{H \in \mathcal{H}, H \cap (a'Z'_J) \neq \emptyset \\ H \not\supseteq (a'Z'_J)}} H + \sum_{\substack{H \in \mathcal{H}, H \cap (aZ'_J) \neq \emptyset \\ H \not\supseteq (aZ'_J)}} H$$

is equal to J. By the definition in (3), we have that

$$w(e_J) = D_{a',Z_J} - D_{a,Z_J} - D_{a',Z'_J} + D_{a,Z'_J}.$$

For any $J \in \mathcal{H}$ such that $J \ni a, J \not\supseteq a'$, and $J \cap Y = \emptyset$ or $Y \subset J$, let R_J, R'_J be such that the sum

$$\sum_{\substack{H \in \mathcal{H}, H \cap (aR_J) \neq \emptyset \\ H \not\supseteq (aR_J)}} H - \sum_{\substack{H \in \mathcal{H}, H \cap (a'R_J) \neq \emptyset \\ H \not\supseteq (a'R_J)}} H - \sum_{\substack{H \in \mathcal{H}, H \cap (aR'_J) \neq \emptyset \\ H \not\supseteq (aR'_J)}} H + \sum_{\substack{H \in \mathcal{H}, H \cap (a'R'_J) \neq \emptyset \\ H \not\supseteq (a'R'_J)}} H$$

is equal to J; by the definition in (3), we have that

$$w(e_J) = D_{a,R_J} - D_{a',R_J} - D_{a,R'_J} + D_{a',R'_J}.$$

So our assertion becomes

$$D_{a,Y} - D_{a',Y} = D_{a,X} - D_{a',X} - \sum_{J \in \mathcal{H}, J \cap Y = \emptyset, J \ni a', J \not\ni a} (D_{a',Z_J} - D_{a,Z_J} - D_{a',Z'_J} + D_{a,Z'_J}) - \sum_{J \in \mathcal{H}, Y \subset J, J \ni a, J \not\ni a'} (D_{a,R_J} - D_{a',R_J} - D_{a,R'_J} + D_{a',R'_J}) + \sum_{J \in \mathcal{H}, Y \subset J, J \ni a', J \not\ni a} (D_{a',Z_J} - D_{a,Z_J} - D_{a',Z'_J} + D_{a,Z'_J}) + \sum_{J \in \mathcal{H}, J \cap Y = \emptyset, J \ni a, J \not\ni a'} (D_{a,R_J} - D_{a',R_J} - D_{a,R'_J} + D_{a',R'_J}),$$
(6)

that is

$$\begin{pmatrix} D_{a,Y} - D_{a',Y} \\ + \sum_{J \in \mathcal{H}, J \cap Y = \emptyset, J \ni a', J \not\ni a} (D_{a,Z'_{J}} - D_{a',Z'_{J}}) \\ + \sum_{J \in \mathcal{H}, Y \subset J, J \ni a, J \not\ni a'} (D_{a,R_{J}} - D_{a',R_{J}}) \\ + \sum_{J \in \mathcal{H}, J \cap Y = \emptyset, J \ni a, J \not\ni a'} (D_{a,R'_{J}} - D_{a',R'_{J}}) \end{pmatrix} = \begin{pmatrix} D_{a,X} - D_{a',X} \\ + \sum_{J \in \mathcal{H}, J \cap Y = \emptyset, J \ni a', J \not\ni a} (D_{a,Z_{J}} - D_{a',Z_{J}}) \\ + \sum_{J \in \mathcal{H}, Y \subset J, J \ni a', J \not\ni a'} (D_{a,R'_{J}} - D_{a',R'_{J}}) \\ + \sum_{J \in \mathcal{H}, J \cap Y = \emptyset, J \ni a, J \not\ni a'} (D_{a,R'_{J}} - D_{a',R'_{J}}) \end{pmatrix}$$

$$= \begin{pmatrix} D_{a,X} - D_{a',X} \\ + \sum_{J \in \mathcal{H}, J \cap Y = \emptyset, J \ni a, J \not\ni a'} (D_{a,R'_{J}} - D_{a',Z_{J}}) \\ + \sum_{J \in \mathcal{H}, J \cap Y = \emptyset, J \ni a, J \not\ni a'} (D_{a,R_{J}} - D_{a',R_{J}}) \\ + \sum_{J \in \mathcal{H}, J \cap Y = \emptyset, J \ni a, J \not\ni a'} (D_{a,R_{J}} - D_{a',R_{J}}). \end{pmatrix}$$

$$(7)$$

Observe that

$$#(\{J \in \mathcal{H} | J \ni a', J \not\ni a\} \cup \{J \in \mathcal{H} | J \ni a, J \not\ni a'\}) \le n-2,$$

in fact: let

$$x := \#\{J \in \mathcal{H} | J \ni a', J \not\ni a\}, \qquad y := \#\{J \in \mathcal{H} | J \ni a, J \not\ni a'\};$$

the set $\{J \in \mathcal{H} | J \ni a', J \not\supseteq a\}$ is a chain, so in its largest \mathcal{H} -cluster, call it A, there are at least x + 1 elements; analogously in the largest \mathcal{H} -cluster contained in $\{J \in \mathcal{H} | J \ni a', J \not\supseteq a\}$, call it B, there are at least y + 1 elements; since A and B are disjoint, we have that

$$(x+1) + (y+1) \le n$$

thus $x + y \le n - 2$, as we wanted to prove. Hence the number of the terms at each member of (7) is at most n - 1. Therefore it is easy to see that our assertion (7) follows from condition (ii): write it as (6) and observe that the sum

$$\sum_{H \in \mathcal{H}, \ H \cap (aX) \neq \emptyset, \ H \not\supset (aX)} H - \sum_{H \in \mathcal{H}, \ H \cap (a'X) \neq \emptyset, \ H \not\supset (a'X)} H$$

is 0 for the definition of X.

So we have defined the weight of e_i for every $i \in [n]$ and the weight of e_J for every $J \in \mathcal{H}$. Let $\mathcal{P} = (P, w)$, where w is the weight we have just defined. We have to show that $D_I(\mathcal{P}) = D_I$ for any $I \in {[n] \choose k}$. First we show that, for any $i, j \in [n]$,

$$w(e_i) - w(e_j) = D_{i,X(j,i)} - D_{j,X(j,i)},$$
(8)

for any X(i, j) such that

$$\sum_{\substack{H \in \mathcal{H}, H \cap (j, X(j, i)) \neq \emptyset \\ H \not\supseteq (j, X(j, i))}} H - \sum_{\substack{H \in \mathcal{H}, H \cap (i, X(j, i)) \neq \emptyset \\ H \not\supseteq (i, X(j, i))}} H = 0.$$

Let us choose the same I in the definition of $w(e_i)$ and $w(e_j)$ (see (5)) and let us choose it containing neither i nor j; so we get

$$w(e_i) - w(e_j) = \frac{1}{k} \left[\sum_{t \in I} \left(D_{i,X(t,i)} - D_{t,X(t,i)} \right) - \sum_{t \in I} \left(D_{j,X(t,j)} - D_{t,X(t,j)} \right) \right] = \frac{1}{k} \left[\sum_{t \in I} \left(D_{i,X(t,i)} - D_{t,X(t,i)} - D_{j,X(t,j)} + D_{t,X(t,j)} \right) \right].$$

For any $t \in I$, take X(t, i) and X(t, j) equal to a set X_t satisfying the conditions of Lemma 20 for the couple t, i, for the couple t, j and for the couple i, j (there exists since $k \geq 5$). So we get

$$w(e_i) - w(e_j) = \frac{1}{k} \left[\sum_{t \in I} (D_{i,X_t} - D_{j,X_t}) \right]$$

Moreover, by condition (ii), we have that $D_{j,X_t} - D_{i,X_t} = D_{j,X(j,i)} - D_{i,X(j,i)}$ for any $t \in I$, since

$$\sum_{\substack{H \in \mathcal{H}, H \cap (j, X_t) \\ H \not\supseteq (j, X_t)}} H - \sum_{\substack{H \in \mathcal{H}, H \cap (i, X_t) \neq \emptyset \\ H \not\supseteq (i, X_t)}} H = 0$$

Hence we get (8). Obviously, for any $I \in {[n] \choose k}$, we have that

$$D_I(\mathcal{P}) = \sum_{l \in I} w(e_l) + \sum_{J \in \mathcal{H}, \ J \cap I \neq \emptyset, \ J \not\supseteq I} w(e_J).$$

So, for any $i \in I$,

$$w(e_i) = \frac{1}{k} \left[D_I(\mathcal{P}) + \sum_{l \in I} \left(w(e_i) - w(e_l) \right) - \sum_{J \in \mathcal{H}, J \cap I \neq \emptyset, J \not\supseteq I} w(e_J) \right],$$

which, by (8), is equal to

$$w(e_i) = \frac{1}{k} \left[D_I(\mathcal{P}) + \sum_{l \in I} \left(D_{i,X(l,i)} - D_{l,X(l,i)} \right) - \sum_{J \in \mathcal{H}, J \cap I \neq \emptyset, J \not\supseteq I} w(e_J) \right].$$

On the other side we have defined $w(e_i)$ to be

$$\frac{1}{k} \left[D_I + \sum_{l \in I} \left(D_{i,X(l,i)} - D_{l,X(l,i)} \right) - \sum_{J \in \mathcal{H}, J \cap I \neq \emptyset, J \not\supset I} w(e_J) \right],$$

so we get $D_I(\mathcal{P}) = D_I$ for any I.

Remark 19. Let $n, k \in \mathbb{N}$ with $2 \leq k \leq n-2$. Let $\{D_I\}_{I \in \binom{[n]}{k}}$ be a family of positive real numbers. Obviously the family $\{D_I\}_I$ is p-l-treelike if and only if there exists a hierarchy \mathcal{H} over [n] such that the conditions (i) and (ii) of Theorem 18 hold, and, in addition, the numbers in (3) and (5) are positive for any $i \in [n], J \in \mathcal{H}$.

4 Appendix

Lemma 20. Let $k, n \in \mathbb{N}$ with $4 \leq k \leq n-2$. Let \mathcal{H} be a hierarchy on [n] such that its clusters have cardinality less than or equal to n-k and greater than or equal to 2. For any $t \in \bigcup_{H \in \mathcal{H}} H$, denote by m_t the minimal \mathcal{H} -cluster containing t and by M_t the maximal \mathcal{H} -cluster containing t. Let $i, l \in [n]$ and $X \in {[n]-\{i,l\} \atop k-1}$ satisfy the following conditions:

- if $i, l \in \bigcup_{H \in \mathcal{H}} H$ and $M_i \cap M_l = \emptyset$, then
 - X contains an element $\overline{i} \in m_i$ different from i,
 - X contains an element $\overline{l} \in m_l$ different from l;
- if $i, l \in \bigcup_{H \in \mathcal{H}} H$ and $M_i \cap M_l \neq \emptyset$ then
 - if $m_i \subset m_l$:

X contains an element $\overline{i} \in m_i$ different from *i*, X contains an element $\hat{i} \in [n] - M_i$;

• if $m_l \subset m_i$:

X contains an element $\overline{l} \in m_l$ different from l, X contains an element $\hat{l} \in [n] - M_l$;

- if $m_i \cap m_l = \emptyset$:
 - X contains an element $\overline{i} \in m_i$ different from i,
 - X contains an element $\overline{l} \in m_l$ different from l,
 - X contains an element $\hat{i} \in [n] M_i$;
- if $i \in \bigcup_{H \in \mathcal{H}} H$ and $l \notin \bigcup_{H \in \mathcal{H}} H$, then
 - X contains an element $\overline{i} \in m_i$ different from *i*, X contains an element $\hat{i} \in [n] - M_i$;
- if $l \in \bigcup_{H \in \mathcal{H}} H$ and $i \notin \bigcup_{H \in \mathcal{H}} H$, then
 - X contains an element $\overline{l} \in m_l$ different from l,
 - X contains an element $\hat{l} \in [n] M_l$.

Then, in the free \mathbb{Z} -module $\oplus_{H \in \mathcal{H}} \mathbb{Z}H$,

$$\sum_{H\in\mathcal{H},\; H\cap(iX)\neq\emptyset,\; H\not\supset(iX)}H \quad -\sum_{H\in\mathcal{H},\; H\cap(lX)\neq\emptyset,\; H\not\supset(lX)}H \;\;=\;\; 0$$

Proof. We have to show that, for every $V \in \mathcal{H}$, we have that $V \cap (iX) \neq \emptyset$ and $V \not\supseteq (iX)$ if and only if $V \cap (lX) \neq \emptyset$ and $V \not\supseteq (lX)$. We have five possible cases. • $V \cap X = \emptyset$.

We want to prove that, in this case, we have that $V \cap (iX) = \emptyset$. Suppose on the contrary that $V \cap (iX) \neq \emptyset$; hence $i \in V$ and then, obviously, $i \in \bigcup_{H \in \mathcal{H}} H$. If $l \in \bigcup_{H \in \mathcal{H}} H$, $M_i \cap M_l \neq \emptyset$ and

 $m_l \subset m_i$, then $\overline{l} \in X$ by assumption; by definition, we have that $\overline{l} \in m_l$ and, since $m_l \subset m_i \subset V$, we have $\overline{l} \in V$ and thus $X \cap V \neq \emptyset$, which is absurd. In the other cases, by assumption we have that $X \ni \overline{i}$; moreover $\overline{i} \in V$, since m_i contains \overline{i} and is contained in V; so we get that $V \cap X \neq \emptyset$, which is absurd. Analogously, we can show that $V \cap (lX) = \emptyset$ and then we can conclude. • $V \cap X \neq \emptyset, V \not\ni i, l.$

In this case, we have obviously that $V \cap (iX) \neq \emptyset$, $V \cap (lX) \neq \emptyset$, $V \not\supseteq (iX)$, $V \not\supseteq (lX)$ and we can conclude.

• $V \cap X \neq \emptyset, V \ni i, l.$

In this case, we have obviously that $V \cap (iX) \neq \emptyset$ and $V \cap (lX) \neq \emptyset$. Furthermore, $V \supset (iX)$ if and only if $V \supset X$ and this holds if and only if $V \supset (lX)$, so we can conclude. • $V \cap X \neq \emptyset, V \ni i, V \not\supseteq l.$

In this case, we have obviously that $i \in \bigcup_{H \in \mathcal{H}} H$; moreover $V \cap (iX) \neq \emptyset$ and $V \cap (lX) \neq \emptyset$. Furthermore, $V \not\supseteq (lX)$ since $V \not\supseteq l$. So we have to prove that $V \not\supseteq (iX)$. Suppose on the contrary

that $V \supset (iX)$; thus $V \supset X$. If $l \notin \bigcup_{H \in \mathcal{H}} H$, then, by assumption, $X \ni \hat{i}$; since $V \supset X$, we have that $V \ni \hat{i}$, and thus $\hat{i} \in M_i$, which is absurd. We can argue analogously in case $l \in \bigcup_{H \in \mathcal{H}} H$, $M_i \cap M_l \neq \emptyset$, and $m_i \subset m_l$ or

 $m_i \cap m_l = \emptyset.$ If $l \in \bigcup_{H \in \mathcal{H}} H$, $M_i \cap M_l \neq \emptyset$, and $m_l \subset m_i$, then, by assumption, $X \ni \hat{l}$; since $V \supset X$ and $M_i \supset V$ (because V contains i), we have that $M_i \ni l$; furthermore observe that $M_i = M_l$, because if two \mathcal{H} -clusters have a nonempty intersection and are maximal, then they are equal; so $M_l \ni l$, which is absurd.

If $l \in \bigcup_{H \in \mathcal{H}} H$ and $M_i \cap M_l = \emptyset$, then by assumption $X \ni \overline{l}$; since $X \subset V$, we have that $\overline{l} \in V$; since $V \subset M_i$ (because V contains i), we get that $\overline{l} \in M_i$ and thus $\overline{l} \in M_i \cap M_l$, which is absurd. • $V \cap X \neq \emptyset, V \ni l, V \not\ni i.$

Analogous to the previous case.

Lemma 21. Let $k, n \in \mathbb{N}$ with $4 \leq k \leq n-2$. Let \mathcal{H} be a hierarchy on [n] such that its clusters have cardinality less than or equal to n-k and greater than or equal to 2. Let $a, a' \in [n]$, $J \in \mathcal{H}$ with $a \in J$, $a' \notin J$. Let denote the maximal cluster containing a' and the minimal cluster containing a' respectively by $M_{a'}$ and $m_{a'}$. Let $X, X' \in \binom{[n]-\{a,a'\}}{k-1}$ satisfy the following conditions:

- 1. if $a' \in \bigcup_{H \in \mathcal{H}} H$, then
 - 1.1 X and X' contain an element b of $m_{a'}$ with $b \neq a'$;
 - 1.2 X contains an element c which is not in $M_{a'}$ and X' contains an element c' which is not in $M_{a'}$;
- 2. if $a' \notin \bigcup_{H \in \mathcal{H}} H$, then X and X' contain an element d which is not in the maximal cluster containing J;
- 3. if there exists \overline{J} in \mathcal{H} with $a \in \overline{J} \subseteq J$, suppose that \overline{J} is maximal among the \mathcal{H} -clusters with these characteristics; then X' contains an element of $J - \overline{J}$ and $X' \cap \overline{J} = \emptyset$; if there does not exist \overline{J} in \mathcal{H} with $a \in \overline{J} \subsetneq J$, then $X' \cap J \neq \emptyset$;

4. $X \cap J = \emptyset$; moreover, if there exists \tilde{J} in \mathcal{H} with $J \subsetneq \tilde{J}$, suppose that \tilde{J} is minimal among the \mathcal{H} -clusters with these characteristics; then X contains an element of $\tilde{J} - J$;

Then, in the free \mathbb{Z} -module $\bigoplus_{H \in \mathcal{H}} \mathbb{Z}H$,

$$J = \sum_{\substack{H \in \mathcal{H}, \\ H \cap (aX) \neq \emptyset, \ H \not\supset (aX)}} H - \sum_{\substack{H \in \mathcal{H}, \\ H \cap (a'X) \neq \emptyset, \ H \not\supset (a'X)}} H - \sum_{\substack{H \in \mathcal{H}, \\ H \cap (aX') \neq \emptyset, \ H \not\supset (aX')}} H + \sum_{\substack{H \in \mathcal{H}, \\ H \cap (a'X') \neq \emptyset, \ H \not\supset (a'X')}} H$$
(9)

Proof. In order to prove (9), we have to show that every \mathcal{H} -cluster V different from J does not appear in the second member of (9) and that J appears with coefficient 1. Let $V \in \mathcal{H}$. • Suppose $V \not\supseteq a, a'$ (so $V \not\supseteq J$).

In this case V does not contain any of aX, a'X, aX', a'X' and we can conclude easily by considering the four possible cases:

- $V \cap X \neq \emptyset, \ V \cap X' \neq \emptyset$,
- $-V \cap X = \emptyset, \ V \cap X' \neq \emptyset,$
- $-V \cap X \neq \emptyset, V \cap X' = \emptyset,$
- $-V \cap X = \emptyset, V \cap X' = \emptyset.$
- Suppose $V \ni a'$ (so $V \neq J$).

Then $a' \in \bigcup_{H \in \mathcal{H}} H$, therefore, by assumption (1), $b \in X, X', c \in X, c' \in X'$. Moreover $V \ni a'$, thus $V \ni b$, so $V \cap X \neq \emptyset$ and $V \cap X' \neq \emptyset$. Since $c \in X, c' \in X'$ and $c, c' \notin V$, we have that $X \notin V$ and $X' \notin V$, so we can conclude.

• Suppose $V \ni a, V \not\supseteq a'$.

There are at most three possible cases: $V \subset \overline{J}, V \supset \tilde{J}, V = J$.

- If $V \subset \overline{J}$, then $V \cap X = \emptyset$ and $V \cap X' = \emptyset$ by assumptions (3) and (4), thus $V \not\supseteq (a'X')$, $V \not\supseteq (aX), V \not\supseteq (aX), V \cap (a'X') = \emptyset$ and $V \cap (a'X) = \emptyset$. Moreover, since $V \ni a$, $V \cap (aX') \neq \emptyset$ and $V \cap (aX) \neq \emptyset$ and we conclude easily.

- If $V \supset \tilde{J}$, then $V \cap X \neq \emptyset$ and $V \cap X' \neq \emptyset$ since $\tilde{J} \cap X \neq \emptyset$ and $\tilde{J} \cap X' \neq \emptyset$ by assumptions (3) and (4).

Suppose $a' \in \bigcup_{H \in \mathcal{H}} H$. Then, if V contained X, then it would contain b and thus $V \cap m_{a'}$ would be nonempty; thus either $m_{a'} \subset V$ or $V \subset m_{a'}$; if $m_{a'} \subset V$, we would have $a' \in V$, which is absurd; if $V \subset m_{a'}$, we would have $c \in X \subset V \subset m_{a'} \subset M_{a'}$, so $c \in M_{a'}$, which is absurd; so Vdoes not contain X. Analogously V does not contain X'. So $V \not\supseteq (a'X'), V \not\supseteq (aX), V \not\supseteq (aX'),$ $V \not\supseteq (aX)$, and we conclude.

Suppose $a' \notin \bigcup_{H \in \mathcal{H}} H$. Hence X and X' contain d by assumption (2). Then, if V contained X, then it would contain d, which is absurd since d is not in the maximal cluster containing J; thus V does not contain X. Analogously V does not contain X'. So $V \not\supseteq (a'X'), V \not\supseteq (a'X), V \not\supseteq (aX'), V \not\supseteq (aX)$, and we conclude.

- Finally consider the cluster J. We have that $J \cap X' \neq \emptyset$ by assumption (3) and $J \ni a$, so $J \cap (aX) \neq \emptyset$, $J \cap (aX') \neq \emptyset$, $J \cap (a'X') \neq \emptyset$. Since $a' \notin J$ and $J \cap X = \emptyset$ by assumption (4), we have that $J \cap (a'X) = \emptyset$. Moreover $J \not\supseteq (aX)$, since $J \cap X = \emptyset$, and $J \not\supseteq (a'X)$ and $J \not\supseteq (a'X')$, since $J \not\supseteq a'$. Finally $J \not\supseteq X'$, in fact: if $a' \in \bigcup_{H \in \mathcal{H}} H$, then $b \in X'$ by assumption (1), so, if J contained X', it would contain b, thus $J \cap m_{a'}$ would be nonempty, hence either

 $J \subset m_{a'}$ or $m_{a'} \subset J$; if $m_{a'} \subset J$, we would have $a' \in J$, which is absurd; if $J \subset m_{a'}$, we would have $c' \in X' \subset J \subset m_{a'} \subset M_{a'}$, thus $c' \in M_{a'}$, which is absurd; if $a' \notin \bigcup_{H \in \mathcal{H}} H$, then $d \in X'$ by assumption (2), so, if J contained X', it would contain d, which is absurd. So $J \not\supseteq X'$, thus $J \not\supseteq (aX')$ and we can conclude.

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