The displacement and split decompositions for a Q-polynomial distance-regular graph^{*}

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Abstract

Let Γ denote a *Q*-polynomial distance-regular graph with diameter at least three and standard module *V*. We introduce two direct sum decompositions of *V*. We call these the *displacement* decomposition for Γ and the *split* decomposition for Γ . We describe how these decompositions are related.

1 Introduction

In this paper $\Gamma = (X, R)$ will denote a Q-polynomial distance-regular graph with diameter $D \geq 3$ and adjacency matrix A (see Section 2 for formal definitions). In order to describe our main results we make a few comments. Fix a vertex $x \in X$. For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\operatorname{Mat}_X(\mathbb{C})$ that represents the projection onto the *i*th subconstituent of Γ with respect to x. Let E_0, E_1, \ldots, E_D denote a Q-polynomial ordering of the primitive idempotents for A and let $A^* = A^*(x)$ denote the corresponding dual adjacency matrix. The subconstituent algebra T = T(x) is the subalgebra of $Mat_X(\mathbb{C})$ generated by A and A^* . Let W denote an irreducible T-module. By the displacement of W we mean $\rho + \tau + d - D$, where $\rho = \min\{i | E_i^* W \neq 0\}, \tau = \min\{i | E_i W \neq 0\}, d = |\{i | E_i W \neq 0\}$ $0\}|-1$. We show the displacement of W is nonnegative and at most D. Let $V = \mathbb{C}^X$ denote the standard module. We show $V = \sum_{\eta=0}^{D} V_{\eta}$ (orthogonal direct sum), where V_{η} denotes the subspace of V spanned by the irreducible T-modules that have displacement η . This is the displacement decomposition with respect to x. For $-1 \leq i, j \leq D$ we define $V_{ij} = (E_0^*V + \dots + E_i^*V) \cap (E_0V + \dots + E_jV)$. We show $V = \sum_{i=0}^{D} \sum_{j=0}^{D} \tilde{V}_{ij}$ (direct sum), where \tilde{V}_{ij} denotes the orthogonal complement of $V_{i,j-1} + V_{i-1,j}$ in V_{ij} with respect to the Hermitean dot product. This direct sum is the *split decomposition* with respect to x. The above decompositions are related as follows. For $0 \leq \eta \leq D$ we show $V_{\eta} = \sum V_{ij}$, where the sum is over all ordered pairs i, j such that $0 \le i, j \le D$ and $i + j = D + \eta$. Using this we obtain the following results. For $0 \le i, j \le D$ we show $V_{ij} = 0$ if i + j < D. For $0 \leq i \leq D$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) for E_i (resp. E_i^*). For $0 \leq i, j \leq D$ we show $(A - \theta_j I) \tilde{V}_{ij} \subseteq \tilde{V}_{i+1,j-1}$ and $(A^* - \theta_i^* I) \tilde{V}_{ij} \subseteq \tilde{V}_{i-1,j+1}$, where $\tilde{V}_{rs} := 0$ unless $r, s \in \{0, 1, \ldots, D\}$. We finish with an application related to the work of Brouwer. Godsil, Koolen and Martin [4] concerning the dual width of a subset of X.

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2 Preliminaries concerning distance-regular graphs

In this section we review some definitions and basic concepts concerning distance-regular graphs. For more background information we refer the reader to [1], [3], [19] and [29].

Let \mathbb{C} denote the complex number field. Let X denote a nonempty finite set. Let $\operatorname{Mat}_X(\mathbb{C})$ denote the \mathbb{C} -algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in \mathbb{C} . Let $V = \mathbb{C}^X$ denote the vector space over \mathbb{C} consisting of column vectors whose coordinates are indexed by X and whose entries are in \mathbb{C} . We observe $\operatorname{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We call V the standard module. We endow V with the Hermitean inner product \langle , \rangle that satisfies $\langle u, v \rangle = u^t \overline{v}$ for $u, v \in V$, where t denotes transpose and $\overline{}$ denotes complex conjugation. For all $y \in X$, let \hat{y} denote the element of V with a 1 in the y coordinate and 0 in all other coordinates. We observe $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set R. Let ∂ denote the path-length distance function for Γ , and set $D := \max\{\partial(x, y) \mid x, y \in X\}$. We call D the *diameter* of Γ . We say Γ is *distance-regular* whenever for all integers $h, i, j \ (0 \le h, i, j \le D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p_{ij}^h = |\{z \in X \mid \partial(x, z) = i, \partial(z, y) = j\}|$$

is independent of x and y. The p_{ij}^h are called the *intersection numbers* of Γ .

For the rest of this paper we assume Γ is distance-regular with diameter $D \geq 3$.

We recall the Bose-Mesner algebra of Γ . For $0 \leq i \leq D$ let A_i denote the matrix in $\operatorname{Mat}_X(\mathbb{C})$ with xy entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \quad (x, y \in X).$$

We call A_i the *i*th distance matrix of Γ . We abbreviate $A := A_1$ and call this the adjacency matrix of Γ . We observe (i) $A_0 = I$; (ii) $\sum_{i=0}^{D} A_i = J$; (iii) $\overline{A_i} = A_i$ $(0 \le i \le D)$; (iv) $A_i^t = A_i \ (0 \le i \le D)$; (v) $A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h \ (0 \le i, j \le D)$, where I (resp. J) denotes the identity matrix (resp. all 1's matrix) in $\operatorname{Mat}_X(\mathbb{C})$. Using these facts we find A_0, A_1, \ldots, A_D is a basis for a commutative subalgebra M of $\operatorname{Mat}_X(\mathbb{C})$. We call M the Bose-Mesner algebra of Γ . It turns out A generates M [1, p. 190]. By [3, p. 45], M has a second basis E_0, E_1, \ldots, E_D such that (i) $E_0 = |X|^{-1}J$; (ii) $\sum_{i=0}^{D} E_i = I$; (iii) $\overline{E_i} = E_i \ (0 \le i \le D)$; (iv) $E_i^t = E_i \ (0 \le i \le D)$; (v) $E_i E_j = \delta_{ij} E_i \ (0 \le i, j \le D)$. We call E_0, E_1, \ldots, E_D the primitive idempotents of Γ .

We recall the eigenvalues of Γ . Since E_0, E_1, \ldots, E_D form a basis for M there exist complex scalars $\theta_0, \theta_1, \ldots, \theta_D$ such that $A = \sum_{i=0}^{D} \theta_i E_i$. Observe $AE_i = E_i A = \theta_i E_i$ for $0 \le i \le D$. By [1, p. 197] the scalars $\theta_0, \theta_1, \ldots, \theta_D$ are in \mathbb{R} . Observe $\theta_0, \theta_1, \ldots, \theta_D$ are mutually distinct since A generates M. We call θ_i the eigenvalue of Γ associated with E_i ($0 \le i \le D$). Observe

$$V = E_0 V + E_1 V + \dots + E_D V \qquad \text{(orthogonal direct sum)}.$$

For $0 \le i \le D$ the space $E_i V$ is the eigenspace of A associated with θ_i .

We now recall the Krein parameters. Let \circ denote the entrywise product in $\operatorname{Mat}_X(\mathbb{C})$. Observe $A_i \circ A_j = \delta_{ij}A_i$ for $0 \le i, j \le D$, so M is closed under \circ . Thus there exist complex scalars q_{ij}^h $(0 \le h, i, j \le D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \qquad (0 \le i, j \le D).$$

By [2, p. 170], q_{ij}^h is real and nonnegative for $0 \le h, i, j \le D$. The q_{ij}^h are called the *Krein* parameters. The graph Γ is said to be *Q*-polynomial (with respect to the given ordering E_0, E_1, \ldots, E_D of the primitive idempotents) whenever for $0 \le h, i, j \le D$, $q_{ij}^h = 0$ (resp. $q_{ij}^h \ne 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two [1, 4, 5, 6, 9, 10, 14, 15, 23, 24]. From now on assume Γ is *Q*-polynomial with respect to E_0, E_1, \ldots, E_D .

We recall the dual Bose-Mesner algebra of Γ . Fix a vertex $x \in X$. We view x as a "base vertex." For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $Mat_X(\mathbb{C})$ with yy entry

$$(E_i^*)_{yy} = \begin{cases} 1, & \text{if } \partial(x, y) = i\\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \quad (y \in X).$$

$$(1)$$

We call E_i^* the *i*th dual idempotent of Γ with respect to x [29, p. 378]. We observe (i) $\sum_{i=0}^{D} E_i^* = I$; (ii) $\overline{E_i^*} = E_i^*$ ($0 \le i \le D$); (iii) $E_i^{*t} = E_i^*$ ($0 \le i \le D$); (iv) $E_i^* E_j^* = \delta_{ij} E_i^*$ ($0 \le i, j \le D$). By these facts $E_0^*, E_1^*, \ldots, E_D^*$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\operatorname{Mat}_X(\mathbb{C})$. We call M^* the dual Bose-Mesner algebra of Γ with respect to x [29, p. 378]. For $0 \le i \le D$ let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\operatorname{Mat}_X(\mathbb{C})$ with yy entry $(A_i^*)_{yy} = |X|(E_i)_{xy}$ for $y \in X$. Then $A_0^*, A_1^*, \ldots, A_D^*$ is a basis for M^* [29, p. 379]. Moreover (i) $A_0^* = I$; (ii) $\overline{A_i^*} = A_i^*$ ($0 \le i \le D$); (iii) $A_i^{*t} = A_i^*$ ($0 \le i \le D$); (iv) $A_i^*A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^*$ ($0 \le i, j \le D$) [29, p. 379]. We call $A_0^*, A_1^*, \ldots, A_D^*$ the dual distance matrices of Γ with respect to x. We abbreviate $A^* := A_1^*$ and call this the dual adjacency matrix of Γ with respect to x. The matrix A^* generates M^* [29, Lemma 3.11].

We recall the dual eigenvalues of Γ . Since $E_0^*, E_1^*, \ldots, E_D^*$ form a basis for M^* , there exist complex scalars $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ such that $A^* = \sum_{i=0}^{D} \theta_i^* E_i^*$. Observe $A^* E_i^* = E_i^* A^* = \theta_i^* E_i^*$ for $0 \le i \le D$. By [29, Lemma 3.11] the scalars $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ are in \mathbb{R} . The scalars $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ are mutually distinct since A^* generates M^* . We call θ_i^* the dual eigenvalue of Γ associated with E_i^* ($0 \le i \le D$).

We recall the subconstituents of Γ . From (1) we find

$$E_i^* V = \operatorname{span}\{\hat{y} \mid y \in X, \quad \partial(x, y) = i\} \qquad (0 \le i \le D).$$

$$(2)$$

By (2) and since $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V we find

$$V = E_0^* V + E_1^* V + \dots + E_D^* V \qquad \text{(orthogonal direct sum)}.$$

For $0 \le i \le D$ the space $E_i^* V$ is the eigenspace of A^* associated with θ_i^* . We call $E_i^* V$ the *ith subconstituent* of Γ with respect to x.

We recall the subconstituent algebra of Γ . Let T = T(x) denote the subalgebra of $Mat_X(\mathbb{C})$ generated by M and M^* . We call T the subconstituent algebra (or Terwilliger algebra) of Γ with respect to x [29, Definition 3.3]. We observe T is generated by A and A^* . We observe T has finite dimension. Moreover T is semi-simple since it is closed under the conjugate transponse map [12, p. 157]. See [7, 8, 11, 16, 17, 18, 20, 26, 29, 30, 31] for more information on the subconstituent algebra.

For the rest of this paper we adopt the following notational convention.

Definition 2.1 We assume $\Gamma = (X, R)$ is a distance-regular graph with diameter $D \geq 3$. We assume Γ is Q-polynomial with respect to the ordering E_0, E_1, \ldots, E_D of the primitive idempotents. We fix $x \in X$ and write $A^* = A^*(x)$, $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$, T = T(x). We abbreviate $V = \mathbb{C}^X$. For notational convenience we define $E_{-1} = 0$, $E_{D+1} = 0$ and $E_{-1}^* = 0$, $E_{D+1}^* = 0$.

We have some comments.

Lemma 2.2 [29, Lemma 3.2] With reference to Definition 2.1, the following (i), (ii) hold.

- (i) $AE_i^*V \subseteq E_{i-1}^*V + E_i^*V + E_{i+1}^*V \ (0 \le i \le D).$
- (*ii*) $A^*E_iV \subseteq E_{i-1}V + E_iV + E_{i+1}V \ (0 \le i \le D).$

Lemma 2.3 With reference to Definition 2.1, the following (i)-(iv) hold.

(i)
$$A \sum_{h=0}^{i} E_{h}^{*}V \subseteq \sum_{h=0}^{i+1} E_{h}^{*}V \ (0 \le i \le D).$$

(ii) $(A - \theta_{i}I) \sum_{h=0}^{i} E_{h}V = \sum_{h=0}^{i-1} E_{h}V \ (0 \le i \le D).$
(iii) $A^{*} \sum_{h=0}^{i} E_{h}V \subseteq \sum_{h=0}^{i+1} E_{h}V \ (0 \le i \le D).$
(iv) $(A^{*} - \theta_{i}^{*}I) \sum_{h=0}^{i} E_{h}^{*}V = \sum_{h=0}^{i-1} E_{h}^{*}V \ (0 \le i \le D).$

Proof: (i) Immediate from Lemma 2.2(i).

- (ii) Recall $AE_j = \theta_j E_j$ for $0 \le j \le D$.
- (iii) Immediate from Lemma 2.2(ii).
- (iv) Recall $A^* E_j^* = \theta_j^* E_j^*$ for $0 \le j \le D$.

3 The irreducible *T*-modules

In this section we recall some results on T-modules for later use.

With reference to Definition 2.1, by a *T*-module we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. Let W denote a *T*-module. Then W is said to be *irreducible* whenever W is nonzero and W contains no *T*-modules other than 0 and W. Let W, W' denote *T*-modules. By an *isomorphism of T*-modules from W to W' we mean an isomorphism of vector spaces $\sigma : W \to W'$ such that $(\sigma B - B\sigma)W = 0$ for all $B \in T$. The modules W, W'

are said to be *isomorphic as* T-modules whenever there exists an isomorphism of T-modules from W to W'. Any two nonisomorphic irreducible T-modules are orthogonal [7, Lemma 3.3].

Let W denote a T-module and let W' denote a T-module contained in W. Then the orthogonal complement of W' in W is a T-module [18, p. 802]. It follows that each T-module is an orthogonal direct sum of irreducible T-modules. In particular V is an orthogonal direct sum of irreducible T-modules.

Let W denote an irreducible T-module. By the endpoint of W we mean $\min\{i|0 \le i \le D, E_i^*W \ne 0\}$. By the diameter of W we mean $|\{i|0 \le i \le D, E_i^*W \ne 0\}| - 1$. By the dual endpoint of W we mean $\min\{i|0 \le i \le D, E_iW \ne 0\}$. By the dual diameter of W we mean $|\{i|0 \le i \le D, E_iW \ne 0\}| - 1$. The diameter of W is equal to the dual diameter of W [23, Corollary 3.3]. There exists a unique irreducible T-module with diameter D. We call this module the primary T-module. The primary T-module has basis $A_0\hat{x}, \ldots, A_D\hat{x}$ [29, Lemma 3.6].

Lemma 3.1 [29, Lemma 3.4, Lemma 3.9, Lemma 3.12] With reference to Definition 2.1, let W denote an irreducible T-module with endpoint ρ , dual endpoint τ , and diameter d. Then ρ, τ, d are nonnegative integers such that $\rho + d \leq D$ and $\tau + d \leq D$. Moreover the following (i)–(iv) hold.

- (i) $E_i^*W \neq 0$ if and only if $\rho \leq i \leq \rho + d$, $(0 \leq i \leq D)$.
- (ii) $W = \sum_{h=0}^{d} E_{\rho+h}^* W$ (orthogonal direct sum).
- (iii) $E_i W \neq 0$ if and only if $\tau \leq i \leq \tau + d$, $(0 \leq i \leq D)$.
- (iv) $W = \sum_{h=0}^{d} E_{\tau+h} W$ (orthogonal direct sum).

Lemma 3.2 With reference to Definition 2.1, let W denote an irreducible T-module with endpoint ρ , dual endpoint τ , and diameter d. Then the following (i), (ii) hold.

(i) $AE_{\rho+i}^*W \subseteq E_{\rho+i-1}^*W + E_{\rho+i}^*W + E_{\rho+i+1}^*W$ ($0 \le i \le d$). (ii) $A^*E_{\tau+i}W \subseteq E_{\tau+i-1}W + E_{\tau+i}W + E_{\tau+i+1}W$ ($0 \le i \le d$).

Proof: (i) Follows from Lemma 2.2(i) and since $E_j^*W = E_j^*V \cap W$ for $0 \le j \le D$. (ii) Follows from Lemma 2.2(ii) and since $E_jW = E_jV \cap W$ for $0 \le j \le D$.

Remark 3.3 With reference to Definition 2.1, let W denote an irreducible T-module. Then A and A^* act on W as a tridiagonal pair in the sense of [21, Definition 1.1]. This follows from Lemma 3.1, Lemma 3.2, and since A, A^* together generate T. See [22, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41] for information on tridiagonal pairs.

Lemma 3.4 [6, Lemma 5.1, Lemma 7.1] With reference to Definition 2.1, let W denote an irreducible T-module with endpoint ρ , dual endpoint τ , and diameter d. Then the following (i), (ii) hold.

- (i) $2\rho + d \ge D$.
- (ii) $2\tau + d \ge D$.

Lemma 3.5 With reference to Definition 2.1, let W denote an irreducible T-module with endpoint ρ , dual endpoint τ , and diameter d. Then

$$W = \sum_{h=0}^{d} W_h \qquad (direct \ sum), \tag{3}$$

where

$$W_h = (E_{\rho}^* W + \dots + E_{\rho+h}^* W) \cap (E_{\tau} W + \dots + E_{\tau+d-h} W) \qquad (0 \le h \le d).$$
(4)

Proof: Immediate from Remark 3.3 and [21, Theorem 4.6].

Remark 3.6 The sum (3) is not orthogonal in general.

4 The displacement decomposition

In this section we introduce the *displacement decomposition* for the standard module.

Definition 4.1 With reference to Definition 2.1, let W denote an irreducible T-module. By the *displacement* of W we mean the integer $\rho + \tau + d - D$, where ρ, τ, d denote respectively the endpoint, dual endpoint, and diameter of W.

Lemma 4.2 With reference to Definition 2.1, let W denote an irreducible T-module with displacement η . Then $0 \le \eta \le D$.

Proof: Let ρ, τ, d denote respectively the endpoint, dual endpoint, and diameter of W. By Lemma 3.4 we have $2\rho+d \ge D$ and $2\tau+d \ge D$; adding these inequalities we find $\rho+\tau+d \ge D$ so $\eta \ge 0$. By Lemma 3.1 we have $\rho \le D$ and $\tau + d \le D$. Combining these inequalities we find $\rho + \tau + d \le 2D$ so $\eta \le D$.

Definition 4.3 With reference to Definition 2.1, For $0 \leq \eta \leq D$ we let V_{η} denote the subspace of V spanned by the irreducible T-modules that have displacement η . We observe V_{η} is a T-module.

Lemma 4.4 With reference to Definition 2.1,

$$V = \sum_{\eta=0}^{D} V_{\eta} \qquad (orthogonal \ direct \ sum). \tag{5}$$

Proof: We mentioned earlier that V is spanned by the irreducible T-modules. By Lemma 4.2 and Definition 4.3, each of these modules is contained in one of V_0, V_1, \ldots, V_D . Therefore $V = \sum_{\eta=0}^{D} V_{\eta}$. To show this sum is orthogonal and direct, it suffices to show V_0, V_1, \ldots, V_D are mutually orthogonal. For distinct integers $i, j \ (0 \le i, j \le D)$ observe V_i, V_j are orthogonal since the isomorphism classes of irreducible T-modules that span V_i are distinct from the isomorphism classes of irreducible T-modules that span V_j . We have now shown V_0, V_1, \ldots, V_D are mutually orthogonal so the sum $\sum_{\eta=0}^{D} V_{\eta}$ is orthogonal and direct. \Box

Definition 4.5 We call the sum (5) the displacement decomposition of V with respect to x.

5 The split decomposition

In this section we introduce the *split decomposition* of the standard module.

Definition 5.1 With reference to Definition 2.1, for $-1 \le i, j \le D$ we define

$$V_{ij} = (E_0^* V + E_1^* V + \dots + E_i^* V) \cap (E_0 V + E_1 V + \dots + E_j V).$$
(6)

We observe $V_{ij} = 0$ if i = -1 or j = -1.

In the following three lemmas we make some observations concerning Definition 5.1. In each case the proof is routine and omitted.

Lemma 5.2 With reference to Definition 2.1, for $0 \le i, j \le D$ the space V_{ij} consists of those vectors $v \in V$ such that $E_h^* v = 0$ for $i < h \le D$ and $E_h v = 0$ for $j < h \le D$.

Lemma 5.3 With reference to Definition 2.1, we have $V_{i-1,j} \subseteq V_{ij}$ and $V_{i,j-1} \subseteq V_{ij}$ for $0 \leq i, j \leq D$.

Lemma 5.4 With reference to Definition 2.1, the following (i)-(iii) hold.

- (i) $V_{iD} = E_0^* V + E_1^* V + \dots + E_i^* V \ (0 \le i \le D).$
- (*ii*) $V_{Dj} = E_0 V + E_1 V + \dots + E_j V \ (0 \le j \le D).$
- (iii) $V_{DD} = V$.

Later in the paper we will show $V_{ij} = 0$ if i + j < D, $(0 \le i, j \le D)$.

Definition 5.5 With reference to Definition 2.1, for $0 \le i, j \le D$ we let \tilde{V}_{ij} denote the orthogonal complement of $V_{i,j-1} + V_{i-1,j}$ in V_{ij} . For notational convenience we define $\tilde{V}_{ij} := 0$ unless $i, j \in \{0, 1, \ldots, d\}$.

Our next goal is to show $V_{rs} = \sum_{i=0}^{r} \sum_{j=0}^{s} \tilde{V}_{ij}$ (direct sum) for $0 \le r, s \le D$. We will use the following lemma.

Lemma 5.6 With reference to Definition 2.1,

$$\dim \tilde{V}_{ij} = \dim V_{ij} - \dim V_{i,j-1} - \dim V_{i-1,j} + \dim V_{i-1,j-1}$$
(7)

for $0 \leq i, j \leq D$.

Proof: Let z denote the dimension of $V_{i,j-1} + V_{i-1,j}$. The space \tilde{V}_{ij} is the orthogonal complement of $V_{i,j-1} + V_{i-1,j}$ in V_{ij} so dim $\tilde{V}_{ij} + z = \dim V_{ij}$. Using Definition 5.1 we find $V_{i,j-1} \cap V_{i-1,j} = V_{i-1,j-1}$ so $z + \dim V_{i-1,j-1} = \dim V_{i,j-1} + \dim V_{i-1,j}$. From these comments we routinely obtain (7).

Theorem 5.7 With reference to Definition 2.1, for $0 \le r, s \le D$ we have

$$V_{rs} = \sum_{i=0}^{r} \sum_{j=0}^{s} \tilde{V}_{ij} \qquad (direct \ sum).$$

Proof: We first show

$$V_{rs} = \sum_{i=0}^{r} \sum_{j=0}^{s} \tilde{V}_{ij}.$$
(8)

The proof is by induction on r + s. The result is trivial for r + s = 0 so assume r + s > 0. Recall \tilde{V}_{rs} is the orthogonal complement of $V_{r,s-1} + V_{r-1,s}$ in V_{rs} . Therefore

$$V_{rs} = V_{rs} + V_{r,s-1} + V_{r-1,s}.$$
(9)

By induction we have both

$$V_{r,s-1} = \sum_{i=0}^{r} \sum_{j=0}^{s-1} \tilde{V}_{ij}, \qquad V_{r-1,s} = \sum_{i=0}^{r-1} \sum_{j=0}^{s} \tilde{V}_{ij}.$$
(10)

Combining (9), (10) we routinely obtain (8). We now show the sum (8) is direct. From Lemma 5.6 we routinely obtain

$$\dim V_{rs} = \sum_{i=0}^{r} \sum_{j=0}^{s} \dim \tilde{V}_{ij}$$

and it follows the sum (8) is direct.

Corollary 5.8 With reference to Definition 2.1,

$$V = \sum_{i=0}^{D} \sum_{j=0}^{D} \tilde{V}_{ij} \qquad (direct \ sum). \tag{11}$$

Proof: Set r = D and s = D in Theorem 5.7 and use Lemma 5.4(iii).

Definition 5.9 We call the sum (11) the *split decomposition* of V with respect to x. This decomposition is not orthogonal in general.

6 The displacement and split decompositions

In this section we describe the relationship between the displacement decomposition and the split decomposition. Our main result is the following. With reference to Definition 2.1, for $0 \le \eta \le D$ we show $V_{\eta} = \sum \tilde{V}_{ij}$, where the sum is over all ordered pairs i, j such that $0 \le i, j \le D$ and $i + j = D + \eta$. We begin with a lemma.

Lemma 6.1 With reference to Definition 2.1, let W denote an irreducible T-module with endpoint ρ , dual endpoint τ , and diameter d. Let the subspaces W_0, W_1, \ldots, W_d be as in Lemma 3.5. Then $W_h \subseteq \tilde{V}_{\rho+h,\tau+d-h}$ for $0 \leq h \leq d$.

Proof: Comparing (4) and (6) we find $W_h \subseteq V_{\rho+h,\tau+d-h}$. We show W_h is orthogonal to $V_{\rho+h-1,\tau+d-h} + V_{\rho+h,\tau+d-h-1}$. For $w \in W_h$ and for $v \in V_{\rho+h-1,\tau+d-h}$ we show $\langle w, v \rangle = 0$. Let W^{\perp} denote the orthogonal complement of W in V. Observe $V = W + W^{\perp}$ (direct sum) and that W^{\perp} is a T-module. Observe there exists $w_1 \in W$ and $v_1 \in W^{\perp}$ such that $v = w_1 + v_1$. By the construction $w \in W$ and $v_1 \in W^{\perp}$ so $\langle w, v_1 \rangle = 0$. We show $w_1 = 0$. By Lemma 5.2 and since $v \in V_{\rho+h-1,\tau+d-h}$ we find $E_i^*v = 0$ for $\rho + h \leq i \leq D$ and $E_jv = 0$ for $\tau + d - h + 1 \leq j \leq D$. Since $V = W + W^{\perp}$ is a direct sum of T-modules we find $E_i^*w_1 = 0$ for $\rho + h \leq i \leq D$ and $E_jw_1 = 0$ for $\tau + d - h + 1 \leq j \leq D$. Since W has endpoint ρ we have $E_i^*w_1 = 0$ for $0 \leq i \leq \rho - 1$. Similarly since W has dual endpoint τ we have $E_jw_1 = 0$ for $0 \leq j \leq \tau - 1$. From these comments we find

$$w_1 \in (E_{\rho}^*W + \dots + E_{\rho+h-1}^*W) \cap (E_{\tau}W + \dots + E_{\tau+d-h}W).$$
(12)

Using (4) we find the intersection on the right in (12) is equal to $W_h \cap W_{h-1}$, where $W_{-1} = 0$. The sum (3) is direct so $W_h \cap W_{h-1} = 0$. We now see $w_1 = 0$. Now $v = v_1$ so $\langle w, v \rangle = 0$. We have now shown W_h is orthogonal to $V_{\rho+h-1,\tau+d-h}$. By a similar argument we find W_h is orthogonal to $V_{\rho+h,\tau+d-h-1}$. We conclude $W \subseteq V_{\rho+h,\tau+d-h}$.

Theorem 6.2 With reference to Definition 2.1, the following (i)-(iii) hold.

- (i) For $0 \le \eta \le D$ we have $V_{\eta} = \sum \tilde{V}_{ij}$, where the sum is over all ordered pairs i, j such that $0 \le i, j \le D$ and $i + j = D + \eta$.
- (*ii*) $\tilde{V}_{ij} = 0$ if i + j < D, $(0 \le i, j \le D)$.
- (*iii*) $V_{ij} = 0$ if i + j < D, $(0 \le i, j \le D)$.

Proof: (i), (ii) For $-D \leq \eta \leq D$ we define $V'_{\eta} = \sum \tilde{V}_{ij}$ where the sum is over all ordered pairs i, j such that $0 \leq i, j \leq D$ and $i + j = D + \eta$. Using (11) we find $V = \sum_{\eta=-D}^{D} V'_{\eta}$ (direct sum). We show $V'_{\eta} = 0$ for $-D \leq \eta < 0$ and $V'_{\eta} = V_{\eta}$ for $0 \leq \eta \leq D$. Since the sums $V = \sum_{\eta=0}^{D} V_{\eta}$ and $V = \sum_{\eta=-D}^{D} V'_{\eta}$ are direct it suffices to show $V_{\eta} \subseteq V'_{\eta}$ for $0 \leq \eta \leq D$. Let η be given. Let W denote an irreducible T-module with displacement η . Combining Lemma 3.5 and Lemma 6.1 we find $W \subseteq V'_{\eta}$. The space V_{η} is spanned by the irreducible T-modules that have displacement η ; therefore $V_{\eta} \subseteq V'_{\eta}$. We have now shown $V_{\eta} \subseteq V'_{\eta}$ for $0 \leq \eta \leq D$. We conclude $V'_{\eta} = 0$ for $-D \leq \eta < 0$ and $V'_{\eta} = V_{\eta}$ for $0 \leq \eta \leq D$. Lines (i), (ii) follow. (iii) Combine (ii) above with Theorem 5.7.

We have some comments.

Theorem 6.3 With reference to Definition 2.1, for $0 \le i, j \le D$ such that $i + j \ge D$, and for $0 \le \eta \le D$,

$$V_{ij} \cap V_{\eta} = \sum \tilde{V}_{rs},$$

where the sum is over all ordered pairs r, s such that $0 \le r \le i$ and $0 \le s \le j$ and $r+s-D=\eta$.

Proof: Combine Theorem 5.7 and Theorem 6.2(i).

Corollary 6.4 With reference to Definition 2.1, for $0 \le i, j \le D$ such that $i + j \ge D$, we have $\tilde{V}_{ij} = V_{ij} \cap V_{\eta}$ where $\eta = i + j - D$.

Proof: Apply Theorem 6.3 with $\eta = i + j - D$.

7 The action of A and A^* on the split decomposition

In this section we describe how the adjacency matrix and the dual adjacency matrix act on the split decomposition.

Theorem 7.1 With reference to Definition 2.1, the following (i), (ii) hold.

- (i) $(A \theta_j I) \tilde{V}_{ij} \subseteq \tilde{V}_{i+1,j-1} \ (0 \le i, j \le D).$
- (*ii*) $(A^* \theta_i^* I) \tilde{V}_{ij} \subseteq \tilde{V}_{i-1,j+1} \ (0 \le i, j \le D).$

Proof: (i) Assume $i + j \ge D$; otherwise $\tilde{V}_{ij} = 0$ and the result is trivial. For convenience we treat the cases i = D and i < D separately. To obtain the result for the case i = D, we show $(A - \theta_j I)\tilde{V}_{Dj} = 0$. From Corollary 6.4 (with i = D and $\eta = j$) we have $\tilde{V}_{Dj} = V_{Dj} \cap V_j$. Using Lemma 2.3(ii) and Lemma 5.4(ii) we find $(A - \theta_j I)V_{Dj} = V_{D,j-1}$. Therefore $(A - \theta_j I)\tilde{V}_{Dj} \subseteq V_{D,j-1}$. Recall V_j is a T-module so $(A - \theta_j I)V_j \subseteq V_j$. Therefore $(A - \theta_j I)\tilde{V}_{Dj} \subseteq V_j$. Now

$$(A - \theta_j I) V_{Dj} \subseteq V_{D,j-1} \cap V_j$$

= 0

in view of Theorem 6.3. We have now shown $(A - \theta_j I)\tilde{V}_{Dj} = 0$ so we are done for the case i = D. Next assume i < D. From Corollary 6.4 we have $\tilde{V}_{ij} = V_{ij} \cap V_{\eta}$ where $\eta = i + j - D$. Using Lemma 2.3 and (6) we find $(A - \theta_j I)V_{ij} \subseteq V_{i+1,j-1}$. Therefore $(A - \theta_j I)\tilde{V}_{ij} \subseteq V_{i+1,j-1}$. Recall V_{η} is a *T*-module so $(A - \theta_j I)V_{\eta} \subseteq V_{\eta}$. Therefore $(A - \theta_j I)\tilde{V}_{ij} \subseteq V_{\eta}$. Now

$$(A - \theta_j I)V_{ij} \subseteq V_{i+1,j-1} \cap V_\eta$$

= $\tilde{V}_{i+1,j-1}$

in view of Corollary 6.4.

(ii) Similar to the proof of (i) above.

8 An application

In this section we give an application of Theorem 6.2(iii). We first give two definitions.

Definition 8.1 Let $\Gamma = (X, R)$ denote a distance-regular graph with standard module V. For $v \in V$, by the *support* of v we mean the subset of X consisting of those vertices y such that coordinate y of v is nonzero.

Definition 8.2 [4, Section 4] Let Γ denote a distance-regular graph with diameter $D \geq 3$. Assume Γ is *Q*-polynomial with respect to the ordering E_0, E_1, \ldots, E_D of the primitive idempotents. Let v denote a nonzero vector in the standard module V. By the *dual width* of v we mean

$$\max\{i|0 \le i \le D, \ E_i v \ne 0\}.$$

Theorem 8.3 Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$. Assume Γ is Q-polynomial with respect to the ordering E_0, E_1, \ldots, E_D of the primitive idempotents. Let v denote a nonzero vector in the standard module V and let g denote the corresponding dual width from Definition 8.2. Then for all $x \in X$ there exists y in the support of v such that

$$\partial(x,y) \ge D - g. \tag{13}$$

Proof: We assume the result is false and obtain a contradiction. By this assumption there exists $x \in X$ such that $\partial(x, y) < D - g$ for all vertices y in the support of v. Abbreviate $E_i^* = E_i^*(x)$ for $0 \le i \le D$. Then $v \in E_0^*V + \cdots + E_f^*V$ where f = D - g - 1. Using Definition 8.2 we find $v \in E_0V + \cdots + E_qV$. Now

$$v \in (E_0^*V + \dots + E_f^*V) \cap (E_0V + \dots + E_gV)$$

= V_{fg} .

We mentioned f = D - g - 1 so f + g < D; combining this with Theorem 6.2(iii) we find $V_{fg} = 0$. Now v = 0 for a contradiction. The result follows.

Remark 8.4 Referring to Theorem 8.3, pick any $x \in X$. If v is not orthogonal to the primary module for T(x) then (13) follows from [25, Equation (2.8)]. See also [4, Lemma 1].

9 Directions for further research

In this section we give some suggestions for further research.

Problem 9.1 With reference to Definition 2.1, recall that for $0 \le i, j \le D$ the space V_{ij} depends on x. Does the dimension of \tilde{V}_{ij} depend on x?

Problem 9.2 With reference to Definition 2.1, let W denote an irreducible T-module and consider the multiplicity with which W appears in V. In general this multiplicity is not determined by the intersection numbers of Γ [26]. Is this multiplicity determined by the intersection numbers of Γ and the scalars $\{\dim \tilde{V}_{ij} \mid 0 \leq i, j \leq D\}$?

Problem 9.3 Let Γ denote a *Q*-polynomial distance-regular graph. In many cases Γ exists on the top fiber of a ranked poset [13], [27], [28]. For this case investigate the relationship between the poset structure and the split decomposition of Γ .

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References

- [1] E. Bannai and T. Ito. Algebraic Combinatorics I: Association Schemes. Benjamin/Cummings, London, 1984.
- [2] N. Biggs Algebraic Graph Theory. Second edition. Cambridge University Press, Cambridge, 1993.
- [3] A. E. Brouwer, A. M. Cohen, and A. Neumaier. Distance-Regular Graphs. Springer-Verlag, Berlin, 1989.
- [4] A. E. Brouwer, C. D. Godsil, J. H. Koolen, W. J. Martin. Width and dual width of subsets in polynomial association schemes. Preprint.
- [5] J. S. Caughman IV. Spectra of bipartite P- and Q-polynomial association schemes. Graphs Combin. 14 (1998), 321–343.
- [6] J. S. Caughman IV. The Terwilliger algebras of bipartite P- and Q-polynomial association schemes. Discrete Math. 196 (1999), 65–95.
- [7] B. Curtin. Bipartite distance-regular graphs I. Graphs Combin. 15 (1999), 143–158.
- [8] B. Curtin. Bipartite distance-regular graphs II. Graphs Combin. 15 (1999), 377–391.
- [9] B. Curtin. 2-homogeneous bipartite distance-regular graphs. Discrete Math. 187 (1998), 39–70.
- [10] B. Curtin. Distance-regular graphs which support a spin model are thin. 16th British Combinatorial Conference (London, 1997). Discrete Math. 197/198 (1999), 205–216.
- [11] B. Curtin and K. Nomura. Distance-regular graphs related to the quantum enveloping algebra of sl(2). J. Algebraic Combin. 12 (2000), 25–36.

- [12] C. Curtis and I. Reiner. Representation Theory of Finite Groups and Associative Algebras. Interscience, New York, 1962.
- [13] P. Delsarte. Association schemes and t-designs in regular semilattices. J. Combin. Theory Ser. A 20 (1976), 230–243.
- [14] G. Dickie. Twice Q-polynomial distance-regular graphs are thin. European J. Combin. 16 (1995), 555–560.
- [15] G. Dickie and P. Terwilliger. A note on thin P-polynomial and dual-thin Q-polynomial symmetric association schemes. J. Algebraic Combin. 7 (1998), 5–15.
- [16] E. Egge. A generalization of the Terwilliger algebra. J. Algebra 233 (2000), 213–252.
- [17] J. T. Go. The Terwilliger algebra of the hypercube. European J. Combin. 23 (2002), 399–429.
- [18] J. T. Go and P. Terwilliger. Tight distance-regular graphs and the subconstituent algebra. European J. Combin. 23 (2002), 793–816.
- [19] C. D. Godsil. Algebraic Combinatorics. Chapman and Hall, Inc., New York, 1993.
- [20] S. A. Hobart and T. Ito. The structure of nonthin irreducible *T*-modules: ladder bases and classical parameters. J. Algebraic Combin. 7 (1998), 53–75.
- [21] T. Ito, K. Tanabe, P. Terwilliger. Some algebra related to P- and Q-polynomial association schemes. Codes and Association Schemes (Piscataway NJ, 1999), 167–192, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 56, Amer. Math. Soc., Providence RI 2001.
- [22] T. Ito and P. Terwilliger. The shape of a tridiagonal pair. J. Pure Appl. Algebra. Submitted.
- [23] A. A. Pascasio. On the multiplicities of the primitive idempotents of a Q-polynomial distance-regular graph. European J. Combin. 23 (2002), 1073–1078.
- [24] A. A. Pascasio. Tight distance-regular graphs and the Q-polynomial property. Graphs Combin. 17 (2001), 149–169.
- [25] C. Roos. On antidesigns and designs in association schemes. Delft Progress Rpt. 7 (1982), 98–109.
- [26] K. Tanabe. The irreducible modules of the Terwilliger algebras of Doob schemes. J. Algebraic Combin. 6 (1997), 173–195.
- [27] P. Terwilliger. Quantum matroids. Progress in Algebraic Combinatorics (Fukuoka, 1993), 323–441, Adv. Stud. Pure Math., 24, Math. Soc. Japan, Tokyo, 1996.
- [28] P. Terwilliger. The incidence algebra of a uniform poset. Coding Theory and Design Theory, Part I, 193–212, IMA Vol. Math. Appl., 20, Springer, New York, 1990.

- [29] P. Terwilliger. The subconstituent algebra of an association scheme I. J. Algebraic Combin. 1 (1992), 363–388.
- [30] P. Terwilliger. The subconstituent algebra of an association scheme II. J. Algebraic Combin. 2 (1993), 73–103.
- [31] P. Terwilliger. The subconstituent algebra of an association scheme III. J. Algebraic Combin. 2 (1993), 177–210.
- [32] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other. *Linear Algebra Appl.* **330** (2001), 149–203.
- [33] P. Terwilliger. Two relations that generalize the q-Serre relations and the Dolan-Grady relations. In *Physics and Combinatorics 1999 (Nagoya)*, 377–398, World Scientific Publishing, River Edge, NJ, 2001.
- [34] P. Terwilliger. Leonard pairs from 24 points of view. Rocky Mountain J. Math. 32 (2002), 827–888.
- [35] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other; the *TD-D* and the *LB-UB* canonical form. *J. Algebra*. Submitted.
- [36] P. Terwilliger. Introduction to Leonard pairs. OPSFA Rome 2001. J. Comput. Appl. Math. 153(2) (2003), 463–475.
- [37] P. Terwilliger. Introduction to Leonard pairs and Leonard systems. Sūrikaisekikenkyūsho Kōkyūroku, (1109):67–79, 1999. Algebraic combinatorics (Kyoto, 1999).
- [38] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other; comments on the split decomposition. *Indag. Math.* Submitted.
- [39] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other; comments on the parameter array. *Geometric and Algebraic Combinatorics* 2, Oisterwijk, The Netherlands 2002. Submitted.
- [40] P. Terwilliger. Leonard pairs and the q-Racah polynomials. *Linear Algebra Appl.* Submitted.
- [41] P. Terwilliger and R. Vidunas. Leonard pairs and the Askey-Wilson relations. Preprint.

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