k – Pyramidal One–Factorizations

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Abstract. We consider one–factorizations of complete graphs which possess an automorphism group fixing $k \geq 0$ vertices and acting regularly (i.e., sharply transitively) on the others. Since the cases $k = 0$ and $k = 1$ are well known in literature, we study the case $k \geq 2$ in some detail. We prove that both k and the order of the group are even and the group necessarily contains $k - 1$ involutions. Constructions for some classes of groups are given. In particular we extend the result of [7]: let G be an abelian group of even order and with $k - 1$ involutions, a one–factorization of a complete graph admitting G as an automorphism group fixing k vertices and acting regularly on the others can be constructed.

Key words. One–factorization, Sharply transitive permutation group, Starter.

1. Introduction

A *one–factor* in a graph is a set of pairwise disjoint edges that partition the set of vertices and a *one–factorization* in a graph is a partition of the set of edges into one–factors.

One–factorizations of complete graphs have been studied from different point of view, we refer to [14] and to the monograph [18] for a survey and for the general notions that will not be explicitly defined here. A complete graph admits a one–factorization if and only if it has an even number of vertices. For this reason, we will only be concerned with the complete graph K_v, v an even integer.

As the number of non-isomorphic one–factorizations of K_v rapidly explodes as v increases, [9], a general classification seems to be not possible. In an attempt to describe one–factorizations which have some degree of symmetry, one can impose conditions on the automorphism group. We recall that an automorphism group is a group of bijections on the vertex-set preserving the one–factorization.

Let k be an integer, with $0 \leq k \leq v$, in this paper we deal with the following

Questions. Does there exist a one–factorization of K_v admitting an automorphism group G fixing k vertices and acting regularly (i.e. sharply transitively) on the others? What can we say about the one–factorization? What can we say about G?

The case $k = 1$ is completely settled. The existence is well known for any group G of odd order, [8], and the one–factorization is said to be $1-$*rotational*, or *pyramidal* under G (see [14] for the terminology).

When $k = 0$, there exists a one–factorization of K_v admitting a cyclic sharply-vertex-transitive automorphism group except when $v = 2^n$, $n \geq 3$, [11]. This result was extended to all abelian groups in [7], and to many other classes of groups, see [3], [4], [5], [6], [16], giving ground to conjecture that for each group G of even order (except for the cyclic groups of order 2^n, $n \geq 3$) a one–factorization of a complete graph admitting G as a sharply-vertex-transitive automorphism group always exists. Further results are also in [13].

When $k = 2$, a one–factorization of K_v admitting a cyclic automorphism group fixing 2 vertices and acting regularly on the others is constructed in [14] for all even v's. Other examples were constructed in [2] in case the automorphism group G is symmetrically sequenceable.

Paralleling the notations introduced in [14], a one–factorization with an automorphism group G which acts regularly on all but k pointwise fixed vertices, will be said *k-pyramidal* under G (when $k = 2$ the term *bipyramidal* is used in [14]). We will also say that G realizes a $k-$pyramidal one–factorization.

When $k \geq 2$ we prove that necessarily k is even and G contains exactly $k - 1$ involutions.

Hence, a group G of even order with $k - 1$ involutions stands as a candidate to realize a sharply-vertex-transitive one–factorization as well as a k-pyramidal one–factorization of a complete graph of suitable order.

In [7], Buratti proved that each abelian group of even order (except for the cyclic group of order 2^n, $n \geq 3$) realizes a sharply-vertex-transitive one–factorization. In [2], Anderson proved that each abelian group of even order with a unique involution realizes a bipyramidal one–factorization, see also Theorem 1 below.

In this note we extend these results and we prove that each abelian group of even order and with $k - 1$ ($k \geq 2$) involutions realizes a $k-$pyramidal one–factorization of a complete graph.

In addition we also examine some other classes of groups and we obtain an analogous result for the class of dihedral groups (thus extending a result of [3]), and for the class of Hamiltonian groups.

2. Preliminaries and Construction

Consider the complete graph K_v and denote respectively by V and E its vertex-set and its edge-set.

Proposition 1. *Let $\mathcal{F}$ be a one–factorization of K_v which is k-pyramidal under the action of a group G. If $k \geq 2$ then k is even and G has even order.*

Proof. Suppose k to be odd, $k \neq 1$. Let $\infty_1, \ldots, \infty_k$ be the fixed vertices and let F_1 be the one–factor containing $[\infty_1, \infty_2]$. F_1 is fixed by G, as well as all its edges containing $\infty_3, \ldots, \infty_k$. This implies the existence of at least one more vertex which is fixed by G: a contradiction. The group G acts regularly on an even number of vertices, therefore it has even order. □

In the rest of the paper set $k = 2t > 0$, $v = 2n + 2t$ and denote by $\mathcal{F}$ a $2t$–pyramidal one–factorization of K_v under the action of a group G. Obviously $2n$ is precisely the order of G in this case. Denote by $X = \{\infty_1, \dots, \infty_{2t}\}$ the set of fixed vertices and identify $V - X$ with the elements of G. The action of G on V can be assumed to occur by right multiplication: an element $g \in G$ fixes each element of X (i.e., $\infty_i g = \infty_i$, $i = 1, \dots, 2t$) and g maps a vertex $v \in V - X$ onto vg. This action extends to edges and one–factors. Hence if R is any subset of V we write: $Rg = \{xg \mid x \in R\}$, in particular if $S = [x, y]$ is an edge then $[x, y]g = [xg, yg]$. Furthermore, if U is a collection of subsets of V, then we write $Ug = \{Sg \mid S \in U\}$. In particular, if U is a collection of edges of K_v then $Ug = \{[xg, yg] \mid [x, y] \in U\}$.

Denote by 1_G the identity of G and set $G = \{g_1 = 1_G, g_2, \dots, g_{2n}\}$.

Proposition 2. *The group G contains exactly $2t - 1$ involutions, it fixes $2t - 1$ one–factors of $\mathcal{F}$ and acts regularly on the others.*

Proof. The edge-set of the complete subgraph K_X is pointwise fixed by G, then the group G fixes at least $2t - 1$ one–factors of $\mathcal{F}$. Consider the set of edges of K_v of type $[\infty_1, g_i]$, as g_i varies in G. They belong to different factors and form an orbit of length $2n$ under the action of G. We conclude that G has exactly $2t - 1$ fixed one–factors and acts regularly on the others. Consider an edge $e = [g_i, g_j]$, $g_i, g_j \in G$. The stabilizer G_e has either cardinality 2 or it is trivial according to whether $g_j^{-1} g_i$ is an involution or not. If $G_e = \{1_G\}$ then $|Orb_G(e)| = 2n$ and each of these $2n$ edges belongs to a different one–factor. In fact if F denotes the one–factor containing e, the existence of $g \in G - \{1_G\}$ mapping e onto an edge of F forces F to be fixed by G and to contain $t + 2n$ distinct edges: a contradiction. If $|G_e| = 2$ then $|Orb_G(e)| = n$. The involution $g_j^{-1} g_i$ fixes e together with the one–factor, say F, to which e belongs. Therefore the one–factor F is fixed by G, it contains t edges with both vertices in X and $Orb_G(e)$ yields the other edges. In particular the element $g_j g_i^{-1}$ is an involution itself, say σ, and $Orb_G(e) = \{[g, \sigma g] \mid g \in T\}$ where T denotes a set of distinct representatives for the right cosets in G of the subgroup $\{1_G, \sigma\}$. We conclude that each involution $\tau \in G$ corresponds to a fixed one–factor, namely the one containing the orbit of the edge $[1_G, \tau]$, and viceversa. We conclude that the number of involutions in G is $2t - 1$. □

Given a group G of even order $2n$ possessing $2t - 1$ involutions, $t \geq 1$, we want to test the existence of a one–factorization which is $2t$-pyramidal under G. Moreover, we want the minimum amount of information which is necessary to reconstruct the one–factorization from G. Use previous notations and set $X = \{\infty_1, \dots, \infty_{2t}\}$, with $G \cap X = \emptyset$ and construct the complete graph with vertex-set $V = G \cup X$. Consider the natural action of G on vertices, edges and factors as before and let J_G be the set of involutions of G. An edge $e = [g_i, g_j]$ with both vertices in G will be called a *proper edge* and there are two possibilities:

- It has trivial stabilizer in G and $|Orb_G(e)| = 2n$. In this case $g_j g_i^{-1}$ is not an involution (otherwise $g_i^{-1} g_j$ fixes e), e is called a *long* edge and we set $\Phi(e) = \{g_i, g_j\}$ and $\partial e = \{g_j g_i^{-1}, g_i g_j^{-1}\}$.
- The element $g_j^{-1} g_i$ is an involution fixing e. The edge is called *short* in this case, $|Orb_G(e)| = n$, the element $g_j g_i^{-1}$ is an involution itself and we set $\partial e = \{g_j g_i^{-1}\}$.

If S is a set of long proper edges, we define $\partial S = \bigcup_{e \in S} \partial e$ and $\Phi(S) = \bigcup_{e \in S} \Phi(e)$. Obviously these unions can contain repeated elements and so, in general, will return a multiset.

Definition 1. *Let $S_G = \{e_1, \dots, e_{n-t}\}$ be a set of $n - t$ distinct long and proper edges. We say that S_G is a* weak-starter *in G if the following conditions are satisfied:*

- $\partial S_G = G - (J_G \cup \{1_G\})$
- $\Phi(e_i) \cap \Phi(e_j) = \emptyset$, *for every $i, j \in \{1, \dots, n-t\}$.*

When $|J_G| = 1$, this definition coincides with the definition of *right even starter* introduced in [2].

Furthermore, if G is an abelian group with an elementary abelian 2–Sylow subgroup P, then a weak-starter in G is the patterned frame starter in $G - P$, see [10] page 473. If G itself is elementary abelian, then a set S_G is a weak-starter in G if and only if S_G is the empty set.

Proposition 3. *Let G be a group of order $2n$ which contains exactly $2t - 1$ involutions. The existence of a weak-starter in G is equivalent to the existence of a one–factorization of K_{2n+2t} which is $2t$-pyramidal under G.*

Proof. Suppose the existence in G of a weak -starter $S_G = \{[g_i, h_i], i = 1, \dots, n-t\}$. We construct a one–factorization $\mathcal{F}$ of K_{2n+2t}. Let $\mathcal{H} = \{H_1, \dots, H_{2t-1}\}$ be a one–factorization of K_X. Pair each factor H_i with an involution $\sigma_i \in G$ in such a way that distinct factors of $\mathcal{H}$ are paired with distinct involutions. Complete each one–factor H_i to a one–factor R_i of K_{2n+2t} adding all proper short edges $[g, \sigma_i g]$, $g \in T_i$, denoting by T_i a set of distinct representatives for the right cosets of $\{1_G, \sigma_i\}$ in G. In this way, we obtain $2t - 1$ distinct one–factors for $\mathcal{F}$ and each of them is fixed by G. Observe also that each proper edge of R_i is short and $\partial e = \{\sigma_i\}$. Let $\{a_1, \dots, a_{2t}\} = G - \Phi(S_G)$. Construct a one–factor F containing all edges $[\infty_i, a_i]$, $i = 1, \dots, 2t$, together with the edges of S_G. Set $\mathcal{F} = \{R_1, \dots, R_{2t-1}\} \cup \{Fg \mid g \in G\}$, $\mathcal{F}$ is a set of one–factors and contains at most $(n + t)(2n + 2t - 1)$ edges. To prove that $\mathcal{F}$ is a one–factorization of K_{2n+2t}, it is sufficient to prove that each edge e of the complete graph with vertex-set $G \cup X$ appears in at least one one–factor of $\mathcal{F}$. If e has both its vertices in X, then it is an edge of K_X and appears in exactly one one–factor $R_i \in \mathcal{F}$. If $e = [\infty_i, h]$, $h \in G$, then $e \in Fa_i^{-1} h$. If e is a short proper edge, then $\partial e = \{\sigma_i\}$ and e lies in R_i, the one–factor associated to the involution σ_i. If $e = [x, y]$ is a long proper edge, let $[g_i, h_i] \in S_G$ such that $\partial e = \partial [g_i, h_i]$. If $yx^{-1} = h_i g_i^{-1}$, then $[x, y] \in Fg$ with $g = g_i^{-1} x$; otherwise if $yx^{-1} = g_i h_i^{-1}$ then

$[x, y] \in Fg$ with $g = g_i^{-1}y$. Obviously $\mathcal{F}$ is $2t$-pyramidal under G. For the converse follow the proof of Proposition 2: a weak-starter S_G is the set $\{e_1, \dots, e_{n-t}\}$ of proper edges of a non-fixed one–factor F. In fact each e_i is long and $\Phi(e_i) \cap \Phi(e_j) = \emptyset$, $i \neq j$. Moreover if $e_i = [g_i, h_i]$ and $e_j = [g_j, h_j]$, $i \neq j$, then it is $\partial e_i \cap \partial e_j = \emptyset$ otherwise either $h_j^{-1}h_i$ or $g_j^{-1}h_i$ maps e_j onto e_i and fixes the one–factor F which is a contradiction. □

In general the one–factorization constructed from a weak-starter is not unique. In the above construction different choices of $\mathcal{H}$ as well as different choices of the edges with a vertex in X and the other in G can lead to non isomorphic one–factorizations.

Proposition 4. *A group G of order $2n$ with $2t - 1$ involutions and such that $2t > n$, realizes a $2t$–pyramidal one–factorization of a complete graph.*

Proof. To prove the statement it is sufficient to prove the existence of a weak-starter in G. If $t = n$ (that is G is an elementary abelian 2–group), then the empty set is a weak-starter in G. Suppose $t < n$ and suppose a weak-starter does not exist in G. Let S be a maximum cardinality set of long edges such that ∂S and $\Phi(S)$ do not contain repeated elements, and denote by m its cardinality. Since S is not a weak-starter, we have $m < n - t$ and there exists a non-identity element $g \in G - \partial S$ such that g is not an involution of G. We also have $|\Phi(S)| = 2m < 2n - 2t$, this implies $|\Phi(S)| < n$ by the hypothesis on $2t$. Consider the edge set $E = \{[x, y] : \partial([x, y]) = \{g, g^{-1}\}\}$. Through each vertex $a \in G$ there are exactly two edges of E, namely $[a, ga]$ and $[a, g^{-1}a]$, so that $|E| = 2n$. Furthermore at most $2|\Phi(S)| < 2n$ edges of E have at least one vertex in $\Phi(S)$ and then there exists at least one edge $e \in E$ with both the vertices in $G - \Phi(S)$. The set $\bar{S} = S \cup \{e\}$ is a set of long edges such that $\partial \bar{S}$ and $\Phi(\bar{S})$ do not contain repeated elements: a contradiction. □

In the next sections we will examine some classes of groups. We will make use of the notions of *sequenceability* and *R-sequenceability* in a finite group G. These notions, together with some relevant results, are briefly summarized below.

Definition 2. *A non-trivial finite group G of order n, with identity 1_G, is said to be sequenceable if its elements can be listed in a sequence $g_1, g_2, \dots, g_n$ in such a way that the quotients $g_2g_1^{-1}, g_3g_2^{-1}, \dots, g_ng_{n-1}^{-1}$ are distinct.*

Definition 3. *A non-trivial finite group G of even order $2n$ with identity 1_G and with a unique involution j is said to be symmetrically sequenceable if its elements can be listed in a sequence (symmetric sequence) $g_1, g_2, \dots, g_{2n}$ in such a way that the quotients $g_2g_1^{-1}, g_3g_2^{-1}, \dots g_{2n}g_{2n-1}^{-1}$ are distinct and $g_{n+i} = g_{n-i+1}j$, $i = 1, \dots, n$.*

Definition 4. *A group G of order n with identity 1_G is said to be R-sequenceable if the elements of $G - \{1_G\}$ can be listed in an R–sequence $g_1, g_2, \dots, g_{n-1}$ such that the quotients $g_2g_1^{-1}, g_3g_2^{-1}, \dots, g_{n-1}g_{n-2}^{-1}, g_1g_{n-1}^{-1}$ are distinct.*

For a recent survey on this topic we refer to [15]. Usually the quotients on a sequence (either symmetric or not) or on an R−sequence $g_1, \dots, g_t$ are defined by $g_i^{-1} g_{i+1}$, $i = 1, \dots, t$, nevertheless our definitions 2, 3 and 4 are equivalent to those given in [15] and are more efficient in our context. Observe also that if $g_1, \dots, g_n$ is a sequence then for each $h \in G$, $g_1 h, \dots, g_n h$ is a sequence itself. Moreover, it is a symmetric sequence if and only if the previous one is symmetric.

We simply recall that each solvable group with a unique involution, except for the quaternion group Q_8, is symmetrically sequenceable, [1].

Each abelian 2−group is R−sequenceable if and only if it is not cyclic, [12].

The only groups known to be non-sequenceable are the abelian groups with more than one involution, the quaternion group Q_8 and the dihedral groups D_6 and D_8. Each dihedral group of order at least 10 is sequenceable.

It was proved in [2] that each symmetrically sequenceable group, together with Q_8, possesses a right even starter (i.e., a weak-starter in our terminology) this, together with Proposition 3, proves the following:

Theorem 1 [2]. *Each symmetrically sequenceable group, together with the quaternion group Q_8, realizes a bipyramidal one–factorization of a complete graph.*

3. Abelian k−Pyramidal One–Factorizations

In this section we prove some preliminary Lemmas and Propositions which will lead to the main Theorem 2.

We will denote by Z_n the cyclic group of order n in additive notation.

Lemma 1. *Let G be a sequenceable group then $G \times Z_2$ admits a weak-starter.*

Proof. Let $g_1, g_2, \dots, g_n$ be a sequence for G. Consider the set of edges $S' = \{e_1, e_2, \dots, e_{n-1}\}$ where $e_i = [(g_i, x_i), (g_{i+1}, y_i)]$, $i = 1, \dots, n-1$ and $x_1 = y_1 = 0$, i.e., $e_1 = [(g_1, 0), (g_2, 0)]$, and each edge e_{i+1} is defined from the previous edge e_i setting $e_{i+1} = \Gamma(e_i)$ where:

$$\Gamma(e_i) = \begin{cases} [(g_{i+1}, y_i + 1), (g_{i+2}, y_i + 1)] & \text{if } g_{i+2} g_{i+1}^{-1} = (g_{r+2} g_{r+1}^{-1})^{-1} \Rightarrow r \geq i \\ \\ [(g_{i+1}, y_i + 1), (g_{i+2}, y_i)] & \text{if } g_{i+2} g_{i+1}^{-1} = (g_{r+2} g_{r+1}^{-1})^{-1} \Rightarrow r < i \end{cases}$$

Denote by I the set of all short edges in S'. The set $S = S' - I$ is a weak-starter in $G \times Z_2$. In fact it is obvious that $\Phi(e_i) \cap \Phi(e_j) = \emptyset$ when $i \neq j$. Moreover, let (g, x) be an element of order greater than 2 in $G \times Z_2$. Let (g_i, g_{i+1}) and (g_r, g_{r+1}) be the two pairs of elements of the sequence for G such that $g_{i+1} g_i^{-1} = g$ and $g_{r+1} g_r^{-1} = g^{-1}$. Denote by m and $\bar{m}$ the minimum and the maximum between i and r, respectively. If $x = 0$ then $(g, x) = \partial([(g_m, y), (g_{m+1}, y)])$ and otherwise, if $x = 1$, then $(g, x) = \partial([(g_{\bar{m}}, y), (g_{\bar{m}+1}, y + 1)])$, where y is a suitable element in Z_2. $\square$

Lemma 2. *Let G be an R-sequenceable group of even order then $G \times Z_2$ admits a weak-starter.*

Proof. Let $g_1, \ldots, g_{n-1}$ be an R-sequence of G. Without loss in generality we can suppose that $g_1 g_{n-1}^{-1}$ is an involution of G. Paralleling the construction of Lemma 1, consider the set $S' = \{e_1, \ldots, e_{n-2}\}$ where $e_i = [(g_i, x_i), (g_{i+1}, y_i)]$, $i = 1, \ldots, n-2$, $x_1 = y_1 = 0$, i.e., $e_1 = [(g_1, 0), (g_2, 0)]$, and each edge e_{i+1} is defined by the previous edge e_i setting $e_{i+1} = \Gamma(e_i)$, where Γ is defined as in Lemma 1. Denote by I the set of all short edges in S'. The set $S = S' - I$ is a weak-starter in $G \times Z_2$. $\square$

Lemma 3. *The group* Z_{8n} *admits a weak-starter* S *such that* $0, 4n \notin \Phi(S)$.

Proof. Consider the following sets:

$$S_1 = \{[1, 2n+1], [5n, n+1]\}$$
$$S_2 = \{[2n+1-i, 2n+1+i]/i = 1, \ldots, n-1\}$$
$$S_3 = \{[n+1-i, 3n+i]/i = 1, \ldots, n-1\}$$
$$S_4 = \{[-i, 4n+i]/i = 1, \ldots, n-1\}$$
$$S_5 = \{[5n+i, 7n+1-i]/i = 1, \ldots, n\}$$

we have:

$\partial S_1 = \{\pm 2n, \pm(4n-1)\}$, $\Phi(S_1) = \{1, n+1, 2n+1, 5n\}$.

$\partial S_2 = \{\pm 2i, i = 1, \ldots, n-1\}$, $\Phi(S_2) = \{n+2, n+3, \ldots, 2n, 2n+2, 2n+3, \ldots, 3n\}$.

$\partial S_3 = \{\pm(2n+2i-1), i = 1, \ldots, n-1\}$,

$\Phi(S_3) = \{2, 3, \ldots, n, 3n+1, 3n+2, \ldots, 4n-1\}$.

$\partial S_4 = \{\pm(4n+2i), i = 1, \ldots, n-1\}$,

$\Phi(S_4) = \{4n+1, 4n+2, \ldots, 5n-1, 7n+1, 7n+2, \ldots, 8n-1\}$.

$\partial S_5 = \{\pm(2n-2i+1), i = 1, \ldots, n\}$, $\Phi(S_5) = \{5n+1, 5n+2, \ldots, 7n\}$.

The union $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$ is a weak-starter in Z_{8n} such that $0, 4n \notin \Phi(S)$. $\square$

Lemma 4. *The group* $Z_{8m} \times Z_{8n}$ *admits a weak-starter.*

Proof. Let $S_1 = \{[a_1, b_1], \ldots, [a_{4m-1}, b_{4m-1}]\}$ be a weak-starter in Z_{8m} and let $S_2 = \{[x_1, y_1], \ldots, [x_{4n-1}, y_{4n-1}]\}$ be a weak-starter in Z_{8n}. By Lemma 3, we can assume $0, 4m \notin \Phi(S_1)$ as well as $0, 4n \notin \Phi(S_2)$, so that $\Phi(S_1) = Z_{8m} - \{0, 4m\}$ and $\Phi(S_2) = Z_{8n} - \{0, 4n\}$. Let $c_1, \ldots, c_{4m}, c_{4m+1}, \ldots, c_{8m}$ be a symmetric sequence in Z_{8m} such that $c_{4m} = 0$ and $c_{4m+1} = 4m$. Similarly, let $d_1, \ldots, d_{4n}, d_{4n+1}, \ldots, d_{8n}$ be a symmetric sequence in Z_{8n} such that $d_{4n} = 4n$ and $d_{4n+1} = 0$.

Recall that both the sets $\pm\{c_{i+1} - c_i \mid i = 1, \ldots, 4m-1\}$ and $\pm\{c_{i+1} - c_i \mid i = 4m+1, \ldots, 8m-1\}$ cover the elements of $Z_{8m} - \{0, 4m\}$ exactly once.

In the same manner, both the sets $\pm\{d_{i+1} - d_i \mid i = 1, \ldots, 4n-1\}$ and $\pm\{d_{i+1} - d_i \mid i = 4n+1, \ldots, 8n-1\}$ cover the elements of $Z_{8n} - \{0, 4n\}$ exactly once.

Consider the following sets:

$$\begin{aligned}
Q &= \{[(a_i, x_j), (b_i, y_j)], [(a_i, y_j), (b_i, x_j)] \mid [a_i, b_i] \in S_1, [x_j, y_j] \in S_2\} \\
R &= \{[(c_i, 0), (c_{i+1}, 4n)] \mid i = 1, \dots, 4m-1\} \\
U_1 &= \{[(c_{4m+2i+1}, 0), (c_{4m+2i+2}, 0)] \mid i = 0, \dots 2m-1\} \\
U_2 &= \{[(c_{4m+2i+2}, 4n), (c_{4m+2i+3}, 4n)] \mid i = 0, \dots, 2m-2\} \\
Z &= \{[(0, d_i), (4n, d_{i+1})] \mid i = 1, \dots, 4n-1\} \\
W_1 &= \{[(0, d_{4n+2i+1}), (0, d_{4n+2i+2})] \mid i = 0, \dots 2n-1\} \\
W_2 &= \{[(4m, d_{4n+2i+2}), (4m, d_{4n+2i+3})] \mid i = 0, \dots, 2n-2\}
\end{aligned}$$

Set $U = U_1 \cup U_2$ and $W = W_1 \cup W_2$, we have:
$\partial Q = (Z_{8m} - \{0, 4m\}) \times (Z_{8n} - \{0, 4n\})$, $\Phi(Q) = \Phi(S_1) \times \Phi(S_2)$ and $\Phi(Q)$ is disjoint from both $\{0, 4m\} \times Z_{8n}$ and $Z_{8m} \times \{0, 4n\}$.
$\partial R = (Z_{8m} - \{0, 4m\}) \times \{4n\}$,
$\Phi(R) = \{(c_j, 0), j = 1, \dots, 4m-1\} \cup \{(c_j, 4n), j = 2, \dots, 4m\}$.
$\partial U = (Z_{8m} - \{0, 4m\}) \times \{0\})$,
$\Phi(U) = \{(c_j, 0), j = 4m+1, \dots, 8m\} \cup \{(c_j, 4n), j = 4m+2, \dots, 8m-1\}$.
$\partial Z = \{4m\} \times (Z_{8n} - \{0, 4n\})$,
$\Phi(Z) = \{(0, d_j), j = 1, \dots, 4n-1\} \cup \{(4m, d_j), j = 2, \dots, 4n\}$.
$\partial W = \{0\} \times (Z_{8n} - \{0, 4n\})$,
$\Phi(W) = \{(0, d_j), j = 4n+1, \dots, 8n\} \cup \{(4m, d_j), j = 4n+2, \dots, 8n-1\}$.
The set $Q \cup R \cup U \cup Z \cup W$ is a weak-starter in $Z_{8m} \times Z_{8n}$. □

Lemma 5. *Let G be an R-sequenceable group of even order then $G \times Z_{8m}$ admits a weak-starter.*

Proof. Let $g_1, \dots, g_{n-1}$ be an R-sequence of G. Let $S' = \{[x_j, y_j] \mid j = 1, \dots, 4n-1\}$ be a starter of Z_{8m} such that $0, 4m \notin \Phi(S')$, take for example the weak-starter described in lemma 3. Set $g_n = g_1$ and consider the following sets:

$$\begin{aligned}
R &= \{[(g_i, x_j), (g_{i+1}, y_j)] \mid i = 1, \dots, n-1, j = 1, \dots, 4m-1\} \\
T &= \{[(1_G, x_j), (1_G, y_j)] \mid j = 1, \dots, 4m-1\}
\end{aligned}$$

The set $G \times Z_{8m} - (\Phi(R) \cup \Phi(T))$ is exactly $G \times \{0, 4m\}$, and $\partial(R \cup T) = G \times (Z_{8m} - \{0, 4m\})$. The set $G \times \{0, 4m\}$ is a subgroup of $G \times Z_{8m}$ and it is isomorphic to $G \times Z_2$, then apply Lemma 2 and construct a weak-starter U in $G \times \{0, 4m\}$. The set $S = R \cup T \cup U$ is a weak-starter in $G \times Z_{8m}$. □

Lemma 6. *Let G be a sequenceable group then $G \times Z_4$ admits a weak-starter.*

Proof. Let $g_1, \dots, g_n$ be a sequence of G. Consider the set

$$S' = \{e_1, \dots, e_{n-1}, f_1, \dots, f_{n-1}\}$$

where

$$e_i = [(g_i, x_i), (g_{i+1}, y_i)], \quad f_i = [(g_i, s_i), (g_{i+1}, t_i)], \quad i = 1, \dots, n-1$$

with $x_1 = t_1 = 0$ and $y_1 = s_1 = 1$, i.e., $e_1 = [(g_1, 0), (g_2, 1)]$, $f_1 = [(g_1, 1), (g_2, 0)]$, and each pair (e_{i+1}, f_{i+1}) is defined from the previous pair (e_i, f_i) setting $(e_{i+1}, f_{i+1}) = \mu((e_i, f_i))$ where:

$$\mu((e_i, f_i)) = \begin{cases} ([(g_{i+1}, y_i + 2), (g_{i+2}, t_i + 2)], [(g_{i+1}, t_i + 2), (g_{i+2}, y_i + 2)]) \\ \qquad \text{if } g_{i+2}g_{i+1}^{-1} = (g_{r+2}g_{r+1}^{-1})^{-1} \text{ implies } r \geq i \\ ([(g_{i+1}, y_i + 2), (g_{i+2}, y_i + 2)], [(g_{i+1}, t_i + 2), (g_{i+2}, t_i)]) \\ \qquad \text{if } g_{i+2}g_{i+1}^{-1} = (g_{r+2}g_{r+1}^{-1})^{-1} \text{ implies } r < i \end{cases}$$

Let I be the set of edges f_i such that $g_{i+1}g_i^{-1}$ is an involution (if G has odd order then I is the empty set), in this case by definition of μ we have $\partial f_i = \partial e_i$, therefore if $\overline{S} = S' - I$, then it is $\partial \overline{S} = \partial S'$. Let $e_0 = [(g_1, 2), (g_1, 3)]$, we prove that the set $S = \overline{S} \cup \{e_0\}$ is a weak-starter in $G \times Z_4$. In fact $\Phi(S')$ does not contain repeated vertices and the same holds for $\Phi(S)$. Moreover, let $u \geq 0$ be the number of involutions in G, therefore I contains u elements, $G \times Z_4$ has $2u + 1$ involutions and the number of elements $(g, x) \in G \times Z_4$ such that $(g^2, 2x) \neq (1_G, 0)$ is $4n - 2u - 2$. Observe that this number is exactly $|\partial S|$. In fact $|\overline{S}| = 2n - u - 2$, each edge of $\overline{S}$ is long as well as e_0 and this implies $|\partial S| = 4n - 2u - 2$. Now, to prove that S is a weak-starter, it is sufficient to prove that each element (g, x) with $(g^2, 2x) \neq (1_G, 0)$ is in ∂S. If g is an involution of G, there is exactly one index $i \in \{1, \dots, n-1\}$ such that $g = g_{i+1}g_i^{-1}$, in this case $(g, x) \in \partial e_i$. Otherwise, if $g^2 \neq 1_G$ there is a pair of indices $1 \leq i < j \leq n-1$ such that $\{g, g^{-1}\} = \{g_{i+1}g_i^{-1}, g_{j+1}g_j^{-1}\}$. We have either $(g, x) \in \partial e_i \cup \partial f_i$ or $(g, x) \in \partial e_j \cup \partial f_j$ according to whether $x \in \{1, -1\}$ or $x \in \{0, 2\}$. Finally, if $g = 1_G$ we have $(g, x) \in \partial e_0 = \{(1_G, \pm 1)\}$. □

Lemma 7. *Let G be an R-sequenceable group of even order then $G \times Z_4$ admits a weak-starter.*

Proof. Let $g_1, \dots, g_{n-1}$ be an R-sequence of G. Without loss of generality we can suppose $g_1 g_{n-1}^{-1}$ is an involution of G. We repeat the construction of Lemma 6 and we consider the set

$$S' = \{e_1, \dots, e_{n-2}, f_1, \dots, f_{n-2}\}$$

where

$$e_i = [(g_i, x_i), (g_{i+1}, y_i)], \quad f_i = [(g_i, s_i), (g_{i+1}, t_i)], \quad i = 1, \dots, n-2$$

with $x_1 = t_1 = 0$ and $y_1 = s_1 = 1$, i.e., $e_1 = [(g_1, 0), (g_2, 1)]$, $f_1 = [(g_1, 1), (g_2, 0)]$, and each pair (e_{i+1}, f_{i+1}) is defined from the previous pair (e_i, f_i) setting $(e_{i+1}, f_{i+1}) = \mu((e_i, f_i))$ where μ is defined as in Lemma 6.

Let I be the set of edges f_i such that $g_{i+1}g_i^{-1}$ is an involution (if $g_1g_{n-1}^{-1}$ is the unique involution in G, then I is the empty set), in this case by definition of μ we have $\partial f_i = \partial e_i$, therefore if $\overline{S} = S' - I$ then $\partial\overline{S} = \partial S'$. Moreover, the edges of $\overline{S}$ are long and $\Phi(\overline{S})$ does not contain repeated elements. Proceeding as in the previous Lemma 6, we can observe that the elements in $\partial\overline{S}$ are distinct and cover all the elements of $G \times Z_4$ which are different from the identity and the involutions, except for $(g_1g_{n-1}^{-1}, \pm 1)$ and $(1_G, \pm 1)$. Let $(g_{n-1}, x) \notin \Phi(\overline{S})$, i.e., $(g_{n-1}, x) \notin \Phi(\{e_{n-2}, f_{n-2}\})$, if $x = \pm 1$ let $e_{n-1} = [(g_{n-1}, x), (g_1, 2)]$, otherwise let $e_{n-1} = [(g_{n-1}, x), (g_1, 3)]$. Observe that $(g_1, 2)$ and $(g_1, 3)$ are not in $\Phi(\overline{S})$ and $\partial e_{n-1} = \{(g_1g_{n-1}^{-1}, \pm 1)\}$. Finally, let $e_0 = [(1_G, 0), (1_G, 1)]$, the set $S = \overline{S} \cup \{e_0\} \cup \{e_{n-1}\}$ is a weak-starter in $G \times Z_4$. □

Proposition 5. *Let G be an R-sequenceable group containing $2t - 1$ involutions and admitting a weak-starter S_G. Let K be a group of odd order. The group $G \times K$ realizes a $2t$–pyramidal one–factorization of a complete graph.*

Proof. To prove the statement it is sufficient to construct a weak-starter in $G \times K$. Denote by $g_1, g_2, \dots, g_{n-1}$ an R-sequence of G and by $S_G = \{[x_i, y_i]/i = 1, \dots, n - t\}$ a weak-starter in G; we express the group K as union of the disjoint sets K_1, K_2 and $\{1_K\}$, defined in such a way that $k \in K_1$ iff $k \neq 1_K$ and $k^{-1} \in K_2$. Consider the sets of edges:

$$Q = \{[(1_G, k), (1_G, k^{-1})] | k \in K_1\}$$
$$S = \{[(x_i, 1_K), (y_i, 1_K)] | i = 1, \dots, n - t\}$$
$$R = \{[(g_{i+1}, k), (g_i, k^{-1})] | k \in K_1, i = 1, \dots, n - 1\} \quad (\text{where } g_n = g_1)$$

It is easy to verify that $Q \cup S \cup R$ is a weak-starter in $G \times K$. □

The above Lemmas 2, 5 and 7 lead to the following:

Proposition 6. *Let G be an R-sequenceable group of even order. For each $t \geq 1$ the group $G \times Z_{2^t}$ admits a weak-starter.*

We are now able to prove the following:

Theorem 2. *Each abelian group G of even order $2n$ and with $2t - 1$ involutions, realizes a $2t$–pyramidal one–factorization of the complete graph K_{2n+2t}.*

Proof. Let P be the 2–Sylow subgroup of G, then $G = P \times K$, where K is an odd order abelian group. From the fundamental theorem on the structure of finite abelian groups, $P = Z_{2^{n_1}} \times Z_{2^{n_2}} \times \dots \times Z_{2^{n_s}}, s \geq 1$. If $s = 1$ then G is symmetrically sequenceable, [1], and the assertion follows from Theorem 1. If $s \geq 2$, the group P is a non cyclic abelian 2–group and then it is R–sequenceable, [12]. Therefore, the proof follows from Proposition 5 as soon as we construct a weak-starter in P. If $P = Z_{2^{n_1}} \times Z_{2^{n_2}}$ (i.e., $s = 2$), there are two possibilities: either $n_1, n_2 \geq 3$ or at least one of them, say n_2, is less than 3. In the former case we construct a weak-starter in P using Lemma 4, in the latter case, since $Z_{2^{n_1}}$ is sequenceable, a weak-starter is constructed applying Lemma 1 or Lemma 6. Now suppose $P = Z_{2^{n_1}} \times Z_{2^{n_2}} \times \ldots \times Z_{2^{n_s}}$, with $s > 2$. Since $Z_{2^{n_1}} \times \ldots \times Z_{2^{n_{s-1}}}$ is an even order R−sequenceable group, we apply proposition 6 to construct a weak-starter in P. □

4. Dihedral and Hamiltonian k-pyramidal One–Factorizations

Non-abelian k−pyramidal one–factorizations are obtained in the previous chapters in some special cases (see for examples Theorem 1, Lemmas 1, 2, 6, 7 and Proposition 6, when G is not abelian). In this section we look at two more classes of groups, namely dihedral and Hamiltonian groups.

As a consequence of Proposition 4 we immediately obtain the following statement:

Theorem 3. *Each dihedral group D_{2n} realizes either a $(n+1)$−pyramidal or a $(n+2)$−pyramidal one–factorization of a complete graph, according to whether n is odd or even.*

We recall that each dihedral group realizes a sharply-vertex transitive one–factorization, see [3], hence the previous Theorem 3 completes this result.

Proposition 7. *Let G be a group of even order admitting a weak-starter. The group $G \times (Z_2)^m$, $m \geq 2$, admits a weak-starter itself.*

Proof. Let $S = \{[u_1, v_1] \ldots, [u_s, v_s]\}$ be a weak-starter in G and let $a_1, \ldots, a_{2^m-1}$ be an R−sequence of the elementary abelian 2−group $(Z_2)^m$. Set $a_{2^m} = a_1$ and for each $j \in \{1, \ldots, 2^m - 1\}$ let $A_j = \{[(u_i, a_j), (v_i, a_{j+1})] \mid i = 1, \ldots, s\}$ and let $A_0 = \{[(u_i, 0), (v_i, 0)], i = 1, \ldots, s\}$. It is easy to check that $A_0 \cup A_1 \cup \ldots \cup A_{2^m-1}$ is a weak-starter in $G \times (Z_2)^m$. □

Recall that a *Hamiltonian group* is defined to be a group in which every subgroup is normal. Apart from the abelian groups, each Hamiltonian group is the direct product of the quaternion group Q_8, together with an elementary abelian 2−group and an odd order group (see [17, p.253]).

Theorem 4. *Each Hamiltonian group G of even order $2n$ and with $2t-1$ involutions, realizes a $2t$−pyramidal one–factorization of the complete graph K_{2n+2t}.*

Proof. If G is abelian, the assertion follows from Theorem 2. Suppose G is not abelian, then $G = Q_8 \times K \times (Z_2)^m$, with K of odd order. From Theorem 1 we know that $Q_8 \times K$ admits a weak-starter, in fact, if K is not trivial, then the group $Q_8 \times K$ is solvable and with a unique involution and by [1] it is symmetrically sequenceable. If $m \geq 2$, the existence of a weak-starter in G follows from Proposition 7. Suppose $m = 1$. If K is not trivial, then the assertion follows from Lemma

1. If K is trivial, we conclude the proof by exhibiting a weak-starter in $Q_8 \times Z_2$. The group Q_8 can be presented as follows: $Q_8 =< a, b : a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} >$. The set $\{[(a,0),(1,0)], [(a^{-1},0),(ab,1)], [(b^{-1},0),(b^2,0)], [(ab,0),(b^2,1)], [(a,1),(b,1)], [(b,0),(ab,1)]\}$ is a weak-starter in $Q_8 \times Z_2$. □

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