# Monochromatic and Heterochromatic Subgraph Problems in a Randomly Colored Graph 

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#### Abstract

Let $K_{n}$ be the complete graph with $n$ vertices and $c_{1}, c_{2}, \cdots, c_{r}$ be $r$ different colors. Suppose we randomly and uniformly color the edges of $K_{n}$ in $c_{1}, c_{2}, \cdots, c_{r}$. Then we get a random graph, denoted by $\mathcal{K}_{n}^{r}$. In the paper, we investigate the asymptotic properties of several kinds of monochromatic and heterochromatic subgraphs in $\mathcal{K}_{n}^{r}$. Accurate threshold functions in some cases are also obtained.


Keywords: monochromatic, heterochromatic, threshold function
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## 1 Introduction

The study of random graphs was begun by P. Erdös and A. Rényi in the 1960s [7-9] and now has a comprehensive literature [3, 6 ].

The most frequently encountered probabilistic model of random graph is $\mathcal{G}_{n, p(n)}$, where $0 \leq p(n) \leq 1$. It consists of all graphs with vertex set $V=\{1,2, \cdots, n\}$ in which the edges are chosen independently and with probability $p(n)$. As $p(n)$ goes from zero to one the random graph $\mathcal{G}_{n, p(n)}$ evolves from empty to full.

[^0]P. Erdös and A. Rényi discovered that for many natural properties A of graphs there was a narrow range in which $\operatorname{Pr}\left[\mathcal{G}_{n, p(n)}\right.$ has property A] moves from near zero to near one. So we introduce the following important definition ( [5], page 14).

Definition 1.1 $A$ function $p(n)$ is a threshold function for property $A$ if the following two conditions are satisfied:

1. If $p^{\prime}(n) \ll p(n)$, then $\lim _{n \rightarrow \infty} \operatorname{Pr}\left[\mathcal{G}_{n, p^{\prime}(n)}\right.$ has property $\left.A\right]=0$.
2. If $p^{\prime}(n) \gg p(n)$, then $\lim _{n \rightarrow \infty} \operatorname{Pr}\left[\mathcal{G}_{n, p^{\prime}(n)}\right.$ has property $\left.A\right]=1$.

In general, if $\operatorname{Pr}\left[\mathcal{G}_{n, p(n)}\right.$ has property A $] \rightarrow 0$, we say almost no $\mathcal{G}_{n, p(n)}$ has property A. Conversely, if $\operatorname{Pr}\left[\mathcal{G}_{n, p(n)}\right.$ has property A] $\rightarrow 1$, we say almost every $\mathcal{G}_{n, p(n)}$ has property A.

In this article, we introduce the following probabilistic model of random graphs. Let $K_{n}$ be the complete graph with vertex set $V=\{1,2, \cdots, n\}$ and $c_{1}, c_{2}, \cdots, c_{r}$ be $r=r(n)$ different colors. We now send $c_{1}, c_{2}, \cdots, c_{r}$ to the edges of $K_{n}$ randomly and equiprobably, which means each edge is colored in $c_{i}(1 \leq i \leq r)$ with probability $\frac{1}{r}$. Thus we get a random graph $\mathcal{K}_{n}^{r}$. The probability space $(\Omega, \mathcal{F}, \mathcal{P})$ of $\mathcal{K}_{n}^{r}$ has a simple form: $\Omega$ has $r^{\binom{n}{2}}$ elements and each one has probability $\frac{1}{\left.r^{n} \begin{array}{c}n \\ 2\end{array}\right)}$ to appear.

The subgraph of $\mathcal{K}_{n}^{r}$ with vertices $1,2, \cdots, n$ and the edges that have color $c_{i}$ is denote by $\mathcal{G}_{i}$. Obviously, it is just the random graph $\mathcal{G}_{n, p(n)}$ ( [3], page 34), where $p(n)=\frac{1}{r}$.

Matching, clique and tree are three kinds of important subgraphs. As to their definitions, please refer to [2]. A $k$-matching is a matching of $k$ independent edges. A $k$-clique is a clique of $k$ vertices. Similar, a $k$-tree is a tree of $k$ vertices. In a $k$-matching ( $k$-clique, $k$-tree), if all of the edges are in a same color, we call it a monochromatic $k$-matching ( $k$-clique, $k$-tree); On the other hand, if any two of edges are of different colors, we call it a heterochromatic $k$-matching ( $k$-clique, $k$-tree).

Having a monochromatic $k$-matching, $k$-clique or $k$-tree or a heterochromatic $k$-matching, $k$-clique or $k$-tree are all properties of $\mathcal{K}_{n}^{r}$. We want to investigate these properties and obtain the threshold functions for them.

Two properties will be especially demonstrated: monochromatic $k$-matching and heterochromatic $k$-matching. For the others, the methods are similar and we list the results in Section 4.

## 2 Monochromatic $k$-Matchings in $\mathcal{K}_{n}^{r}$

Let $k$ be an integer. Obviously, in $\mathcal{K}_{n}^{r}$, there are altogether

$$
q=\frac{\binom{n}{2}\binom{n-2}{2} \cdots\binom{n-2 k+2}{2}}{k!}
$$

sets of $k$ independent edges. Arrange them in an order and the $i$-th one is denoted by $M_{i}$.

Let $A_{i}$ be the event that the edges in $M_{i}$ are monochromatic and $X_{i}$ be the indicator variable for $A_{i}$. That is,

$$
X_{i}= \begin{cases}1 & \text { if } A_{i} \text { happens },  \tag{2.1}\\ 0 & \text { otherwise }\end{cases}
$$

Then the random variable

$$
X=X_{1}+X_{2}+\cdots+X_{q}
$$

denotes the number of monochromatic $k$-matchings in $\mathcal{K}_{n}^{r}$.
For each $1 \leq i \leq q$,

$$
E\left(X_{i}\right)=\operatorname{Pr}\left[X_{i}=1\right]=\frac{r}{r^{k}} .
$$

From the linear of the expectation [1],

$$
\begin{align*}
E(X) & =E\left(X_{1}+X_{2}+\cdots+X_{q}\right) \\
& =\frac{r}{r^{k}} q \\
& =\frac{n!}{(n-2 k)!2^{k} k!r^{k-1}} . \tag{2.2}
\end{align*}
$$

By careful calculation, the following assertions (*) for (2.2) are true, which will be used later:

1. If $r$ is fixed, then for every $1 \leq k \leq \frac{n}{2}, E(X) \rightarrow \infty$.
2. If $k$ is fixed and $r \ll\left(\frac{n!}{(n-2 k)!2^{k} k!}\right)^{\frac{1}{k-1}}$, then $E(X) \rightarrow \infty$;
3. If $k$ is fixed and $r \gg\left(\frac{n!}{(n-2 k)!2^{k} k!}\right)^{\frac{1}{k-1}}$, then $E(X) \rightarrow 0$;
4. If $k$ is fixed and $r=c^{(0)}\left(\frac{n!}{(n-2 k)!2^{k} k!}\right)^{\frac{1}{k-1}}$, where $c^{(0)}>0$ is a constant, then $E(X) \rightarrow \frac{1}{\left(c^{(0)}\right)^{k-1}}$.

Though $k$ and $r$ can be both functions of $n$, if they are both variables, the situation becomes very complicated. So we illustrate monochromatic $k$-matching problem from three aspects: $r$ is fixed, $k$ is fixed and $k=\left\lfloor\frac{n}{2}\right\rfloor$. The last case is the perfect matching case or the nearly perfect matching case. Since we focus on the asymptotic properties, we will not distinguish $\left\lfloor\frac{n}{2}\right\rfloor$ from $\frac{n}{2}$. That is, we always suppose $n$ is an even.

## $2.1 r$ is fixed

Assertion ( ${ }^{*}$ ) 1 says that $E(X) \rightarrow \infty$ for every $1 \leq k \leq \frac{n}{2}$ if $r$ is fixed. We certainly expect that $\operatorname{Pr}[X>0] \rightarrow 1$ holds. In fact, it does.

Theorem 2.1 If $r \geq 1$ is fixed, then almost every $\mathcal{K}_{n}^{r}$ has a monochromatic $k$-matching for any $1 \leq k \leq \frac{n}{2}$.

Proof. We have mentioned in Section 1 that the subgraph $\mathcal{G}_{i}$ of $\mathcal{K}_{n}^{r}$ is actually the random graph $\mathcal{G}_{n, p(n)}$, where $p(n)=\frac{1}{r}$. There is a result saying that the threshold function for $\mathcal{G}_{n, p}$ has a perfect matching is $\frac{\operatorname{logn}}{n}$ ([6], page 85). If $r$ is fixed, then $\frac{1}{r} \gg \frac{\operatorname{logn} n}{n}$, which implies that almost every $\mathcal{G}_{i}$ has a perfect matching. Then almost every $\mathcal{K}_{n}^{r}$ has a monochromatic $k$-matching for every $1 \leq k \leq \frac{n}{2}$.

## $2.2 k$ is fixed

In this case, we prove the following theorem.

Theorem 2.2 If $k$ is fixed ( $k=1$ is a trivial case so suppose $k \geq 2$ ), then

$$
\lim _{n \rightarrow \infty} \operatorname{Pr}[X>0]= \begin{cases}0 & \text { if } r \gg\left(\frac{n!}{\left(n-2 k!!2^{k} k!\right.}\right)^{\frac{1}{k-1}},  \tag{2.3}\\ 1 & \text { if } r \ll\left(\frac{n!}{(n-2 k)!2^{k} k!}\right)^{\frac{1}{k-1}} .\end{cases}
$$

That is to say, $\left(\frac{n!}{(n-2 k)!2^{k} k!}\right)^{\frac{1}{k-1}}$ is the threshold function for the property that $\mathcal{K}_{n}^{r}$ has a monochromatic $k$-matching.

Proof. From Markov's inequality [4]

$$
\operatorname{Pr}[X>0] \leq E(X)
$$

and assertion (*) 3, we have

$$
\operatorname{Pr}[X>0] \rightarrow 0 \text { if } r \gg\left(\frac{n!}{(n-2 k)!2^{k} k!}\right)^{\frac{1}{k-1}} .
$$

For the other half, we estimate $\frac{\Delta}{(E(X))^{2}}$, where $\Delta=\sum_{i \sim j} \operatorname{Pr}\left[A_{i} \cap A_{j}\right]$. $A_{i}\left(A_{j}\right)$ denotes the event that the edges in $M_{i}\left(M_{j}\right)$ are monochromatic and $i \sim j$ means the ordered pair of $A_{i}$ and $A_{j}$ that are not independent from each other.

Our goal is to prove that if $r \ll\left(\frac{n!}{(n-2 k)!2^{k} k!}\right)^{\frac{1}{k-1}}$, then $\frac{\Delta}{(E(X))^{2}} \rightarrow 0$. Because

$$
\begin{aligned}
\Delta & =\sum_{i \sim j} \operatorname{Pr}\left[A_{i} \cap A_{j}\right] \\
& =\sum_{s=1}^{k-1} \sum_{(i, j)_{s}} \frac{r}{r^{2 k-s}} \\
& \text { (where }(i, j)_{s} \text { means the ordered pair of } M_{i} \text { and } M_{j} \text { that have } s \text { common edges) } \\
\leq & \sum_{s=1}^{k-1} \frac{\binom{n}{2}\binom{n-2}{2} \cdots\binom{n-2(s-1)}{2}}{s!}\left(\frac{\binom{n-2 s}{2}\binom{n-2 s-2}{2} \cdots\binom{n-2 k+2}{2}}{(k-s)!}\right)^{2} \frac{1}{r^{2 k-s-1}} \\
= & \frac{n!}{2^{2 k}(n-2 k)!(n-2 k)!r^{2 k-1}} \sum_{s=1}^{k-1} \frac{(n-2 s)!2^{s} r^{s}}{s!(k-s)!(k-s)!}
\end{aligned}
$$

then we have

$$
\begin{equation*}
\frac{\Delta}{(E(X))^{2}} \leq \frac{k!k!}{n!} \sum_{s=1}^{k-1} \frac{(n-2 s)!2^{s} r^{s-1}}{s!(k-s)!(k-s)!} \tag{2.4}
\end{equation*}
$$

If $r \ll\left(\frac{n!}{(n-2 k)!2^{k} k!}\right)^{\frac{1}{k-1}} \sim\left(\frac{1}{2^{k} k!}\right)^{\frac{1}{k-1}} n^{\frac{2 k}{k-1}}$, then there are 3 possible cases: (i) $r \ll n^{2}$, (ii) $r=c^{(1)} n^{2}$, where $c^{(1)}>0$ is a constant and (iii) $n^{2} \ll r \ll$ $\left(\frac{n!}{(n-2 k)!2^{k} k!}\right)^{\frac{1}{k-1}}$.

In case (i),

$$
\begin{equation*}
\sum_{s=1}^{k-1} \frac{(n-2 s)!2^{s} r^{s}}{s!(k-s)!(k-s)!}=(1+\circ(1)) \frac{2(n-2)!}{(k-1)!(k-1)!} \tag{2.5}
\end{equation*}
$$

Then submit (2.5) to (2.4), we get

$$
\begin{equation*}
\frac{\Delta}{(E(X))^{2}} \leq 2(1+\circ(1)) \frac{k^{2}}{n(n-1)} \rightarrow 0 \tag{2.6}
\end{equation*}
$$

In case (ii)

$$
\begin{equation*}
\sum_{s=1}^{k-1} \frac{(n-2 s)!2^{s} r^{s}}{s!(k-s)!(k-s)!}=c^{(2)} \frac{2(n-2)!}{(k-1)!(k-1)!} \tag{2.7}
\end{equation*}
$$

where $c^{(2)}$ is a sufficiently large constant.
Then submit (2.7) to (2.4), we get

$$
\begin{equation*}
\frac{\Delta}{(E(X))^{2}} \leq 2 c^{(2)} \frac{k^{2}}{n(n-1)} \rightarrow 0 \tag{2.8}
\end{equation*}
$$

In case (iii)

$$
\begin{equation*}
\sum_{s=1}^{k-1} \frac{(n-2 s)!2^{s} r^{s}}{s!(k-s)!(k-s)!}=(1+\circ(1)) \frac{(n-2 k+2)!2^{k-1} r^{k-2}}{(k-1)!} \tag{2.9}
\end{equation*}
$$

Then submit (2.9) to (2.4), we get

$$
\begin{equation*}
\frac{\Delta}{(E(X))^{2}} \leq c^{(3)} \frac{n^{\frac{2 k(k-2)}{k-1}}}{n^{2 k-2}} \rightarrow 0 \tag{2.10}
\end{equation*}
$$

where $c^{(3)}$ is a sufficiently large constant.
Summarizing (2.6) (2.8) and (2.10), we end the proof of $\frac{\Delta}{(E(X))^{2}} \rightarrow 0$ with the condition $r \ll\left(\frac{n!}{(n-2 k)!2^{k} k!}\right)^{\frac{1}{k-1}}$.

A corollary of the Chebyshev's inequality [4] asserts that if $E(X) \rightarrow \infty$ and $\Delta=\circ\left((E(X))^{2}\right)$, then almost surely $X>0([1]$,page 46$)$. So from assertion $\left(^{*}\right) 2$ and the above discuss, we obtain

$$
\operatorname{Pr}[X>0] \rightarrow 1 \text { if } r \ll\left(\frac{n!}{(n-2 k)!2^{k} k!}\right)^{\frac{1}{k-1}} .
$$

From the definition of the threshold function (Definition 1.1), we can say that $\left(\frac{n!}{(n-2 k)!2^{k} k!}\right)^{\frac{1}{k-1}}$ is the threshold function for the property that $\mathcal{K}_{n}^{r}$ has a monochromatic $k$-matching.

## $2.3 k=\frac{n}{2}$

When $k=\frac{n}{2}$, a monochromatic $k$-matching is a monochromatic perfect matching.

Replace $k$ with $\frac{n}{2}$ in (2.2), we have

$$
\begin{equation*}
E(X)=\frac{n!}{\left(\frac{n}{2}\right)!2^{\frac{n}{2}} r^{\frac{n}{2}-1}} . \tag{2.11}
\end{equation*}
$$

By calculation of (2.11), we get $E(X) \rightarrow 0$ if $r \geq \frac{n}{c^{(4)}}$, where $c^{(4)}<e$ is a constant; $E(X) \rightarrow \infty$ if $r \leq \frac{n}{e}$.

The following assertion is true as a direct corollary of Markov's inequality and the threshold function for the property that $\mathcal{G}_{n, p}$ having a perfect matching ( [6], page 85). Here we omit its proof.

Theorem 2.3 If $r \geq \frac{n}{c^{(4)}}$, where $c^{(4)}<e$ is a constant, then almost no $\mathcal{K}_{n}^{r}$ has a monochromatic perfect matching. On the other hand, if $r \leq$ $\frac{n}{\operatorname{logn}+c^{(5)}(n)}$, where $c^{(5)}(n) \rightarrow \infty$, then almost every $\mathcal{K}_{n}^{r}$ has a monochromatic perfect matching.

## 3 Heterochromatic $k$-Matchings in $\mathcal{K}_{n}^{r}$

Following the symbols in the previous section, let $B_{i}$ be the event that the edges in $M_{i}$ are heterochromatic and $Y_{i}$ be the indicator variable for the
event $B_{i}$. That is,

$$
Y_{i}= \begin{cases}1 & \text { if } B_{i} \text { happens },  \tag{3.1}\\ 0 & \text { otherwise }\end{cases}
$$

Then for each $1 \leq i \leq q$,

$$
\operatorname{Pr}\left[Y_{i}=1\right]=\frac{\binom{r}{k} k!}{r^{k}}
$$

Then the random variable

$$
Y=Y_{1}+Y_{2}+\cdots+Y_{q}
$$

denotes the number of heterochromatic $k$-matchings in $\mathcal{K}_{n}^{r}$.
From the linear of the expectation [1],

$$
\begin{align*}
E(Y) & =E\left(Y_{1}+Y_{2}+\cdots+Y_{q}\right) \\
& =\frac{r}{r^{k}} q \\
& =\frac{n!}{(n-2 k)!2^{k} k!} \frac{r!}{(r-k)!r^{k}} . \tag{3.2}
\end{align*}
$$

Since $r \geq k$ is a necessary condition in the heterochromatic $k$-matching problem, we have the following assertion for $E(Y)$ by calculation of (3.2).

Lemma 3.1 For every $1 \leq k \leq n^{1-\epsilon}$ and $r \geq k, E(Y) \rightarrow \infty$, where $0<$ $\epsilon<1$ is a constant that can be arbitrarily small.

The main result of this section is the following theorem:

Theorem 3.2 If $1 \leq k \leq n^{1-\epsilon}$ and $r \geq k$, where $0<\epsilon<1$ is a constant that can be arbitrarily small, then almost every $\mathcal{K}_{n}^{r}$ contains a heterochromatic $k$-matching.

Proof. Similar to Theorem [2.2, for heterochromatic $k$-matchings, the following estimate is for $\Delta^{\prime}=\sum_{i \sim j} \operatorname{Pr}\left[B_{i} \cap B_{j}\right]$.

$$
\begin{aligned}
\Delta^{\prime} & =\sum_{i \sim j} \operatorname{Pr}\left[B_{i} \cap B_{j}\right] \\
& =\sum_{s=1}^{k-1} \sum_{(i, j)_{s}} \frac{\left.\binom{r}{k} k!\begin{array}{c}
r-s \\
k-s
\end{array}\right)(k-s)!}{r^{2 k-s}}
\end{aligned}
$$

(where $(i, j)_{s}$ means the ordered pair of $M_{i}$ and $M_{j}$ that have $s$ common edges)
$\leq \sum_{s=1}^{k-1} \frac{\binom{n}{2}\binom{n-2}{2} \cdots\binom{n-2(s-1)}{2}}{s!}\left(\frac{\binom{n-2 s}{2}\binom{n-2(s+1)}{2} \cdots\binom{n-2(k-1)}{2}}{(k-s)!}\right)^{2} \frac{\binom{r}{k} k!\binom{r-s}{k-s}(k-s)!}{r^{2 k-s}}$
$=\frac{n!r!}{2^{2 k}(n-2 k)!(n-2 k)!(r-k)!(r-k)!r^{2 k}} \sum_{s=1}^{k-1} \frac{(n-2 s)!(r-s)!2^{s} r^{s}}{s!(k-s)!(k-s)!}$.

Then

$$
\begin{equation*}
\frac{\Delta^{\prime}}{(E(Y))^{2}} \leq \frac{k!k!}{n!r!} \sum_{s=1}^{k-1} \frac{(n-2 s)!(r-s)!(2 r)^{s}}{s!(k-s)!(k-s)!} \tag{3.3}
\end{equation*}
$$

By careful calculation of (3.3), we get if $k \ll n$, then

$$
\begin{align*}
\frac{\Delta^{\prime}}{(E(Y))^{2}} & \leq \frac{k!k!}{n!r!}(1+\circ(1)) \frac{2 r!(n-2)!}{(k-1)!(k-1)!} \\
& =(1+\circ(1)) \frac{k^{2}}{n(n-1)} \rightarrow 0 \tag{3.4}
\end{align*}
$$

From (3.4), Lemma 3.1 and the assertion that if $E(Y) \rightarrow \infty$ and $\Delta^{\prime}=$ $\circ\left((E(Y))^{2}\right)$, then almost surely $Y>0([1]$, page 46), we have

$$
\operatorname{Pr}[Y>0] \rightarrow 1,
$$

which finishes the proof.

Remark 3.3 As a corollary of Theorem[3.2, if one of $k$ and $r(\geq k)$ is fixed, then almost every $\mathcal{K}_{n}^{r}$ has a heterochromatic $k$-matching. The only left case that we can not deal with is that $k=c^{(6)} n$, where $0<c^{(6)} \leq \frac{1}{2}$ is a constant.

## 4 Results on Other Subgraphs

Completely similar to Section 2 and Section 3, we can study monochromatic $k$-clique, $k$-tree and heterochromatic $k$-clique, $k$-tree in $\mathcal{K}_{n}^{r}$. We list our results here.

Theorem 4.1 If $r$ is fixed, then
$\lim _{n \rightarrow \infty} \operatorname{Pr}\left[\mathcal{K}_{n}^{r}\right.$ contains a monochromatic $k$-clique $]= \begin{cases}0 & \text { if } k \geq 2 \log _{r} n, \\ 1 & \text { if } k \leq \frac{\log _{r} n}{1.704 \times 10^{9}} .\end{cases}$
Theorem 4.2 If $k$ is fixed, then
$\lim _{n \rightarrow \infty} \operatorname{Pr}\left[\mathcal{K}_{n}^{r}\right.$ contains a monochromatic $k$-clique $]= \begin{cases}0 & \text { if } r \gg n^{\frac{k}{\left(\frac{k}{k}\right)-1}}, \\ 1 & \text { if } r \leq\left(\frac{1}{2 k!)^{\left.\frac{1}{k}\right)^{\frac{k}{2}-1}} n^{\frac{k}{\left(\frac{k}{k}\right)-1}} .} .\right.\end{cases}$
That is to say, $n^{\frac{k}{\binom{k}{2}-1}}$ is the threshold function for the property that $\mathcal{K}_{n}^{r}$ has a monochromatic $k$-clique.

Theorem 4.3 If $r \geq n^{4+\epsilon}$, where $\epsilon>0$ is a constant that can be arbitrarily small, then for every $k \leq n$, there almost surely exists a heterochromatic $k$-clique in $\mathcal{K}_{n}^{r}$.

Theorem 4.4 If $k$ is fixed, then

$$
\lim _{n \rightarrow \infty} \operatorname{Pr}\left[\mathcal{K}_{n}^{r} \text { contains a monochromatic } k \text {-tree }\right]= \begin{cases}0 & \text { if } r \gg k\binom{n}{k}^{\frac{1}{k-2}} \\ 1 & \text { if } r \leq \frac{k}{n}\binom{n}{k}^{\frac{1}{k-2}} .\end{cases}
$$

Theorem 4.5 If $r \geq c^{(7)} n$, where $c^{(7)}>1$ is a constant, then almost no $\mathcal{K}_{n}^{r}$ contains a monochromatic spanning tree.

Theorem 4.6 If $r$ is fixed, then almost every $\mathcal{K}_{n}^{r}$ contains a monochromatic $k$-tree for any $2 \leq k \leq n$.

Theorem 4.7 If $2 \leq k \leq \operatorname{logn}$ and $r \geq k-1$, then almost every $\mathcal{K}_{n}^{r}$ contains a heterochromatic $k$-tree.

## References

[1] N. Alon and J. Spencer, The Probabilistic Method, 2nd ed., John Wiley \& Sons, Inc. 2000.
[2] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, The Macmillan Press LTD., 1976.
[3] B. Bollobás, Random Graphs, 2nd ed., Cambridge University Press, 2001.
[4] P. Erdös and J. Spencer, Probabilistic Methods in Combinatorics, Academic Press, 1974.
[5] J. Spencer, The Strange Logic of Random Graphs, Springer, 2001.
[6] S. Janson, T. Łuczak and A. Rucinski, Random Graphs, John Wiley \& Sons, Inc., 2000.
[7] P. Erdös and A. Rényi, On random graphs I, Publ. Math. Debrecen 6, 290-297.
[8] P. Erdös and A. Rényi, On the evolution of random graphs, Publ. Math. Inst. Hungar. Acad. Sci. 5, 17-61.
[9] P. Erdös and A. Rényi, On the evolution of random graphs, Bull. Inst. Int. Statist. Tokyo 38, 343-347.


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