A comment on Ryser's conjecture for intersecting hypergraphs

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Abstract

Let $\tau(\mathcal{H})$ be the cover number and $\nu(\mathcal{H})$ be the matching number of a hypergraph $\mathcal{H}$. Ryser conjectured that every r-partite hypergraph $\mathcal{H}$ satisfies the inequality $\tau(\mathcal{H}) \le (r-1)\nu(\mathcal{H})$. This conjecture is open for all $r \ge 4$. For intersecting hypergraphs, namely those with $\nu(\mathcal{H}) = 1$, Ryser's conjecture reduces to $\tau(\mathcal{H}) \le r-1$. Even this conjecture is extremely difficult and is open for all $r \ge 6$. For infinitely many r there are examples of intersecting r-partite hypergraphs with $\tau(\mathcal{H}) = r-1$, demonstrating the tightness of the conjecture for such r. However, all previously known constructions are not optimal as they use far too many edges. How sparse can an intersecting r-partite hypergraph be, given that its cover number is as large as possible, namely $\tau(\mathcal{H}) \ge r-1$? In this paper we solve this question for $r \le 5$, give an almost optimal construction for $r = 6$, prove that any r-partite intersecting hypergraph with $\tau(H) \ge r-1$ must have at least $(3-\frac{1}{\sqrt{18}})r(1-o(1)) \approx 2.764r(1-o(1))$ edges, and conjecture that there exist constructions with $\Theta(r)$ edges.

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1 Introduction

For a hypergraph $\mathcal{H} = (V, E)$, the (vertex) *cover number*, denoted by $\tau(\mathcal{H})$, is the minimum size of a vertex set that intersects every edge. The *matching number*, denoted by $\nu(\mathcal{H})$, is the maximum size of a subset of edges whose elements are pairwise-disjoint.

Clearly, $\tau(\mathcal{H}) \le \nu(\mathcal{H})$ for any hypergraph. In the graph-theoretic case, König's Theorem [2] asserts that the converse non-trivial inequality also holds for bipartite graphs. Thus, if $\mathcal{H}$ is a bipartite graph then $\tau(\mathcal{H}) = \nu(\mathcal{H})$. Ryser conjectured the following hypergraph generalization of König's Theorem for hypergraphs. A hypergraph is called *r-partite* if its vertex set can be partitioned into r parts, and every edge contains precisely one vertex from each part. In particular, r-partite hypergraphs are r-uniform. Ryser conjectured that every r-partite hypergraph $\mathcal{H}$ satisfies $\nu(\mathcal{H}) \le (r-1)\tau(\mathcal{H})$. This conjecture turns out to be notoriously difficult. Indeed only the case $r = 3$ has been proved by Aharoni [1] using topological methods.

A hypergraph is called *intersecting* if any two edges have nonempty intersection. Clearly, $\mathcal{H}$ is intersecting if and only if $\nu(\mathcal{H}) = 1$. For intersecting hypergraphs, Ryser's conjecture amounts to:

Conjecture 1 *If $\mathcal{H}$ is an r-partite intersecting hypergraph then $\tau(\mathcal{H}) \leq r - 1$.*

Conjecture 1 is still wide open. It has been proved for $r = 4, 5$ by Tuza [3, 4]. We note that the case $r = 3$ of Conjecture 1 was first proved by Tuza in [4], before Aharoni's general proof for the case $r = 3$.

Conjecture 1 (if true) is tight in the sense that for infinitely many r there are constructions of intersecting r-partite hypergraphs with $\tau(\mathcal{H}) = r - 1$. Indeed, whenever $r = q + 1$ and q is a prime power, consider the finite projective plane of order q as a hypergraph. This hypergraph is r-uniform and intersecting. To make it r-partite one just needs to delete one point from the projective plane. This truncated projective plane gives an intersecting r-partite hypergraph with cover number $r - 1$, with $q^2 + q = r(r - 1)$ vertices, and with $q^2 = (r - 1)^2$ edges.

However, the projective plane construction is not the "correct" extremal construction, not only because it does not apply to all r, but also because it is not the smallest possible. Although the projective plane construction only contains $r(r-1)$ vertices (and this is clearly optimal since otherwise some vertex class would have size less than $r - 1$ resulting in a cover number less than $r - 1$), should an extremal example contain so many (namely, $(r - 1)^2$) edges? In order to understand the extremal behavior of intersecting r-partite hypergraphs, it is desirable to construct the *sparsest* possible intersecting r-partite hypergraph with cover number as *large* as possible, namely at least $r - 1$.

More formally, let $f(r)$ be the minimum integer so that there exists an r-partite intersecting hypergraph $\mathcal{H}$ with $\tau(\mathcal{H}) \geq r - 1$ and with $f(r)$ edges. (we write $\tau(\mathcal{H}) \geq r - 1$ instead of $\tau(\mathcal{H}) = r - 1$ to allow for the possibility that Conjecture 1 is false; also note that trivially $\tau(\mathcal{H}) \leq r$ since the set of vertices of any edge forms a cover). A trivial lower bound for $f(r)$ is $2r - 3$. Indeed, the edges of an intersecting hypergraph with at most $2r - 4$ edges can greedily be covered with $r - 2$ vertices. We prove, however, the following non-trivial lower bound.

Theorem 2 *$f(r) \geq (3 - \frac{1}{\sqrt{18}})r(1 - o(1)) \approx 2.764r(1 - o(1))$.*

Although we do not have a matching upper bound, we conjecture that a linear (in r) number of edges indeed suffice.

Conjecture 3 *$f(r) = \Theta(r)$.*

Computing precise values of $f(r)$ seems to be a difficult problem. Trivially, $f(2) = 1$. It is also easy to see that $f(3) = 3$. Indeed, a 3-partite intersecting hypergraph with only two edges has cover number 1. The hypergraph whose edges are $(a_1, b_1, c_1), (a_1, b_2, c_2), (a_2, b_1, c_2)$ is a 3-partite intersecting hypergraph with cover number 2. The next theorem establishes the first non-trivial values of $f(r)$, namely $r = 4, 5$, in addition to upper and lower bounds in the case $r = 6$. More specifically, we prove:

Theorem 4 *$f(4) = 6$, $f(5) = 9$, and $12 \le f(6) \le 15$.*

Comparing our constructions with the projective plane construction, we see that in the case $r = 4, 5, 6$ the latter has $9, 16$ and 25 edges respectively. Thus, the projective plane construction is far from being optimal. Our constructions also have the property that the number of vertices they contain is $r(r-1)$, which, as mentioned earlier, is optimal.

In the rest of this paper we prove Theorems 2 and 4.

2 Proof of Theorem 2

Throughout this section we assume that $\mathcal{H}$ is an r-partite intersecting hypergraph with $\tau(\mathcal{H}) \ge r-1$. Recall that the *degree* of a vertex v in a hypergraph is the number of edges containing v.

Consider the following greedy procedure, starting with the original hypergraph $\mathcal{H}$. As long as there is a vertex x of degree at least 4 in the current hypergraph, we delete x and all of the edges containing x from the current hypergraph, thereby obtaining a smaller hypergraph. Vertices that become isolated are also deleted. Denote by $\mathcal{H}_3$ the hypergraph obtained at the end of the greedy procedure and denote by X_4 the set of vertices deleted by the greedy procedure. Notice that $\mathcal{H}_3$ is either the empty hypergraph or else it is an r-partite intersecting hypergraph, every vertex of which has degree at most 3. We then continue in the same manner, where as long as there is a vertex x of degree 3 in the current hypergraph, we delete x and all of the edges containing x from the current hypergraph. Again, vertices that become isolated are also deleted. Denote by $\mathcal{H}_2$ the hypergraph obtained at the end of this second greedy procedure and denote by X_3 the set of vertices deleted in the second greedy procedure. Notice that $\mathcal{H}_2$ is either the empty hypergraph or else it is an r-partite intersecting hypergraph, every vertex of which has degree at most 2.

We first claim that $\mathcal{H}_3$ contains at most $2r+1$ edges. Indeed, if H is any edge, then every vertex of H appears in at most two other edges. Thus, there are at most $2r$ other edges in addition to H. Similarly, $\mathcal{H}_2$ contains at most $r+1$ edges. Let, therefore, the number of $\mathcal{H}_3$ be denoted by γr and hence $0 \le \gamma \le 2 + 1/r$.

Consider first the case $\gamma \le 1$. In this case we can cover the edges of $\mathcal{H}_3$ greedily with a set U of at most $\lceil \gamma r/2 \rceil$ vertices. Now, since $U \cup X_4$ is a cover of $\mathcal{H}$ and since $\tau(\mathcal{H}) \ge r-1$, we have that $|X_4| \ge r - 1 - \lceil \gamma r/2 \rceil$. As every vertex of X_4 was greedily selected to appear in four *distinct* edges of $\mathcal{H} - \mathcal{H}_3$ we have that the number of edges of $\mathcal{H}$ is at least

$$4|X_4| + \gamma r \ge 4(r - 1 - \lceil \gamma r/2 \rceil) + \gamma r = (4-\gamma)r - 6 \ge 3r - 6$$

which is even better than the bound in the statement of the theorem.

We may now assume that $1 < \gamma \le 2 + 1/r$. Since $\mathcal{H}_2$ has at most $r+1$ edges, we have that $|X_3| \ge (\gamma r - r - 1)/3$. The number of edges of $\mathcal{H}_2$ is $\gamma r - 3|X_3|$. It follows that there is a cover of $\mathcal{H}_3$ whose size is at most

$$|X_3| + \left\lceil \frac{\gamma r - 3|X_3|}{2} \right\rceil \le \left(\frac{1}{6} + \frac{\gamma}{3} \right) r(1 + o(1)).$$

As such a cover, together with X_4, is a cover of $\mathcal{H}$, and since $\tau(\mathcal{H}) \geq r-1$, we have that

$$|X_4| \geq \left(\frac{5}{6} - \frac{\gamma}{3}\right) r(1-o(1)).$$

We therefore have that the number of edges of $\mathcal{H}$, which is at least $4|X_4| + \gamma r$, is at least

$$\frac{10-\gamma}{3} r(1-o(1)). \tag{1}$$

There is, however, another way to bound from below the number of edges of $\mathcal{H}$. For $i = 1, 2, 3$ let $\alpha_i r^2$ denote the number of vertices of $\mathcal{H}_3$ having degree i. Since the sum of the degrees in $\mathcal{H}_3$ is γr^2 we have:

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = \gamma.$$

Consider a specific edge H of $\mathcal{H}_3$ and let $r\beta_i^H$ be the number of vertices in H with degree i for $i = 1, 2, 3$. Clearly, $\beta_1^H + \beta_2^H + \beta_3^H = 1$. As H intersects every edge we must have

$$r\beta_2^H + 2r\beta_3^H \geq \gamma r - 1.$$

It follows that:

$$2\beta_1^H + \beta_2^H = 2 - \beta_2^H - 2\beta_3^H \leq 2 - \gamma + 1/r.$$

In particular,

$$\sum_{H \in \mathcal{H}_3} (2\beta_1^H + \beta_2^H) \leq \gamma r(2 - \gamma + 1/r).$$

On the other hand, by definition we have that

$$\sum_{H \in \mathcal{H}_3} r\beta_1^H = \alpha_1 r^2 \ , \qquad \sum_{H \in \mathcal{H}_3} r\beta_2^H = 2\alpha_2 r^2.$$

It follows that

$$2\alpha_1 + 2\alpha_2 \leq \gamma(2 - \gamma + 1/r).$$

Hence,

$$\alpha_1 + \alpha_2 + \alpha_3 = \frac{\gamma}{3} + \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2 \leq \frac{\gamma}{3} + \frac{2}{3}\alpha_1 + \frac{2}{3}\alpha_2 \leq$$
$$\frac{\gamma}{3} + \frac{1}{3}\gamma(2 - \gamma + 1/r) = \gamma - \frac{1}{3}\gamma^2 + \frac{\gamma}{3r}.$$

Since $r^2(\alpha_1 + \alpha_2 + \alpha_3)$ is the number of vertices of $\mathcal{H}_3$ we have that $\mathcal{H}_3$ has at most

$$(\gamma - \frac{1}{3}\gamma^2) r^2 (1 + o(1))$$

vertices. In particular, there is a vertex class consisting of at most

$$(\gamma - \frac{1}{3}\gamma^2) r (1 + o(1))$$

vertices. Since any vertex class of $\mathcal{H}_3$, together with X_4, form a cover of $\mathcal{H}$, we have that

$$|X_4| \geq (1 - \gamma + \frac{1}{3}\gamma^2) r (1 - o(1)).$$

We therefore have that the number of edges of $\mathcal{H}$, which is at least $4|X_4| + \gamma r$, is at least

$$(4 - 3\gamma + \frac{4}{3}\gamma^2)r(1 - o(1)). \tag{2}$$

Comparing (1) and (2) we see that the minimum of the maximum of both of them is attained when $\gamma = 1+1/\sqrt{2}$, and in this case the number of edges of $\mathcal{H}$ is at least $(3-\frac{1}{\sqrt{18}})r(1-o(1))$, as required. $\square$

3 Proof of Theorem 4

3.1 The case $r = 4$

We need to show first that $f(4) > 5$. Assume the contrary and let $\mathcal{H}$ be a 4-partite intersecting hypergraph with only 5 edges and with $\tau(\mathcal{H}) \geq 3$. No vertex can appear in three or more edges, since such a vertex v, and a vertex u intersecting the (at most two) edges in which v does not appear form a cover of size 2, a contradiction. Thus, every vertex has degree at most 2. Now, since there are $\binom{5}{2}$ nonempty intersections of pairs of edges of $\mathcal{H}$, we have, by the inclusion-exclusion principle that $\mathcal{H}$ contains at most $5 \cdot 4 - \binom{5}{2} = 10$ vertices. But this means that some vertex class contains at most two vertices, again resulting in $\tau(\mathcal{H}) \leq 2$, a contradiction.

We construct a 4-partite intersecting hypergraph with 6 edges and with $\tau(\mathcal{H}) = 3$. Consider the four vertex classes $V_1 = \{a_1, a_2, a_3\}$, $V_2 = \{b_1, b_2, b_3\}$, $V_3 = \{c_1, c_2, c_3\}$, and $V_4 = \{d_1, d_2, d_3\}$. The 6 edges are (a_1, b_1, c_1, d_1), (a_1, b_2, c_2, d_2), (a_2, b_1, c_2, d_3), (a_2, b_2, c_3, d_1), (a_3, b_3, c_2, d_1), and (a_3, b_1, c_3, d_2). It is easy to check that any two edges intersect and that two vertices cannot cover all 6 edges.

3.2 The case $r = 5$

We need to show first that $f(5) > 8$. Assume the contrary and let $\mathcal{H}$ be a 5-partite intersecting hypergraph with only 8 edges and with $\tau(\mathcal{H}) \geq 4$.

Notice first that there is no vertex with degree 4 or greater, since if v is such a vertex, then the (at most) four remaining edges not containing v can always be greedily covered with two additional vertices, resulting in cover number at most 3, a contradiction. Thus, we may assume that the degree of each vertex is at most 3. Clearly we can assume that $\mathcal{H}$ has at least 20 vertices, as otherwise there is a vertex class with at most three vertices, again contradicting the assumption that $\tau(\mathcal{H}) \geq 4$.

Let x_i denote the number of vertices with degree i, for $i = 1, 2, 3$. Since the sum of the degrees of all vertices is $8 \cdot 5 = 40$ we have that $x_1 + 2x_2 + 3x_3 = 40$. We claim also that each vertex class has at most one vertex with degree 3. Indeed, if there were two such vertices in the same vertex class, then they both cover 6 edges, and the remaining two edges can be covered by an additional vertex, contradicting the assumption that $\tau(\mathcal{H}) \geq 4$. Thus, we have that $x_3 \leq 5$.

As there are $28 = \binom{8}{2}$ intersections, we have that $x_2 + 3x_3 \geq 28$. Finally, since there are at least 20 vertices, we have $x_1 + x_2 + x_3 \geq 20$. Now, it follows that $x_1 + 2x_2 + 4x_3 \geq 48$ which implies that $x_3 \geq 8$ which contradicts $x_3 \leq 5$.

Next we construct a 5-partite intersecting hypergraph with 9 edges and with $\tau(\mathcal{H}) = 4$. Consider the five vertex classes $V_1 = \{1,2,3,4\}$, $V_2 = \{5,6,7,8\}$, $V_3 = \{9,10,11,12\}$, $V_4 = \{13,14,15,16\}$, and $V_5 = \{17,18,19,20\}$. The 9 edges are divided into two parts:

$$A = \{(1,5,9,13,17), (2,6,10,14,17), (3,7,10,13,18), (1,6,11,15,18), (2,7,9,15,19)\},$$

$$B = \{(4,5,10,15,20), (4,7,11,16,17), (4,8,9,14,18), (4,6,12,13,19)\}.$$

First, notice that the constructed hypergraph is, indeed, 5-partite, and intersecting. Also note that $\tau(\mathcal{H}) \leq 4$ by considering, for example, the cover $\{4,17,18,19\}$. It remains to show that that there is no cover of size 3. There is only one vertex with degree 4, and it is vertex 4. Vertex 4 precisely covers the set B. Notice, however, that no vertex covers three edges of A. This means that any cover containing 4 must have size at least 4. We now only need to rule out the possibility of a cover of size 3, each vertex of which has degree 3. The vertices of degree 3 are $\{1,2,6,7,9,10,13,15,17,18\}$. However, each of them appears at most one time in B, hence any three of them cannot cover all the vertices of B.

3.3 The case $r = 6$

We first show that $f(6) > 11$. Like before, assume the contrary and let $\mathcal{H}$ be a 6-partite intersecting hypergraph with only 11 edges and with $\tau(\mathcal{H}) \geq 5$.

Notice first that there is no vertex with degree 5 or greater, since if v is such a vertex, then the (at most) 6 remaining edges not containing v can always be greedily covered with 3 additional vertices, resulting in cover number at most 4, a contradiction. Thus, we may assume that the degree of each vertex is at most 4. Clearly we can assume that $\mathcal{H}$ has at least 30 vertices, as otherwise there is a vertex class with at most 4 vertices, again contradicting the assumption that $\tau(\mathcal{H}) \geq 5$.

Let x_i denote the number of vertices with degree i, for $i = 1,2,3,4$. Since the sum of the degrees of all vertices is $11 \cdot 6 = 66$ we have that $x_1 + 2x_2 + 3x_3 + 4x_4 = 66$. Again, notice that the assumption that $\tau(\mathcal{H}) \geq 5$ implies that each vertex class has at most one vertex with degree 4, at most two vertices of degree 3, and if vertex class contains a vertex of degree 4, it does not contain a vertex of degree 3. Thus, $x_4 \leq 6$ and $x_3 \leq 12 - 2x_4$. Hence $x_3 + 3x_4 \leq 18$.

As there are $55 = \binom{11}{2}$ intersections, we have that $x_2 + 3x_3 + 6x_4 \geq 55$. Combining this with the fact $x_1 + x_2 + x_3 + x_4 \geq 30$ we have that $x_1 + 2x_2 + 4x_3 + 7x_4 \geq 85$. This implies that $x_3 + 3x_4 \geq 19$, contradicting the fact that $x_3 + 3x_4 \leq 18$.

Next we create a 6-partite intersecting hypergraph $\mathcal{H} = (V,E)$ with 30 vertices and 15 edges as follows. The six vertex classes of V are:

$$V_1 = \{a_1, a_3, a_4, a_6, a_8\}, \quad V_2 = \{b_1, b_2, b_4, b_8, b_{12}\}, \quad V_3 = \{c_1, c_2, c_4, c_7, c_{11}\},$$
$$V_4 = \{d_1, d_2, d_3, d_9, d_{11}\}, \quad V_5 = \{e_1, e_2, e_3, e_5, e_7\}, \quad V_6 = \{f_1, f_2, f_3, f_5, f_7\}.$$

(The fact that the selection of indices in each set is not consecutive simplifies the description that follows.) We construct E in several steps so as to guarantee that

1. $|H \cap V_i| = 1, \forall H \in E, 1 \leq i \leq 6$, so that $\mathcal{H}$ is a 6-partite.

2. $H \cap H' \neq \emptyset, \forall H, H' \in E$.

3. $\tau(\mathcal{H}) \geq 5$.

Step 1: A "cyclic" construction. Throughout the whole procedure we try to let the edges, albeit intersecting, repeat as little as possible.

$$\begin{aligned} &H_1 = \{a_1, b_1, c_1, d_1, e_1, f_1\}, \quad H_2 = \{a_1, b_2, c_2, d_2, e_2, f_2\}, \quad H_3 = \{a_3, b_1, c_2, d_3, e_3, f_3\}, \\ &H_4 = \{a_4, b_4, c_4, d_1, e_2, f_3\}, \quad H_5 = \{a_3, b_4, c_1, d_2, e_5, f_5\}, \quad H_6 = \{a_6, b_2, c_4, d_3, e_5, f_1\}. \end{aligned}$$

Note that by construction,

$$\begin{aligned} &|H_i \cap H_j| = 1, && 1 \leq i < j \leq 6, \\ &H_i \cap H_j \cap H_k = \emptyset, && 1 \leq i < j < k \leq 6. \end{aligned}$$

Therefore, a minimum cover of $H_1, \ldots, H_6$ has size 3. Now we consider the pairwise intersections and take the union of every three mutually disjoint pairs (the union of the three pairs forms exactly $\bigcup_{i=1}^{6} H_i$), for instance $\{H_1 \cap H_2, H_3 \cap H_4, H_5 \cap H_6\} = \{a_1, f_3, e_5\}$. The following list $\mathcal{L}_1$ thus contains all the minimum covers of $H_1, \ldots, H_6$:

$$\begin{aligned} \mathcal{L}_1 = \big\{ &\{a_1, f_3, e_5\}, \{a_1, a_3, c_4\}, \{a_1, d_3, b_4\}, \{b_1, e_2, e_5\}, \{b_1, d_2, c_4\}, \\ &\{b_1, b_2, b_4\}, \{d_1, c_2, e_5\}, \{d_1, d_2, d_3\}, \{d_1, b_2, a_3\}, \{c_1, c_2, c_4\}, \\ &\{c_1, e_2, d_3\}, \{c_1, b_2, f_3\}, \{f_1, c_2, b_4\}, \{f_1, e_2, a_3\}, \{f_1, d_2, f_3\} \big\}. \end{aligned}$$

Step 2: Any additional edge must contain a cover of $H_1, \ldots, H_6$. Selecting carefully an element from $\mathcal{L}_1$ each time, we construct the edges H_7 through H_{10}.

$$\begin{aligned} &H_7 = \{a_1, b_4, c_7, d_3, e_7, f_7\}, \quad H_8 = \{a_8, b_8, c_2, d_1, e_5, f_7\}, \\ &H_9 = \{a_8, b_2, c_1, d_9, e_7, f_3\}, \quad H_{10} = \{a_3, b_8, c_7, d_9, e_2, f_1\}. \end{aligned}$$

Up to now, $|H_i \cap H_j| = 1, for 1 \leq i < j \leq 10$. Moreover, since the (unique) intersecting element of any pair of edges constructed in Step 2 has a subscript index in $\{7, 8, 9\}$, $H_i \cap H_j \cap H_k = \emptyset, 1 \leq i < j < k \leq 10,$ if either $|\{i, j, k\} \cap [6]| = 3$ or $|\{i, j, k\} \cap \{7, 8, 9, 10\}| \geq 2$. Notice also that any minimum cover that covers $H_1 - H_{10}$ consists of at least four vertices.

Step 3: Five additional edges that force an increase in the size of the minimum cover.

$$\begin{aligned} &H_{11} = \{a_1, b_8, c_{11}, d_{11}, e_5, f_3\}, \quad H_{12} = \{a_8, b_{12}, c_1, d_3, e_2, f_3\}, \quad H_{13} = \{a_8, b_4, c_2, d_{11}, e_1, f_1\}, \\ &H_{14} = \{a_4, b_8, c_1, d_3, e_2, f_1\}, \quad H_{15} = \{a_8, b_8, c_1, d_3, e_2, f_5\}. \end{aligned}$$

Notice that the 15 constructed edges indeed form an intersecting 6-partite hypergraph. It remains to show that:

Proposition 5 $\tau(\mathcal{H}) = 5$.

Proof. Suppose the proposition is false, and let $C = \{x, y, z, w\}$ be a cover of size 4. We use a sequence of arguments to deduce that this is impossible.

For convenience, for $7 \le i \le 15$, let $H_i{}^*$ be the collection of those vertices of H_i with index subscript no bigger than 6. That is,

$$\begin{array}{lll} H_7^* = \{a_1, b_4, d_3\}, & H_8^* = \{c_2, d_1, e_5\}, & H_9^* = \{b_2, c_1, f_3\}, \\ H_{10}^* = \{a_3, e_2, f_1\}, & H_{11}^* = \{a_1, e_5, f_3\}, & H_{12}^* = \{c_1, d_3, e_2, f_3\}, \\ H_{13}^* = \{b_4, c_2, e_1, f_1\}, & H_{14}^* = \{a_4, c_1, d_3, e_2, f_1\}, & H_{15}^* = \{c_1, d_3, e_2, f_5\}. \end{array}$$

First, notice that if $C \cap \{b_{12}, c_{11}, d_{11}\} \neq \emptyset$, then $|C \backslash \{b_{12}, c_{11}, d_{11}\}| \le 3$. So $C \backslash \{b_{12}, c_{11}, d_{11}\}$ must be a triple, and furthermore it must be one of the triples in $\mathcal{L}_1$. But no element of $\mathcal{L}_1$ may cover H_7, H_8, H_9 and H_{10} since $H_7^* - H_{10}^*$ are pairwise disjoint. Hence, $C \cap \{b_{12}, c_{11}, d_{11}\} = \emptyset$.

Second, assume $C \cap \{a_8, b_8\} \neq \emptyset$. Without loss of generality, let $x = C \cap \{a_8, b_8\}$. Then $\{y, z, w\} \in \mathcal{L}_1$.

- Case 1: $x = a_8$. The fact that $\{y, z, w\}$ must cover H_7^* implies that $\{y, z, w\} \cap \{a_1, b_4, d_3\} \neq \emptyset$; $\{y, z, w\}$ must cover H_{10}^* implies that $\{y, z, w\} \cap \{a_3, e_2, f_1\} \neq \emptyset$; $\{y, z, w\}$ must cover H_{11}^* implies that $\{y, z, w\} \cap \{a_1, e_5, f_3\} \neq \emptyset$. The only triple in $\mathcal{L}_1$ satisfying these three requirements is $\{a_1, a_3, c_4\}$. But then $H_{14}^* = \{a_4, c_1, d_3, e_2, f_1\}$ is left uncovered. Impossible.

- Case 2: $x = b_8$. The fact that $\{y, z, w\}$ must cover H_7^* implies that $\{y, z, w\} \cap \{a_1, b_4, d_3\} \neq \emptyset$; $\{y, z, w\}$ must cover H_9^* implies that $\{y, z, w\} \cap \{b_2, c_1, f_3\} \neq \emptyset$; $\{y, z, w\}$ must cover H_{13}^* implies that $\{y, z, w\} \cap \{b_4, c_2, e_1, f_1\} \neq \emptyset$. The only triple in $\mathcal{L}_1$ satisfying these three requirements is $\{b_1, b_2, b_4\}$. But then $H_{12}^* = \{c_1, d_3, e_2, f_3\}$ is left uncovered. Impossible.

Hence, $C \cap \{a_8, b_8\} = \emptyset$.

Third, assume $C \cap \{c_7, e_7, f_7, d_9\} \neq \emptyset$. Let $C \cap \{c_7, e_7, f_7, d_9\} = x$. Then $\{y, z, w\}$ must be a triple in $\mathcal{L}_1$, and it must also cover $H_{11}^* - H_{15}^*$. The fact that $\{y, z, w\}$ must cover H_{11}^* implies that $\{y, z, w\} \cap \{a_1, e_5, f_3\} \neq \emptyset$; $\{y, z, w\}$ must cover H_{13}^* implies that $\{y, z, w\} \cap \{b_4, c_2, e_1, f_1\} \neq \emptyset$; $\{y, z, w\}$ must cover H_{15}^* implies that $\{y, z, w\} \cap \{c_1, d_3, e_2, f_5\} \neq \emptyset$. The only triple in $\mathcal{L}_1$ satisfying these three requirements is $\{a_1, b_4, d_3\}$. But then H_8, H_9, H_{10} are not yet covered. This can not be fixed by any additional one vertex as $H_8 \cap H_9 \cap H_{10} = \emptyset$. Hence we must have $C \cap \{c_7, e_7, f_7, d_9\} = \emptyset$.

Last, we have by now established that the index of any vertex in C is in $\{1, 2, 3, 4, 5, 6\}$. Furthermore, $|C \cap H_i^*| = 1$, for $7 \le i \le 10$. In particular, let $x = C \cap H_7^*$, we discuss each of the three possibilities.

(i) $x = a_1$. Then $\{y, z, w\}$ must cover $H_3 - H_6, H_8^* - H_{10}^*, H_{12}^* - H_{15}^*$. Let $y \in H_8^*$.

- Case 1: $y = c_2$. The fact $\{z, w\}$ needs to cover $H_4 - H_6 \Rightarrow \{z, w\} \cap \{b_4, e_5, c_4\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_9^* \Rightarrow \{z, w\} \cap \{b_2, c_1, f_3\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_{10}^* \Rightarrow \{z, w\} \cap \{a_3, e_2, f_1\} \neq \emptyset$. But this is clearly impossible.

- Case 2: $y = d_1$. The fact $\{z, w\}$ needs to cover H_3, H_5 and $H_6 \Rightarrow \{z, w\} \cap \{a_3, e_5, d_3\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_9^* \Rightarrow \{z, w\} \cap \{b_2, c_1, f_3\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_{13}^* \Rightarrow \{z, w\} \cap \{b_4, c_2, e_1, f_1\} \neq \emptyset$. Impossible.

- Case 3: $y = e_5$. The fact $\{z, w\}$ needs to cover $H_9^* \Rightarrow \{z, w\} \cap \{b_2, c_1, f_3\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_{10}^* \Rightarrow \{z, w\} \cap \{a_3, e_2, f_1\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_{13}^* \Rightarrow \{z, w\} \cap \{b_4, c_2, e_1, f_1\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_{15}^* \Rightarrow \{z, w\} \cap \{c_1, d_3, e_2, f_5\} \neq \emptyset$. Putting the four requirements all together, $\{z, w\}$ is forced to equal $\{c_1, f_1\}$. But then H_4 is left uncovered. Impossible.

(ii) $x = b_4$. Then $\{y, z, w\}$ must cover $H_1 - H_3, H_6, H_8^* - H_{12}^*, H_{14}^*, H_{15}^*$. Let $y \in H_8^*$.

- Case 1: $y = c_2$. The fact $\{z, w\}$ needs to cover $H_9^* \Rightarrow \{z, w\} \cap \{b_2, c_1, f_3\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_{10}^* \Rightarrow \{z, w\} \cap \{a_3, e_2, f_1\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_{11}^* \Rightarrow \{z, w\} \cap \{a_1, e_5, f_3\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_{15}^* \Rightarrow \{z, w\} \cap \{c_1, d_3, e_2, f_5\} \neq \emptyset$. The four requirements force $\{z, w\}$ to equal $\{f_3, e_2\}$. But then H_1 is left uncovered. Impossible.

- Case 2: $y = d_1$. The fact $\{z, w\}$ needs to cover H_2, H_3 and $H_6 \Rightarrow \{z, w\} \cap \{c_2, b_2, d_3\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_{10}^* \Rightarrow \{z, w\} \cap \{a_3, e_2, f_1\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_{11}^* \Rightarrow \{z, w\} \cap \{a_1, e_5, f_3\} \neq \emptyset$. Impossible.

- Case 3: $y = e_5$. The fact $\{z, w\}$ needs to cover $H_1 - H_3 \Rightarrow \{z, w\} \cap \{a_1, b_1, c_2\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_9^* \Rightarrow \{z, w\} \cap \{b_2, c_1, f_3\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_{10}^* \Rightarrow \{z, w\} \cap \{a_3, e_2, f_1\} \neq \emptyset$. Impossible.

(iii) $x = d_3$. Then $\{y, z, w\}$ must cover $H_1, H_2, H_4, H_5, H_8^* - H_{11}^*, H_{13}^*$. Let $y \in H_8^*$.

- Case 1: $y = c_2$. The fact $\{z, w\}$ needs to cover H_1, H_4 and $H_5 \Rightarrow \{z, w\} \cap \{d_1, b_4, c_1\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_9^* \Rightarrow \{z, w\} \cap \{b_2, c_1, f_3\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_{10}^* \Rightarrow \{z, w\} \cap \{a_3, e_2, f_1\} \neq \emptyset$. Impossible.

- Case 2: $y = d_1$. The fact $\{z, w\}$ needs to cover $H_9^* \Rightarrow \{z, w\} \cap \{b_2, c_1, f_3\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_{10}^* \Rightarrow \{z, w\} \cap \{a_3, e_2, f_1\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_{11}^* \Rightarrow \{z, w\} \cap \{a_1, e_5, f_3\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_{13}^* \Rightarrow \{z, w\} \cap \{b_4, c_2, e_1, f_1\} \neq \emptyset$. The four requirements force $\{z, w\}$ to be $\{f_3, e_2\}$. But then H_2 is left uncovered. Impossible.

- Case 3: $y = e_5$. The fact $\{z, w\}$ needs to cover H_1, H_2 and $H_4 \Rightarrow \{z, w\} \cap \{a_1, d_1, e_2\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_9^* \Rightarrow \{z, w\} \cap \{b_2, c_1, f_3\} \neq \emptyset$; $\{z, w\}$ needs to cover $H_{13}^* \Rightarrow \{z, w\} \cap \{b_4, c_2, e_1, f_1\} \neq \emptyset$. Impossible.

In conclusion, our assumption is contradicted. Hence, $\tau(\mathcal{H}) = 5$. □

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