Some combinatorics related to central binomial coefficients: Grand-Dyck paths, coloured noncrossing partitions and signed pattern avoiding permutations

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Abstract

We give some interpretations to certain integer sequences in terms of parameters on Grand-Dyck paths and coloured noncrossing partitions, and we find some new bijections relating Grand-Dyck paths and signed pattern avoiding permutations. Next we transfer a natural distributive lattice structure on Grand-Dyck paths to coloured noncrossing partitions and signed pattern avoiding permutations, thus showing, in particular, that it is isomorphic to the structure induced by the (strong) Bruhat order on a certain set of signed pattern avoiding permutations.

1 Introduction

Let $\mathcal{P}$ be a set of paths in the discrete plane, having both the starting and ending points in common. Then it is natural to consider the partial order on $\mathcal{P}$ defined by declaring that the path $P \in \mathcal{P}$ is less than the path $Q \in \mathcal{P}$ when P lies weakly below Q (weakly meaning that the two paths can have some points in common). This point of view has been considered in a series of papers [BBFP, BF, FP], where some order properties of certain classical sets of lattice paths are exploited. In particular, it is shown that the class of Dyck paths of the same length endowed with such a partial order is actually a distributive lattice, and the same happens for Motzkin and Schröder paths.

The motivation of the present work comes from [BBFP], where it is shown that the lattices of Dyck paths are order isomorphic to the sets of 312-avoiding permutations with the induced (strong) Bruhat order. As a byproduct, we then have that 312-avoiding permutations of any given length

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possess a distributive lattice structure. To obtain this result, a new distributive lattice structure is introduced and studied on noncrossing partitions. Subsequently, similar results are proved in [BF] starting from Motzkin and Schröder paths. The aim of the present work is to consider this order structure on Grand-Dyck paths (which are, by definition, like Dyck paths, except that they are allowed to cross the x-axis), and to study some of its properties, as well as to find some related order structure on some kind of noncrossing partitions and pattern avoiding permutations. In trying to accomplish this project, we will come across some bijections and formulas which we believe to be new; in particular, in the spirit of the last section of [BBFP], we will give certain number sequences a combinatorial interpretation in terms of parameters on (Grand-)Dyck paths. This is essentially the content of section 2.

However, our main result is contained in section 3, and consists of the proof that our order structure on Grand-Dyck paths is isomorphic to the Bruhat order on some classes of signed pattern avoiding permutations. This generalizes the above recalled result on Dyck paths, which in fact can be seen as a specialization of the present one. As a byproduct, we have determined a family of pattern avoiding signed permutations such that the induced Bruhat order gives rise to a distributive lattice structure (and not merely a poset structure): to the best of our knowledge, this is the first result of this nature for signed permutations.

Before starting, we recall a recursive construction of Grand-Dyck paths based on the ECO method which will be useful in section 2.

Let $\mathcal{GD}_n$ be the set of Grand-Dyck paths of length $2n$, that is, by definition, the set of all lattice paths starting at the origin $(0,0)$, ending on the x-axis at $(2n,0)$ and using only two kinds of steps, namely $U=(1,1)$ and $D=(1,-1)$. Dyck paths are a special subclass of Grand-Dyck paths, which can be obtained by adding the constraint of remaining weakly above the x-axis.

It is possible to generate Grand-Dyck paths (according to the semilength) using the so-called ECO method. We will not give a description of this method here, but we refer to the very detailed survey [BDLPP]. The following construction can be found in [PPR].

Given $P \in \mathcal{GD}_n$, we construct a set of paths of $\mathcal{GD}_{n+1}$ as follows:

-) if the last step of P is a down step, then we insert a peak into any point of the last descent of G or a valley into the last point of G;

-) otherwise, we insert a valley into any point of the last ascent of P or a peak into the last point of P.

The succession rule Ω describing the above construction is:

$$\Omega : \begin{cases} (2) \\ (2) \rightsquigarrow (3)(3) \\ (k) \rightsquigarrow (3)^2(4)(5)\cdots(k)(k+1) \end{cases} . \tag{1}$$

We close this section by providing a series of notations and definitions we will frequently need throughout all the paper.

An infinite lower triangular matrix A is called a *Riordan array* [SGWW] when its column k ($k = 0, 1, 2, \ldots$) has generating function $d(x)(xh(x))^k$, where $d(x)$ and $h(x)$ are formal power series with $d(0) \neq 0$.

We will usually denote lattice paths using capital letters, such as $P, Q, R, \ldots$. We will also make some use of a *functional notation* for paths starting and ending on the x-axis: the notation $P(k)$ stands for the ordinate of the path P having abscissa k.

In a lattice path P, a *peak* is a sequence of two consecutive steps, the first one being an up step and the second one being a down step. Dually, a *valley* is defined by interchanging the role of up and down steps in the definition of a peak.

Moreover, a *descent* is a sequence of consecutive down steps, whereas an *ascent* is a sequence of consecutive up steps.

For a permutation π, we use the term *rise* to mean a sequence of consecutive and increasing entries of π, whereas a *fall* is a sequence of consecutive and decreasing entries of π.

We will denote with B_n the *hyperoctahedral group of size* n, i.e. the set of all permutations of $\{1, 2, \ldots n\}$ whose elements can be possibly signed. Signed elements will simply be overlined. A signed element will be often interpreted as a *negative* element. In this sense, we say that the *absolute value* $|x|$ of an element x is that element without its sign. Moreover, given $\pi \in B_n$, we will denote by $|\pi|$ the permutation of S_n obtained from π by taking the absolute values of all its elements.

In every poset we will deal with, the covering relation will be denoted $\prec$.

The linear order on n elements, also called *chain* of cardinality n, will be denoted $\mathcal{C}_n$.

A *join-irreducible* of a distributive lattice D is any element x which is not the minimum of the lattice and with the property that, if $x = u \vee v$, then $x = u$ or $x = v$.

The *spectrum* of a distributive lattice D is the poset $\mathbf{Spec}(D)$ of the join-irreducibles of D.

2 Bijections and numbers

Given a double indexed sequence $(\alpha_{n,k})_{n,k\in\mathbf{N}}$, its *coloured version* is defined to be $(\beta_{n,k})_{n,k\in\mathbf{N}} = (2^k\alpha_{n,k})_{n,k\in\mathbf{N}}$. All the formulas we will get in the present section can be interpreted in the same way, namely we will provide some combinatorial interpretation for (the row sums of) the coloured versions of a series of (not always well known) double indexed sequences using Grand-Dyck paths.

Given a Dyck path P, a *factor* of P is a minimal subpath of P which is itself a Dyck path. In figure 1 a Dyck path having 4 factors is shown.

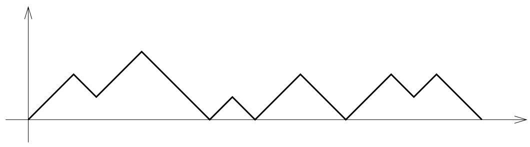

Figure 1: A Dyck path of length 20 having 4 factors.

Now denote by $\overline{\mathcal{D}}_n$ the set of coloured Dyck paths of length $2n$, i.e. Dyck paths whose steps can be coloured in two different ways, say black and white. There is an obvious bijection between $\mathcal{GD}_n$ and a special subset of $\overline{\mathcal{D}}_n$. More precisely, we have the following, simple proposition.

Proposition 2.1 *The set $\mathcal{GD}_n$ of Grand-Dyck paths of length $2n$ is in bijection with the subset of $\overline{\mathcal{D}}_n$ consisting of all coloured Dyck paths in which steps belonging to the same factor occur with the same colour.*

Proof. For any given Grand-Dyck path, just reverse the pieces of the path which lie below the x-axis and colour their steps black (whereas the remaining steps are taken to be white). ■

The subset of $\overline{\mathcal{D}}_n$ mentioned in the above proposition will be denoted $\widetilde{\mathcal{D}}_n$ and its elements will be called *factor-bicoloured Dyck paths.* For an example, see figure 2.

The very easy observation expressed in the above proposition yields the first, obvious enumerative result. Indeed, since it is well known that Grand-Dyck path are counted by the central binomial coefficients $\binom{2n}{n}$, we have the following.

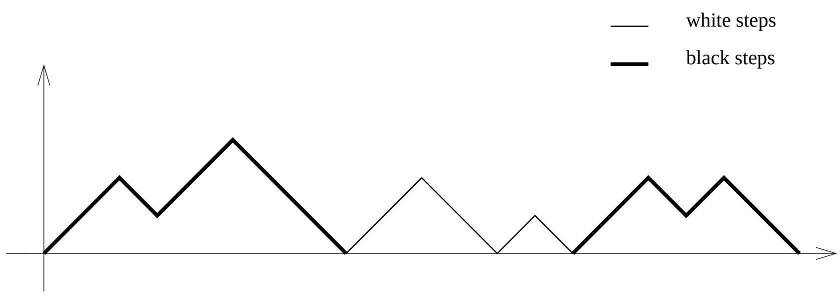

Figure 2: A factor-bicoloured Dyck path of length 20.

Proposition 2.2 *If $b_{n,k} = \frac{k}{2n-k}\binom{2n-k}{n}$ are the ballot numbers, for any $n \geq 1$, we have*

$$\binom{2n}{n} = \sum_{k=1}^{n} 2^k b_{n,k}. \tag{2}$$

Proof. To count Grand-Dyck paths of length $2n$ we can just count the elements of $\widetilde{\mathcal{D}}_n$. Given a Dyck path of length $2n$, the number of its factors clearly coincides with the number of returns of the path, that is how many times the path touches the x-axis except for the starting point. It is well known (see, for example, [D]) that the number of Dyck paths of length $2n$ having precisely k returns is given by the ballot number $b_{n,k} = \frac{k}{2n-k}\binom{2n-k}{n}$. Since each factor can be coloured in two different ways, the thesis immediately follows. ∎

The result of the above proposition is not new, and can be found, for example, in [Sl], among the formulas for sequence A000984 (i.e. central binomial coefficients). Therefore identity (2) provides a trivial combinatorial interpretation for the row sums of the coloured ballot numbers, that is

$$\sum_{k=1}^{n} 2^k b_{n,k} = \sum_{P \in \mathcal{GD}_n} 1.$$

A definitely more interesting result can be obtained by generalizing a result described in the last section of [BBFP]. To this aim, we need first of all to introduce a bijection between Grand-Dyck paths and a special class of set partitions.

If we denote by Π_n the set of partitions of $\{1, 2, \ldots, n\}$, given $\pi = B_1|B_2|\cdots|B_h \in \Pi_n$, we will always represent it in such a way that (i) the elements inside each block B_i are listed in decreasing order and (ii) the blocks are listed in increasing order of their maxima. This will be called the *standard representation, or standard form* of π. Thus, for instance, the partition

$\{4\}|\{5,3,1\}|\{6,2\}$ in Π_6 is represented here in its standard form. To improve readability, in the sequel we will also delete all the parentheses and commas, so that the above partition will be written $4|531|62$. Each partition can be factored as follows. Given $\pi = B_1|B_2|\cdots|B_h \in \Pi_n$ represented in its standard form, we say that π *has k components* when its (linearly ordered) set of blocks can be partitioned into k nonempty intervals with the property that the union of the blocks inside the same interval is an interval of $\{1,2,\ldots,n\}$, and k is maximum with respect to this property. Each of the above intervals of blocks will be called a *component* of π. For instance, the partition $2|43|651|8|97$ in Π_9 has 2 components, which are $2|43|651$ and $8|97$. We will denote $t_{n,k}$ the number of partitions of an n-set having k components. Set partitions having a single component are also called *atomic partitions* (see [BZ]).

Proposition 2.3 *The infinite triangular array $(t_{n,k})_{n,k\in\mathbf{N}}$ is the Riordan array $(p(x),p(x))$, where $p(x)$ is the generating function of atomic partitions. In particular, $t_{n,k}$ is the coefficient of x^n in $x^k p(x)^k$.*

Proof. Observe that a partition of an n-set having k components can be uniquely recovered by the subpartition constituted by its first $k-1$ components and the atomic partition isomorphic to its k-th component. This argument can be translated into the following recurrence relation:

$$t_{n,k} = \sum_{h=1}^{n} t_{n-h,k-1} p_h,$$

where p_n denotes the number of atomic partitions of an n-set. If $C_k(x)$ is the generating function of the k-th column of the array $T = (t_{n,k})_{n,k\in\mathbf{N}}$, the above recurrence becomes:

$$C_k(x) = xp(x)C_{k-1}(x),$$

where $p(x) = \sum_{n>1} p_n x^n$ is the generating function of atomic partitions. Iterating we then get:

$$C_k(x) = (xp(x))^k,$$

which is precisely our thesis. ■

The sequence of atomic partitions is also recorded in [Sl] (it is sequence A074664), and its generating function is $p(x) = 1 - \frac{1}{B(x)}$, where $B(x)$ is the (ordinary) generating function of Bell numbers (see, for example, [Kl]). The infinite matrix T is in [Sl] too (sequence A127743), but the combinatorial interpretation given here is different.

Remark. Observe that the generating function $p(x)$ can also be determined using a species-theoretic argument. Indeed, the fact that any nonempty partition can be decomposed into the (possibly empty) partition constituted by all but the last of its components and the atomic partition constituted by the last component alone means that the species of nonempty partitions is obtained as the Hadamard product of the species of partitions and the species of atomic partitions, and so we get the generating function relation:

$$B(x) - 1 = B(x) \cdot p(x),$$

whence the equality $p(x) = 1 - \frac{1}{B(x)}$ follows.

To generalize the last result of [BBFP], which gives a combinatorial interpretation of Bell numbers in terms of natural parameters on Dyck paths, we now need to introduce the notion of *component-bicoloured partition*. As the name itself suggests, a *component-bicoloured partition* is a set partitions whose components can be coloured using two different colours, say black and white. The total number of component-bicoloured partitions of an n-set is clearly given by $\sum_{k=1}^{n} 2^k t_{n,k}$. This is sequence A059279 in [Sl], but also in this case the present interpretation is not recorded. We now find a further combinatorial interpretation of this sequence in terms of natural parameters on Grand-Dyck paths.

A *bicoloured Dyck word* is a Dyck word (i.e. a word on the alphabet $\{U, D\}$ such that, interpreting each U as an up step and each D as a down step, the resulting path is a Dyck path) whose letters can be coloured either black or white. We call *factor-bicoloured Dyck word* any bicoloured Dyck word corresponding to a factor-bicoloured Dyck path.

Define a *bicoloured Bell matching* of a factor-bicoloured Dyck word to be any Bell matching of the associated Dyck word (i.e. the word obtained by "forgetting colours"). Following [BBFP], a Bell matching of a Dyck word ω is a matching between the U's and the D's of ω such that

1. for any set of consecutive D's, the leftmost D is matched with the adjacent U on its left;

2. every other D is matched with a U on its left, in such a way that there are no crossings among the arcs originated from a set of consecutive D's.

Observe that, in a bicoloured Bell matching, if a U and a D are matched, then they have the same colour.

Adapting the argument given for proposition 6.1 in [BBFP], the reader can now easily prove the following.

Proposition 2.4 *There is a bijection between bicoloured Bell matchings of factor-bicoloured Dyck words of length $2n$ and component-bicoloured partitions of an n-set.*

Given two bicoloured Bell matchings, we say that they are *equivalent* when they are bicoloured Bell matchings of the same factor-bicoloured Dyck word. Since it is clear that, for each fixed factor-bicoloured Dyck word there is exactly one bicoloured Bell matching without crossings among its arcs, as an immediate consequence of the last proposition we have that in the equivalence classes of the above defined equivalence relation there is precisely one bicoloured Bell matching corresponding to a component-bicoloured noncrossing partition. It is convenient to record this fact in a proposition.

Proposition 2.5 *There is a bijection between $\mathcal{GD}_n$ (or, which is the same, $\widetilde{\mathcal{D}}_n$) and the set $\widetilde{NC}(n)$ of component-bicoloured noncrossing partitions of an n-set.*

Now we are ready to state the main result of this section, which provides a combinatorial interpretation for the coloured version of the row sums of the sequence $(t_{n,k})_{n,k\in\mathbf{N}}$ using Grand-Dyck paths. In the following theorem, the word "positive" means "above the x-axis", and "negative" stands for "below the x-axis". Moreover, the *(absolute) height* of a peak (or a valley) is given by the (absolute value of) the ordinate if its vertex.

Theorem 2.1 *Given a Grand-Dyck path P, let $\mathcal{A}$ be the set of positive peaks and negative valleys of P and $\mathcal{B}$ the set of non-positive peaks, non-negative valleys and returns on the x-axis of P. If we linearly order the elements of $\mathcal{A}$ and $\mathcal{B}$ using their abscissas, and denote by $p_1, p_2, \ldots, p_h$ the absolute heights of the elements of $\mathcal{A}$ and by $v_1, v_2, \ldots, v_h$ the absolute heights of the elements of $\mathcal{B}$ (with the convention $v_h = 0$), then we have*

$$\sum_{k=1}^{n} 2^k t_{n,k} = \sum_{P\in\mathcal{GD}_n} \prod_{i=1}^{h} \binom{p_i - 1}{v_i}. \tag{3}$$

Proof. First of all, we observe that $|\mathcal{B}| = |\mathcal{A}|$, since each element of $\mathcal{A}$ is followed by precisely one element of $\mathcal{B}$. Therefore the statement of the theorem is proved to be consistent.

Now let π be a component-bicoloured noncrossing partition of an n-set; proposition 2.5 implies that π is uniquely associated with a Grand-Dyck path P, and so with a factor-bicoloured Dyck path $\widetilde{P}$ of length $2n$. If we denote by Q the Dyck path obtained from $\widetilde{P}$ by "forgetting" colours, we observe that the p_i's are the heights of the peaks of Q, whereas the v_i's are the heights of the valleys (except for v_h, which is the height of the last return of Q, and so $v_h = 0$). Therefore, recalling the definition of Bell matching and the last theorem of [BBFP], if $|[\pi]|$ is the equivalence class of π, we obtain:

$$|[\pi]| = \prod_{i=1}^{h} \binom{p_i - 1}{v_i}.$$

Summing over all Grand-Dyck paths, we then get formula (3), and the theorem is proved. ■

To conclude the part of this section devoted to partitions, we observe that the notion of component-bicoloured partition is a specialization of the well-known notion of *bicoloured partition*, i.e. partition with bicoloured blocks (see, for instance, [LLPP]). As it is obvious, such partitions are enumerated by the sequence $\sum_{k=1}^{n} 2^k S_{n,k}$, where the $S_{n,k}$'s are the Stirling numbers of the second kind; this is sequence A001861 in [Sl]. Also in this case there is a combinatorial interpretation of this numbers in terms of Grand-Dyck paths. The key result is the following proposition, which refines the formula found in [BBFP] to express Bell numbers in terms of parameters on Dyck paths and whose proof (which can be carried out by suitably generalizing the argument of [BBFP]) is left to the reader.

Proposition 2.6 *If $\mathcal{D}_n(k)$ denotes the set of Dyck paths of length $2n$ having exactly k peaks, it is*

$$S_{n,k} = \sum_{P \in \mathcal{D}_n(k)} \prod_{i=1}^{k} \binom{p_i - 1}{v_i},$$

where p_i and v_i are the heights of the peaks and the valleys of P, respectively (with $v_k = 0$ by convention).

As an immediate consequence, we have the following alternative interpretation for the row sums of the coloured Stirling numbers of the second kind.

Corollary 2.1 *With the same notations as in the above proposition, we have:*

$$\sum_{k=1}^{n} 2^k S_{n,k} = \sum_{k=1}^{n} 2^k \sum_{P \in \mathcal{D}_n(k)} \prod_{i=1}^{k} \binom{p_i - 1}{v_i}.$$

The second part of the present section is devoted to the description of some new bijections between Grand-Dyck paths and signed pattern avoiding permutations. More precisely, we propose here two bijections: the former has been found with the help of the ECO method, whereas the latter will be useful in the next section to define an interesting distributive lattice structure.

The first bijection involves the classes of signed pattern avoiding permutations $B_n(21, \overline{21})$, where B_n is the hyperoctahedral group on n elements. It is known [Si] that $|B_n(21, \overline{21})| = \binom{2n}{n}$. Moreover, combining some results in

[Si] and in [Re], an explicit bijection between $\mathcal{GD}_n$ and $B_n(21, \overline{21})$ can be described. However, our bijection is different from the one so obtained. Before starting, observe that a permutation in $B_n(21, \overline{21})$ is a shuffle of the signed and unsigned elements, each of which are ordered increasingly by absolute value. Thus, given $\pi \in B_n(21, \overline{21})$, we can consider the two elements a and b such that $|a|$ and $|b|$ are the maximum of the absolute values of the signed and of the unsigned elements, respectively. It is clear that $|a| = n$ or $|b| = n$. We call *quasi maximum* of π the one between a and b whose absolute value is different from n.

To describe our first bijection, we represent permutations by a graphical device used, for instance, in [BFP]. We represent the elements of the permutations as dots placed on horizontal lines, in such a way that elements with greater absolute values lie on higher lines. It is an extremely natural representation, so we deem it is not necessary to give a more formal definition (see figure 3 for an example).

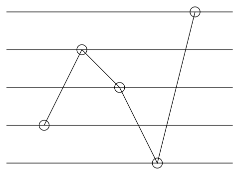

Figure 3: A graphical representation of the permutation 24315

Let $\pi \in B_n(21, \overline{21})$. Starting from π we will construct a set of permutations belonging to $B_{n+1}(21, \overline{21})$, by adding a new element in the last position of π and then suitably renaming some of the elements of π. From a graphical point of view, we simply add a new horizontal line in the representation of π, and we place on such a line the new element. In performing this operation, we have to take care that the resulting permutations still avoids the two patterns 21 and $\overline{21}$. To make sure that this happens, we can distinguish two cases.

1. Suppose that the quasi maximum $\overline{a}$ of π is signed. In this case, we can add at the end of π any signed element having absolute value greater than a, as well as the unsigned element $n+1$. Figure 4 describes how this construction works.

2. On the other hand, if the quasi maximum a of π is unsigned, we are allowed to add at the end of π any unsigned element greater than a, as well as the signed element $\overline{n+1}$ (see figure 5).

In the first case, if $\overline{a} = \overline{n+2-k}$, then π produces k sons, whose quasi maximums are easily seen to be $\overline{n+2-k}, \overline{n+3-k}, \ldots, \overline{n}, n$, and so the numbers of their sons are, respectively, $k+1, k, \ldots 4, 3, 3$. Similarly, in the

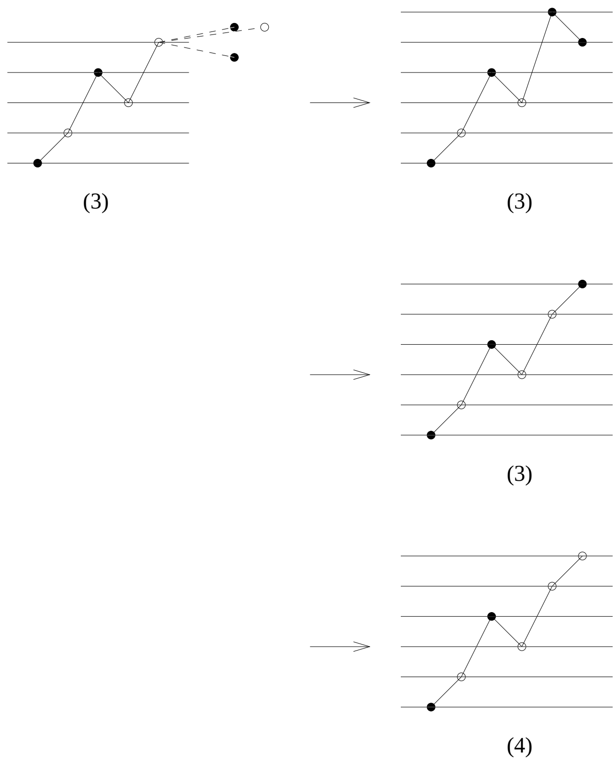

Figure 4: Our ECO construction performed on $\overline{1}2\overline{4}35$. Here signed elements are represented using black bullets.

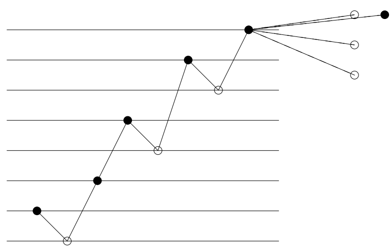

Figure 5: Our ECO construction performed on $\overline{2}1\overline{3}\overline{5}4\overline{7}6\overline{8}$.

second case, we have the same statement as above, with signed elements replaced by unsigned ones and vice versa. Thus, also in this case the numbers of sons are the same as above.

Therefore, we observe that we have an ECO construction for signed permutations avoiding 21 and $\overline{21}$ which is isomorphic to the ECO construction for Grand-Dyck paths recalled in the introduction, and described by the succession rule Ω given in (1). Such an isomorphism defines a bijection between $\mathcal{GD}_n$ and $B_n(21, \overline{21})$. Moreover, the underlying ECO construction allows to easily translate many natural statistics on Grand-Dyck paths into specific parameters defined on permutations. To give just a glimpse of this fact, we propose a couple of examples.

Given a signed permutation $\pi = \pi_1 \cdots \pi_n \in B_n$, we define the following two sets:

$$\begin{aligned} A(\pi) = \{i \leq n \mid \pi_i \text{ is unsigned and } \pi_1 \cdots \pi_{i-1} \text{ has a signed maximum}\} \cup \\ \{i \leq n \mid \pi_i \text{ is signed and } \pi_1 \cdots \pi_i \text{ has an unsigned maximum}\}; \\ B(\pi) = \{i \leq n \mid \text{the two greatest elements of } \pi_1 \cdots \pi_i \text{ have different signs}\}. \end{aligned}$$

By convention, we assume that $1 \in A(\pi)$ if and only if π_1 is unsigned and that $1 \in B(\pi)$ for any permutation π.

Proposition 2.7 *The statistic "number of peaks" on Grand-Dyck paths corresponds to the A-statistic on $B_n(21, \overline{21})$.*

Proof. Observe that, in the ECO construction of Grand-Dyck paths proposed in the introduction (and encoded by the rule in (1)), the sons of a Grand-Dyck path P having h peaks can have either h peaks or $h+1$ peaks. More precisely:

$path_1$) if P ends with a sequence of down steps, then all the sons of P have $h+1$ peaks but the one obtained by adding a valley at the end of P and the one obtained by adding a peak at the beginning of the last descent of P; if P has label (k), the two sons of P having h peaks are then labelled (3) and $(k+1)$;

$path_2$) if P ends with a sequence of up steps, then all the sons of P have $h+1$ peaks but the one obtained by adding a valley at the beginning of the last ascent of P (if P has label (k), then the label of such a son is $(k+1)$).

Transferring the above considerations on permutations by means of our bijection, we have the following two cases:

$perm_1$) if $\pi \in B_n(21, \overline{21})$ has an unsigned maximum, then, in the above described ECO construction of $B_n(21, \overline{21})$, the two sons of π having labels (3) and $(k+1)$ corresponding to the paths mentioned in $path_1$) are avoided by adding a signed element a whose absolute value is not greater than the absolute values of all the elements of π; this means that πa has an unsigned maximum;

$perm_2$) if $\pi \in B_n(21, \overline{21})$ has a signed maximum, then the son of π labelled $(k+1)$ is avoided by adding any signed element.

The above considerations immediately implies that, if π corresponds to P in our bijection, then $h = |A(\pi)|$, which is the thesis. ■

Proposition 2.8 *The statistic "number of returns" on Grand-Dyck paths corresponds to the B-statistic on* $B_n(21, \overline{21})$.

Proof (sketch). The arguments to be used here are completely analogous to those of the previous proposition. We just observe that, in the ECO construction of Grand-Dyck paths, a new return is produced whenever either a valley or a peak is appended at the end of the path. This translates on permutations into the addition to the right of π of an element a whose sign is different from the sign of the maximum of π and such that the maximum of π and a are the two greatest elements of πa. ■

In passing through, we notice that it is possible to define another (presumably new) bijection between $\mathcal{GD}_n$ and $B_n(21, \overline{21})$. Also in this case, we start by considering the usual ECO construction of Grand-Dyck paths and then we translate it into permutations avoiding the two patterns 21 and $\overline{21}$. Without going into details, the idea is to generate permutations by adding a new maximum (instead of adding the rightmost element). This can be represented by using a graphical device similar to the one above: just replace horizontal lines with vertical lines. In figure 6 an example of how this construction works is shown. We entirely leave to the interest reader the accomplishment of all the details of this alternative approach, as well as the task of translating some statistics on Grand-Dyck paths (such as the number of peaks and the number of returns considered above) into permutations.

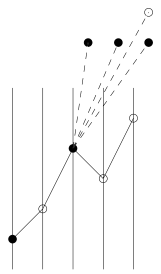

Figure 6: An alternative ECO construction performed on $\overline{1}2\overline{4}35$.

Our second bijection involves a different type of pattern avoiding permutations, namely permutations which avoid the four patterns $312, \overline{312}, 2\overline{1}, \overline{2}1$. The classes of pattern-avoiding permutations $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$ are counted by the central binomial coefficients $\binom{2n}{n}$, and this can be easily proved by exhibiting a completely trivial bijection with component-bicoloured noncrossing partitions.

Proposition 2.9 *(Bar-removing bijection). There is a bijection between* $\widetilde{NC}(n)$ *and* $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$.

Proof. Taken $\pi \in \widetilde{NC}(n)$, written as usual in its standard form, delete the vertical bars so to obtain a permutation belonging to B_n (still denoted by π). The presence of a pattern $2\overline{1}$ or of a pattern $\overline{2}1$ in π would imply that, in the associated partition, the elements of such a pattern should belong to two different components and the greatest of them should belong to a block with a lesser index. But this is impossible in our standard representation of partitions. Moreover, the fact that $|\pi|$ is noncrossing implies that every signed version of the pattern 312 cannot appear in the associated signed permutation. Finally, it is immediate that, avoiding any signed version of 312 and the two patterns $\overline{2}1$ and $2\overline{1}$ is equivalent to avoiding $312, \overline{312}, \overline{2}1$ and $2\overline{1}$.

To prove that this is actually a bijection, it is sufficient to observe that, given $\pi \in B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$, the associated partition can be uniquely recovered by inserting a vertical bar between the elements of each rise of $|\pi|$ ■

Now, using propositions 2.5 and 2.9, we get the following corollary.

Corollary 2.2 *There is a bijection between* $\mathcal{GD}_n$ *and* $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$.

This last bijection is a sort of signed analog of a well known bijection between Dyck paths and 312-avoiding permutations, which can be found for example in [BK, Kr]. It also has the remarkable feature of translating many natural statistics on paths into natural statistics on permutations. For instance, the number of unsigned (resp. signed) left-to-right maxima in a permutation of $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$ is equal to the number of positive peaks (resp. negative valley) of the associated Grand-Dyck path.

3 Posets

The set $\mathcal{GD}_n$ of Grand-Dyck paths of length $2n$ can be naturally ordered by declaring $P \leq Q$ whenever $P(k) \leq Q(k)$, for all k. This means that the path P lies weakly below Q (see the example in figure 7). This very natural partial order is easily seen to yield a distributive lattice structure, in which the join and meet of two paths are taken coordinatewise. The minimum and maximum of these lattices will be denoted $\mathbf{0}$ and $\mathbf{1}$, respectively. See figure 8 for the Hasse diagram of $\mathcal{GD}_3$. The resulting lattice structures have already appeared in the literature (see [NF]), but they have been considered on different combinatorial objects. It is immediate the following fact, whose easy proof is left to the reader.

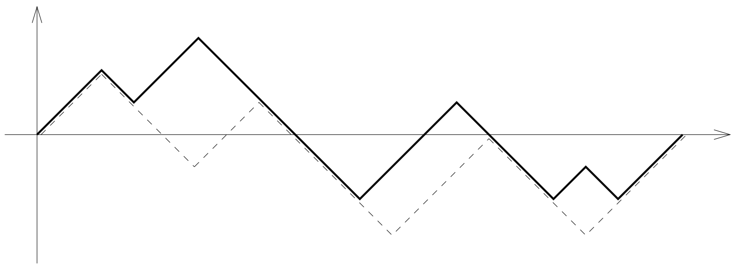

Figure 7: Two comparable Grand-Dyck paths of length 20.

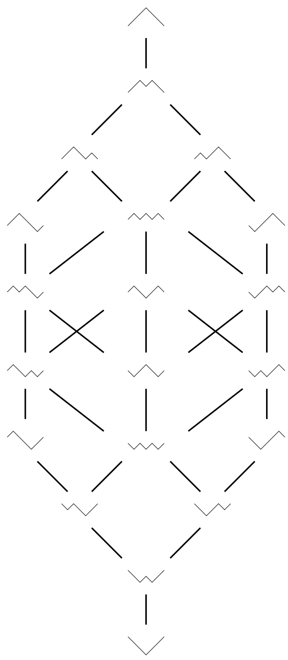

Figure 8: The lattice $\mathcal{GD}_3$.

Proposition 3.1 *The lattice $\mathcal{GD}_n$ of Grand-Dyck paths of length $2n$ is isomorphic to the Young lattice of integer partitions which lie inside the $n \times n$ square.*

Young lattices of partitions have been intensively investigated since many years, see for instance [St] for an interesting study on the unimodality properties of such lattices. Of course, as an immediate consequence of the above proposition, we have that Grand-Dyck lattices are rank-unimodal.

However, even if the abstract lattice structure we are considering is not new, we claim that the study of order and lattice properties arising from the special representation in terms of Grand-Dyck paths is worth being carried out.

Our first result concerns the shape of join-irreducible elements.

Proposition 3.2 *A Grand-Dyck path $P \in \mathcal{GD}_n$ is join-irreducible if and only if it has precisely one peak. Therefore,* $\mathbf{Spec}(\mathcal{GD}_n) \simeq \mathcal{C}_n^2$.

Proof (sketch). The covering relation on Grand-Dyck paths works as follows: Q covers P if and only if Q can be obtained from P by changing a valley into a peak. Then the first part of the thesis immediately follows. As far as the second part is concerned, just observe that, thanks to the last proposition, join-irreducibles corresponds to integer partitions of rectangular shape. ■

We have already mentioned that the lattices $\mathcal{GD}_n$ are rank-unimodal. The rank function r_n of $\mathcal{GD}_n$ is clearly related to the area function. More precisely, we have the following proposition.

Proposition 3.3 *If $P \in \mathcal{GD}_n$, then*

$$r_n(P) = \frac{A(P) + n^2}{2},$$

where $A(P)$ denotes the area (with sign) of the region included between the path and the x-axis.

Proof. By induction, suppose that $P \prec Q$ in $\mathcal{GD}_n$ and that $r_n(P) = \frac{A(P)+n^2}{2}$. Since Q is obtained from P by simply reversing a valley, we have that $A(Q) = A(P) + 2$, and so $r_n(Q) = \frac{A(Q)+n^2}{2} = \frac{A(P)+2+n^2}{2} = \frac{A(P)+n^2}{2} + 1 = r_n(P)$. Since $r_n(\mathbf{0}) = \frac{A(\mathbf{0})+n^2}{2} = \frac{-n^2+n^2}{2} = 0$, the proof is completed. ■

Our next goal will be to translate the above described lattice structure on partitions. To accomplish this task we make use of the bijection between the set $\widetilde{\mathcal{D}}_n$ of factor-bicoloured Dyck paths of length n and component-bicoloured noncrossing partitions stated in proposition 2.5. For the sake of simplicity, from now on we will refer to black (resp., white) items (steps, factors,...) as to coloured (resp., noncoloured) items.

We start by observing that the Grand-Dyck lattice structure can be read off on factor-bicoloured Dyck paths as follows: if $P, Q \in \widetilde{\mathcal{D}}_n$, we say that $P \leq Q$ when, for every $k \in \mathbf{N}$, one of the following holds:

1. if $P(k)$ and $Q(k)$ both belong to coloured factors, then $P(k) \geq Q(k)$;

2. if $P(k)$ and $Q(k)$ both belong to noncoloured factors, then $P(k) \leq Q(k)$;

3. $P(k)$ belongs to a coloured factor and $Q(k)$ belongs to a noncoloured one.

Now, if we transport this lattice structure along the above recalled bijection, we obtain a lattice structure on component-bicoloured noncrossing partitions. In the sequel, we denote these lattices of partitions $\widetilde{NC}(n)$ (where n is the size of the ground set, of course).

Our next results concerns the description of the order relation on $\widetilde{NC}(n)$.

Given $\pi \in \widetilde{NC}(n)$, define the *max-vector* of π to be the vector $max(\pi)$ whose i-th component equals the maximum of the first i element of $|\pi|$, when π is written in its standard form; moreover, each component of $max(\pi)$ appears coloured when it is coloured in π. Therefore, for instance, taken $\pi = 2|431|\overline{6}|\overline{7}|\overline{985} \in \widetilde{NC}(9)$, we have $max(\pi) = (2, 4, 4, 4, \overline{6}, \overline{7}, \overline{9}, \overline{9}, \overline{9})$.

For any given $n \in \mathbf{N}$, denote by $M(n)$ the set of max-vectors of $\widetilde{NC}(n)$, that is $M(n) = \{v = (v_1, \ldots, v_n) \mid \exists \pi \in \widetilde{NC}(n) : v = max(\pi)\}$. It is not difficult to see that $M(n)$ consists of all vectors with n bicoloured components having increasing absolute values and such that, for any $i < n$, $|v_i| \geq i$ and, if v_i and v_{i+1} have different colours, then $v_i = i$.

Now define on $M(n)$ a partial order as follows. Given two max-vectors $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$, we say that $v \leq w$ when, for every $i \leq n$ either

(i) v_i and w_i are both coloured and $v_i \geq w_i$, or

(ii) v_i and w_i are both noncoloured and $v_i \leq w_i$, or

(iii) v_i is coloured and w_i is noncoloured.

Clearly, the covering relation of the poset $[M(n); \leq]$ can be described by saying that precisely one of the above situations $(i), (ii)$ or (iii) holds for a specific i, whereas all the other components are equal, and in each of the three cases we have respectively:

(i) $v_i = w_i + 1$,

(ii) $w_i = v_i + 1$,

(iii) $|v_i| = |w_i|$.

Using max-vectors it is now possible to characterize the covering relation of $\widetilde{NC}(n)$.

Proposition 3.4 *Given $\pi, \rho \in \widetilde{NC}(n)$, it is $\pi \prec \rho$ if and only if $max(\pi) \prec max(\rho)$.*

Proof. Saying that $\pi \prec \rho$ in $\widetilde{NC}(n)$ means that, if we consider the two associated factor-bicoloured Dyck paths $P = P(\pi)$ and $R = R(\rho)$, they differ precisely in two steps, namely either:

(i) a coloured peak of P is changed into a coloured valley of R, or

(ii) a noncoloured valley of P is changed into a noncoloured peak of R, or

(iii) a coloured peak of P is changed into a noncoloured peak of R (in this case the two peaks necessarily lies on the x-axis).

Now observe that, given $v = (v_1, \dots, v_n) = max(\pi) \in M(n)$, the position $d_i(\pi)$ of the i-th down step in the associated factor-bicoloured Dyck path is given by $v_i + i$, and of course the same happens for the max-vector $w = (w_1, \dots, w_n) = max(\rho) \in M(n)$ (i.e., $d_i(\rho) = w_i + i$). Therefore, in the above three cases, we have:

(i) P and R coincide, except for a pair of adjacent steps, which is $\overline{UD}$ in P and $\overline{DU}$ in R. If the down step involved is the i-th, then we have $w_i = d_i(\rho) - i = d_i(\pi) - 1 - i = v_i - 1$.

(ii) P and R coincide, except for a pair of adjacent steps, which is DU in P and UD in R. If the down step involved is the i-th, then we have $w_i = d_i(\rho) - i = d_i(\pi) + 1 - i = v_i + 1$.

(iii) P and R coincide, except for a pair of adjacent steps, which is $\overline{UD}$ in P and UD in R, and such a peak lies on the x-axis. In this last case, if the down step involved is the i-th, then we have that v_i and w_i have the same absolute value, but v_i is coloured whereas w_i is noncoloured.

Thus, the fact that $\pi \prec \rho$ in $\widetilde{NC}(n)$ is equivalent to the fact that $max(\pi) \prec max(\rho)$ in $M(n)$, which is our thesis. ∎

Corollary 3.1 *Given $\pi, \rho \in \widetilde{NC}(n)$, it is $\pi \leq \rho$ if and only if $max(\pi) \leq max(\rho)$.*

Our last goal is to transfer the above order on signed pattern avoiding permutations. This can be done by simply applying the bar-removing bijection of proposition 2.9, thanks to which we obtain a partial order on $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$. The following proposition gives a characterization of the associated covering relation.

We will use the term *signed* (respectively, *unsigned*) *inversion* of π to mean a pair (π_i, π_j), with $i < j$, $\pi_i = \overline{a}$ (resp., $\pi_i = a$), $\pi_j = \overline{b}$ (resp., $\pi_j = b$) and $a > b$. An analogous definition is given for the term *signed (unsigned) noninversion.*

Proposition 3.5 *Let $\pi, \rho \in B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$. Then $\pi \prec \rho$ if and only if ρ is obtained from π by either:*

(i) interchanging the elements of a signed inversion, or

(ii) interchanging the elements of an unsigned noninversion, or

(iii) changing a signed element belonging to a rise of $|\pi|$ into an unsigned one.

Proof. The condition $\pi \prec \rho$ in $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$ can be translated on factor-bicoloured paths by saying that exactly one of the three conditions listed in the proof of proposition 3.4 holds. Using the results of [BBFP], it is not difficult to conclude that the first two conditions corresponds, on permutations, to conditions (*i*) and (*ii*) of the present proposition. Concerning the third condition, if, in a path P, a coloured peak at height 0 is changed into a noncoloured one, then, on the associated permutation π, we have a signed element belonging to a rise of $|\pi|$ which is changed into an unsigned one, and the proof is completed. ■

The distributive lattice structure on $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$ obtained via the bar-removing bijection turns out to have a very interesting alternative combinatorial description. Before giving it explicitly, we determine a formula to compute its rank function. Given a permutation $\pi \in B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$, we denote by $inv(\pi)$ the number of *unsigned* inversions of π, by $ninv(|\pi|)$ the number of non-inversions of $|\pi|$ and by $\#(\pi)$ the number of *unsigned* elements of π. Thus, for instance, given $\pi = 2431\overline{67}58\overline{9} \in \widetilde{NC}(9)$, we have $inv(\pi) = 4$, $ninv(|\pi|) = 30$ and $\#(\pi) = 5$.

Proposition 3.6 *If $\pi \in B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$, denoting by $r_n(\pi)$ its rank, we have:*

$$r_n(\pi) = ninv(|\pi|) + 2inv(\pi) + \#(\pi).$$

Proof. The minimum $\mathbf{0}$ of the lattice $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$ is the permutation $\overline{n(n-1)\cdots 21}$, and it is clear that $ninv(|\overline{n(n-1)\cdots 21}|) + 2inv(\overline{n(n-1)\cdots 21}) + \#(\overline{n(n-1)\cdots 21}) = 0 = r_n(\mathbf{0})$.

Using an induction argument, suppose that, in $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$, we have $\pi \prec \rho$. Thanks to proposition 3.5, we have three possible cases, which we deal with using the same numeration as in the statement of such a proposition.

(i) In this case, in the associated coloured Dyck path there is a coloured peak which is changed into a coloured valley. Referring to [BBFP], we can then say that, in the permutation π, a new signed noninversion is produced, and so:

$$inv(\rho) = inv(\pi), \quad ninv(|\rho|) = ninv(|\pi|) + 1, \quad \#(\rho) = \#(\pi),$$

whence

$$\begin{aligned} ninv(|\rho|) + 2inv(\rho) + \#(\rho) &= ninv(|\pi|) + 2inv(\pi) + \#(\pi) + 1 \\ &= r_n(\pi) + 1 = r_n(\rho). \end{aligned}$$

(ii) This situation corresponds to changing a valley in a peak in a non-coloured factor of the associated coloured Dyck path. But is it known from [BBFP] that this produces one more unsigned inversion in ρ (leaving unchanged all the signed elements), and so

$$inv(\rho) = inv(\pi) + 1, \quad ninv(|\rho|) = ninv(|\pi|) - 1, \quad \#(\rho) = \#(\pi),$$

and an analogous computation as above immediately yields

$$ninv(|\rho|) + 2inv(\rho) + \#(\rho) = r_n(\rho).$$

(iii) Finally, in this last case the difference between π and ρ consists of the fact that ρ contains one more unsigned element, whence (using analogous arguments as those for the preceding two cases) the thesis follows. ■

Our last result, which we deem is the main one of the present paper, is the determination of an isomorphism between our lattices $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$ and an important and well known poset structure on permutations. To this aim, we need to introduce a few notations and definitions.

Call $S_{\pm n}$ the set of permutation of the set $\{\overline{n}, \overline{n-1}, \dots, \overline{2}, \overline{1}, 1, 2, \dots, n-1, n\}$, linearly ordered as indicated. As already remarked, this corresponds to setting $\overline{k} = -k$ and then considering the usual linear order. In what follows, we will often tacitly make the above identification, but we keep on writing $\overline{k}$ instead of $-k$ in order to gain a better readability. Given π in the hyperoctahedral group B_n, denote by $\hat{\pi}$ the permutation of $S_{\pm n}$ defined by $\hat{\pi}(i) = \overline{\pi(n+1-i)}$ and $\hat{\pi}(\overline{i}) = \overline{\hat{\pi}(i)}$, for every $i \in \{1, \dots, n\}$. Thus, for instance, if $\pi = 3\overline{2}5\overline{4}\overline{1} \in B_5$, then $\hat{\pi} = 3\overline{2}5\overline{4}\overline{1}14\overline{5}2\overline{3} \in S_{\pm 5}$. Observe, in particular, that, given $\hat{\pi}$ expressed, as usual, in one-line notation, π is obtained, again in one-line notation, by taking the first half of the elements of $\hat{\pi}$. Moreover, let $\hat{S}_{\pm n}[312, \overline{312}, 2\overline{1}, \overline{2}1] = \{\hat{\pi} \mid \pi \in B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)\}$ (here we use square brackets in order to avoid confusion with the notation for pattern avoidance).

Theorem 3.1 *The poset* $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$ *is isomorphic to* $\hat{S}_{\pm n}[312, \overline{312}, 2\overline{1}, \overline{2}1]$ *endowed with the (strong) Bruhat order.*

Proof. Suppose first that $\pi \prec \rho$ in $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$. From proposition 3.5, we have three cases to analyze.

(i) Suppose that ρ is obtained from π by interchanging the elements of a signed inversion. To fix the notations, we can denote by $i, j \leq n$ the two elements such that $i < j$, $\pi(i) = \rho(j)$ and $\pi(j) = \rho(i)$ are both signed and $\pi(i) > \pi(j)$ (and so $\rho(i) < \rho(j)$). Then, from the definition of $\hat{\pi}$ and $\hat{\rho}$, it follows that $\hat{\pi}(\overline{n+1-i}) = \hat{\rho}(\overline{n+1-j})$ and $\hat{\pi}(\overline{n+1-j}) = \hat{\rho}(\overline{n+1-i})$ are both signed and $\hat{\pi}(\overline{n+1-i}) > \hat{\pi}(\overline{n+1-j})$ (and so $\hat{\rho}(\overline{n+1-i}) < \hat{\rho}(\overline{n+1-j})$). Since $n+1-i > n+1-j$, this means that $\hat{\rho}$ possesses at least one more inversion than $\hat{\pi}$. An analogous argument on the elements $\hat{\pi}(n+1-i) = \hat{\rho}(n+1-j)$ and $\hat{\pi}(n+1-j) = \hat{\rho}(n+1-i)$ shows that $\hat{\rho}$ has one further inversion more than $\hat{\pi}$. It is then easy to realize that these are the only inversions of $\hat{\rho}$ which are not also in $\hat{\pi}$, and so we can conclude that $\hat{\rho}$ has two more inversions than $\hat{\pi}$ in the Bruhat order of $\hat{S}_{\pm n}[312, \overline{312}, 2\overline{1}, \overline{2}1]$, which is enough to say that $\hat{\pi} \leq \hat{\rho}$ in such a Bruhat poset.

(ii) An analogous argument can be developed when ρ is obtained from π by interchanging the elements of an unsigned noninversion. Also in this case, following the same lines, it is possible to show that $\hat{\rho}$ has two more inversions than $\hat{\pi}$ in the Bruhat order of $\hat{S}_{\pm n}[312, \overline{312}, 2\overline{1}, \overline{2}1]$.

(iii) If ρ is obtained from π by changing a signed element belonging to a rise into an unsigned one, then $\hat{\pi}$ and $\hat{\rho}$ coincide except for two elements; more precisely, there exists a positive $i \leq n$ such that $\hat{\pi}(\overline{i}) = \hat{\rho}(i)$ is signed (and so $\hat{\pi}(i) = \hat{\rho}(\overline{i})$ is unsigned). Thus, the pair $(\hat{\pi}(\overline{i}), \hat{\pi}(i))$ is a noninversion in $\hat{\pi}$, whereas $(\hat{\rho}(\overline{i}), \hat{\rho}(i))$ is an inversion in $\hat{\rho}$. From this we deduce that $\hat{\rho}$ has one more inversion than $\hat{\pi}$.

The above arguments allows us to conclude that, if $\pi \prec \rho$ in $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$, then $\hat{\pi} \leq \hat{\rho}$ in the Bruhat order of $\hat{S}_{\pm n}[312, \overline{312}, 2\overline{1}, \overline{2}1]$. As an obvious consequence, we have that, if $\pi \leq \rho$ in $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$, then $\hat{\pi} \leq \hat{\rho}$ in the Bruhat order of $\hat{S}_{\pm n}[312, \overline{312}, 2\overline{1}, \overline{2}1]$, which is the first part of the theorem.

Vice versa, suppose that $\hat{\pi} \prec \hat{\rho}$ in $\hat{S}_{\pm n}[312, \overline{312}, 2\overline{1}, \overline{2}1]$ with the induced Bruhat order. This means that $\hat{\rho}$ is obtained from $\hat{\pi}$ by performing as little inversions as possible (and, of course, remaining inside the class $\hat{S}_{\pm n}[312, \overline{312}, 2\overline{1}, \overline{2}1]$). We can distinguish some cases.

- Given $i > j > 0$, if $\hat{\pi}(\overline{i}) < \hat{\pi}(\overline{j})$ and both $\hat{\pi}(\overline{i}), \hat{\pi}(\overline{j})$ are unsigned, then we can exchange $\hat{\pi}(\overline{i})$ and $\hat{\pi}(\overline{j})$, provided that there is no k, $i > k > j$, such that $\hat{\pi}(\overline{i}) < \hat{\pi}(\overline{k}) < \hat{\pi}(\overline{j})$. However, in this case, we also need to exchange the two elements $\hat{\pi}(i)$ and $\hat{\pi}(j)$ in order to remain inside $\hat{S}_{\pm n}[312, \overline{312}, 2\overline{1}, \overline{2}1]$. Thus, we have obtained a permutation $\hat{\rho}$ having two more inversions than $\hat{\pi}$. Translating all this on $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$, we have that ρ is obtained from π by interchanging the elements of an unsigned noninversion. Of course, we have, by symmetry, exactly the same situation if we start by considering $0 < i < j$ such that $\hat{\pi}(i) < \hat{\pi}(j)$ and $\hat{\pi}(i), \hat{\pi}(j)$ both signed.

- With a completely analogous argument, we can prove that a permutation $\hat{\rho}$ which covers $\hat{\pi}$ in the induced Bruhat order of $\hat{S}_{\pm n}[312, \overline{312}, 2\overline{1}, \overline{2}1]$ can be obtained by interchanging $\hat{\pi}(i)$ and $\hat{\pi}(j)$, when $0 < i < j$, $\hat{\pi}(i)$ and $\hat{\pi}(j)$ are both unsigned and $\hat{\pi}(i) < \hat{\pi}(j)$ (or, equivalently, when $i > j > 0$, and $\hat{\pi}(\bar{i}) < \hat{\pi}(\bar{j})$). In $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$, this means that ρ is obtained from π by interchanging the elements of a signed inversion.

- If $\hat{\pi}(i)$ and $\hat{\pi}(j)$ have different signs, for $i, j > 0$, we cannot exchange $\hat{\pi}(i)$ and $\hat{\pi}(j)$, since a $\overline{2}1$ pattern would arise. And the same would happen for $\hat{\pi}(\bar{i})$ and $\hat{\pi}(\bar{j})$.

- The only case which does not fit into one of the above is when $\hat{\rho}$ is obtained from $\hat{\pi}$ by interchanging two elements $\hat{\pi}(\bar{i})$ and $\hat{\pi}(j)$, with $i, j > 0$. In this case, it is easy to see that, if $\hat{\pi}(\bar{i})$ and $\hat{\pi}(j)$ had different absolute values, then, in $\rho \in B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$ we would have two elements having the same absolute value, which is clearly not allowed. The only possibility we have to perform an interchange is to have $i = j$ (i.e., to interchange two elements of the kind $a, \overline{a}$). In this case, ρ is obtained from π by changing a signed element into an unsigned one, and, in order that the inversion in $\hat{S}_{\pm n}[312, \overline{312}, 2\overline{1}, \overline{2}1]$ is minimal, it is necessary that, for every $0 < k < i$, $\hat{\pi}(k) > \hat{\pi}(i) > 0$. This means that, in B_n, $\pi(n + i - 1)$ is signed and belongs to a rise of $|\pi|$.

Now, putting things together, thanks to proposition 3.5, we have shown that, if $\hat{\pi} \prec \hat{\rho}$ in $\hat{S}_{\pm n}(312, \overline{312}, 2\overline{1}, \overline{2}1)$, then $\pi \prec \rho$ in $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$, which is enough to conclude. ■

This last result is a “signed generalization” of the fact (proved in [BBFP]) that the lattices of Dyck paths are isomorphic to the lattices of 312−avoiding permutations under the Bruhat order. Indeed, consider the hyperoctahedral group B_n endowed with the Bruhat order, as it is defined, for instance, in [BB]. Using our language, it can be described as follows. Given $\pi', \rho' \in B_n$, consider the permutations $\hat{\pi}, \hat{\rho} \in S_{\pm n}$ defined by the juxtaposition of π and π' and of ρ and ρ', respectively, where π (resp., ρ) is defined by reversing and changing all the signs of π' (resp., ρ). For instance, if $\pi' = 14\overline{5}2\overline{3} \in B_5$, then $\pi = 3\overline{2}5\overline{41}$ and $\hat{\pi} = 3\overline{2}5\overline{41}14\overline{5}2\overline{3} \in S_{\pm 5}$. Then $\pi' \leq \rho'$ in the Bruhat order of B_n if and only if $\hat{\pi} \leq \hat{\rho}$ in the Bruhat order of the symmetric group $S_{\pm n}$. Therefore, as a consequence of the last theorem, we get our final results, which states that $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$ is isomorphic to a set of signed pattern avoiding permutation under the Bruhat order.

Corollary 3.2 *The poset* $B_n(312, \overline{312}, 2\overline{1}, \overline{2}1)$ *is isomorphic to* $B_n(213, \overline{213}, 1\overline{2}, \overline{1}2)$ *endowed with the Bruhat order.*

Proof. Just observe that, if π and π' are related as above, then π avoids a pattern σ if and only if π' avoids the pattern obtained by reversing σ and changing the signs of all its elements. ■

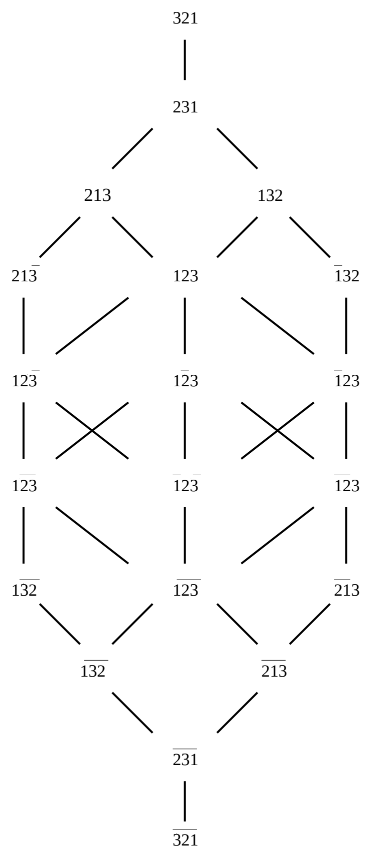

Figure 9: The lattice $B_3(312, \overline{312}, 2\overline{1}, \overline{2}1)$

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