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ORIGINAL PAPER

Pairs of Chromatically Equivalent Graphs

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Abstract Two graphs are said to be *chromatically equivalent* if they have the same chromatic polynomial. In this paper we give the means to construct infinitely many pairs of chromatically equivalent graphs where one graph in the pair is *clique-separable*, that is, can be obtained by identifying an r-clique in some graph H_1 with an r-clique in some graph H_2, and the other graph is non-clique-separable. There are known methods for finding pairs of chromatically equivalent graphs where both graphs are clique-separable or both graphs are non-clique-separable. Although examples of pairs of chromatically equivalent graphs where only one of the graphs is clique-separable are known, a method for the construction of infinitely many such pairs was not known. Our method constructs such pairs of graphs with odd order $n \geq 9$.

Keywords Chromatic equivalence · Chromatic polynomial

1 Introduction

Two graphs, G and G', are said to be *chromatically equivalent* if they have the same chromatic polynomial. We write $G \sim G'$. A graph G is an *r-gluing* of graphs H_1 and H_2 if G can be obtained by identifying an r-clique in H_1 with an r-clique in H_2. We say that G is *clique-separable* if it is isomorphic to an r-gluing of two graphs. In this article we give a method for constructing infinitely many pairs of chromatically equivalent graphs where one of the pair is clique-separable but the other is non-clique-separable.

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The chromatic polynomial of a clique-separable graph G is

$$P(G,\lambda) = \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_r,\lambda)} \tag{1}$$

where H_1 and H_2 are graphs, $r \geq \min\{\chi(H_1), \chi(H_2)\}$ and $\chi(\_)$ is the chromatic number of a graph. In [7,8] we showed that there exist graphs whose chromatic polynomials satisfy (1) but are not the chromatic polynomial of any clique-separable graph. Any graph that satisfies (1) is said to have a *chromatic factorisation* with *chromatic factors* H_1 and H_2.

There are some known techniques for constructing pairs of chromatically equivalent clique-separable graphs. We will briefly consider some of these.

If non-isomorphic graphs, G and G', can both be obtained by different r-gluings of graphs H_1 and H_2, we say that G' is a *re-gluing* of G. Any re-gluing of a graph is chromatically equivalent, but not necessarily isomorphic, to the original graph. Hence the re-gluing operation gives a pair of chromatically equivalent clique-separable graphs. The smallest pair of chromatically equivalent clique-separable graphs consists of the two non-isomorphic trees of order four. One of these trees is P_4, the path on four vertices. In this case the second tree, the complete bipartite graph $K_{1,3}$, can be obtained by a re-gluing of P_4.

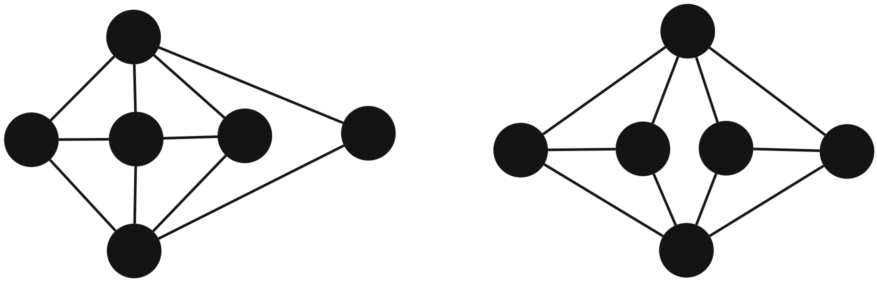
If G is an r-gluing of H_1 and H_2, and G' is the graph obtained by an r-gluing of K_s and H where H is an s-gluing of H_1 and H_2 and $0 \leq r \leq s$, then it is clear that G and G' are chromatically equivalent as

$$P(G',\lambda) = \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_s,\lambda)}\frac{P(K_s,\lambda)}{P(K_r,\lambda)} = \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_r,\lambda)} = P(G,\lambda).$$

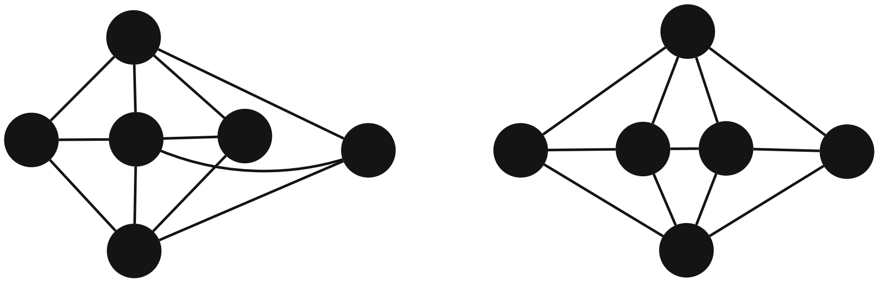
If G is a graph with a separating set $\{u_0, u_1\}$ where G can be obtained by identifying vertex u_i in graph H_1 with vertex u_i' in graph H_2, $i = 0, 1$, then the graph G' is said to be obtained by *twisting* G at vertices u_0 and u_1 if G' can be obtained by identifying vertex u_i in H_1 with vertex $u'_{i+1 \bmod 2}$ in H_2, $i = 0, 1$ [9, pp.148–149]. The graphs G and G' are said to be *2-isomorphic* [12]. Graphs that are 2-isomorphic are chromatically equivalent. In some cases the twisted graph is isomorphic to the original graph. However, in many cases twisting gives a pair of non-isomorphic, chromatically equivalent graphs. If G is twisted on an edge u_0u_1, we have a re-gluing and both G and G' are clique-separable graphs. If G is twisted at non-adjacent vertices u_0 and u_1, then the twisted graph is clique-separable if and only if G is clique-separable.

Pairs of non-clique-separable chromatically equivalent graphs that are not 2-isomorphic have been found. Examples include those given in [2], in [1, Figs. 4–9], in [3, Figs. 4–5] and in [10, Fig. 2]. The smallest pairs of chromatically equivalent non-clique-separable graphs are of order six (see Fig. 1).

Although there are many clique-separable graphs, such as trees, that are not chromatically equivalent to any non-clique-separable graph, pairs of chromatically equivalent graphs exist where one of the pair is clique-separable but the other is not. Read and Tutte [11] give an example of a non-clique-separable graph that is chromatically equivalent to a chordal graph. This pair of graphs has order 7 which is the smallest



(a)



(b)

Fig. 1 Pairs of chromatically equivalent non-clique-separable graphs. **a** Graphs with chromatic polynomial $\lambda(\lambda-1)(\lambda-2)^2(\lambda^2-5\lambda+8)$. **b** Graphs with chromatic polynomial $\lambda(\lambda-1)(\lambda-2)(\lambda^3-8\lambda^2+23\lambda-23)$

odd order for which a pair of chromatically equivalent graphs exist where one of the graphs is clique-separable and the other is not. The smallest even order for which a pair of chromatically equivalent graphs exist where one of the graphs is clique-separable and the other is not is order 6. A pair of graphs satisfying these properties and having order 6 is given in [5, p.61] (credited to Chee and Royle). The wheel with five spokes, W_5, is shown to be chromatically equivalent to the clique-separable graph X_5^* given in [4, Fig. 6] and the wheel W_7 is shown to be chromatically equivalent to the clique-separable graph in [13, Fig. 3]. However, unlike the cases where gluings and twistings can be used to obtain chromatically equivalent graphs that are both clique-separable or (sometimes in the case of twistings) both non-clique-separable, there appears to be no general technique that can be used to construct pairs of chromatically equivalent graphs where one graph is clique-separable and the other is not.

We show that if $H_1 \ncong K_s$, $s \geq 3$, is a graph containing at least two (not necessarily disjoint) cycles, one of length three, then there exists a non-clique-separable graph G' that is chromatically equivalent to a clique-separable graph G where both these graphs have H_1 as a chromatic factor. We give a construction of G' and show that it is chromatically equivalent to a 3-gluing of H_1 and some graph H_2. This construction gives us the means to find infinitely many pairs of chromatically equivalent graphs where one graph is clique-separable and the other is non-clique-separable.

In Sect. 2 we give an overview of our work on chromatic factors and its role in motivating this work on chromatic equivalence. In particular we present a certificate of factorisation from [6] which we will use in our proof of chromatic equivalence.

Section 3 then gives the construction of a non-clique-separable graph G' which is shown to be chromatically equivalent to a clique-separable graph G.

2 Certificates of Factorisation

A graph G is said to have a *chromatic factorisation* if the chromatic polynomial $P(G, \lambda)$ can be expressed as in (1). Any clique-separable graph or graph that is chromatically equivalent to a clique-separable graph has a chromatic factorisation. In [7,8] we showed that there exist graphs that have chromatic factorisations, but are not clique-separable nor chromatically equivalent to any clique-separable graph. We introduced the notion of a certificate of factorisation in [8] to provide an explanation for these chromatic factorisations. A certificate of factorisation is a sequence of expressions $P_0, P_1, \ldots, P_k$ where $P_0 = P(G, \lambda)$, $P_k = P(H_1, \lambda)P(H_2, \lambda)/P(K_r, \lambda)$ and each P_i, $i > 0$, can be obtained from P_{i-1} by applying basic properties of the chromatic polynomial or algebraic rules. The shortest certificate of factorisation is for clique-separable graphs where (1) is the certificate. If a graph G' is known to be chromatically equivalent to a graph G, then the following is a certificate of factorisation for G':

$$\begin{aligned} P(G', \lambda) &= P(G, \lambda) \\ &= \frac{P(H_1, \lambda)P(H_2, \lambda)}{P(K_r, \lambda)}. \end{aligned} \tag{2}$$

In [6] we proved that any triangle-free graph $H_1 \ncong K_3$ that has chromatic number at least 3 can be a chromatic factor in a chromatic factorisation where $r = 3$, that is,

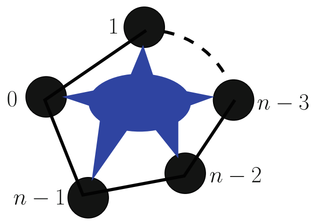
$$P(G, \lambda) = \frac{P(H_1, \lambda)P(H_2, \lambda)}{P(K_3, \lambda)}. \tag{3}$$

Our proof constructed a graph G by inserting copies of H_1 into the graph in Fig. 4a and gave the following certificate of factorisation to show that G has a chromatic factorisation with H_1 as a chromatic factor:

$$\begin{aligned} P(G, \lambda) &= P(G + (n-2, n-1), \lambda) + P(G/(n-2, n-1), \lambda) \\ &= \frac{P(H_1, \lambda)P(H_3, \lambda)}{P(K_2, \lambda)} + \frac{P(H_1/(n-2, n-1), \lambda)P(H_1, \lambda)}{P(K_1, \lambda)} \\ &= P(H_1, \lambda)\left(\frac{P(H_3, \lambda)}{P(K_2, \lambda)} + \frac{P(H_1/(n-2, n-1), \lambda)}{P(K_1, \lambda)}\right) \\ &= \frac{P(H_1, \lambda)}{P(K_3, \lambda)}\left(\frac{P(H_3, \lambda)P(K_3, \lambda)}{P(K_2, \lambda)} + \frac{P(H_1/(n-2, n-1), \lambda)P(K_3, \lambda)P(K_2, \lambda)}{P(K_1, \lambda)P(K_2, \lambda)}\right) \\ &= \frac{P(H_1, \lambda)}{P(K_3, \lambda)}\left(P(H_2 + (n-1, c_{n-2}), \lambda) + P(H_2/(n-1, c_{n-2}), \lambda)\right) \\ &= \frac{P(H_1, \lambda)P(H_2, \lambda)}{P(K_3, \lambda)} \end{aligned}$$

Certificate 1.

Fig. 2 Graph H_1



A natural question arising from this work is what happens if we construct G by inserting copies of H_1 where H_1 **has** a triangle. In this paper we use the same general method of construction as shown in [6]. However in this case H_1 has at least two cycles, one of which is a triangle. The graph obtained by this construction has a chromatic factorisation with chromatic factors H_1 and H_2 that satisfies Certificate 1. However, a much shorter certificate exists. As H_1 and H_2 both contain at least one triangle, we can construct a graph G' by identifying a triangle in H_1 with a triangle in H_2. As $G \sim G'$ a shorter certificate of factorisation is:

$$\begin{aligned} P(G,\lambda) &\sim P(G',\lambda) \\ &= \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_3,\lambda)}. \end{aligned} \tag{4}$$

We show that for any non-clique-separable graph, $H_1 \ncong K_s$, $s \geq 3$, that has at least two cycles one of which is a triangle, we can construct a non-clique-separable graph G that is chromatically equivalent to a clique-separable graph G'. These graphs have a chromatic factorisation with chromatic factors H_1 (see Fig. 2) and H_2 (see Fig. 3c).

3 Construction of Chromatically Equivalent Graphs

A graph A is said to be *inserted* into a graph B by *identifying* a cycle C_A in A with a cycle C_B in B under the bijection ϕ, if each vertex v in C_A is identified with $\phi(v)$ in C_B.

We denote the graph obtained by inserting A into B in this way, $I(A, B, C_A, C_B, \phi)$. The cycle with vertex set $\{v_0, v_1, \ldots, v_n\}$ and edge set $\{(v_i, v_{i+1}) : 0 \leq i \leq n-1\} \cup \{(v_0, v_n)\}$ is denoted $(v_0, v_1, \ldots, v_n)$.

Let $C = (0, \ldots, n-1)$ be a cycle of length ≥ 4 in H_1 where at least one triangle in H_1 does not contain the edge $(n-1, n-2)$, and let $C_{H'}$ be a cycle $(c_0, c_1, c_2, \ldots, c_{n-2}, n-1)$ in H' (see Fig. 3a). The graph H_2 can be obtained by inserting a copy of H_1 into the graph H' (see Fig. 3b) as follows: First we obtain $H_2' \cong I(H_1, H', C, C_{H'}, \phi)$ where $\phi(n-1) = n-1$ and $\phi(i) = c_i$ for all $0 \leq i \leq n-2$. The graph $H_2 \cong H_2' \backslash (c_{n-2}, n-1)$ (see Fig. 3c).

Theorem 1 *If $H_1 \ncong K_s$, $s \geq 3$, has a cycle $(0, \ldots, n-1)$, $n \geq 4$, and at least one triangle that does not contain the edge $(n-2, n-1)$ and H_2 is the graph obtained by inserting a copy of H_1 into H' and deleting the edge $(c_{n-2}, n-1)$ (see Fig. 3), then there exists a clique-separable graph G' with chromatic factors H_1 and H_2.*

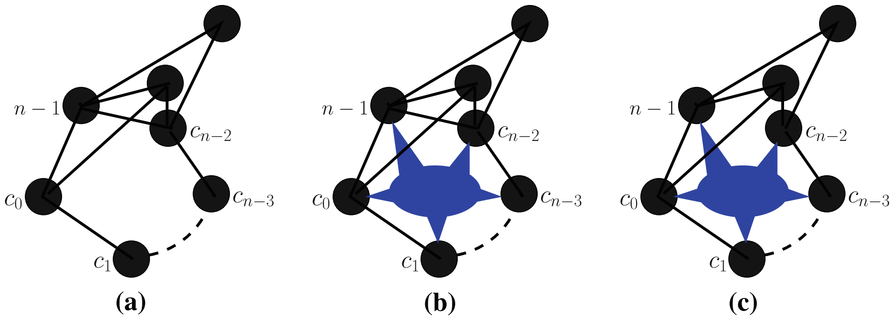


Fig. 3 The graph H_2' is obtained by inserting a copy of H_1 into H'. The graph $H_2 \cong H_2' \backslash (n-1, c_{n-2})$. **a** Graph H', **b** graph H_2', **c** graph H_2

Proof As $H_1 \backslash (n-2, n-1)$ contains at least one triangle, H_2 contains at least one triangle. Let G' be the graph obtained by identifying a triangle in H_1 with a triangle in H_2. Then G' is a clique-separable graph with

$$P(G', \lambda) = \frac{P(H_1, \lambda) P(H_2, \lambda)}{P(K_3, \lambda)}.$$

□

The following theorem uses our construction in [6] to obtain a non-clique-separable graph that is chromatically equivalent to G'.

Theorem 2 *If G' is a clique-separable graph obtained by overlaying the graphs H_1 (see Fig. 2) and H_2 (see Fig. 3c) on a 3-clique, and $H_1 \ncong K_s$, $s \geq 3$, is not clique-separable and contains at least two cycles, one of which is a triangle, then there exists a non-clique-separable graph $G \sim G'$.*

Proof Let G_0 be the graph in Fig. 4a, and let $G_1 = I(H_1, G_0, C, A, \iota)$ be the graph in Fig. 4b where $C = (0, 1, \ldots, n-1)$ is a cycle in H_1, the cycle $A = (0, 1, \ldots, n-1)$ is a cycle in G_0 and ι is the identify map. By inserting a copy of H_1 into G_1 we obtain $G_2 = I(H_1, G_1, C, B, \phi)$, where $B = (c_0, c_1, c_2, \ldots, c_{n-2}, n-1)$ is a cycle in G_2 and $\phi(n-1) = n-1$ and $\phi(i) = c_i$, $i \neq n-1$ (see Fig. 4c). We show that $G \cong G_2 \backslash (n-2, n-1)$ (see Fig. 4d) is chromatically equivalent, but not isomorphic, to G'. First we show that G and G' have the same chromatic factorisation, and thus have the same chromatic polynomial. We then show that G is not isomorphic to G'. Now G has a chromatic factorisation that satisfies Certificate 1 where H_3 is the graph in Fig. 5 and the chromatic factors H_1 and H_2 are the chromatic factors of G'. This certificate of factorisation for G is illustrated in Fig. 6. It is clear that G and G' have the same chromatic factorisation, and thus $G \sim G'$. If we can show that $G \ncong G'$, then we are done.

As G' is a clique-separable graph, it suffices to show that G is not clique-separable. Suppose, in order to obtain a contradiction, that G is a clique-separable graph.

Let J be the graph obtained replacing the edge $(n-2, n-1)$ in H_1 by the path $n-1, c_{n-2}, n-2$.

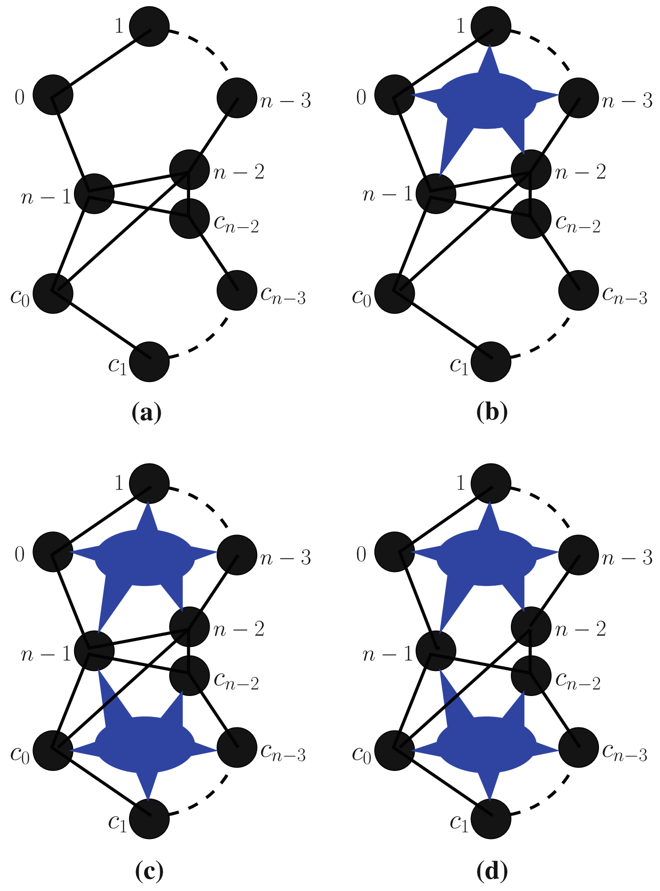
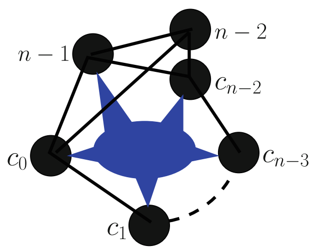


Fig. 4 Graph G_1 is obtained by inserting a copy of H_1 into G_0. Graph G_2 is obtained by inserting another copy of H_1 into G_2. Graph $G \cong G_2\backslash(n-2, n-1)$. **a** Graph G_0, **b** graph G_1, **c** graph G_2, **d** graph G

Fig. 5 Graph H_3



Now $G\backslash(c_0, n-2)$ is isomorphic to the graph obtained by identifying the vertices $n-1$ and c_{n-2} in J with vertices $n-1$ and $n-2$ in H_1 respectively. As H_1 is not clique-separable, J is not clique-separable. Furthermore, H_1 is not a complete graph. Thus the only separating clique in $G\backslash(c_0, n-2)$ is the edge $(n-1, c_{n-2})$. But this clique is not a separating clique in G. Thus the only possible clique separating G must

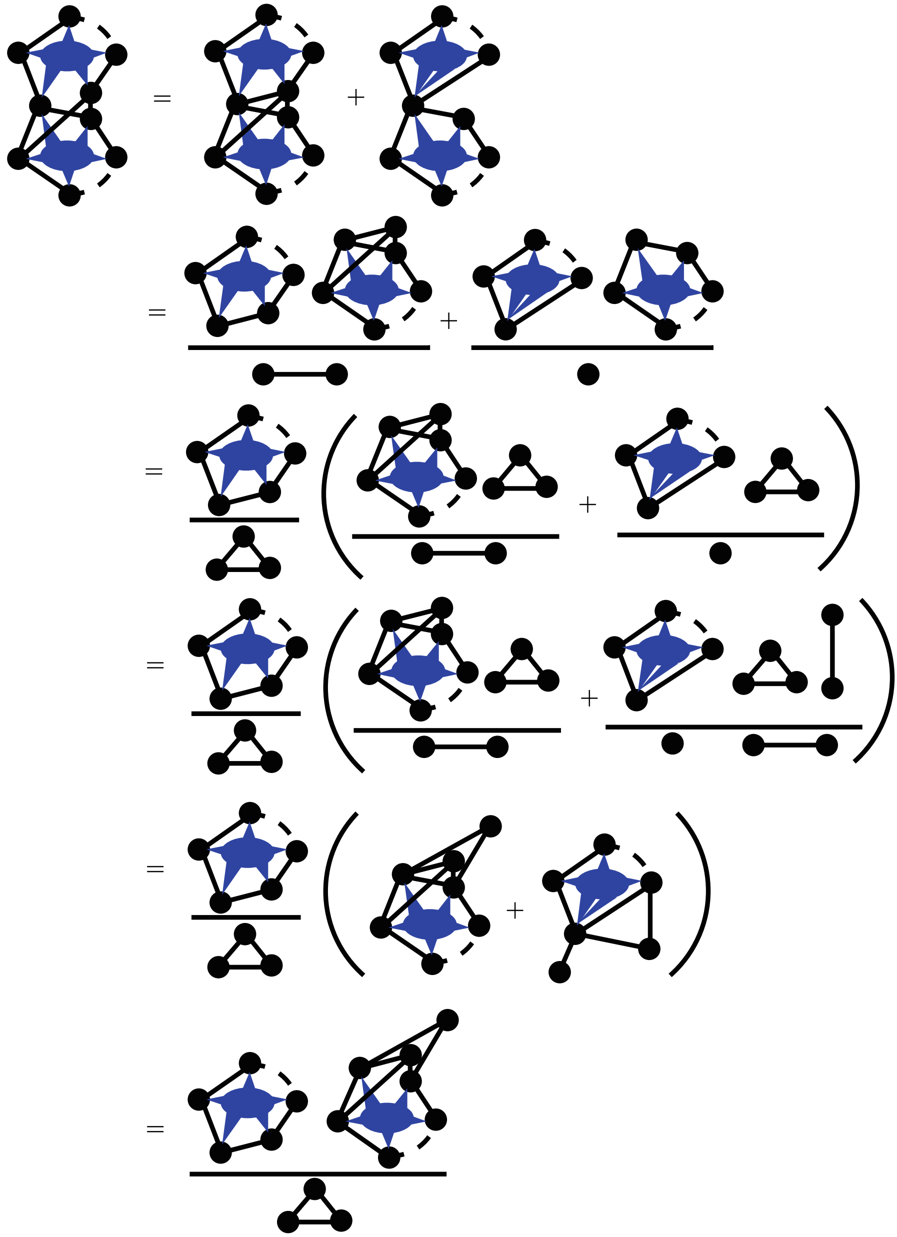


Fig. 6 Graph G has chromatic factorisation $P(H_1, \lambda)P(H_2, \lambda)/P(K_3, \lambda)$

contain the edge $(c_0, n-2)$. Now $n-2$ has at most one neighbour common with c_0, so there are two possible cliques containing this edge: the edge itself, or the 3-clique $(c_0, n-2, c_{n-2})$. It is clear that the removal of the edge $(c_0, n-2)$ does not separate the graph, as this edge is not a cut-edge. On the other hand, suppose G contains the

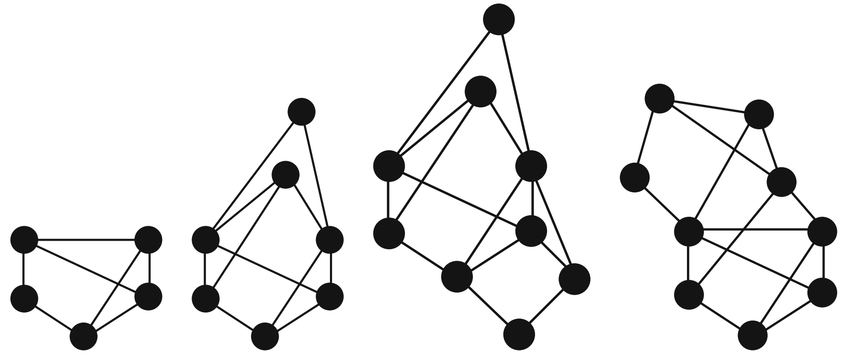


Fig. 7 Graph G is chromatically equivalent to G'. **a** Graph H_1, **b** graph H_2, **c** graph G', a 3-gluing of H_1 and H_2, **d** graph G

3-clique $(c_0, n-2, c_{n-2})$ that separates the graph. In this case vertex $n-1$ is only adjacent to the vertices 0, c_0 and c_{n-2}. But if this is the case, then the H_1 used in the construction of G has an edge, $(0, n-2)$ which separates the graph, a contradiction. □

An example of a pair of chromatically equivalent graphs obtained by our construction is given in Fig. 7. Graph G' is a 3-gluing of the graphs H_1 and H_2 where H_2 is obtained by insertion process illustrated in Fig. 3. Graph G is a non-clique-separable graph obtained by the insertion process illustrated in Fig. 4.

In cases where H_1 contains more than one cycle that can be used when inserting copies of H_1 to obtain G and H_2, then it may be possible to find sets of more than two chromatically equivalent graphs or more than one pair of chromatically equivalent graphs. If G and H_2 are the graphs obtained by inserting copies of H_1 on cycle C_1 and J and H_2' are the graphs by inserting copies of H_1 on cycle C_2, then if H_2 is chromatically equivalent but not isomorphic to H_2' we have a quartet of chromatically equivalent graphs. On the otherhand if $H_2 \not\sim H_2'$ we have two pairs of chromatically equivalent graphs, $\{G, G'\}$ and $\{J, J'\}$, where $G \not\sim J$.

4 Conclusion

In this article we give the means to construct pairs of chromatically equivalent graphs where one of the pair is clique-separable and the other is not. The author does not know of any other explicit methods for constructing such pairs. This method gives at least one pair of chromatically equivalent graphs for each graph that has at least two cycles, one of which is a triangle. Applying techniques such as re-gluing, twisting and insertion of copies of H_1 on different cycles may produce sets of graphs that are chromatically equivalent to a pair produced by this method.

Graphs constructed by our method have $2N-1$ vertices where N is the number of vertices in H_1. Thus our method gives pairs of chromatically equivalent graphs of odd order ≥ 9. It would be interesting to give a method to find pairs of chromatically equivalent graphs of even order where one of the pair is clique-separable and the other is non-clique-separable. Such pairs exist: a pair of order 6 is given in [5, p.61] (credited to Chee and Royle), another pair of order 6 is given in [4, Fig. 6] and a pair of order 8 is given in [13]. However, in general there is no known method to construct pairs of chromatically equivalent graphs of even order where one of the pair is clique-separable and the other is not.

An interesting problem is that of characterising non-clique-separable graphs that are chromatically equivalent to clique-separable graphs. It is an open problem even when restricted to characterising *quasi-chordal graphs*, that is, graphs that are chromatically equivalent to chordal graphs. The family of quasi-chordal graphs are of particular interest as they are precisely the graphs that have chromatic polynomials with only integer roots.

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