# On parsimonious edge-colouring of graphs with maximum degree three 

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#### Abstract

In a graph $G$ of maximum degree $\Delta$ let $\gamma$ denote the largest fraction of edges that can be $\Delta$ edge-coloured. Albertson and Haas showed that $\gamma \geq \frac{13}{15}$ when $G$ is cubic. We show here that this result can be extended to graphs with maximum degree 3 with the exception of a graph on 5 vertices. Moreover, there are exactly two graphs with maximum degree 3 (one being obviously the Petersen graph) for which $\gamma=\frac{13}{15}$. This extends a result given by Steffen. These results are obtained by using structural properties of the so called $\delta$-minimum edge colourings for graphs with maximum degree 3 .


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## 1 Introduction

Throughout this paper, we shall be concerned with connected graphs with maximum degree 3. We know by Vizing's theorem 15] that these graphs can be edge-coloured with 4 colours. Let $\phi: E(G) \rightarrow\{\alpha, \beta, \gamma, \delta\}$ be a proper edgecolouring of $G$. It is often of interest to try to use one colour (say $\delta$ ) as few as possible. When it is optimal following this constraint, we shall say that such a parsimonious edge-colouring $\phi$ is $\delta$-minimum. In [3] we gave without proof (in French, see [6] for a translation) results on $\delta$ - minimum edge-colourings of cubic graphs. Some of them have been obtained later and independently by Steffen [13, 14]. Some other results which were not stated formally in [4] are needed here. The purpose of Section 2 is to give those results as structural properties of $\delta$-minimum edge-colourings as well as others which will be useful in Section 3 ,

An edge colouring of $G$ using colours $\alpha, \beta, \gamma, \delta$ is said to be $\delta$-improper provided that adjacent edges having the same colours (if any) are coloured with $\delta$. It is clear that a proper edge colouring (and hence a $\delta$-minimum edge-colouring) of $G$ is a particular $\delta$-improper edge colouring. For a proper or $\delta$-improper edge colouring $\phi$ of $G$, it will be convenient to denote $E_{\phi}(x)(x \in\{\alpha, \beta, \gamma, \delta\})$ the set of edges coloured with $x$ by $\phi$. For $x, y \in\{\alpha, \beta, \gamma, \delta\}, x \neq y, \phi(x, y)$ is the partial subgraph of $G$ spanned by these two colours, that is $E_{\phi}(x) \cup E_{\phi}(y)$ (this subgraph being a union of paths and even cycles where the colours $x$ and $y$ alternate). Since any two $\delta$-minimum edge-colourings of $G$ have the same number
of edges coloured $\delta$ we shall denote by $s(G)$ this number (the colour number as defined by Steffen in [13]).

As usual, for any undirected graph $G$, we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges and we suppose that $|V(G)|=n$ and $|E(G)|=m$. Moreover, $V_{i}(G)$ denotes the set of vertices of $G$ of degree $i$, and when no confusion is possible we shall write $V_{i}$ instead of $V_{i}(G)$. A strong matching $C$ in a graph $G$ is a matching $C$ such that there is no edge of $E(G)$ connecting any two edges of $C$, or, equivalently, such that $C$ is the edge-set of the subgraph of $G$ induced on the vertex-set $V(C)$.

## 2 On $\delta$-minimum edge-colouring

The graph $G$ considered in this section will have maximum degree 3 .
Lemma 1 Let $\phi$ be a $\delta$-improper colouring of $G$ then there exists a proper colouring of $G \phi^{\prime}$ such that $E_{\phi^{\prime}}(\delta) \subseteq E_{\phi}(\delta)$

Proof Let $\phi$ be a $\delta$-improper edge colouring of $G$. If $\phi$ is a proper colouring, we are done. Hence, assume that $u v$ and $u w$ are coloured $\delta$. If $d(u)=2$ we can change the colour of $u v$ to $\alpha, \beta$ or $\gamma$ since $v$ is incident to at most two colours in this set.

If $d(u)=3$ assume that the third edge $u z$ incident to $u$ is also coloured $\delta$, then we can change the colour of $u v$ for the same reason as above.

If $u z$ is coloured with $\alpha, \beta$ or $\gamma$, then $v$ and $w$ are incident to the two remaining colours of the set $\{\alpha, \beta, \gamma\}$ otherwise one of the edges $u v, u w$ can be recoloured with the missing colour. W.l.o.g., consider that $u z$ is coloured $\alpha$ then $v$ and $w$ are incident to $\beta$ and $\gamma$. Since $u$ has degree 1 in $\phi(\alpha, \beta)$ let $P$ be the path of $\phi(\alpha, \beta)$ which ends on $u$. We can assume that $v$ or $w$ (say $v$ ) is not the other end vertex of $P$. Exchanging $\alpha$ and $\beta$ along $P$ does not change the colours incident to $v$. But now $u z$ is coloured $\alpha$ and we can change the colour of $u v$ with $\beta$.

In each case, we get hence a new $\delta$-improper edge colouring $\phi_{1}$ with $E_{\phi_{1}}(\delta) \subsetneq$ $E_{\phi}(\delta)$. Repeating this process leads us to construct a proper edge colouring of $G$ with $E_{\phi^{\prime}}(\delta) \subseteq E_{\phi}(\delta)$ as claimed.

Proposition 2 Let $v_{1}, v_{2}, \ldots, v_{k} \in V(G)$ be such that $G-\left\{v_{1}, v_{2}, \ldots, v_{k}\right\}$ is 3 -edge colourable. Then $s(G) \leq k$.

Proof Let us consider a 3 -edge colouring of $G-\left\{v_{1}, v_{2}, \ldots v_{k}\right\}$ with $\alpha, \beta$ and $\gamma$ and let us colour the edges incident to $v_{1}, v_{2}, \ldots, v_{k}$ with $\delta$. We get a $\delta$-improper edge colouring $\phi$ of $G$. Lemma 1 gives a proper colouring of $G \phi^{\prime}$ such that $E_{\phi^{\prime}}(\delta) \subseteq E_{\phi}(\delta)$. Hence $\phi^{\prime}$ has at most $k$ edges coloured with $\delta$ and $s(G) \leq k$.

Proposition 2 above has been obtained by Steffen [13] for cubic graphs.
Lemma 3 Let $\phi$ be a $\delta$-improper colouring of $G$ then $\left|E_{\phi}(\delta)\right| \geq s(G)$.

Proof Applying Lemma 1 let $\phi^{\prime}$ be a proper edge colouring of $G$ such that $E_{\phi^{\prime}}(\delta) \subseteq E_{\phi}(\delta)$. We clearly have $\left|E_{\phi}(\delta)\right| \geq\left|E_{\phi^{\prime}}(\delta)\right| \geq s(G)$.

Theorem 4 [6] Let $G$ be a graph of maximum degree 3 and $\phi$ be a $\delta$-minimum colouring of $G$. Then the following hold.

1. $E_{\phi}(\delta)=A_{\phi} \cup B_{\phi} \cup C_{\phi}$ where an edge e in $A_{\phi}$ ( $B_{\phi}, C_{\phi}$ respectively) belongs to a uniquely determined cycle $C_{A_{\phi}}(e)\left(C_{B_{\phi}}(e), C_{C_{\phi}}(e)\right.$ respectively) with precisely one edge coloured $\delta$ and the other edges being alternately coloured $\alpha$ and $\beta$ ( $\beta$ and $\gamma, \alpha$ and $\gamma$ respectively).
2. Each edge having exactly one vertex in common with some edge in $A_{\phi}$ ( $B_{\phi}, C_{\phi}$ respectively) is coloured $\gamma$ ( $\alpha, \beta$, respectively).
3. The multiset of colours of edges of $C_{A_{\phi}}(e)\left(C_{B_{\phi}}(e), C_{C_{\phi}}(e)\right.$ respectively $)$ can be permuted to obtain a (proper) $\delta$-minimum edge-colouring of $G$ in which the colour $\delta$ is moved from e to an arbitrarily prescribed edge.
4. No two consecutive vertices of $C_{A_{\phi}}(e)\left(C_{B_{\phi}}(e), C_{C_{\phi}}(e)\right.$ respectively) have degree 2.
5. The cycles from 1 that correspond to distinct edges of $E_{\phi}(\delta)$ are vertexdisjoint.
6. If the edges $e_{1}, e_{2}, e_{3} \in E_{\phi}(\delta)$ all belong to $A_{\phi}$ ( $B_{\phi}, C_{\phi}$ respectively), then the set $\left\{e_{1}, e_{2}, e_{3}\right\}$ induces in $G$ a subgraph with at most 4 edges.

Lemma 5 [4] Let $\phi$ be a $\delta$-minimum edge-colouring of $G$. For any edge $e=$ $u v \in E_{\phi}(\delta)$ with $d(v) \leq d(u)$ there is a colour $x \in\{\alpha, \beta, \gamma\}$ present at $v$ and $a$ colour $y \in\{\alpha, \beta, \gamma\}-\{x\}$ present at $u$ such that one of connected components of $\phi(x, y)$ is a path of even length joining the two ends of $e$. Moreover, if $d(v)$ $=2$, then both colours of $\{\alpha, \beta, \gamma\}-\{x\}$ satisfy the above assertion.

An edge of $E_{\phi}(\delta)$ is in $A_{\phi}$ when its ends can be connected by a path of $\phi(\alpha, \beta), B_{\phi}$ by a path of $\phi(\beta, \gamma)$ and $C_{\phi}$ by a path of $\phi(\alpha, \gamma)$. From Lemma 5 it is clear that if $d(u)=3$ and $d(v)=2$ for an edge $e=u v \in E_{\phi}(\delta)$, the $A_{\phi}, B_{\phi}$ and $C_{\phi}$ are not pairwise disjoint; indeed, if the colour $\gamma$ is present at the vertex $v$, then $e \in A_{\phi} \cap B_{\phi}$.

When $e \in A_{\phi}$ we can associate to $e$ the odd cycle $C_{A_{\phi}}(e)$ obtained by considering the path of $\phi(\alpha, \beta)$ together with $e$. We define in the same way $C_{B_{\phi}}(e)$ and $C_{C_{\phi}}(e)$ when $e$ is in $B_{\phi}$ or $C_{\phi}$.

For each edge $e \in E_{\phi}(\delta)$ (where $\phi$ is a $\delta$-minimum edge-colouring of $G$ ) we can associate one or two odd cycles following the fact that $e$ is in one or two sets among $A_{\phi}, B_{\phi}$ or $C_{\phi}$. Let $\mathcal{C}$ be the set of odd cycles associated to edges in $E_{\phi}(\delta)$.

By Theorem 4 any two cycles in $\mathcal{C}$ corresponding to edges in distinct sets $A_{\phi}, B_{\phi}$ or $C_{\phi}$ are at distance at least 2. Assume that $C_{1}=C_{A_{\phi}}\left(e_{1}\right)$ and $C_{2}=C_{A_{\phi}}\left(e_{2}\right)$ for some edges $e_{1}$ and $e_{2}$ in $A_{\phi}$. Can we say something about the subgraph of $G$ whose vertex set is $V\left(C_{1}\right) \cup V\left(C_{2}\right)$ ? In general, we have no answer to this problem. However, when $G$ is cubic and any vertex of $G$
lies on some cycle of $\mathcal{C}$ (we shall say that $\mathcal{C}$ is spanning), we have a property which will be useful later. Let us remark first that whenever $\mathcal{C}$ is spanning, we can consider that $G$ is edge-coloured in such a way that the edges of the cycles of $\mathcal{C}$ are alternatively coloured with $\alpha$ and $\beta$ (except one edge coloured $\delta$ ) and the remaining perfect matching is coloured with $\gamma$. For this $\delta$-minimum edge-colouring of $G$ we have $B_{\phi}=\emptyset$ as well as $C_{\phi}=\emptyset$.

Lemma 6 Assume that $G$ is cubic and $\mathcal{C}$ is spanning. Let $e_{1}, e_{2} \in A_{\phi}$ and let $C_{1}, C_{2} \in \mathcal{C}$ such that $C_{1}=C_{A_{\phi}}\left(e_{1}\right)$ and $C_{2}=C_{A_{\phi}}\left(e_{2}\right)$. Then at least one of the following is true:
(i) $C_{1}$ and $C_{2}$ are at distance at least 2.
(ii) $C_{1}$ and $C_{2}$ are joined by at least 3 edges.
(iii) $C_{1}$ and $C_{2}$ have at least two chords each.

Proof Since $e_{1}, e_{2} \in A_{\phi}$ and $\mathcal{C}$ is spanning we have $B_{\phi}=C_{\phi}=\emptyset$. Let $C_{1}=v_{0} v_{1} \ldots v_{2 k_{1}}$ and $C_{2}=w_{0} w_{1} \ldots w_{2 k_{2}}$. Assume that $C_{1}$ and $C_{2}$ are joined by the edge $v_{0} w_{0}$. By Theorem4, up to a re-colouring of the edges in $C_{1}$ and $C_{2}$, we can consider a $\delta$-minimum edge-colouring $\phi$ such that $\phi\left(v_{0} v_{1}\right)=\phi\left(w_{0} w_{1}\right)=\delta$, $\phi\left(v_{1} v_{2}\right)=\phi\left(w_{1} w_{2}\right)=\beta$ and $\phi\left(v_{0} v_{2 k_{1}}\right)=\phi\left(w_{0} w_{2 k_{2}}\right)=\alpha$. Moreover each edge of $G$ (in particular $v_{0} w_{0}$ ) incident with these cycles is coloured $\gamma$. We can change the colour of $v_{0} w_{0}$ in $\beta$. We obtain thus a new $\delta$-minimum edge-colouring $\phi^{\prime}$. Performing that exchange of colours on $v_{0} w_{0}$ transforms the edges coloured $\delta$ $v_{0} v_{1}$ and $w_{0} w_{1}$ in two edges of $C_{\phi^{\prime}}$ lying on odd cycles $C_{1}^{\prime}$ and $C_{2}^{\prime}$ respectively. We get hence a new set $\mathcal{C}^{\prime}=\mathcal{C}-\left\{C_{1}, C_{2}\right\} \cup\left\{C_{1}^{\prime}, C_{2}^{\prime}\right\}$ of odd cycles associated to $\delta$-coloured edges in $\phi^{\prime}$.

From Theorem $4 C_{1}^{\prime}$ ( $C_{2}^{\prime}$ respectively) is at distance at least 2 from any cycle in $\mathcal{C}-\left\{C_{1}, C_{2}\right\}$. Hence $V\left(C_{1}^{\prime}\right) \cup V\left(C_{2}^{\prime}\right) \subseteq V\left(C_{1}\right) \cup V\left(C_{2}\right)$. It is an easy task to check now that (ii) or (iii) above must be verified.

Lemma 7 [4] Let $e_{1}=u v_{1}$ be an edge of $E_{\phi}(\delta)$ such that $v_{1}$ has degree 2 in $G$. Then $v_{1}$ is the only vertex in $N(u)$ of degree 2 and for any other edge $e_{2} \in E_{\phi}(\delta)$, $\left\{e_{1}, e_{2}\right\}$ induces a $2 K_{2}$.

## 3 Applications

### 3.1 On a result by Payan

In 10 Payan showed that it is always possible to edge-colour a graph of maximum degree 3 with three maximal matchings (with respect to the inclusion) and introduced henceforth a notion of strong-edge colouring where a strong edge-colouring means that one colour is a strong matching while the remaining colours are usual matchings. Payan conjectured that any $d$-regular graph has $d$ pairwise disjoint maximal matchings and showed that this conjecture holds true for graphs with maximum degree 3 .

The following result has been obtained first by Payan [10], but his technique does not exhibit explicitly the odd cycles associated to the edges of the strong matching and their properties.

Theorem 8 Let $G$ be a graph with maximum degree at most 3. Then $G$ has a $\delta$-minimum edge-colouring $\phi$ where $E_{\phi}(\delta)$ is a strong matching and, moreover, any edge in $E_{\phi}(\delta)$ has its two ends of degree 3 in $G$.

Proof Let $\phi$ be a $\delta$-minimum edge-colouring of $G$. From Theorem 4 any two edges of $E_{\phi}(\delta)$ belonging to distinct sets from among $A_{\phi}, B_{\phi}$ and $C_{\phi}$ are at least at distance 2 and thus induce a strong matching. Hence, we have to find a $\delta$-minimum edge-colouring where each set $A_{\phi}, B_{\phi}$ or $C_{\phi}$ induces a strong matching (with the supplementary property that the end vertices of these edges have degree 3). That means that we can work on each set $A_{\phi}, B_{\phi}$ and $C_{\phi}$ independently. Without loss of generality, we only consider $A_{\phi}$ here.

Assume that $A_{\phi}=\left\{e_{1}, e_{2}, \ldots e_{k}\right\}$ and $A_{\phi}^{\prime}=\left\{e_{1}, \ldots e_{i}\right\}(1 \leq i \leq k-1)$ is a strong matching and each edge of $A_{\phi}^{\prime}$ has its two ends with degree 3 in $G$. Consider the edge $e_{i+1}$ and let $C=C_{e_{i+1}}(\phi)=u_{0}, u_{1} \ldots u_{2 p}$ be the odd cycle associated to this edge (Theorem (4)).

Let us mark any vertex $v$ of degree 3 on $C$ with a + whenever the edge of colour $\gamma$ incident to this vertex has its other end which is a vertex incident to an edge of $A_{\phi}^{\prime}$ and let us mark $v$ with - otherwise. By Theorem 4, no consecutive vertices on $C$ have degree 2 , that means that a vertex of degree 2 on $C$ has its two neighbours of degree 3 and by Lemma 7 these two vertices are marked with a - . By Theorem 4 we cannot have two consecutive vertices marked with a + , otherwise we would have three edges of $E_{\phi}(\delta)$ inducing a subgraph with more than 4 edges, a contradiction. Hence, $C$ must have two consecutive vertices of degree 3 marked with - whatever is the number of vertices of degree 2 on $C$.

Let $u_{j}$ and $u_{j+1}$ be two vertices of $C$ of degree 3 marked with $-(j$ being taken modulo $2 p+1$ ). We can transform the edge colouring $\phi$ by exchanging colours on $C$ uniquely, in such a way that the edge of colour $\delta$ of this cycle is $u_{j} u_{j+1}$. In the resulting edge colouring $\phi_{1}$ we have $A_{\phi_{1}}=A_{\phi}-\left\{e_{i+1}\right\} \cup\left\{u_{j} u_{j+1}\right\}$ and $A_{\phi_{1}}^{\prime}=A_{\phi}^{\prime} \cup\left\{u_{j} u_{j+1}\right\}$ is a strong matching where each edge has its two ends of degree 3. Repeating this process we are left with a new $\delta$-minimum colouring $\phi^{\prime}$ where $A_{\phi^{\prime}}$ is a strong matching.

Corollary 9 Let $G$ be a graph with maximum degree 3 then there are $s(G)$ vertices of degree 3 pairwise non-adjacent $v_{1} \ldots v_{s(G)}$ such that $G-\left\{v_{1} \ldots v_{s(G)}\right\}$ is 3 -colourable.

Proof Pick a vertex on each edge coloured $\delta$ in a $\delta$-minimum colouring $\phi$ of $G$ where $E_{\phi}(\delta)$ is a strong matching (Theorem 8). We get a subset $S$ of vertices satisfying our corollary.

Steffen [13] obtained Corollary 9 for bridgeless cubic graphs.

### 3.2 Parsimonious edge colouring

Let $\chi^{\prime}(G)$ be the classical chromatic index of $G$. For convenience let

$$
\begin{gathered}
c(G)=\max \left\{|E(H)|: H \subseteq G, \chi^{\prime}(H)=3\right\} \\
\gamma(G)=\frac{c(G)}{|E(G)|}
\end{gathered}
$$

Staton 12 (and independently Locke (9) showed that whenever $G$ is a cubic graph distinct from $K_{4}$ then $G$ contains a bipartite subgraph (and hence a 3edge colourable graph, by König's theorem [8]) with at least $\frac{7}{9}$ of the edges of $G$. Bondy and Locke [2] obtained $\frac{4}{5}$ when considering graphs with maximum degree at most 3.

In [1] Albertson and Haas showed that whenever $G$ is a cubic graph, we have $\gamma(G) \geq \frac{13}{15}$ while for graphs with maximum degree 3 they obtained $\gamma(G) \geq \frac{26}{31}$. Our purpose here is to show that $\frac{13}{15}$ is a lower bound for $\gamma(G)$ when $G$ has maximum degree 3, with the exception of the graph $G_{5}$ depicted in Figure 1 below.


Figure 1: $G_{5}$

Lemma 10 Let $G$ be a graph with maximum degree 3 then $\gamma(G)=1-\frac{s(G)}{m}$.
Proof Let $\phi$ be a $\delta$-minimum edge-colouring of $G$. The restriction of $\phi$ to $E(G)-E_{\phi}(\delta)$ is a proper 3-edge-colouring, hence $c(G) \geq m-s(G)$ and $\gamma(G) \geq 1-\frac{s(G)}{m}$.

If $H$ is a subgraph of $G$ with $\chi(H)=3$, consider a proper 3-edge-colouring $\phi: E(H) \rightarrow\{\alpha, \beta, \gamma\}$ and let $\psi: E(G) \rightarrow\{\alpha, \beta, \gamma, \delta\}$ be the continuation of $\phi$ with $\psi(e)=\delta$ for $e \in E(G)-E(H)$. By Lemma 1 there is a proper edgecolouring $\psi^{\prime}$ of $G$ with $E_{\psi^{\prime}}(\delta) \subseteq E_{\psi}(\delta)$ so that $|E(H)|=\left|E(G)-E_{\psi}(\delta)\right| \leq$ $\left|E(G)-E_{\psi^{\prime}}(\delta)\right| \leq m-s(G), c(G) \leq m-s(G)$ and $\gamma(G) \leq 1-\frac{s(G)}{m}$.

In [11], Rizzi shows that for triangle-free graphs of maximum degree 3, $\gamma(G) \geq 1-\frac{2}{3 g_{o}(G)}$ (where the odd girth of a graph $G$, denoted by $g_{o}(G)$, is the minimum length of an odd cycle in $G$ ).

Theorem 11 Let $G$ be a graph with maximum degree 3 then $\gamma(G) \geq 1-\frac{2}{3 g_{o}(G)}$.
Proof Let $\phi$ be a $\delta$-minimum edge-colouring of $G$ and $E_{\phi}(\delta)=\left\{e_{1}, e_{2} \ldots, e_{s(G)}\right\}$. $\mathcal{C}$ being the set of odd cycles associated to edges in $E_{\phi}(\delta)$, we write $\mathcal{C}=$ $\left\{C_{1}, C_{2} \ldots, C_{s(G)}\right\}$ and suppose that for $i=1,2 \ldots, s(G), e_{i}$ is an edge of $C_{i}$. We know by Theorem 4 that the cycles of $\mathcal{C}$ are vertex-disjoint.

Let us write $\mathcal{C}=\mathcal{C}_{2} \cup \mathcal{C}_{3}$, where $\mathcal{C}_{2}$ denotes the set of odd cycles of $\mathcal{C}$ which have a vertex of degree 2 , while $\mathcal{C}_{3}$ is for the set of cycles in $\mathcal{C}$ whose all vertices have degree 3 . Let $k=\left|\mathcal{C}_{2}\right|$, obviously we have $0 \leq k \leq s(G)$ and $\mathcal{C}_{2} \cap \mathcal{C}_{3}=\emptyset$.

If $C_{i} \in \mathcal{C}_{2}$ we suppose without loss of generality that $C_{i} \in A_{\phi}$ and we have $\left|C_{i}\right| \geq g_{o}(G)$. Moreover, since any edge in $C_{i}$ can be coloured $\delta$ (Theorem (4),
we may assume that $e_{i}$ has a vertex of degree 2 . We can associate to $e_{i}$ another odd cycle say $C_{i}^{\prime} \in B_{\phi}$ (Lemma (5) whose edges distinct from $e_{i}$ form an even path of $\phi(\alpha, \gamma)$ using at least $\frac{g_{o}(G)-1}{2}$ edges, coloured $\gamma$, which are not edges of $C_{i}$.

When $\left|C_{i}\right|>g_{o}(G)$ or $\left|C_{i}^{\prime}\right|>g_{o}(G)$ there are either at least $g_{o}(G)+2$ edges in $C_{i}$ or at least $\frac{g_{o}(G)-1}{2}+1$ edges coloured $\gamma$ in $C_{i}^{\prime}$. If $\left|C_{i}\right|=\left|C_{i}^{\prime}\right|=g_{o}(G)$ there is at least one edge coloured $\alpha$ in $C_{i}^{\prime}$ that is not an edge of $C_{i}$, otherwise all the edges coloured $\gamma$ of $C_{i}^{\prime}$ would be chords of $C_{i}$, a contradiction since a such chord would form with vertices of $C_{i}$ an odd cycle of length smaller than $g_{o}(G)$.

Hence, $C_{i} \cup C_{i}^{\prime}$ contains at least $g_{o}(G)+\frac{g_{o}(G)-1}{2}+1>\frac{3}{2} g_{o}(G)$ edges.
Consequently there are at least $\frac{3}{2} \times k \times g_{o}(G)$ edges in $\bigcup_{C_{i} \in \mathcal{C}_{2}}\left(C_{i} \cup C_{i}^{\prime}\right)$.
When $C_{i} \in \mathcal{C}_{3}, C_{i}$ contains at least $g_{o}(G)$ edges, moreover, each vertex of $C_{i}$ being of degree 3, there are at least $\frac{s(G)-k}{2} \times g_{o}(G)$ additionnal edges which are incident to a vertex of $\bigcup_{C_{i} \in \mathcal{C}_{3}} C_{i}$.

Since $C_{i} \cap C_{j}=\emptyset$ and $C_{i}^{\prime} \cap C_{j}=\emptyset(1 \leq i, j \leq s(G), i \neq j)$, we have
$m \geq \frac{3}{2} g_{o}(G) \times k+(s(G)-k) \times g_{o}(G)+\frac{s(G)-k}{2} \times g_{o}(G)=\frac{3}{2} \times s(G) \times g_{o}(G)$.
Consequently $\gamma(G)=1-\frac{s(G)}{m} \geq 1-\frac{2}{3 g_{o}(G)}$.
As a matter of fact, $\gamma(G)>1-\frac{2}{3 g_{o}(G)}$ when the graph $G$ contains vertices of degree 2. In a further work (see [5]) we refine the bound and prove that $\gamma(G) \geq 1-\frac{2}{3 g_{o}(G)+2}$ when $G$ is a graph of maximal degree 3 distinct from the Petersen graph.

Lemma 12 [1] Let $G$ be a graph with maximum degree 3. Assume that $v \in$ $V(G)$ is such that $d(v)=1$ then $\gamma(G)>\gamma(G-v)$.

A triangle $T=\{a, b, c\}$ is said to be reducible whenever its neighbours are distinct. When $T$ is a reducible triangle in $G$ ( $G$ having maximum degree 3 ) we can obtain a new graph $G^{\prime}$ with maximum degree 3 by shrinking this triangle into a single vertex and joining this new vertex to the neighbours of $T$ in $G$.

Lemma 13 [1] Let $G$ be a graph with maximum degree 3. Assume that $T=$ $\{a, b, c\}$ is a reducible triangle and let $G^{\prime}$ be the graph obtained by reduction of this triangle. Then $\gamma(G)>\gamma\left(G^{\prime}\right)$.

Theorem 14 Let $G$ be a graph with maximum degree 3. If $G \neq G_{5}$ then $\gamma(G) \geq$ $1-\frac{\frac{2}{15}}{1+\frac{2}{3} \frac{\| V_{2}}{\left|V_{3}\right|}}$.
Proof From Lemma 12 and Lemma 13 we can consider that $G$ has only vertices of degree 2 or 3 and that $G$ contains no reducible triangle.

Assume that we can associate a set $P_{e}$ of at least 5 distinct vertices of $V_{3}$ for each edge $e \in E_{\phi}(\delta)$ in a $\delta$-minimum edge-colouring $\phi$ of $G$. Assume moreover that

$$
\begin{equation*}
\forall e, e^{\prime} \in E_{\phi}(\delta) \quad P_{e} \cap P_{e^{\prime}}=\emptyset \tag{1}
\end{equation*}
$$

Then

$$
\gamma(G)=1-\frac{s(G)}{m}=1-\frac{s(G)}{\frac{3}{2}\left|V_{3}\right|+\left|V_{2}\right|} \geq 1-\frac{\frac{\left|V_{3}\right|}{5}}{\frac{3}{2}\left|V_{3}\right|+\left|V_{2}\right|}
$$

Hence

$$
\gamma(G) \geq 1-\frac{\frac{2}{15}}{1+\frac{2}{3} \frac{\left|V_{2}\right|}{\left|V_{3}\right|}}
$$

It remains to see how to construct the sets $P_{e}$ satisfying Property (11). Let $\mathcal{C}$ be the set of odd cycles associated to edges in $E_{\phi}(\delta)$. Let $e \in E_{\phi}(\delta)$, assume that $e$ is contained in a cycle $C \in \mathcal{C}$ of length 3. By Theorem 4 the edges incident to that triangle have the same colour in $\{\alpha, \beta, \gamma\}$. This triangle is hence reducible, impossible. We can thus consider that each cycle of $\mathcal{C}$ has length at least 5 . By Lemma 7 we know that whenever such a cycle contains vertices of $V_{2}$, their distance on this cycle is at least 3 . Which means that every cycle $C \in \mathcal{C}$ contains at least 5 vertices in $V_{3}$ as soon as $C$ has length at least 7 or $C$ has length 5 but does not contain a vertex of $V_{2}$. For each edge $e \in E_{\phi}(\delta)$ contained in such a cycle we associate $P_{e}$ as any set of 5 vertices of $V_{3}$ contained in the cycle.

There may exist edges in $E_{\phi}(\delta)$ contained in a 5 -cycle of $\mathcal{C}$ having exactly one vertex in $V_{2}$. Let $C=a_{1} a_{2} a_{3} a_{4} a_{5}$ be such a cycle and assume that $a_{1} \in V_{2}$. By Lemma 7 $a_{1}$ is the only vertex of degree 2 and by exchanging colours along this cycle, we can suppose that $a_{1} a_{2} \in E_{\phi}(\delta)$. Since $a_{1} \in V_{2}, e=a_{1} a_{2}$ is contained in a second cycle $C^{\prime}$ of $\mathcal{C}$ (see Remark ??). If $C^{\prime}$ contains a vertex $x \in V_{3}$ distinct from $a_{2}, a_{3}, a_{4}$ and $a_{5}$ then we set $P_{e}=\left\{a_{2}, a_{3}, a_{4}, a_{5}, x\right\}$. Otherwise $C^{\prime}=a_{1} a_{2} a_{4} a_{3} a_{5}$ and $G$ is isomorphic to $G_{5}$, impossible.

The sets $\left\{P_{e} \mid e \in E_{\phi}(\delta)\right\}$ are pairwise disjoint since any two cycles of $\mathcal{C}$ associated to distinct edges in $E_{\phi}(\delta)$ are disjoint. Hence Property 1 holds and the proof is complete.

Albertson and Haas [1] proved that $\gamma(G) \geq \frac{26}{31}$ when $G$ is a graph with maximum degree 3 and Rizzi [11] obtained $\gamma(G) \geq \frac{6}{7}$. From Theorem [14] we get immediately for all graphs $G \neq G_{5} \gamma(G) \geq \frac{13}{15}$, a better bound. Let us remark that we get also $\gamma(G) \geq \frac{13}{15}$ by Theorem 11 as soon as $g_{o}(G) \geq 5$.

Lemma 15 Let $G$ be a cubic graph which can be factored into $s(G)$ cycles of length 5 and has no reducible triangle. Then every 2 -factor of $G$ contains $s(G)$ cycles of length 5 .

Proof Since $G$ has no reducible triangle, all cycles in a 2-factor have length at least 4 . Let $\mathcal{C}$ be any 2 -factor of $G$. Let us denote $n_{4}$ the number of cycles of length $4, n_{5}$ the number of cycles of length 5 and $n_{6+}$ the number of cycles on at least 6 vertices in $\mathcal{C}$. We have $5 n_{5}+6 n_{6+} \leq 5 s(G)-4 n_{4}$.

If $n_{4}+n_{6_{+}}=0$, then $n_{5}=s(G)$. If $n_{4}+n_{6_{+}}>0$, then the number of odd cycles in $\mathcal{C}$ is at most $n_{5}+n_{6_{+}} \leq \frac{5 s(G)-4 n_{4}-n_{6_{+}}}{5}=\frac{5 s(G)-\left(n_{4}+n_{6_{+}}\right)-3 n_{4}}{5}<s(G)$. A contradiction since a 2 -factor of $G$ contains at least $s(G)$ odd cycles.

Corollary 16 Let $G$ be a graph with maximum degree 3 such that $\gamma(G)=\frac{13}{15}$. Then $G$ is a cubic graph which can be factored into $s(G)$ cycles of length 5 . Moreover every 2 -factor of $G$ has this property.

Proof The optimum for $\gamma(G)$ in Theorem[14 is obtained whenever $s(G)=\frac{\left|V_{3}\right|}{5}$ and $\left|V_{2}\right|=0$. That is, $G$ is a cubic graph admitting a 2-factor of $s(G)$ cycles of length 5 . Moreover by Lemma $13 G$ has no reducible triangle, the result comes from Lemma 15

As pointed out by Albertson and Haas [1], the Petersen graph with $\gamma(G)=$ $\frac{13}{15}$ supplies an extremal example for cubic graphs. Steffen 14 proved that the only cubic bridgeless graph with $\gamma(G)=\frac{13}{15}$ is the Petersen graph. In fact, we can extend this result to graphs with maximum degree 3 where bridges are allowed (excluding the graph $G_{5}$ ). Let $P^{\prime}$ be the cubic graph on 10 vertices obtained from two copies of $G_{5}$ (Figure 1) by joining by an edge the two vertices of degree 2 .

Theorem 17 Let $G$ be a connected graph with maximum degree 3 such that $\gamma(G)=\frac{13}{15}$. Then $G$ is isomorphic to the Petersen graph or to $P^{\prime}$.
Proof Let $G$ be a graph with maximum degree 3 such that $\gamma(G)=\frac{13}{15}$.
From Corollary 16, we can consider that $G$ is cubic and $G$ has a 2 -factor of cycles of length 5 . Let $\mathcal{C}=\left\{C_{1} \ldots C_{s(G)}\right\}$ be such a 2 -factor ( $\mathcal{C}$ is spanning). Let $\phi$ be a $\delta$-minimum edge-colouring of $G$ induced by this 2 -factor.

Without loss of generality consider two cycles in $\mathcal{C}$, namely $C_{1}$ and $C_{2}$, and let us denote $C_{1}=v_{1} v_{2} v_{3} v_{4} v_{5}$ while $C_{2}=u_{1} u_{2} u_{3} u_{4} u_{5}$ and assume that $v_{1} u_{1} \in G$. From Lemma 6. $C_{1}$ and $C_{2}$ are joined by at least 3 edges or each of them has two chords. If $s(G)>2$ there is a cycle $C_{3} \in \mathcal{C}$. Without loss of generality, $G$ being connected, we can suppose that $C_{3}$ is joined to $C_{1}$ by an edge. Applying one more time Lemma 6, $C_{1}$ and $C_{3}$ have two chords or are joined by at least 3 edges, contradiction with the constraints imposed by $C_{1}$ and $C_{2}$. Hence $s(G)=2$ and $G$ has 10 vertices and no 4 -cycle, which leads to a graph isomorphic to $P^{\prime}$ or the Petersen graph as claimed.

We can construct cubic graphs with chromatic index 4 (snarks in the litterature) which are cyclically 4 - edge connected and having a 2-factor of $C_{5}$ 's.

Indeed, let $G$ be a cubic cyclically 4 -edge connected graph of order $n$ and $M$ be a perfect matching of $G, M=\left\{x_{i} y_{i} \left\lvert\, i=1 \ldots \frac{n}{2}\right.\right\}$. Let $P_{1} \ldots P_{\frac{n}{2}}$ be $\frac{n}{2}$ copies of the Petersen graph. For each $P_{i}\left(i=1 \ldots \frac{n}{2}\right)$ we consider two edges at distance 1 apart $e_{i}^{1}$ and $e_{i}^{2}$. Let us observe that $P_{i}-\left\{e_{i}^{1}, e_{i}^{2}\right\}$ contains a 2-factor of two $C_{5}$ 's $\left(C_{i}^{1}\right.$ and $\left.C_{i}^{2}\right)$.

We construct then a new cyclically 4 -edge connected cubic graph $H$ with chromatic index 4 by applying the well known operation dot-product (see Figure 2 see Isaacs [7] for a description and for a formal definition) on $\left\{e_{i}^{1}, e_{i}^{2}\right\}$ and the edge $x_{i} y_{i}\left(i=1 \ldots \frac{n}{2}\right)$. We remark that the vertices of $G$ vanish in the operation and the resulting graph $H$ has a 2 factor of $C_{5}$, namely $\left\{C_{1}^{1}, C_{1}^{2}, \ldots C_{i}^{1}, C_{i}^{2}, \ldots C_{\frac{n}{2}}^{1}, C_{\frac{n}{2}}^{2}\right\}$.

We do not know an example of a cyclically 5 -edge connected snark (except the Petersen graph) with a 2 -factor of induced cycles of length 5 .

Problem 18 Is there any cyclically 5-edge connected snark distinct from the Petersen graph with a 2 -factor of $C_{5}$ 's ?


Figure 2: The dot product operation on graphs $G_{1}, G_{2}$.

As a first step towards the resolution of this Problem we propose the following Theorem. Recall that a permutation graph is a cubic graph having some 2 -factor with precisely 2 odd cycles.

Theorem 19 Let $G$ be a cubic graph which can be factored into $s(G)$ induced odd cycles of length at least 5 , then $G$ is a permutation graph. Moreover, if $G$ has girth 5 then $G$ is the Petersen graph.

Proof Let $\mathcal{F}$ be a 2 -factor of $s(G)$ cycles of length at least 5 in $G$, every cycle of $\mathcal{F}$ being an induced odd cycle of $G$. We consider the $\delta$-minimum edge-colouring $\phi$ such that the edges of all cycles of $\mathcal{F}$ are alternatively coloured $\alpha$ and $\beta$ except for exactly one edge per cycle which is coloured with $\delta$, all the remaining edges of $G$ being coloured $\gamma$. By construction we have $B_{\phi}=C_{\phi}=\emptyset$ and $A_{\phi}=\mathcal{F}$.

Let $x y$ be an edge connecting two distinct cycles of $\mathcal{F}$, say $C_{1}$ and $C_{2}\left(x \in C_{1}\right.$, $y \in C_{2}$ ). By Theorem 4, since any edge in $C_{1}$ or $C_{2}$ can be coloured $\delta$, we may assume that there is an edge in $C_{1}$, say $e_{1}$, adjacent to $x$ and coloured with $\delta$, similarly there is on $C_{2}$ an edge $e_{2}$ adjacent to $y$ and coloured with $\delta$. Let $z$ be the neighbour of $y$ on $C_{2}$ such that $e_{2}=y z$ and let $t$ be the neighbour of $z$ such that $z t$ is coloured with $\gamma$. If $t \notin C_{1}$, there must be $C_{3} \neq C_{1}$ such that $t \in C_{3}$, by Theorem 4 again there is an edge $e_{3}$ of $C_{3}$, adjacent to $t$ and coloured with $\delta$. But now $\left\{e_{1}, e_{2}, e_{3}\right\}$ induces a subgraph with at least 5 edges, a contradiction with Theorem 4

It follows that $\mathcal{C}$ contains exactly two induced odd cycles and they are of equal length. Consequently $G$ is a permutation graph. When these cycles have length 5 , since $G$ has girth $5 G$ is obviously the Petersen graph.

When $G$ is a cubic bridgeless planar graph, we know from the Four Colour Theorem that $G$ is 3-edge colourable and hence $\gamma(G)=1$. Albertson and Haas [1] gave $\gamma(G) \geq \frac{6}{7}-\frac{2}{35 m}$ when $G$ is a planar bridgeless graph with maximum degree 3. Our Theorem 14 improves this lower bound (allowing moreover bridges). On the other hand, they exhibit a family of planar graphs with maximum degree 3 (bridges are allowed) for which $\gamma(G)=\frac{8}{9}-\frac{2}{9 n}$.

As Steffen in [14] we denote $g(\mathcal{F})$ the minimum length of an odd cycle in a 2 -factor $\mathcal{F}$ and $g^{+}(G)$ the maximum of these numbers over all 2-factors. We suppose that $g^{+}(G)$ is defined, that is $G$ has at least one 2 -factor (when $G$ is a cubic bridgeless graphs this condition is obviously fulfilled).

When $G$ is cubic bridgeless, Steffen [14] showed that we have :

$$
\gamma(G) \geq \quad \max \left\{1-\frac{2}{3 g^{+}(G)}, \frac{11}{12}\right\}
$$

The difficult part being to show that $\gamma(G) \geq \frac{11}{12}$.
Theorem 20 Let $G$ be a graph with maximum degree 3. Then $\gamma(G) \geq 1-$ $\frac{2 n}{\left(3 n-\left|V_{2}\right|\right) g^{+}(G)}$.

Proof By Lemma 12, we may assume $V_{1}=\emptyset$. Hence, $m=\frac{1}{2}\left(2\left|V_{2}\right|+3\left|V_{3}\right|\right)$, moreover $n=\left|V_{2}\right|+\left|V_{3}\right|$, henceforth $m=\frac{3 n-\left|V_{2}\right|}{2}$. We have $\gamma(G)=1-\frac{s(G)}{m}$, obviously, $s(G) \leq \frac{n}{g^{+}(G)}$. The result follows.

Theorem 21 Let $G$ be a graph with maximum degree 3 having at least one 2-factor. Assume that $\left|V_{2}\right| \leq \frac{n}{3}$ and $g^{+}(G) \geq 11$ then $\gamma(G) \geq \max \{1-$ $\left.\frac{3}{4 g^{+}(G)}, \frac{11}{12}\right\}$.

Proof By assumption we have $V_{1}=\emptyset$. From Theorem[20 we have just to prove that $\gamma(G) \geq \frac{11}{12}$. Following the proof of Theorem [14. we try to associate a set $P_{e}$ of at least 8 distinct vertices of $V_{3}$ for each edge $e \in E_{\phi}(\delta)$ in a $\delta$-minimum edge-colouring $\phi$ of $G$ such that

$$
\begin{equation*}
\forall e, e^{\prime} \in E_{\phi}(\delta) \quad P_{e} \cap P_{e^{\prime}}=\emptyset \tag{2}
\end{equation*}
$$

Indeed, let $\mathcal{F}$ be a 2 -factor of $G$ where each odd cycle has length at least 11 and let $C_{1}, C_{2} \ldots C_{2 k}$ be its set of odd cycles. We have, obviously $s(G) \leq 2 k$. Let $V_{3}^{\prime}$ and $V_{2}^{\prime}$ be the sets of vertices of degree 3 and 2 respectively contained in these odd cycles. As soon as $\left|V_{3}^{\prime}\right| \geq 8 s(G)$ we have

$$
\begin{equation*}
\gamma(G)=1-\frac{s(G)}{m}=1-\frac{s(G)}{\frac{3}{2}\left|V_{3}\right|+\left|V_{2}\right|} \geq 1-\frac{\frac{\left|V_{3}^{\prime}\right|}{8}}{\frac{3}{2}\left|V_{3}\right|+\left|V_{2}\right|} \tag{3}
\end{equation*}
$$

which leads to

$$
\gamma(G) \geq 1-\frac{\frac{2\left|V_{3}^{\prime}\right|}{24\left|V_{1}\right|}}{1+\frac{2}{3} \frac{\left|V_{2}\right|}{\left|V_{3}\right|}}
$$

Since $\left|V_{3}\right| \geq\left|V_{3}^{\prime}\right|$, we have

$$
\gamma(G) \geq 1-\frac{\frac{2}{24}}{1+\frac{2}{3} \frac{\left|V_{2}\right|}{\left|V_{3}\right|}}
$$

and

$$
\gamma(G) \geq \frac{11}{12}
$$

as claimed.
It remains the case where $\left|V_{3}^{\prime}\right|<8 s(G)$. Since each odd cycle has at least 11 vertices we have $\left|V_{2}^{\prime}\right|>11 \times 2 k-\left|V_{3}^{\prime}\right|>3 s(G)$.

$$
\gamma(G)=\frac{m-s(G)}{m} \geq \frac{m-\frac{\left|V_{2}^{\prime}\right|}{3}}{m}
$$

We have

$$
\frac{m-\frac{\left|V_{2}^{\prime}\right|}{3}}{m} \geq \frac{11}{12}
$$

when

$$
\begin{equation*}
m \geq 4\left|V_{2}^{\prime}\right| \tag{4}
\end{equation*}
$$

Since $\left|V_{2}\right| \leq \frac{n}{3}$ we have $\left|V_{3}\right| \geq \frac{2 n}{3}$ and

$$
\begin{equation*}
m=3 \frac{\left|V_{3}\right|}{2}+\left|V_{2}\right|=3 \frac{n-\left|V_{2}\right|}{2}+\left|V_{2}\right|=3 \frac{n}{2}-\frac{\left|V_{2}\right|}{2} \geq 4 \frac{n}{3} \geq 4\left|V_{2}^{\prime}\right| \tag{5}
\end{equation*}
$$

and the result holds.
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