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SOME INVERSION FORMULAS AND FORMULAS FOR STIRLING NUMBERS

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ABSTRACT. In the paper we present some new inversion formulas and two new formulas for Stirling numbers.

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1. Introduction.

Let \mathbb{N} be the set of positive integers. Let $a(x) = x + a_2x^2 + a_3x^3 + \dots$ and $\frac{a(x)^m}{m!} = \sum_{n=m}^{\infty} a(n, m) \frac{x^n}{n!}$ for $m \in \mathbb{N}$. In Section 2 we show that for any $k, n \in \mathbb{N}$,

$$a(n+k, n) = \sum_{r=1}^k \binom{k-n}{k-r} \binom{k+n}{k+r} a(k+r, r).$$

Let $f(x) = c_0 + c_1x + c_2x^2 + \dots$ with $c_0 \neq 0$. In Section 3 we establish the following general inversion formula:

$$\begin{aligned} a_n &= n \sum_{m=1}^n [x^{n-m}] f(x)^m \cdot b_m \quad (n = 1, 2, 3, \dots) \\ \iff b_n &= \frac{1}{n} \sum_{m=1}^n [x^{n-m}] f(x)^{-n} \cdot a_m \quad (n = 1, 2, 3, \dots), \end{aligned}$$

where $[x^k]g(x)$ is the coefficient of x^k in the power series expansion of $g(x)$. As a consequence, for a given complex number t we have the following inversion formula:

$$a_n = n \sum_{m=1}^n \binom{mt}{n-m} b_m \quad (n \geq 1) \iff b_n = \frac{1}{n} \sum_{m=1}^n \binom{-nt}{n-m} a_m \quad (n \geq 1).$$

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Let $\alpha^{-1}(x)$ be the inverse function of $\alpha(x)$. In Section 4 we derive a general formula for $[x^{m+n}]\alpha(x)^m$ by using the power series expansion of $\alpha^{-1}(x)$. As a consequence, we deduce a symmetric inversion formula, see Theorem 4.3.

Suppose $n \in \mathbb{N}$ and $k \in \{0, 1, \dots, n\}$. Let $s(n, k)$ be the unsigned Stirling number of the first kind and $S(n, k)$ be the Stirling number of the second kind defined by

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^n (-1)^{n-k} s(n, k) x^k$$

and

$$x^n = \sum_{k=0}^n S(n, k) x(x-1)\cdots(x-k+1).$$

In the paper we obtain new formulas for Stirling numbers, see Theorems 2.3 and 4.2.

2. The formula for $[x^m]f(x)^t$.

Lemma 2.1. *Let t be a variable and $m \in \mathbb{N}$. Then*

$$\begin{aligned} & [x^m](1 + a_1x + a_2x^2 + \cdots + a_mx^m + \cdots)^t \\ &= \sum_{k_1+2k_2+\cdots+mk_m=m} \frac{t(t-1)\cdots(t-(k_1+\cdots+k_m)+1)}{k_1! \cdots k_m!} a_1^{k_1} \cdots a_m^{k_m}. \end{aligned}$$

Proof. Using the binomial theorem and the multinomial theorem we see that

$$\begin{aligned} & [x^m](1 + a_1x + a_2x^2 + \cdots + a_mx^m + \cdots)^t \\ &= [x^m](1 + a_1x + a_2x^2 + \cdots + a_mx^m)^t \\ &= [x^m] \sum_{n=0}^{\infty} \binom{t}{n} (a_1x + a_2x^2 + \cdots + a_mx^m)^n \\ &= \sum_{n=0}^m \binom{t}{n} [x^m](a_1x + a_2x^2 + \cdots + a_mx^m)^n \\ &= \sum_{n=0}^m \binom{t}{n} [x^m] \sum_{k_1+k_2+\cdots+k_m=n} \frac{n!}{k_1! \cdots k_m!} (a_1x)^{k_1} \cdots (a_mx^m)^{k_m} \\ &= \sum_{n=0}^m \binom{t}{n} \sum_{\substack{k_1+\cdots+k_m=n \\ k_1+2k_2+\cdots+mk_m=m}} \frac{n!}{k_1! \cdots k_m!} a_1^{k_1} \cdots a_m^{k_m} \\ &= \sum_{k_1+2k_2+\cdots+mk_m=m} \frac{t(t-1)\cdots(t-(k_1+\cdots+k_m)+1)}{k_1! \cdots k_m!} a_1^{k_1} \cdots a_m^{k_m}. \end{aligned}$$

Theorem 2.1. Let t be a variable, $m \in \mathbb{N}$ and $f(x) = 1 + a_1x + a_2x^2 + \dots$. Then

$$[x^m]f(x)^t = \sum_{r=1}^m \binom{m-t}{m-r} \binom{t}{r} [x^m]f(x)^r.$$

Proof. From Lemma 2.1 we see that $[x^m]f(x)^t$ is a polynomial of t with degree $\leq m$. Hence

$$P_m(t) = [x^m]f(x)^t - \sum_{r=1}^m \binom{m-t}{m-r} \binom{t}{r} [x^m]f(x)^r$$

is also a polynomial of t with degree $\leq m$. If $r \in \{1, 2, \dots, m\}$ and $t \in \{0, 1, \dots, m\}$ with $t \neq r$, then $t < r$ or $m-t < m-r$ and hence $\binom{m-t}{m-r} \binom{t}{r} = 0$. Thus $P_m(t) = 0$ for $t = 0, 1, \dots, m$. Therefore $P_m(t) = 0$ for all t . This yields the result.

Corollary 2.1. Let $m \in \mathbb{N}$ and let a be a complex number. Then

$$\sum_{r=1}^m \binom{m+a}{m-r} (-1)^{m-r} \binom{a+r-1}{r} r^m = a^m.$$

Proof. Clearly $[x^m](e^x)^t = \frac{t^m}{m!}$. Thus, by Theorem 2.1 we have

$$\frac{t^m}{m!} = \sum_{r=1}^m \binom{m-t}{m-r} \binom{t}{r} \frac{r^m}{m!}.$$

Now taking $t = -a$ and noting that $\binom{-a}{r} = (-1)^r \binom{a+r-1}{r}$ we deduce the result.

Theorem 2.2. Let $a(x) = x + a_2x^2 + a_3x^3 + \dots$. For $m \in \mathbb{N}$ let $\frac{a(x)^m}{m!} = \sum_{n=m}^{\infty} a(n, m) \frac{x^n}{n!}$. Then for any $k, n \in \mathbb{N}$ we have

$$a(n+k, n) = \sum_{r=1}^k \binom{k-n}{k-r} \binom{k+n}{k+r} a(k+r, r).$$

Proof. Set $\alpha(x) = a(x)/x$. Then for $m \in \mathbb{N}$ we have

$$\alpha(x)^m = \frac{a(x)^m}{x^m} = \sum_{k=0}^{\infty} a(m+k, m) \cdot \frac{m!}{(m+k)!} x^k.$$

Thus,

$$[x^k]\alpha(x)^n = a(n+k, n) \frac{n!}{(n+k)!} \quad \text{and} \quad [x^k]\alpha(x)^r = a(k+r, r) \frac{r!}{(k+r)!}.$$

Since $\alpha(0) = 1$, by Theorem 2.1 we have

$$[x^k]\alpha(x)^n = \sum_{r=1}^k \binom{k-n}{k-r} \binom{n}{r} [x^k]\alpha(x)^r.$$

Hence

$$a(n+k, n) \frac{n!}{(n+k)!} = \sum_{r=1}^k \binom{k-n}{k-r} \binom{n}{r} \frac{r!}{(k+r)!} a(k+r, r)$$

and so

$$a(n+k, n) = \sum_{r=1}^k \binom{k-n}{k-r} \frac{(n+k)!}{(n-r)!(k+r)!} a(k+r, r).$$

This is the result.

Theorem 2.3. *Let $k, n \in \mathbb{N}$. Then*

$$S(n+k, n) = \sum_{r=1}^k \binom{k-n}{k-r} \binom{k+n}{k+r} S(k+r, r)$$

and

$$s(n+k, n) = \sum_{r=1}^k \binom{k-n}{k-r} \binom{k+n}{k+r} s(k+r, r).$$

Proof. It is well known that ([1])

$$\frac{(e^x - 1)^m}{m!} = \sum_{n=m}^{\infty} S(n, m) \frac{x^n}{n!} \quad \text{and} \quad \frac{(\log(1+x))^m}{m!} = \sum_{n=m}^{\infty} (-1)^{n-m} s(n, m) \frac{x^n}{n!}.$$

Thus the result follows from Theorem 2.2.

3. A general inversion formula involving $[x^k]f(x)^t$.

Lemma 3.1. *Let $\alpha^{-1}(x)$ be the inverse function of $\alpha(x)$. Then for any two sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ we have:*

$$\begin{aligned} a_n &= \sum_{m=0}^{\infty} [x^n]\alpha(x)^m b_m \quad (n = 0, 1, 2, \dots) \\ \iff b_n &= \sum_{m=0}^{\infty} [x^n]\alpha^{-1}(x)^m a_m \quad (n = 0, 1, 2, \dots). \end{aligned}$$

Proof. Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$ and $b(x) = \sum_{n=0}^{\infty} b_n x^n$. Then clearly

$$\begin{aligned} a_n &= \sum_{m=0}^{\infty} [x^n] \alpha(x)^m b_m \quad (n = 0, 1, 2, \dots) \\ \iff a(x) &= \sum_{m=0}^{\infty} b_m \sum_{n=0}^{\infty} [x^n] \alpha(x)^m x^n = \sum_{m=0}^{\infty} b_m \alpha(x)^m \\ \iff a(x) &= b(\alpha(x)) \iff b(x) = a(\alpha^{-1}(x)) \\ \iff b_n &= \sum_{m=0}^{\infty} [x^n] \alpha^{-1}(x)^m a_m \quad (n = 0, 1, 2, \dots). \end{aligned}$$

So the lemma is proved.

Theorem 3.1. *Let $k \in \mathbb{N}$. For nonnegative integers m and n let*

$$\alpha_k(n, m) = \begin{cases} (-1)^{\frac{n}{k}} \binom{\frac{m}{n}}{\frac{k}{n}} & \text{if } k \mid n, \\ 0 & \text{if } k \nmid n. \end{cases}$$

Then we have the following inversion formula:

$$\begin{aligned} a_n &= \sum_{m=0}^{\infty} \alpha_k(n, m) b_m \quad (n = 0, 1, 2, \dots) \\ \iff b_n &= \sum_{m=0}^{\infty} \alpha_k(n, m) a_m \quad (n = 0, 1, 2, \dots). \end{aligned}$$

Proof. Let $\alpha(x) = (1 - x^k)^{\frac{1}{k}}$ ($0 < x < 1$). Then clearly $\alpha^{-1}(x) = \alpha(x)$ and $\alpha(x)^m = (1 - x^k)^{\frac{m}{k}} = \sum_{r=0}^{\infty} \binom{\frac{m}{k}}{r} (-1)^r x^{kr} = \sum_{n=0}^{\infty} \alpha_k(n, m) x^n$. Thus applying Lemma 3.1 we deduce the theorem.

Lemma 3.2 (Lagrange inversion formula ([1, p.148], [3, pp.36-44])). *Let $\alpha(x) = \alpha_1 x + \alpha_2 x^2 + \dots$ with $\alpha_1 \neq 0$, and let $k, n \in \mathbb{N}$ with $k \leq n$. Then*

$$[x^n](\alpha^{-1}(x))^k = \frac{k}{n} [x^{n-k}] \left(\frac{\alpha(x)}{x} \right)^{-n}.$$

Theorem 3.2. *Let $f(x) = c_0 + c_1 x + c_2 x^2 + \dots$ with $c_0 \neq 0$. Then for any two sequences $\{a_n\}$ and $\{b_n\}$ we have the following inversion formula:*

$$\begin{aligned} a_n &= n \sum_{m=1}^n [x^{n-m}] f(x)^m \cdot b_m \quad (n = 1, 2, 3, \dots) \\ \iff b_n &= \frac{1}{n} \sum_{m=1}^n [x^{n-m}] f(x)^{-n} \cdot a_m \quad (n = 1, 2, 3, \dots). \end{aligned}$$

Proof. Set $\alpha(x) = xf(x)$. Then clearly $[x^n]\alpha(x)^m = 0$ for $m > n$. As $\alpha^{-1}(xf(x)) = \alpha^{-1}(\alpha(x)) = x$ we see that $\alpha^{-1}(0) = 0$ and so $\alpha^{-1}(x) = d_1x + d_2x^2 + \dots$ for some d_1, d_2, \dots . Thus $[x^n]\alpha^{-1}(x)^m = 0$ for $m > n$. Set $a_0 = b_0 = 0$. From Lemma 3.1 we see that

$$\begin{aligned} a_n &= \sum_{m=0}^{\infty} [x^n]\alpha(x)^m \cdot b_m = \sum_{m=1}^n [x^n]\alpha(x)^m \cdot b_m \quad (n \geq 1) \\ \iff b_n &= \sum_{m=0}^{\infty} [x^n]\alpha^{-1}(x)^m \cdot a_m = \sum_{m=1}^n [x^n]\alpha^{-1}(x)^m \cdot a_m \quad (n \geq 1). \end{aligned}$$

For $m \leq n$ we see that $[x^n]\alpha(x)^m = [x^n]x^m f(x)^m = [x^{n-m}]f(x)^m$ and $[x^n]\alpha^{-1}(x)^m = \frac{m}{n}[x^{n-m}]f(x)^{-n}$ by Lemma 3.2. Thus

$$a_n = \sum_{m=1}^n [x^{n-m}]f(x)^m \cdot b_m \quad (n \geq 1) \iff b_n = \sum_{m=1}^n \frac{m}{n} [x^{n-m}]f(x)^{-n} \cdot a_m \quad (n \geq 1).$$

Now substituting a_n by a_n/n we obtain the result.

As $e^{cx} = \sum_{k=0}^{\infty} \frac{(cx)^k}{k!}$, we see that

$$[x^{n-m}](e^x)^m = \frac{m^{n-m}}{(n-m)!} \quad \text{and} \quad [x^{n-m}](e^x)^{-n} = \frac{(-n)^{n-m}}{(n-m)!}.$$

Thus, putting $f(x) = e^x$ in Theorem 3.2 we have the following inversion formula:

$$a_n = n \sum_{m=1}^n \frac{m^{n-m}}{(n-m)!} b_m \quad (n \geq 1) \iff b_n = \frac{1}{n} \sum_{m=1}^n \frac{(-n)^{n-m}}{(n-m)!} a_m \quad (n \geq 1).$$

Substituting a_n by $a_n/(n-1)!$, and b_n by $b_n/n!$ we obtain

$$a_n = \sum_{m=1}^n \binom{n}{m} m^{n-m} b_m \quad (n \geq 1) \iff b_n = \sum_{m=1}^n \binom{n-1}{m-1} (-n)^{n-m} a_m \quad (n \geq 1).$$

This is a known result. See [2, p.96].

As $(1+x)^{ct} = \sum_{k=0}^{\infty} \binom{ct}{k} x^k$ ($|x| < 1$), we see that

$$[x^{n-m}](1+x)^{mt} = \binom{mt}{n-m} \quad \text{and} \quad [x^{n-m}](1+x)^{-nt} = \binom{-nt}{n-m}.$$

Now putting $f(x) = (1+x)^t$ in Theorem 3.2 and applying the above we deduce the following result.

Theorem 3.3. Let t be a complex number. For any two sequences $\{a_n\}$ and $\{b_n\}$ we have the following inversion formula:

$$a_n = n \sum_{m=1}^n \binom{mt}{n-m} b_m \quad (n \geq 1) \iff b_n = \frac{1}{n} \sum_{m=1}^n \binom{-nt}{n-m} a_m \quad (n \geq 1).$$

Theorem 3.4. Let $f(x) = c_0 + c_1x + c_2x^2 + \dots$ with $c_0 \neq 0$. For $k, n \in \mathbb{N}$ with $k < n$ we have

$$\sum_{m=k}^n \frac{1}{m} [x^{n-m}] f(x)^m \cdot [x^{m-k}] f(x)^{-m} = \sum_{m=k}^n m [x^{m-k}] f(x)^k \cdot [x^{n-m}] f(x)^{-n} = 0.$$

Proof. For $m \in \mathbb{N}$ let $b_m = \frac{1}{m} \sum_{k=1}^m [x^{m-k}] f(x)^{-m} \cdot y^k$. Applying Theorem 3.2 we see that

$$\sum_{m=1}^n [x^{n-m}] f(x)^m \cdot b_m = \frac{y^n}{n}.$$

On the other hand,

$$\begin{aligned} \sum_{m=1}^n [x^{n-m}] f(x)^m \cdot b_m &= \sum_{m=1}^n [x^{n-m}] f(x)^m \cdot \frac{1}{m} \sum_{k=1}^m [x^{m-k}] f(x)^{-m} \cdot y^k \\ &= \sum_{k=1}^n \left(\sum_{m=k}^n [x^{n-m}] f(x)^m \cdot \frac{1}{m} [x^{m-k}] f(x)^{-m} \right) y^k. \end{aligned}$$

Thus,

$$\sum_{k=1}^n \left(\sum_{m=k}^n \frac{1}{m} [x^{n-m}] f(x)^m [x^{m-k}] f(x)^{-m} \right) y^k = \frac{y^n}{n}$$

and hence

$$\sum_{m=k}^n \frac{1}{m} [x^{n-m}] f(x)^m \cdot [x^{m-k}] f(x)^{-m} = 0 \quad \text{for } k < n.$$

For $m \in \mathbb{N}$ let $a_m = m \sum_{k=1}^m [x^{m-k}] f(x)^k \cdot y^k$. Applying Theorem 3.2 we have

$$\sum_{m=1}^n [x^{n-m}] f(x)^{-n} \cdot a_m = ny^n.$$

On the other hand,

$$\begin{aligned} \sum_{m=1}^n [x^{n-m}] f(x)^{-n} \cdot a_m &= \sum_{m=1}^n [x^{n-m}] f(x)^{-n} \cdot m \sum_{k=1}^m [x^{m-k}] f(x)^k \cdot y^k \\ &= \sum_{k=1}^n \left(\sum_{m=k}^n [x^{n-m}] f(x)^{-n} \cdot m [x^{m-k}] f(x)^k \right) y^k. \end{aligned}$$

Thus,

$$\sum_{k=1}^n \left(\sum_{m=k}^n m[x^{m-k}]f(x)^k \cdot [x^{n-m}]f(x)^{-n} \right) y^k = ny^n$$

and hence

$$\sum_{m=k}^n m[x^{m-k}]f(x)^k \cdot [x^{n-m}]f(x)^{-n} = 0 \quad \text{for } k < n.$$

This completes the proof.

Corollary 3.1. *For $k, n \in \mathbb{N}$ with $k < n$ we have*

$$\sum_{m=k}^n \frac{1}{m} \binom{mt}{n-m} \binom{-mt}{m-k} = \sum_{m=k}^n m \binom{kt}{m-k} \binom{-nt}{n-m} = 0.$$

Proof. Since $(1+x)^{rt} = \sum_{s=0}^{\infty} \binom{rt}{s} x^s$, taking $f(x) = (1+x)^t$ in Theorem 3.4 we deduce the result.

4. A formula for $[x^{m+n}]\alpha(x)^m$.

Theorem 4.1. *Let $\beta(x) = x \sum_{n=0}^{\infty} \beta_n x^n$ with $\beta_0 \neq 0$. Let $\alpha(x)$ be the inverse function of $\beta(x)$. For $m, n \in \mathbb{N}$ we have*

$$[x^{m+n}]\alpha(x)^m = \frac{m}{(n+m)!} \sum_{\substack{k_1+2k_2+\dots+nk_n=n}} \frac{(n+m-1+k_1+\dots+k_n)!}{k_1! \dots k_n!} \\ \times (-1)^{k_1+k_2+\dots+k_n} \beta_0^{-n-m-k_1-\dots-k_n} \beta_1^{k_1} \beta_2^{k_2} \dots \beta_n^{k_n}.$$

Proof. By the multinomial theorem we have

$$\left(\sum_{k=1}^n \frac{\beta_k}{\beta_0} x^k \right)^s = \sum_{\substack{k_1+\dots+k_n=s}} \frac{s!}{k_1! \dots k_n!} \prod_{i=1}^n \left(\frac{\beta_i}{\beta_0} x^i \right)^{k_i}.$$

Thus

$$[x^n] \left(\sum_{k=1}^{\infty} \frac{\beta_k}{\beta_0} x^k \right)^s = [x^n] \left(\sum_{k=1}^n \frac{\beta_k}{\beta_0} x^k \right)^s = \sum_{\substack{k_1+\dots+k_n=s \\ k_1+2k_2+\dots+nk_n=n}} \frac{s!}{k_1! \dots k_n!} \prod_{i=1}^n \left(\frac{\beta_i}{\beta_0} \right)^{k_i}.$$

As

$$\begin{aligned} & \beta_0^{m+n} \left(\frac{x}{\beta(x)} \right)^{m+n} - 1 \\ &= \beta_0^{m+n} \left(\beta_0 + \sum_{k=1}^{\infty} \beta_k x^k \right)^{-n-m} - 1 = \left(1 + \sum_{k=1}^{\infty} \frac{\beta_k}{\beta_0} x^k \right)^{-n-m} - 1 \\ &= \sum_{s=1}^{\infty} \frac{(-n-m)(-n-m-1)\dots(-n-m-s+1)}{s!} \left(\sum_{k=1}^{\infty} \frac{\beta_k}{\beta_0} x^k \right)^s. \end{aligned}$$

From the above we see that

$$\begin{aligned}
& [x^n] \beta_0^{m+n} \left(\frac{x}{\beta(x)} \right)^{m+n} \\
&= \sum_{s=1}^{\infty} \frac{(-n-m)(-n-m-1) \cdots (-n-m-s+1)}{s!} \\
&\quad \times \sum_{\substack{k_1+\cdots+k_n=s \\ k_1+2k_2+\cdots+nk_n=n}} \frac{s!}{k_1! \cdots k_n!} \prod_{i=1}^n \left(\frac{\beta_i}{\beta_0} \right)^{k_i} \\
&= \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(n+m)(n+m+1) \cdots (n+m+k_1+\cdots+k_n-1)}{k_1! \cdots k_n!} \\
&\quad \times \left(-\frac{1}{\beta_0} \right)^{k_1+\cdots+k_n} \beta_1^{k_1} \cdots \beta_n^{k_n}.
\end{aligned}$$

Thus applying Lemma 3.2 we have

$$\begin{aligned}
[x^{m+n}] \alpha(x)^m &= \frac{m}{n+m} [x^n] \left(\frac{x}{\beta(x)} \right)^{m+n} \\
&= \frac{m}{n+m} \beta_0^{-m-n} \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n+n+m-1)!}{k_1! \cdots k_n! (n+m-1)!} \\
&\quad \times (-1)^{k_1+\cdots+k_n} \beta_0^{-(k_1+\cdots+k_n)} \beta_1^{k_1} \cdots \beta_n^{k_n}.
\end{aligned}$$

This yields the result.

Corollary 4.1. *For $m, n \in \mathbb{N}$ we have*

$$\sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n+n+m-1)!}{(m+n-1)! k_1! \cdots k_n!} (-1)^{k_1+\cdots+k_n} = (-1)^n \binom{m+n}{m}.$$

Proof. Let $\beta(x) = x \sum_{r=0}^{\infty} x^r = \frac{x}{1-x}$. Then the inverse function of $\beta(x)$ is given by $\alpha(x) = \frac{x}{1+x}$. Using the binomial theorem we see that $[x^{m+n}] \alpha(x)^m = [x^n] (1+x)^{-m} = \binom{-m}{n} = (-1)^n \binom{m+n-1}{n}$. Now applying Theorem 4.1 we deduce the result.

Corollary 4.2. *For $n \in \mathbb{N}$ we have*

$$\begin{aligned}
& \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n+n)!}{k_1! \cdots k_n!} (-1)^{k_1+\cdots+k_n} 2^{k_1} 3^{k_2} \cdots (n+1)^{k_n} \\
&= (-1)^n \cdot (n+1)! \cdot \frac{1}{n+2} \binom{2n+2}{n+1}.
\end{aligned}$$

Proof. Let

$$\beta(x) = \frac{x}{(1+x)^2} \quad \text{and} \quad \alpha(x) = \frac{1 - \sqrt{1 - 4x}}{2x} - 1 \quad (0 < x < \frac{1}{4}).$$

It is easily seen that $\alpha(x) = \beta^{-1}(x)$. From the binomial theorem we know that

$$\alpha(x) = x \sum_{n=0}^{\infty} \frac{1}{n+2} \binom{2n+2}{n+1} x^n \quad \text{and} \quad \beta(x) = x \sum_{n=0}^{\infty} (-1)^n (n+1) x^n.$$

Now applying Theorem 4.1 (with $m = 1$) we deduce the result.

Corollary 4.3. *For $n \in \mathbb{N}$ we have*

$$\sum_{k_1+2k_2+\dots+nk_n=n} \frac{(k_1 + \dots + k_n + 2n)!}{k_1! \dots k_n!} \cdot \frac{(-1)^{k_1+k_2+\dots+k_n+n}}{3!^{k_1} 5!^{k_2} \dots (2n+1)!^{k_n}} = (2n-1)!!^2.$$

Proof. It is well known that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

and

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n+1) \cdot (2n)!!} x^{2n+1} \quad (|x| \leq 1).$$

Set $\beta(x) = \sin x = x \sum_{n=0}^{\infty} \beta_n x^n$. Then $\beta^{-1}(x) = \arcsin x$ and

$$\beta_i = \begin{cases} 0 & \text{if } 2 \nmid i, \\ \frac{(-1)^{i/2}}{(i+1)!} & \text{if } 2 \mid i. \end{cases}$$

Thus, taking $m = 1$ in Theorem 4.1 and substituting n by $2n$ we obtain

$$\begin{aligned} & (2n+1)! \cdot [x^{2n+1}] \arcsin x \\ &= \sum_{k_1+2k_2+\dots+2nk_{2n}=2n} \frac{(2n+k_1+k_2+\dots+k_{2n})!}{k_1!k_2!\dots k_{2n}!} (-1)^{k_1+k_2+\dots+k_{2n}} \beta_1^{k_1} \beta_2^{k_2} \dots \beta_{2n}^{k_{2n}} \\ &= \sum_{k_2+2k_4+\dots+nk_{2n}=n} \frac{(2n+k_2+k_4+\dots+k_{2n})!}{k_2!k_4!\dots k_{2n}!} (-1)^{k_2+k_4+\dots+k_{2n}} \prod_{i=1}^n \left(\frac{(-1)^i}{(2i+1)!} \right)^{k_{2i}}. \end{aligned}$$

Replacing k_{2i} by k_i in the above formula and observing that

$$(2n+1)! \cdot [x^{2n+1}] \arcsin x = (2n+1)! \cdot \frac{(2n-1)!!}{(2n+1) \cdot (2n)!!} = (2n-1)!!^2$$

we deduce the result.

Theorem 4.2. For $m, n \in \mathbb{N}$ we have

$$S(n+m, m) = \frac{(-1)^n}{(m-1)!} \sum_{k_1+2k_2+\dots+nk_n=n} (-1)^{k_1+\dots+k_n} \frac{(k_1 + \dots + k_n + n + m - 1)!}{2^{k_1} k_1! \cdot 3^{k_2} k_2! \cdots (n+1)^{k_n} k_n!}$$

and

$$s(n+m, m) = \frac{(-1)^n}{(m-1)!} \sum_{k_1+2k_2+\dots+nk_n=n} (-1)^{k_1+\dots+k_n} \frac{(k_1 + \dots + k_n + n + m - 1)!}{2^{k_1} k_1! \cdot 3^{k_2} k_2! \cdots (n+1)^{k_n} k_n!}.$$

Proof. Clearly $e^x - 1$ and $\log(1+x)$ are a pair of inverse functions. As

$$\frac{(e^x - 1)^m}{m!} = \sum_{n=0}^{\infty} S(n+m, m) \frac{x^{n+m}}{(n+m)!} \quad \text{and} \quad \log(1+x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i+1} x^{i+1},$$

putting $\alpha(x) = e^x - 1$, $\beta(x) = \log(1+x)$ and $\beta_i = \frac{(-1)^i}{i+1}$ in Theorem 4.1 we see that

$$\begin{aligned} \frac{m! S(n+m, m)}{(n+m)!} &= [x^{m+n}] (e^x - 1)^m \\ &= \frac{m}{(n+m)!} \sum_{k_1+2k_2+\dots+nk_n=n} \frac{(k_1 + \dots + k_n + n + m - 1)!}{k_1! \cdots k_n!} \\ &\quad \times (-1)^{k_1+k_2+\dots+k_n} \cdot (-1)^{k_1+2k_2+\dots+nk_n} \frac{1}{2^{k_1} \cdot 3^{k_2} \cdots (n+1)^{k_n}}. \end{aligned}$$

Since

$$\frac{(\log(1+x))^m}{m!} = \sum_{n=0}^{\infty} (-1)^n s(n+m, m) \frac{x^{n+m}}{(n+m)!} \quad \text{and} \quad e^x - 1 = \sum_{i=0}^{\infty} \frac{x^{i+1}}{(i+1)!},$$

putting $\alpha(x) = \log(1+x)$, $\beta(x) = e^x - 1$ and $\beta_i = \frac{1}{(i+1)!}$ in Theorem 4.1 we see that

$$\begin{aligned} &(-1)^n \frac{m! s(n+m, m)}{(n+m)!} \\ &= [x^{m+n}] (\log(1+x))^m \\ &= \frac{m}{(n+m)!} \sum_{k_1+2k_2+\dots+nk_n=n} \frac{(k_1 + \dots + k_n + n + m - 1)!}{k_1! \cdots k_n!} \cdot \frac{(-1)^{k_1+\dots+k_n}}{2^{k_1} \cdot 3^{k_2} \cdots (n+1)^{k_n}}. \end{aligned}$$

By the above, the theorem is proved.

We remark that Theorem 4.2 provides a straightforward method to calculate $s(n+m, m)$ and $S(n+m, m)$ for small n . For example, we have
(4.1)

$$S(m+3, m) = \binom{m+1}{2} \binom{m+3}{4} \quad \text{and} \quad s(m+3, m) = \binom{m+3}{2} \binom{m+3}{4}.$$

Corollary 4.4. For $m, n \in \mathbb{N}$ we have

$$\begin{aligned} & \sum_{r=0}^m \binom{m}{r} (-1)^{m-r} r^{m+n} \\ &= m \sum_{\substack{k_1+2k_2+\dots+nk_n=n}} (-1)^{k_1+\dots+k_n+n} \frac{(k_1 + \dots + k_n + n + m - 1)!}{2^{k_1} k_1! \cdot 3^{k_2} k_2! \cdots (n+1)^{k_n} k_n!}. \end{aligned}$$

Proof. It is well known that ([1, p.204])

$$\sum_{r=0}^m \binom{m}{r} (-1)^{m-r} r^{m+n} = m! S(n+m, m).$$

Combining this with Theorem 4.2 we obtain the result.

Let $\alpha(x) = -x + \alpha_1 x^2 + \alpha_2 x^3 + \dots$ and $\beta(x) = -x + \beta_1 x^2 + \beta_2 x^3 + \dots$ be a pair of inverse functions. Taking $m = 1$ in Theorem 4.1 we deduce:

Theorem 4.3. We have the following inversion formula:

$$\begin{aligned} \alpha_n &= \frac{(-1)^{n+1}}{(n+1)!} \sum_{\substack{k_1+2k_2+\dots+nk_n=n}} \frac{(k_1 + \dots + k_n + n)!}{k_1! \cdots k_n!} \beta_1^{k_1} \cdots \beta_n^{k_n} (n \geq 1) \\ \iff \beta_n &= \frac{(-1)^{n+1}}{(n+1)!} \sum_{\substack{k_1+2k_2+\dots+nk_n=n}} \frac{(k_1 + \dots + k_n + n)!}{k_1! \cdots k_n!} \alpha_1^{k_1} \cdots \alpha_n^{k_n} (n \geq 1). \end{aligned}$$

Definition 4.1. If $\alpha(x) = \alpha^{-1}(x)$, we say that $\alpha(x)$ is a self-inverse function.

For example, $\alpha(x) = \frac{rx+s}{tx-r}$ ($(r^2 + t^2)(r^2 + st) \neq 0$) and $\alpha(x) = (1 - x^k)^{\frac{1}{k}}$ are self-inverse functions.

Theorem 4.4. Let $\alpha(x) = -x + \alpha_1 x^2 + \alpha_2 x^3 + \dots$ be a self-inverse function. Then $\alpha_2, \alpha_4, \dots$ depend only on $\alpha_1, \alpha_3, \dots$. Moreover, for $n \in \mathbb{N}$,

$$\begin{aligned} (4.2) \quad & \sum_{\substack{k_1+2k_2+\dots+(n-1)k_{n-1}=n}} \frac{(k_1 + \dots + k_{n-1} + n)!}{k_1! \cdots k_{n-1}!} \alpha_1^{k_1} \cdots \alpha_{n-1}^{k_{n-1}} \\ &= \begin{cases} 0 & \text{if } 2 \nmid n, \\ -2 \cdot (n+1)! \alpha_n & \text{if } 2 \mid n. \end{cases} \end{aligned}$$

Proof. By Theorem 4.3 we have

$$\begin{aligned} \alpha_n &= \frac{(-1)^{n+1}}{(n+1)!} \sum_{\substack{k_1+2k_2+\dots+nk_n=n}} \frac{(k_1 + \dots + k_n + n)!}{k_1! \cdots k_n!} \alpha_1^{k_1} \cdots \alpha_n^{k_n} \\ &= \frac{(-1)^{n+1}}{(n+1)!} \sum_{\substack{k_1+2k_2+\dots+(n-1)k_{n-1}=n}} \frac{(k_1 + \dots + k_{n-1} + n)!}{k_1! \cdots k_{n-1}!} \alpha_1^{k_1} \cdots \alpha_{n-1}^{k_{n-1}} + (-1)^{n+1} \alpha_n. \end{aligned}$$

Thus (4.2) is true. Using (4.2) and induction we deduce that $\alpha_2, \alpha_4, \dots$ depend only on $\alpha_1, \alpha_3, \dots$. This completes the proof.

If $\alpha(x) = -x + \alpha_1 x^2 + \alpha_2 x^3 + \dots$ is a self-inverse function, from (4.2) we deduce

$$(4.3) \quad \begin{aligned} \alpha_2 &= -\alpha_1^2, \quad \alpha_4 = 2\alpha_1^4 - 3\alpha_1\alpha_3, \\ \alpha_6 &= -13\alpha_1^6 - 4\alpha_1\alpha_5 - 2\alpha_3^2 + 18\alpha_1^3\alpha_3, \\ \alpha_8 &= 145\alpha_1^8 - 221\alpha_1^5\alpha_3 + 50\alpha_1^2\alpha_3^2 + 35\alpha_1^3\alpha_5 - 5\alpha_3\alpha_5 - 5\alpha_1\alpha_7. \end{aligned}$$

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