Silver block intersection graphs of Steiner 2-designs

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Abstract

For a block design $\mathcal{D}$, a series of block intersection graphs G_i, or i-BIG($\mathcal{D}$), $i = 0, \ldots, k$ is defined in which the vertices are the blocks of $\mathcal{D}$, with two vertices adjacent if and only if the corresponding blocks intersect in exactly i elements. A silver graph G is defined with respect to a maximum independent set of G, called an α-set. Let G be an r-regular graph and c be a proper $(r+1)$-coloring of G. A vertex x in G is said to be rainbow with respect to c if every color appears in the closed neighborhood $N[x] = N(x) \cup \{x\}$. Given an α-set I of G, a coloring c is said to be silver with respect to I if every $x \in I$ is rainbow with respect to c. We say G is silver if it admits a silver coloring with respect to some I. Finding silver graphs is of interest, for a motivation and progress in silver graphs see [7] and [15]. We investigate conditions for 0-BIG($\mathcal{D}$) and 1-BIG($\mathcal{D}$) of Steiner 2-designs $\mathcal{D} = S(2, k, v)$ to be silver.

keywords: Silver coloring, Block intersection graph, Steiner 2-design, and Steiner triple system

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1 Introduction and preliminaries

We follow standard notations and concepts from design theory. For these, one may refer to, for example, [5] and [14].

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A 2-(v, k, λ) design $(2 < k < v)$ is a pair $(V, \mathcal{B})$ where V is a v-set and $\mathcal{B}$ is a collection of b k-subsets of V (blocks) such that any 2-subset of V is contained in exactly λ blocks. A 2-$(v, k, 1)$ design is called Steiner 2-design and is denoted by $S(2, k, v)$. An $S(2, 3, v)$ is a Steiner triple system or STS(v). A design with $b = v$ is a symmetric (v, k, λ)-design. A symmetric $S(2, k, v)$ is called a projective plane. If k is the size of the blocks then $n := k - 1$ is called the order of the plane. This design is usually denoted by PG$(2, n)$. A 2-$(n^2, n, 1)$ design is called an affine plane. For such design we use the notation AG$(2, n)$.

A partial parallel class is a set of blocks that contains no element of the design more than once. A parallel class (PC) or a resolution class in a design is a set of blocks that partition the set of elements V. A near parallel class is a partial parallel class missing a single element. A resolvable balanced incomplete block design is a 2-(v, k, λ) design whose blocks can be partitioned into parallel classes. The notation RBIBD(v, k, λ) is commonly used. An affine plane of order n is an RBIBD$(n^2, n, 1)$. A resolvable STS(v) together with a resolution of its blocks is called a Kirkman triple system, KTS(v).

Given a design $\mathcal{D}$, a series of block intersection graphs G_i, or i-BIG, $i = 0, \ldots, k$ can be defined in which the vertices are the blocks of $\mathcal{D}$, with two vertices are adjacent if and only if the corresponding blocks intersect in exactly i elements.

Example 1 *For* STS(7)*,* 0-BIG *is empty graph and* 1-BIG *is* K_7*. For* STS(9)*,* 0-BIG *is disconnected and consists of four disjoint* K_3*'s and* 1-BIG *is* $K_{3,3,3,3}$.

The study of i-BIG$(\mathcal{D})$ is useful in characterizing block designs. Some researchers have studied properties of various kinds of block intersection graphs, see for example [1], [2], [4], [8], [9], [10], [16], and [17].

A graph of order v is strongly regular, denoted by SRG(v, k, λ, μ), whenever it is not complete or edgeless and, (i) each vertex is adjacent to k vertices, (ii) for each pair of adjacent vertices there are λ vertices adjacent to both, (iii) for each pair of non-adjacent vertices there are μ vertices adjacent to both.

Remark 1 *Let* G_i *be the* i*-block intersection graph of an* $S(2, k, v)$*. Then for each* $i = 2, 3, \ldots, k$*, the graph* G_i *is empty. So we consider only* G_0 *and* G_1*. Graphs* G_0 *and* G_1 *are complements of each other.* G_1 *is an* SRG$(b, k(r-1), r-2+(k-1)^2, k^2)$ *and* G_0 *is an* SRG$(b, b-k(r-1)-1, b-2k(r-1)+k^2-2, b-2kr+k^2+r-1)$ *(see Chapter 21 of [14]).*

In a graph $G = (V, E)$ an independent set is a subset of vertices no two of which are adjacent. The independence number $\alpha(G)$ is the cardinality of a largest set of independent vertices. We refer to any maximum independent set of a graph as an α-set. Let c be a proper $(r+1)$-coloring of an r-regular graph G. A vertex x in G is said to be rainbow with respect to c if every color appears in the closed neighborhood $N[x] = N(x) \cup \{x\}$. Given an α-set I of G the coloring c is said to be silver with respect to I if every $x \in I$ is rainbow with respect to c. We say G is silver if it admits a silver coloring with respect to some α-set. If all vertices of G are rainbow, then c is called a totally silver coloring of G and G is said to be totally silver. Note that the definition of silver coloring depends on the chosen α-set. For example in Figure 1, a graph G is shown which is silver when the α-set (the bold vertices) is taken as in the left, but it does not have any silver coloring with the α-set taken as on the right hand side.

Figure 1: A silver coloring of a graph

There are many different version of rainbow colorings in the literature, for example see [3], [11], [12], and [13]. For a motivation and progress in silver graphs see [7] and [15]. In fact silver graphs are closely related to a concept in graph coloring, called defining set. Let c be a proper k-coloring of a graph G and let $S \subseteq V(G)$. If c is the only extension of $c|_S$ to a proper k-coloring of G, then S is called a defining set of c. The minimum size of a defining set among all k-colorings of G is called a defining number and denoted by $\mathrm{def}(G, k)$. A more general survey of defining sets in combinatorics appears in [6]. Let G be an r-regular graph, then G is silver if and only if $\mathrm{def}(G, r+1) = |V(G)| - \alpha(G)$. In [15] an open problem is raised:

Question 1 *Find classes of r-regular graphs G, for which* $\mathrm{def}(G, r+1) = |V(G)| - \alpha(G)$, *i.e. determine classes of all silver graphs.*

A silver cube is a silver graph $G = K_n^d$, the Cartesian power of the complete graph K_n. Silver cubes are generalizations of silver matrices, which are $n \times n$ matrices where each symbol in $\{1, 2, \ldots, 2n-1\}$ appears in either the i-th row or the i-th column of the matrix. In [7] some algebraic constructions and a product construction of silver cubes are given. They show the relation of these cubes to codes over finite fields, dominating sets of a graph, Latin squares, and finite geometry. In particular the Hamming codes are used to produce a totally silver cube and the bound for the best binary codes is used to prove the non-existence of silver cubes for a large class of parameters with $n = 2$.

To study Question 1, here we consider i-BIGs of designs. First we give some examples of designs with silver i-BIGs.

Example 2 *In any symmetric (v, k, λ)-design $\mathcal{D}$, every two distinct blocks have exactly λ elements in common, so for $0 \leq i \leq k$, $i \neq \lambda$, i-BIG$(\mathcal{D})$ is empty graph, and λ-BIG$(\mathcal{D})$ is complete graph. Hence all of these graphs are totally silver. Specifically for each k and $0 \leq i \leq k+1$, i-BIG$(S(2, k+1, k^2+k+1))$ is totally silver.*

If $\mathcal{D}$ is an AG$(2, n)$, then $G_0 =$ 0-BIG$(\mathcal{D})$ consists of $(n+1)$ disjoint K_n's, so it is totally silver, and $G_1 =$ 1-BIG$(\mathcal{D}) = K_{\underbrace{n, n, \ldots, n}_{n+1}}$, is silver.

In this paper we prove the following results: If an $S(2, k, v)$ contains a parallel class, then a necessary condition for 1-BIG$(S(2, k, v))$ to be silver is $k^2 \mid v$. For each admissible $v = 9m$ we construct a $\mathcal{D}_1$= KTS(v), such that 1-BIG$(\mathcal{D}_1)$ is silver. And in general for each k and v where an AG$(2, k)$ and an RBIBD$(v, k, 1)$ exist we construct a $\mathcal{D}^* =$ RBIBD$(kv, k, 1)$ such that 1-BIG$(\mathcal{D}^*)$ is silver. Also a lower bound for $\alpha(G_1)$ is given in order for a 1-BIG$(S(2, k, v))$ to be silver. For any admissible v, the existence of a silver 1-BIG$(S(2, k, v))$ which possesses a maximum possible independent set, i.e. of size $\frac{v}{k}$ or $\frac{v-1}{k}$, is settled. We prove that for $v > k^3 - 2k^2 + 2k$ there is no silver 0-BIG$(S(2, k, v))$. Also we settle the question of existence of silver 0-BIG(STS(v)) for all admissible v.

Since every vertex of i-BIG$(\mathcal{D})$ corresponds to a block of $\mathcal{D}$, we will mostly refer to them as "blocks" rather than vertices. The following notation will be used in our discussion. Let G be a graph and I be an α-set of G. For each $i = 1, \ldots, |I|$, we let

$$X_i := \{u | u \in V(G) \setminus I,\ u \text{ is adjacent to exactly } i \text{ vertices of } I\}.$$

2 One block intersection graphs

The following is a necessary condition for 1-BIG$(\mathcal{D})$ of a Steiner system $\mathcal{D} = S(2, k, v)$ with $\alpha(G_1) = \frac{v}{k}$, to be silver.

Theorem 1 *Let $\mathcal{D}$ be an $S(2, k, v)$, which has a parallel class, and let G_1 be 1-BIG$(\mathcal{D})$. A necessary condition for G_1 to be silver is $k^2 \mid v$.*

Proof. G_1 is a $\frac{k(v-k)}{(k-1)}$–regular graph. Let I be an α-set, and assume that G_1 has a silver coloring with respect to I with C as the set of colors. We have $|I| = \frac{v}{k}$, and $|C| = \frac{k(v-k)}{k-1} + 1$. Since $|C| > |I|$, a color like ι exists that is not used in I. The vertices of I are rainbow, and each vertex with color ι from $V(G_1) \setminus I$, must be adjacent to k distinct vertices of I. Therefore $|I|$ must be a multiple of k, which implies $k^2 \mid v$. ■

Example 3 *There are* 80 *nonisomorphic* STS(15)*s, where* 70 *of them have parallel class* (see [5], page 32)*. So by Theorem 1, none of those* 70 *has silver* G_1*.*

By Theorem 1, if v is not a multiple of 9, then no silver 1-BIG(KTS(v)) exists. In the next lemma we show that for the case $9 \mid v$, when a KTS(v) exists, i.e. $v = 18q + 9$, there exists a silver 1-BIG(KTS(v)). This lemma is an illustration of a general structure which will be discussed in Theorem 2.

Lemma 1 *If* $v \equiv 3 \pmod 6$*, then a* $\mathcal{K} = \mathrm{KTS}(3v)$ *exists such that* 1-BIG($\mathcal{K}$) *is silver.*

Proof. Let $\mathcal{A} = \mathrm{AG}(2,3) = \mathrm{STS}(9)$ with $V(\mathcal{A}) = \{(i,j) \mid 1 \le i,j \le 3\}$, and denote its parallel classes by:

$$\begin{array}{c} \Theta_0 \\ \{(1,1),(2,1),(3,1)\} \\ \{(1,2),(2,2),(3,2)\} \\ \{(1,3),(2,3),(3,3)\} \end{array}$$

$$\begin{array}{ccc} \Theta_1 & \Theta_2 & \Theta_3 \\ a_1 = \{(1,1),(1,2),(1,3)\} & a_4 = \{(1,1),(2,2),(3,3)\} & a_7 = \{(1,1),(2,3),(3,2)\} \\ a_2 = \{(2,1),(2,2),(2,3)\} & a_5 = \{(1,3),(2,1),(3,2)\} & a_8 = \{(1,2),(2,1),(3,3)\} \\ a_3 = \{(3,1),(3,2),(3,3)\} & a_6 = \{(1,2),(2,3),(3,1)\} & a_9 = \{(1,3),(2,2),(3,1)\} \end{array}$$

Consider a KTS(v) $\mathcal{D} = (V, \mathcal{B})$, $V = \{x_1, x_2, \ldots, x_v\}$ with parallel classes $\pi_1, \pi_2, \ldots, \pi_{\frac{v-1}{2}}$. Using its blocks we construct $\mathcal{K} = (V^*, \mathcal{B}^*)$, a KTS($3v$) in the following manner.

The set of elements of $\mathcal{K}$ is $V^* = \{1,2,3\} \times V$, and the blocks are introduced in the following 4 types of parallel classes, $\Omega_{0,\beta}$, $\Omega_{1,\beta}$, $\Omega_{2,\beta}$ and $\Omega_{3,\beta}$.

- $\Omega_{0,\beta}$: $\Big\{\{(1,x_i),(2,x_i),(3,x_i)\} \mid 1 \le i \le v\Big\}$.

We denote every block of $\mathcal{D}$ by $\{x_i, x_j, x_k\}$, where $i < j < k$. In the following a label (m, β) for each block is its color, the block with label (m,β) is obtained by using the block a_m of $\mathcal{A}$.

- $\Omega_{1,\beta}$: $\Big\{\{(1,x_i),(1,x_j),(1,x_k)\}_{(1,\beta)},\ \{(2,x_i),(2,x_j),(2,x_k)\}_{(2,\beta)},\ \{(3,x_i),(3,x_j),(3,x_k)\}_{(3,\beta)} \mid \{x_i,x_j,x_k\} \in \pi_\beta\Big\}$, for $1 \le \beta \le \frac{v-1}{2}$,
- $\Omega_{2,\beta}$: $\Big\{\{(1,x_i),(2,x_j),(3,x_k)\}_{(4,\beta)}, \{(1,x_k),(2,x_i),(3,x_j)\}_{(5,\beta)}, \{(1,x_j),(2,x_k),(3,x_i)\}_{(6,\beta)} \mid \{x_i,x_j,x_k\} \in \pi_\beta\Big\}$, for $1 \le \beta \le \frac{v-1}{2}$,

- $\Omega_{3,\beta}$: $\Big\{\{(1,x_i),(2,x_k),(3,x_j)\}_{(7,\beta)},\{(1,x_j),(2,x_i),(3,x_k)\}_{(8,\beta)},\{(1,x_k),(2,x_j),(3,x_i)\}_{(9,\beta)} \mid \{x_i,x_j,x_k\}\in\pi_\beta\Big\}$, for $1\le\beta\le\frac{v-1}{2}$.

Figures 2 and 3 demonstrate the 4 types of blocks.

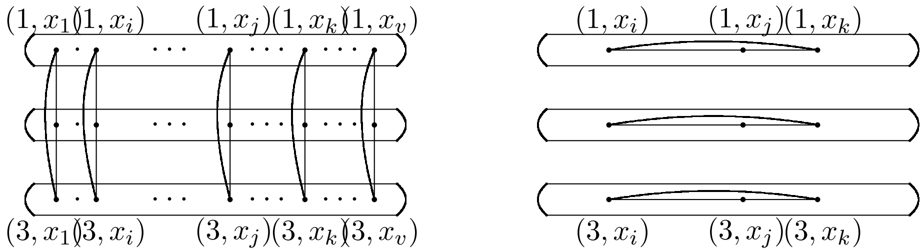

Figure 2: Blocks of $\Omega_{0,\beta}$ and $\Omega_{1,\beta}$

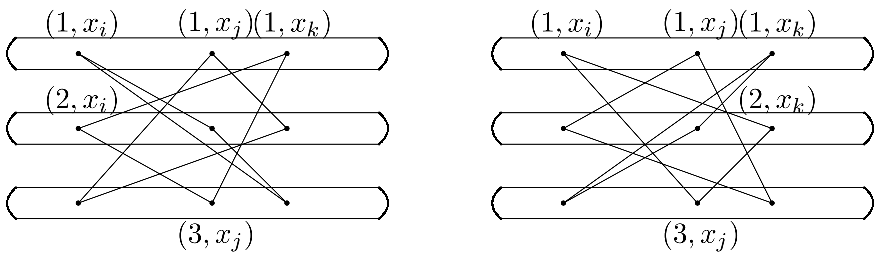

Figure 3: Blocks of $\Omega_{2,\beta}$ and $\Omega_{3,\beta}$

We note that there is only one parallel class in $\Omega_{0,\beta}$, but there are $\frac{v-1}{2}$ parallel classes in each of other types, so we have $\frac{3v-1}{2}$ parallel classes and each class has v blocks.

Clearly, $\mathcal{K}$ is a KTS$(3v)$. The number of colors needed in a silver coloring of 1-BIG$(\mathcal{K})$ is equal to $\frac{9v-7}{2}$. We color 0 the vertices corresponding to the blocks in $\Omega_{0,\beta}$ class. The label of each block in other classes, which is shown as its index, is the color of its corresponding vertex in 1-BIG$(\mathcal{K})$: (m,β), $1\le m\le 9$, $1\le\beta\le\frac{v-1}{2}$. It is easy to check that this is a proper coloring and all vertices in $\Omega_{0,\beta}$ class, i.e. the α-set, are rainbow. ■

Next theorem is a generalization of the construction introduced in Lemma 1.

Theorem 2 *Assume there exist an affine plane* $\mathcal{A}=\mathrm{AG}(2,k)$*, and a resolvable balanced incomplete block design* $\mathcal{D}=\mathrm{RBIBD}(v,k,1)$*. Then there exists a* $\mathcal{D}^*=\mathrm{RBIBD}(kv,k,1)$ *where* 1-BIG$(\mathcal{D}^*)$ *is silver.*

Proof. Let $V(\mathcal{A})=\{(i,j)\mid 1\le i,j\le k\}$ and denote its parallel classes by $\Theta_0,\Theta_1,\ldots,\Theta_k$. Specifically we let

$$\Theta_0=\Big\{\{(1,j),(2,j),\ldots,(k,j)\}\mid j=1,2,\ldots,k\Big\}.$$

Also we let $V(\mathcal{D}) = \{x_1, x_2, \ldots, x_v\}$ with parallel classes $\pi_1, \pi_2, \ldots, \pi_{\frac{v-1}{k-1}}$.

For each block $b = \{x_{s_1}, x_{s_2}, \ldots, x_{s_k}\}$ of $\mathcal{D}$ we consider an ordering on b such that

$$x_{s_i} \prec x_{s_j} \iff s_i < s_j,$$

and define a function:

$$\Psi_b : V(\mathcal{A}) \to \{1, 2, \ldots, k\} \times \{x_{s_1}, x_{s_2}, \ldots, x_{s_k}\}$$

$$\Psi_b(i, j) = (i, x_{s_j}).$$

We extend Ψ_b for each block a of $\mathcal{A}$ as $\Psi_b(a) = \{\Psi_b(i, j) | \ (i, j) \in a\}$.

Now we construct a design $\mathcal{D}^* = (V^*, \mathcal{B}^*)$, as in the following:

$V^* = \{1, 2, \ldots, k\} \times V(\mathcal{D})$.

$\mathcal{B}^* = \{\Psi_b(a) | \ b$ and a are blocks of $\mathcal{D}$ and $\mathcal{A}$, respectively$\}$.

See Figure 4.

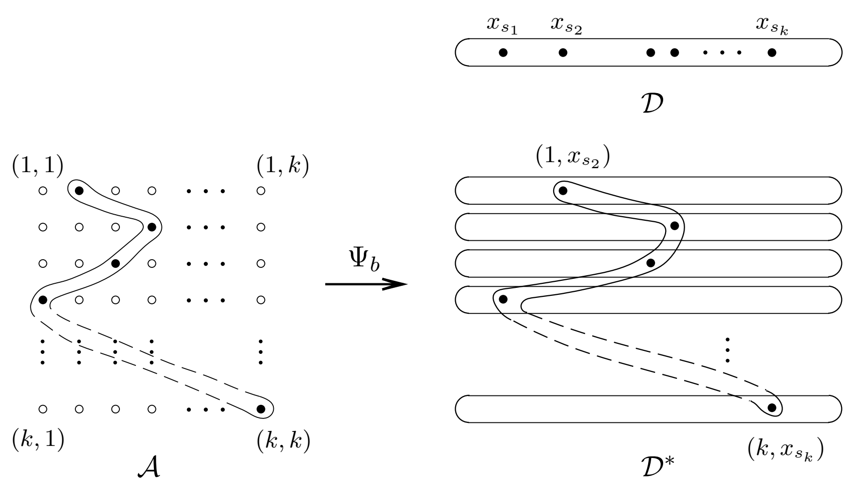

Figure 4: Blocks of $\mathcal{D}^*$ are constructed by using blocks of $\mathcal{A}$

$\mathcal{D}^*$ is an RBIBD with the following parallel classes:

$$\Omega_{\alpha,\beta} = \{\Psi_b(a) | \ a \in \Theta_\alpha, b \in \pi_\beta\}, \text{ for each } 0 \leq \alpha \leq k \text{ and } 1 \leq \beta \leq \tfrac{v-1}{k-1}.$$

Note that:

$$\Omega_{0,1} = \Omega_{0,2} = \cdots = \Omega_{0,\frac{v-1}{k-1}} = \Big\{\{(1, x_s), (2, x_s), \ldots, (k, x_s)\} | \ s = 1, 2, \ldots, v\Big\}.$$

We show that 1-BIG($\mathcal{D}^*$) is silver with respect to the α-set

$$\begin{aligned} I^* &= \{\Psi_b(a) | \ a \in \Theta_0 \text{ and } b \text{ is a block of } \mathcal{D}\} \\ &= \Big\{\{(1, x_s), (2, x_s), \ldots, (k, x_s)\} | \ s = 1, 2, \ldots, v\Big\}, \end{aligned}$$

by the following coloring:

$$c : \mathcal{B}^* \longrightarrow \{0\} \cup \{(a,\beta)|\ a \text{ is a block of } \mathcal{A} \setminus \Theta_0 \text{ and } 1 \leq \beta \leq \frac{v-1}{k-1}\}$$

$$\Psi_b(a) \longmapsto \begin{cases} 0 & \text{if } a \in \Theta_0, \\ \\ (a,\beta) & \text{if } a \notin \Theta_0, \text{ and } b \in \pi_\beta. \end{cases}$$

We show that c is a proper coloring and any vertex $b^* \in I^*$ is rainbow. Note that all the vertices of I^* have color 0. Let $\Psi_{b_1}(a_1)$ and $\Psi_{b_2}(a_2)$ be two blocks of $\mathcal{D}^*$ with the same color (a,β). Then we have $b_1, b_2 \in \pi_\beta$. Therefore b_1 and b_2 are disjoint blocks of $\mathcal{D}$, so $\Psi_{b_1}(a_1)$ and $\Psi_{b_2}(a_2)$ are disjoint. Thus c is proper.

To show silverness, for a fixed s let $b_s^* = \{(1,x_s),(2,x_s),\ldots,(k,x_s)\}$ be a block of I^*. By definition, for any given nonzero color like (a,β) we have $a \notin \Theta_0$, and there exists a unique block b of π_β which contains x_s and the color of $\Psi_b(a)$ is (a,β). Since in $\mathcal{A}$, the block a intersects each block of Θ_0, thus by definition of $\mathcal{B}^*$, $\Psi_b(a)$ intersects b_s^* in $\mathcal{D}^*$, so the color (a,β) appears in the neighborhood of b_s^*. ∎

In the next theorem for any $\mathcal{D} = S(2,k,v)$, we show a lower bound for $\alpha(G_1)$, in order $G_1 = $ 1-BIG$(\mathcal{D})$ to be silver.

Theorem 3 *Let $\mathcal{D}$ be an $S(2,k,v)$, and $G_1 = $ 1-BIG$(\mathcal{D})$. If $\alpha(G_1) > k\lfloor\frac{v(v-1)}{k^2v-k^3+k^2-k}\rfloor$, then G_1 is not silver.*

Proof. G_1 is a $\frac{k(v-k)}{(k-1)}$–regular graph with $\frac{v(v-1)}{k(k-1)}$ vertices. Let I be an α-set, and assume that G_1 has a silver coloring with respect to I with C as the set of colors, $|C| = \frac{k(v-k)}{k-1} + 1$. A color like ι exists that is used in the coloring of at most $\lfloor\frac{|V(G_1)|}{|C|}\rfloor = \lfloor\frac{v(v-1)}{k^2v-k^3+k^2-k}\rfloor$ vertices of G_1. For a set $X \subseteq V(G_1)$ we denote the set of vertices with color ι in X by $X(\iota)$. By counting the number of appearances of color ι in I and in the neighborhood of I we obtain,

$$\begin{aligned} \alpha(G_1) &= |I(\iota)| + |X_1(\iota)| + 2|X_2(\iota)| + \cdots + k|X_k(\iota)| \\ &\leq k(|I(\iota)| + |X_1(\iota)| + |X_2(\iota)| + \cdots + |X_k(\iota)|) \\ &\leq k\lfloor\tfrac{v(v-1)}{k^2v-k^3+k^2-k}\rfloor \\ &< \alpha(G_1). \end{aligned}$$

A contradiction. ∎

Example 4 *It is easy to check that for any of two* STS(13)*s, $\alpha(G_1) = 4$. For* 80 *nonisomorphic* STS(15)*s, we have $\alpha(G_1) = 4$ or* 5 (see [5], page 32)*. Also there are* 18 *nonisomorphic* $S(2,4,25)$ (see [5], page 34)*, by a computer search they have* $\alpha(G_1) = 5$ or 6*. So by Theorem 3 none of them has a silver G_1.*

Remark 2 *Let G_1 be the 1-block intersection graph of an $S(2,k,v)$ with a parallel class. Then $\alpha(G_1) = \frac{v}{k}$, and all the elements of V appear in the blocks corresponding to each α-set. Let I be an α-set for G_1, therefore any vertex of $V(G_1) \setminus I$ is adjacent to k vertices of I. Thus $|X_1| = |X_2| = \cdots = |X_{k-1}| = 0$, $|X_k| = \frac{v(v-k)}{k(k-1)}$.*

If an $S(2,k,v)$ has a near parallel class, then $\alpha(G_1) = \frac{v-1}{k}$, and each α-set contains all the elements of V except one. Hence in this case any vertex of $V(G_1) \setminus I$ is adjacent to either $(k-1)$ or k vertices of I, and $|X_1| = |X_2| = \cdots = |X_{k-2}| = 0$, $|X_{k-1}| = \frac{v-1}{k-1}$, $|X_k| = \frac{(v-1)(v-2k+1)}{k(k-1)}$.

Theorem 4 *Let $\mathcal{D}$ be an $S(2,k,v)$, with a near parallel class. Then $G_1 = 1\text{-BIG}(\mathcal{D})$ is not silver.*

Proof. Let I be an α-set for G_1. Assume that G_1 has a silver coloring with respect to I and C is the set of colors. G_1 is $\frac{k(v-k)}{k-1}$–regular, $|C| = \frac{k(v-k)}{k-1} + 1$ and $|I| = \frac{v-1}{k}$. By Remark 2, $|X_{k-1}| = \frac{v-1}{k-1}$ and $|X_k| = \frac{(v-1)(v-2k+1)}{k(k-1)}$. Since $|C| > |I \cup X_{k-1}|$, a color like ι exists that is used only in the coloring of vertices of X_k. The vertices of I are rainbow, so each of the vertices of X_k that have color ι, must be adjacent to k different vertices of I. Thus $|I|$ is a multiple of k, say $|I| = mk$.

Since $|X_{k-1}| = \frac{v-1}{k-1} > |I|$, a color like ι' exists that is used in the coloring of vertices of X_{k-1} but is not used in I. The induced subgraph on X_{k-1} is a clique, so ι' appears only in one vertex of X_{k-1} and it has $(k-1)$ neighbors in I. Thus $|I| - k + 1$ vertices of I, each must have a neighbor in X_k with color ι'. Again vertices from X_k that have color ι', each must be adjacent to k different vertices of I. Therefore $|I| - k + 1 = (m-1)k + 1$ is also a multiple of k. This is impossible. ■

Example 5 *The 1-block intersection graph of any Hanani triple system* (see [5], page 67 for the definition) *is not silver.*

Note that by Theorems 1, 2, 3, and 4, for any admissible v the problem of existence of a silver 1-BIG($S(2,k,v)$) which possesses maximum possible independent set is settled.

3 Zero block intersection graphs

In this section we discuss 0-block intersection graphs of $S(2,k,v)$.

Notation 1 *Let x be a given element of $S(2,k,v)$, and denote by $T(x)$ the set of $\frac{v-1}{k-1}$ blocks containing x.*

It is trivial that $T(x)$ is an independent set for G_0, thus $\alpha(G_0) \geq \frac{v-1}{k-1}$.

Lemma 2 *Let $\mathcal{D}$ be an $S(2,k,v)$, and $G_0 = 0\text{-BIG}(\mathcal{D})$. If $v > k^3 - 2k^2 + 2k$ then any maximum independent set of G_0 is of the form $T(x)$, therefore $\alpha(G_0) = \frac{v-1}{k-1}$.*

Proof. Let I be an α-set of G_0. Suppose I is not of the form $T(x)$. There exists an element x_0 of $\mathcal{D}$ which appears in at least two blocks of I. Let $I_1 = \{B_1, B_2, \ldots, B_p\} = \{B \mid B \in I \cap T(x_0)\}$, and $I \setminus I_1 = \{B_{p+1}, B_{p+2}, \ldots, B_{p+q}\}$. Since $\lambda = 1$, for $1 \leq i < j \leq p$, $(B_i \setminus \{x_0\}) \cap (B_j \setminus \{x_0\}) = \emptyset$. Every two blocks in I have one intersection. So, for each block $B \in I \setminus I_1$ we have $B \cap B_i = \{a_i\}$, $i = 1, 2, \ldots, p$. So $p \leq |B| = k$.

Now suppose $B_1, B_2 \in I_1$. There exist exactly $(k-1)^2$ pairs $\{x, y\}$ where $x \in B_1 \setminus \{x_0\}$ and $y \in B_2 \setminus \{x_0\}$, and each of these pairs appears at most in one of the blocks of $I \setminus I_1$. Thus $q \leq (k-1)^2$.

So $|I| = p + q \leq k + (k-1)^2$. But since $v > k^3 - 2k^2 + 2k$, for each x we have $|T(x)| = \frac{v-1}{k-1} > k + (k-1)^2 \geq |I|$. Hence the statement follows. ■

Theorem 5 *Let $\mathcal{D}$ be an $S(2,k,v)$. For $v > k^3 - 2k^2 + 2k$, $G_0 = 0\text{-BIG}(\mathcal{D})$ is not silver.*

Proof. G_0 is a $\frac{v^2+k^3-v(k^2+1)-k^2+k}{k(k-1)}$–regular graph (Remark 1). Let I be any α-set for G_0. By Lemma 2, $I = T(x)$ and $|I| = \alpha(G_0) = \frac{v-1}{k-1}$. Since each block out of I intersects exactly k blocks of I, each vertex of $V(G_0) \setminus I$ is adjacent to $\frac{v-1}{k-1} - k = \frac{v-1-k^2+k}{k-1}$ vertices of I. Then $V(G_0) = I \cup X_{\frac{v-1-k^2+k}{k-1}}$ and $|X_{\frac{v-1-k^2+k}{k-1}}| = \frac{(v-1)(v-k)}{k(k-1)}$.

To the contrary, G_0 has a silver coloring with respect to I. Let C be the set of colors, $|C| = \frac{v^2-v-k^2v+k^3}{k(k-1)}$. Since $|C| > \frac{v-1}{k-1}$, a color like ι exists that is not used in the coloring of I. The vertices of I are rainbow, and the vertices from $X_{\frac{v-1-k^2+k}{k-1}}$ that have color ι, each must be adjacent to $\frac{v-1-k^2+k}{k-1}$ different vertices of I. Therefore $|I|$ must be divisible by $\frac{v-1-k^2+k}{k-1}$, then $(v - k^2 + k - 1) \mid (v-1)$ which is impossible, since $v > k^3 - 2k^2 + 2k$. Therefore graph G_0 is not silver with respect to any α-set. ■

3.1 0-BIG for Steiner triple systems

Both 0-BIG(STS(v)) for $v = 7$ and $v = 9$, by Example 2, are totally silver.

Theorem 6 *For any admissible $v > 9$, $G_0 = 0\text{-BIG(STS}(v))$ is not silver.*

Proof. For $v > 15$, it follows by Theorem 5.
If $v \leq 15$, then suppose I is an α-set of G_0, and I is not of the form $T(x)$. Then it is easy to check that, each element of STS(v) appears at most in 3 blocks of I. If it has 3 blocks containing an element x, then such a set has at most 7 blocks, and they are contained in I_1, where:

$$I_1 = \{\{x,a,b\},\{x,c,d\},\{x,e,f\},\{a,c,f\},\{a,d,e\},\{b,c,e\},\{b,d,f\}\} \approx \mathrm{STS}(7).$$

Now we discuss possible cases.

$v = 15$:

For $v = 15$ an α-set, I, may be of the form $T(x)$ or it may come from a subsystem STS(7), in either case $\alpha(G_0) = 7$. From 80 non-isomorphic STS(15)s, 23 of them have a subsystem STS(7) ([5], page 32). It is straightforward to check that in all of STS(15)s for any α-set I, each block out of I has intersection with exactly three blocks of I. So each vertex in $V(G_0) \setminus I$ is adjacent to exactly four vertices of I. In any silver coloring with C as the set of colors of G_0, we have $|C| = 17 > 7 = |I|$. So there exists a color ι which is not used in I. Every vertex with the color ι has exactly 4 neighbors in I, therefore 7 must be a multiple of 4. So G_0 does not have a silver coloring.

$v = 13$:

For $v = 13$ there are two non-isomorphic STS(13)s. No STS(13) has a subsystem of STS(7), even no STS(13) has 6 blocks of an STS(7). So, in G_0 for both of them, the sets of the form $T(x)$, are the only α-sets and $\alpha(G_0) = 6$. Suppose I is any α-set.
First, we show that it is always possible to find three vertices in I with no common neighbor:

- One of two STS(13)s, Type 1, has a cyclic automorphism, and we can construct its blocks on $\{1, 2, \ldots, 13\}$ by the following base blocks:

$$\{1,2,5\}, \quad \{1,3,8\} \quad \mod 13.$$

  If $I = T(1)$, then $B_1 = \{1,2,5\}$, $B_2 = \{1,3,8\}$, and $B_3 = \{1,10,11\}$ do not have common neighbor. Let $x \neq 1$ be a given element of STS(v), and $I = T(x)$. Three vertices of I, B'_1, B'_2, B'_3 are obtained by adding $(x-1)$ to all members of blocks B_1, B_2, B_3, do not have common neighbor.

- The other STS(13) is non-cyclic and we can construct its blocks from Type 1 by replacing four blocks of trade T_1 with four blocks of trade T_2 as follows:

$$T_1: \begin{array}{ccc} 1 & 2 & 5 \\ 1 & 3 & 8 \\ 10 & 2 & 8 \\ 10 & 3 & 5 \end{array} \qquad T_2: \begin{array}{ccc} 1 & 2 & 8 \\ 1 & 3 & 5 \\ 10 & 2 & 5 \\ 10 & 3 & 8 \end{array}$$

Let $I = T(x)$ for some x. If x is an element of T_2, i.e. $x \in \{1, 2, 3, 5, 8, 10\}$, then there are two blocks say B_1 and B_2 of T_2 which contain x. There exists one element y, such that $y \in T_2$ but $y \notin B_1 \cup B_2$. We consider B_3, the block containing x and y. Then these three blocks do not have common neighbor. If x is not in T_2, then we consider several cases for $I = T(x)$, and show that there exist three vertices of I, which do not have common neighbor.

Now, assume for some STS(13), $G_0 = $ 0-BIG(STS(13)) is silver with respect to some α-set $I = T(x) = \{B_1, B_2, B_3, B_4, B_5, B_6\}$. The color of all neighbors of B_i, $i = 1, \dots, 6$, must be distinct. Assume $\{B_1, B_2, B_3\} \subset I$ do not have common neighbor. Let $N(B_i)$ be the set of neighbors of B_i. $G_0 = SRG(26, 10, 3, 4)$, so $|N(B_1) \cap N(B_2)| + |N(B_2) \cap N(B_3)| + |N(B_1) \cap N(B_3)| = 12$. Thus the color of these vertices must be distinct, while we have only 11 colors. Therefore G_0 does not have a silver coloring. ■

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