

A generalization of Opsut's lower bounds for the competition number of a graph

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Abstract

The notion of a competition graph was introduced by J. E. Cohen in 1968. The *competition graph* $C(D)$ of a digraph D is a (simple undirected) graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if there exists a vertex v in D such that (x, v) and (y, v) are arcs of D . For any graph G , G together with sufficiently many isolated vertices is the competition graph of some acyclic digraph. In 1978, F. S. Roberts defined the *competition number* $k(G)$ of a graph G as the minimum number of such isolated vertices. In general, it is hard to compute the competition number $k(G)$ for a graph G and it has been one of the important research problems in the study of competition graphs to characterize a graph by its competition number. In 1982, R. J. Opsut gave two lower bounds for the competition number of a graph. In this paper, we give a generalization of these two lower bounds for the competition number of a graph.

Keywords: Competition graph; Competition number; Vertex clique cover number; Edge clique cover number

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1. Introduction

Throughout this paper, all graphs G are finite, simple, and undirected. The notion of a competition graph was introduced by J. E. Cohen [1] in connection with a problem in ecology. The *competition graph* $C(D)$ of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if there exists a vertex v in D such that (x, v) and (y, v) are arcs of D . For any graph G , G together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. From this observation, F. S. Roberts [14] defined the *competition number* $k(G)$ of a graph G to be the minimum number k such that G together with k isolated vertices is the competition graph of an acyclic digraph:

$$k(G) := \min\{k \in \mathbb{Z}_{\geq 0} \mid G \cup I_k = C(D) \text{ for some acyclic digraph } D\}, \quad (1.1)$$

where I_k denotes a set of k isolated vertices.

A digraph is said to be *acyclic* if it contains no directed cycles. For a digraph D , an ordering v_1, v_2, \dots, v_n of the vertices of D is called an *acyclic ordering* of D if $(v_i, v_j) \in A(D)$ implies $i < j$. It is well-known that a digraph D is acyclic if and only if there exists an acyclic ordering of D . For a vertex v in a digraph D , the *out-neighborhood* of v in D is defined to be the set $\{w \in V(D) \mid (v, w) \in A(D)\}$ and is denoted by $N_D^+(v)$, and the *in-neighborhood* of v in D is defined to be the set $\{u \in V(D) \mid (u, v) \in A(D)\}$ and is denoted by $N_D^-(v)$.

For a vertex v in a graph G , the *(open) neighborhood* of v in G is defined to be the set $\{u \in V(G) \mid uv \in E(G)\}$ and is denoted by $N_G(v)$. A subset $S \subseteq V(G)$ of the vertex set of a graph G is called a *clique* of G if the subgraph $G[S]$ of G induced by S is a complete graph. For a clique S of a graph G and an edge e of G , we say e is *covered by* S if both of the endpoints of e are contained in S . An *edge clique cover* of a graph G is a family of cliques of G such that each edge of G is covered by some clique in the family (see [15] for applications of edge clique covers). The *edge clique cover number* $\theta_E(G)$ of a graph G is the minimum size of an edge clique cover of G . A *vertex clique cover* of a graph G is a family of cliques of G such that each vertex of G is contained in some clique in the family. The *vertex clique cover number* $\theta_V(G)$ of a graph G is the minimum size of a vertex clique cover of G .

R. D. Dutton and R. C. Brigham [2] characterized the competition graphs of acyclic digraphs in terms of edge clique covers. (F. S. Roberts and J. E. Steif [16] characterized the competition graphs of loopless digraphs. J. R. Lundgren and J. S. Maybee [9] gave a characterization of graphs whose competition number is at most m .)

However, R. J. Opsut [10] showed that the problem of determining whether a graph is the competition graph of an acyclic digraph or not is NP-complete. It follows that the computation of the competition number of a graph is an NP-hard problem, and thus it does not seem to be easy in general to compute $k(G)$ for an arbitrary graph G (see

[3] and [8] for graphs whose competition numbers are known). It has been one of the important research problems in the study of competition graphs to characterize a graph by its competition number.

R. J. Opsut gave the following two lower bounds for the competition number of a graph.

Theorem 1.1 (Opsut [10, Proposition 5]). *For any graph G ,*

$$k(G) \geq \theta_E(G) - |V(G)| + 2. \quad (1.2)$$

Theorem 1.2 (Opsut [10, Proposition 7]). *For any graph G ,*

$$k(G) \geq \min\{\theta_V(N_G(v)) \mid v \in V(G)\}. \quad (1.3)$$

These seem to be the only known sharp lower bounds for an arbitrary graph G .

In this paper, we give a generalization of these two lower bounds which contains both as special cases. In particular, our main result contains both lower bounds given in Theorems 1.1 and 1.2 as special cases. The proof of our main result is elementary, but the new lower bound given in this paper would be a strong tool in the study of the competition number of a graph.

2. Main Result

Let G be a graph and $F \subseteq E(G)$ be a subset of the edge set of G . An *edge clique cover* of F in G is a family of cliques of G such that each edge in F is covered by some clique in the family. We define the *edge clique cover number* $\theta_E(F; G)$ of $F \subseteq E(G)$ in G as the minimum size of an edge clique cover of F in G :

$$\theta_E(F; G) := \min\{|\mathcal{S}| \mid \mathcal{S} \text{ is an edge clique cover of } F \text{ in } G\}. \quad (2.1)$$

By definition, it follows that the edge clique cover number $\theta_E(E(G); G)$ of $E(G)$ in a graph G is equal to the edge clique cover number $\theta_E(G)$ of the graph G .

Let G be a graph and $U \subseteq V(G)$ be a subset of the vertex set of G . We define

$$N_G[U] := \{v \in V(G) \mid v \text{ is adjacent to a vertex in } U\} \cup U, \quad (2.2)$$

$$E_G[U] := \{e \in E(G) \mid e \text{ has an endpoint in } U\}. \quad (2.3)$$

We denote by the same symbol $N_G[U]$ the subgraph of G induced by $N_G[U]$. Note that $E_G[U]$ is contained in the edge set of the subgraph $N_G[U]$. We denote by $\binom{V}{m}$ the set of all m -subsets of a set V .

Now we are ready to state our main result.

Theorem 2.1. *Let $G = (V, E)$ be a graph. Then*

$$k(G) \geq \max_{m \in \{1, \dots, |V|\}} \min_{U \in \binom{V}{m}} \left(\theta_E(E_G[U]; N_G[U]) - |U| + 1 \right). \quad (2.4)$$

To prove our main theorem, we show the following lemma.

Lemma 2.2. *Let $G = (V, E)$ be a graph. Let m be an integer such that $1 \leq m \leq |V|$. Then*

$$k(G) \geq \min_{U \in \binom{V}{m}} \theta_E(E_G[U]; N_G[U]) - m + 1. \quad (2.5)$$

Proof. Let $k := k(G)$ for convenience. Fix an integer m such that $1 \leq m \leq |V|$. Let D be a minimal acyclic digraph with respect to the number of arcs such that $C(D) = G \cup I_k$, where $I_k := \{z_1, \dots, z_k\}$ is a set of k isolated vertices. Let $v_1, \dots, v_n, z_1, \dots, z_k$ be an acyclic ordering of D , and let $W := \{v_{n-m+1}, \dots, v_n\}$. Note that $|W| = m$. Let

$$\mathcal{S} := \{N_D^-(w) \cap N_G[W] \mid w \in (W \cup I_k) \setminus \{v_{n-m+1}\}\}.$$

For each $w \in (W \cup I_k) \setminus \{v_{n-m+1}\}$, since $N_D^-(w)$ forms a clique of the graph G , the set $N_D^-(w) \cap N_G[W]$ forms a clique of the induced subgraph $N_G[W]$ of G . Thus \mathcal{S} is a family of cliques of $N_G[W]$.

Take any edge $e = uv \in E_G[W]$, where $u \in W$ and $v \in N_G(u)$. Since u and v are adjacent, there exists a common out-neighbor $w \in N_D^+(u) \cap N_D^+(v)$. Then $\{u, v\} \subseteq N_D^-(w)$. Since $v_1, \dots, v_n, z_1, \dots, z_k$ is an acyclic ordering of D , the out-neighborhood $N_D^+(u)$ of u in D is contained in the set $(W \cup I_k) \setminus \{v_{n-m+1}\}$ for each vertex $u \in W$. Therefore $N_D^+(u) \cap N_D^+(v) \subseteq (W \cup I_k) \setminus \{v_{n-m+1}\}$ and so $w \in (W \cup I_k) \setminus \{v_{n-m+1}\}$. Then it follows that the edge e is covered by $N_D^-(w) \cap N_G[W] \in \mathcal{S}$.

Thus the family \mathcal{S} is an edge clique cover of $E_G[W]$ in $N_G[W]$. This implies that $\theta_E(E_G[W]; N_G[W]) \leq |\mathcal{S}| = m + k - 1$, that is, $\theta_E(E_G[W]; N_G[W]) - m + 1 \leq k$. Hence

$$\min_{U \in \binom{V}{m}} \theta_E(E_G[U]; N_G[U]) - m + 1 \leq \theta_E(E_G[W]; N_G[W]) - m + 1 \leq k(G),$$

and the lemma holds. \square

Proof of Theorem 2.1. Since the inequality (2.5) holds for any $m \in \{1, \dots, |V|\}$, it follows that the inequality (2.4) holds. \square

Remark 2.3. Consider the case $m = 1$ in the inequality (2.5). Then we obtain

$$k(G) \geq \min_{v \in V(G)} \theta_E(E_G[v]; N_G[v]).$$

Since a family $\{S_1, \dots, S_r\}$ of cliques is an edge clique cover of $E_G[v]$ in G if and only if $\{S_1 \cap N_G[v], \dots, S_r \cap N_G[v]\}$ is an edge clique cover of $E_G[v]$ in $N_G[v]$, it holds that $\theta_E(E_G[v]; N_G[v]) = \theta_E(E_G[v]; G)$. Since a family $\{S_1, \dots, S_r\}$ of cliques is an edge clique cover of $E_G[v]$ in G if and only if $\{S_1 \setminus \{v\}, \dots, S_r \setminus \{v\}\}$ is a vertex clique cover of $N_G(v)$ in G , it holds that $\theta_E(E_G[v]; G) = \theta_V(N_G(v))$. Therefore we have $\theta_E(E_G[v]; N_G[v]) = \theta_V(N_G(v))$. Hence the above inequality coincides with the lower bound (1.3) in Theorem 1.2.

Remark 2.4. Consider the case $m = |V| - 1$ in the inequality (2.5). Then we obtain

$$k(G) \geq \min_{v \in V} \theta_E(E_G[V \setminus \{v\}]; N_G[V \setminus \{v\}]) - |V| + 2.$$

Since $G = (V, E)$ has no loops, it holds that $E_G[V \setminus \{v\}] = E$. If the vertex v is not isolated in G , then we have $N_G[V \setminus \{v\}] = V$ and thus $\theta_E(E_G[V \setminus \{v\}]; N_G[V \setminus \{v\}]) = \theta_E(E; G) = \theta_E(G)$. If v is an isolated vertex, then we have $N_G[V \setminus \{v\}] = V \setminus \{v\}$ and thus $\theta_E(E_G[V \setminus \{v\}]; N_G[V \setminus \{v\}]) = \theta_E(E; G - \{v\}) = \theta_E(E; G) = \theta_E(G)$. Hence the above inequality coincides with the lower bound (1.2) in Theorem 1.1.

Remark 2.5. The new lower bound given in Theorem 2.1 is a strong tool to compute the exact values of the competition numbers of graphs, especially for symmetric graphs such as complete multipartite graphs, Johnson graphs, Hamming graphs, etc (see [4], [5], [6], [7], [11], [12], [13], [17]).

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