# Rainbow connection in 3-connected graphs* 

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#### Abstract

An edge-colored graph $G$ is rainbow connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection number of a connected graph $G$, denoted by $r c(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow connected. In this paper, we proved that $r c(G) \leq 3(n+1) / 5$ for all 3 -connected graphs.


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## 1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the notation and terminology of Bondy and Murty [1]. An edgecolored graph $G$ is rainbow connected if any two vertices are connected by a path whose edges have distinct colors. Obviously, if $G$ is rainbow connected, then it is also connected. This concept of rainbow connection in graphs was introduced by Chartrand et al. in [5]. The rainbow connection number of a connected graph $G$, denoted by $r c(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow connected. An easy observation is

[^0]that if $G$ is of order $n$ then $\operatorname{rc}(G) \leq n-1$, since one may color the edges of one spanning tree of $G$ with different colors and the remaining edges with colors already used. It is easy to verify that $r c(G)=1$ if and only if $G$ is a complete graph, that $r c(G)=n-1$ if and only if $G$ is a tree. Notice that for the cycle $C_{n}$ of order $n, r c\left(C_{n}\right)=\lceil n / 2\rceil$. It was shown that computing the rainbow connection number of an arbitrary graph is NP-hard [2].

There are some approaches to study the bounds of $r c(G)$ with respect to the minimum degree $\delta(G)$. In [2] Caro et al. have shown that if $G$ is a graph of order $n$ with minimum degree $\delta$, then $r c(G)<\min \{(\ln \delta / \delta) n(1+$ $\left.\left.o_{\delta}(1)\right),(4 \ln \delta+3) n / \delta\right\}$. By employing the method of 2 -step dominating set, Krivelevich and Yuster [6] have shown that a connected graph $G$ with $n$ vertices and minimum degree $\delta$ has $r c(G)<20 n / \delta$. Schiermeyer [7] proved that $r c(G)<3 n / 4$ for graphs with minimum degree three. Very recently, Chandran et al. [4] have improved the upper bound of Krivelevich and Yuster by showing that for every connected graph $G$ of order $n$ and minimum degree $\delta, r c(G) \leq 3 n /(\delta+1)+3$.

With respect to the the relation between $r c(G)$ and the connectivity $\kappa(G)$, in [7], the author mentioned that Hajo Broersma asked a question at the IWOCA workshop:

Problem 1 What happens with the value $r c(G)$ for graphs with higher connectivity.

Schiermeyer [7] have shown that if $G$ is a graph of order $n$ with $\kappa(G)=1$ and $\delta \geq 3$, then $r c(G) \leq(3 n-1) / 4$. In [2] Caro et al. proved that if $\kappa(G)=2$ then $r c(G) \leq 2 n / 3$. From the result of Chandran et al. [4], we can easily obtain an upper bound of the rainbow connection number:

$$
r c(G) \leq \frac{3 n}{\delta+1}+3 \leq \frac{3 n}{\kappa(G)+1}+3 .
$$

Therefore, for $\kappa(G)=3, r c(G) \leq 3 n / 4+3$, and $\kappa(G)=4, r c(G) \leq 3 n / 5+3$. In this paper, motivated by the results in [2], we will improve this bound by showing the following result.

Theorem 1 If $G$ is a 3 -connected simple graph with $n$ vertices, then $r c(G) \leq$ $3(n+1) / 5$.

Before proceeding, we recall the fan lemma, which will be used frequently in the sequel.

Lemma 1 (The Fan Lemma) Let $G$ be a $k$-connected graph, $x$ a vertex of $G$, and let $Y \subseteq V-\{x\}$ be a set of at least $k$ vertices of $G$. Then there exists a $k$-fan in $G$ from $x$ to $Y$, namely there exists a family of $k$ internally disjoint $(x, Y)$-paths whose terminal vertices are distinct in $Y$.

## 2 Proof of Theorem 1.

Let $H$ be a maximal connected subgraph of $G$ satisfying that $r c(H) \leq$ $3 h / 5-1 / 5$, where $h$ is the number of vertices of $H$.

We first claim the existence of $H$. If $G$ contains a triangle $C_{3}$, then we can choose the triangle as $H$, since $r c\left(C_{3}\right)=1<8 / 5$. If $G$ contains $C_{k}$ $(k \geq 4$ and $k \neq 5)$ as a subgraph, then we take $H=C_{k}$, since $r c\left(C_{k}\right)=$ $\lceil k / 2\rceil \leq 3 k / 5-1 / 5$. Now suppose all the cycles contained in $G$ are of length 5 , then we can take $H$ as the graph obtained by adding one pendent edge to $C_{5}$. Observe that $h=6$ and $r c(H)=3<17 / 5$.

We next claim that $h \geq n-3$. By contradiction. Suppose there are four distinct vertices outside of $H$, denoted by $x_{1}, x_{2}, x_{3}, x_{4}$. Then by the fan lemma, each of them has three internally disjoint paths to $H$.

We assume first each of $x_{1}, x_{2}, x_{3}, x_{4}$ has three neighbors in $H$. Let $f_{i j}$ be the edges incident to the vertex $x_{i}, j=1,2,3$. We can add $x_{1}, x_{2}, x_{3}, x_{4}$ to $H$, and form a lager subgraph $H^{\prime}$ with $h+4$ vertices. Now we use only two new colors 1 and 2 to color the 12 edges. Assigning color 1 to edges $f_{i 1}$ for $i=1,2,3$ and color 2 to other 9 edges. Then we have

$$
r c\left(H^{\prime}\right) \leq r c(H)+2 \leq 3 h / 5-1 / 5+2<3(h+4) / 5-1 / 5
$$

contradicting to the choice of $H$.
It follows that at least one of these four vertices, say $x$, has the property that one of the three internally disjoint $(x, H)$-paths $P_{0}, P_{1}, P_{2}$ has length at least two. Furthermore, among all vertices satisfying the above property, we choose vertex $x$ such that one of the three paths has length one, say $P_{0}=e_{0}$, and that the sum of lengths of $P_{1}$ and $P_{2}$ is as large as possible. Denote $P_{1}=a u_{1} u_{2} \ldots u_{s} x$ and $P_{2}=x v_{1} v_{2} \ldots v_{t} b$ with $a, b \in H$ and $u_{i}, v_{j} \notin H$ for all $i$ and $j$. With loss of generality, we assume $t \geq s$, and then $t \geq 1$. We first assume $s+t \geq 3$. We can add $v_{1}, v_{2}, \ldots, v_{s}, x, u_{1}, u_{2}, \ldots, u_{t}$ to $H$ and form a larger subgraph $H^{\prime}$ with $h+s+t+1$ vertices. If $s+t$ is even, then we can color the $s+t+2$ edges of path $a u_{1} u_{2} \ldots u_{s} x v_{1} v_{2} \ldots v_{t} b$ with $(s+t+2) / 2$ new
colors. In the first half of the path the colors are all distinct, and the same ordering of colors is repeated in the second half of the path. We can color edge $e_{0}$ with any color already appeared in $H$, and then it is straightforward to verify that $H^{\prime}$ is rainbow connected. If $s+t$ is odd, then we can color the $s+t+2$ edges of path $a u_{1} u_{2} \ldots u_{s} x v_{1} v_{2} \ldots v_{t} b$ with $(s+t+1) / 2$ new colors as follows. The middle edge of the path and edge $e_{0}$ get any color that already used in $H$. The first $(s+t+1) / 2$ edges of the path all receive distinct new colors, and in the last $(s+t+1) / 2$ edges of the path this coloring is repeated in the same order. Again it is straightforward to verify that $H^{\prime}$ is rainbow connected. We now have

$$
\begin{aligned}
r c\left(H^{\prime}\right) & \leq r c(H)+\lceil(s+t+1) / 2\rceil \\
& \leq 3 h / 5-1 / 5+\lceil(s+t+1) / 2\rceil \leq 3(h+s+t+1) / 5-1 / 5,
\end{aligned}
$$

contradicting the maximality of $H$. Hence, we only assume $1 \leq s+t \leq 2$. We consider three cases as follows.


Figure 1: $s+t=2$ and $P_{1}=a u_{1} x, P_{2}=x v_{1} b$

Case 1. $s+t=2$ and $P_{1}=a u_{1} x, P_{2}=x v_{1} b$ (see Figure 1(A)).
Since there are at least 4 vertices outside of $H$, there exists at least one vertex different from $x, u_{1}$ and $v_{1}$, say $x_{1}$. By the choice of $x$, there is no $\left(x_{1}, x\right)$-path, $\left(x_{1}, u_{1}\right)$-path and $\left(x_{1}, v_{1}\right)$-path without using any vertex of $H$ except one case: there is one path of length two joining $x$ to $H$ through $x_{1}$, say $P_{3}=x x_{1} c$ with $c \in H$. In this case, we only consider the three paths $P_{1}$, $P_{2}$ and $P_{3}$ (as shown in Figure 2(A)). We can add vertices $x, u_{1}, v_{1}$ and $x_{1}$ to $H$ and form a larger subgraph $H^{\prime}$ with $h+4$ vertices. By assigning color 1 to edges $a u_{1}, b v_{1}$, color 2 to edges $u_{1} x, c x_{1}$, and one color already appeared in $H$ to edges $v_{1} x, x x_{1}$, we have a contradiction as

$$
r c\left(H^{\prime}\right) \leq r c(H)+2 \leq 3 h / 5-1 / 5+2<3(h+4) / 5-1 / 5 .
$$



Figure 2: Graphs used in Case 1 and Case 2.

Now by the fan lemma, there are three internally disjoint $\left(x_{1}, H\right)$-paths $P_{0}^{\prime}, P_{1}^{\prime}, P_{2}^{\prime}$. By the choice of $x$, the lengths of $P_{0}^{\prime}, P_{1}^{\prime}, P_{2}^{\prime}$ only have four possibilities:

Subcase 1.1. $1,1,1$. Let $P_{0}^{\prime}=e_{0}^{\prime}, P_{1}^{\prime}=e_{1}^{\prime}, P_{2}^{\prime}=e_{2}^{\prime}$. We can add $x, u_{1}, v_{1}$ and $x_{1}$ to $H$, and form a larger graph $H^{\prime}$ of order $h+4$. By assigning color 1 to $a u_{1}, e_{0}, x v_{1}, e_{0}^{\prime}, e_{1}^{\prime}$, and color 2 to $u_{1} x, v_{2} b, e_{2}^{\prime}$, as shown in Figure $1(1)$, we can obtain that $r c\left(H^{\prime}\right) \leq r c(H)+2 \leq 3 h / 5-1 / 5+2<$ $3(h+4) / 5-1 / 5$, a contradiction.

Subcase 1.2. 1, 1, 2. Let $P_{0}^{\prime}=e_{0}^{\prime}, P_{1}^{\prime}=e_{1}^{\prime}, P_{2}^{\prime}=x_{1} v_{1}^{\prime} b^{\prime}$. We can add $x, u_{1}, v_{1}, x_{1}$ and $v_{1}^{\prime}$ to $H$, and form a larger graph $H^{\prime}$ of order $h+5$. By coloring all edges of paths $P_{0}, P_{1}, P_{2}$ the same as Subcase 1.1 and assigning color 1 to $e_{0}^{\prime}, x_{1} v_{1}^{\prime}$, color 2 to $e_{1}^{\prime}$, and color 3 to $v_{1}^{\prime} b^{\prime}$, as shown in Figure $1(2)$, we can obtain that $r c\left(H^{\prime}\right) \leq r c(H)+3 \leq 3 h / 5-1 / 5+3=3(h+5) / 5-1 / 5$, a contradiction.

Subcase 1.3. 1, 2, 2. Let $P_{0}^{\prime}=e_{0}^{\prime}, P_{1}^{\prime}=a^{\prime} u_{1}^{\prime} x_{1}, P_{2}^{\prime}=x_{1} v_{1}^{\prime} b^{\prime}$. We can add $x, u_{1}, v_{1}, x_{1}, u_{1}^{\prime}$ and $v_{1}^{\prime}$ to $H$, and form a larger graph $H^{\prime}$ of order $h+6$. By coloring all edges of paths $P_{0}, P_{1}, P_{2}$ the same as Subcase 1.1 and assigning color 1 to $a^{\prime} u_{1}^{\prime}, x_{1} v_{1}^{\prime}$, color 2 to $u_{1}^{\prime} x_{1}$, and color 3 to $e_{0}^{\prime}, v_{1}^{\prime} b^{\prime}$, as shown in Figure $1(3)$, we can obtain that $r c\left(H^{\prime}\right) \leq r c(H)+3 \leq 3 h / 5-1 / 5+3<$ $3(h+6) / 5-1 / 5$, a contradiction.

Subcase 1.4. 1, 1, 3. Let $P_{0}^{\prime}=e_{0}^{\prime}, P_{1}^{\prime}=e_{1}^{\prime}, P_{2}^{\prime}=x_{1} v_{1}^{\prime} v_{2}^{\prime} b^{\prime}$. We can add $x, u_{1}, v_{1}, x_{1}, v_{1}^{\prime}$ and $v_{2}^{\prime}$ to $H$, and form a larger graph $H^{\prime}$ of order $h+6$. By coloring all edges of paths $P_{0}, P_{1}, P_{2}$ the same as Subcase 1.1 and assigning color 1 to $v_{1}^{\prime} v_{2}^{\prime}$, color 2 to $x_{1} v_{1}^{\prime}$, and color 3 to $e_{0}^{\prime}, e_{1}^{\prime}, v_{2}^{\prime} b^{\prime}$, as shown in Figure (4), we can obtain that $r c\left(H^{\prime}\right) \leq r c(H)+3 \leq 3 h / 5-1 / 5+3<$ $3(h+6) / 5-1 / 5$, a contradiction.

Case 2. $s+t=2$ and $P_{1}=a x, P_{2}=x v_{1} v_{2} b$ (see Figure 2(B)).
Since $v_{1} \notin H$, there are three disjoint $\left(v_{1}, H\right)$-paths by the fan lemma.

Then there is at least one additional $\left(v_{1}, H\right)$-path $P_{3}$ except paths $v_{1} x a$ and $v_{1} v_{2} b$. By the choice of $x$, the length of $P_{3}$ must be at most two. If $P_{3}$ is of length two, then this is the case of Figure 2(A), we have done. If $P_{3}$ is of length one, then paths $a x v_{1}, v_{1} v_{2} b$ and $P_{3}$ build the same structure as Case 1 , and thus we have done.

Case $3 s+t=1$.
Since $t \geq s$, we have $t=1$. Now we can assume $P_{1}=e_{1}$ and $P_{2}=x v_{1} b$. Then there are at least two distinct vertices outside of $H$ different from $x$ and $v_{1}$, say $x_{1}$ and $x_{2}$. Similarly, for $i=1,2$, there is no $\left(x_{i}, x\right)$-path and $\left(x_{i}, v_{1}\right)$-path without using any vertex of $H$. So there are also three internally disjoint $\left(x_{1}, H\right)$-paths $P_{0}^{\prime}, P_{1}^{\prime}, P_{2}^{\prime}$ and $\left(x_{2}, H\right)$-paths $P_{0}^{\prime \prime}, P_{1}^{\prime \prime}, P_{2}^{\prime \prime}$, respectively. If all these paths are of length one, then we can add $x, v_{1}, x_{1}$ and $x_{2}$ to $H$, and form a larger graph $H^{\prime}$ of order $h+4$. By assigning color 1 to edges $e_{0}, x v_{1}, P_{0}^{\prime}, P_{1}^{\prime}, P_{0}^{\prime \prime}, P_{1}^{\prime \prime}$, color 2 to edges $e_{1}, v_{1} b, P_{2}^{\prime}, P_{2}^{\prime \prime}$, we can obtain that $r c\left(H^{\prime}\right) \leq r c(H)+2 \leq 3 h / 5-1 / 5+2<3(h+4) / 5-1 / 5$, a contradiction. Otherwise, without loss of generality, we assume one of the three $\left(x_{1}, H\right)$ paths $P_{0}^{\prime}, P_{1}^{\prime}, P_{2}^{\prime}$ has length 2. Let $P_{0}^{\prime}=e_{0}^{\prime}, P_{1}^{\prime}=e_{1}^{\prime}, P_{2}^{\prime}=x_{1} v_{1}^{\prime} b^{\prime}$. We can add $x, v_{1}, x_{1}$ and $v_{1}^{\prime}$ to $H$, and form a larger graph $H^{\prime}$ of order $h+4$. By assigning color 1 to edges $e_{0}, e_{1}, x v_{1}, v_{1} b$, color 2 to edges $e_{0}^{\prime}, e_{1}^{\prime}, x_{1} v_{1}^{\prime}, v_{1}^{\prime} b^{\prime}$, we can obtain that $r c\left(H^{\prime}\right) \leq r c(H)+2 \leq 3 h / 5-1 / 5+2<3(h+4) / 5-1 / 5$, a contradiction.

Now we have proved that $h \geq n-3$. By considering some cases, we can easily obtain that $r c(G) \leq 3(n+1) / 5$ : if $h=n-3$, then $r c(G) \leq r c(H)+2 \leq$ $3(h-3) / 5-1 / 5+2<3(n+1) / 5$; if $h=n-2$, then $r c(G) \leq r c(H)+2 \leq$ $3(h-2) / 5-1 / 5+2=3(n+1) / 5$; if $h=n-1$, then $r c(G) \leq r c(H)+1 \leq$ $3(h-1) / 5-1 / 5+1<3(n+1) / 5$.

The proof is completed.

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