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Universal H-colourable graphs

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Abstract

Rado constructed a (simple) denumerable graph R with the positive integers as vertex set with the following edges: For given m and n with $m < n$, m is adjacent to n if n has a 1 in the m'th position of its binary expansion. It is well known that R is a universal graph in the set $\mathcal{I}_c$ of all countable graphs (since every graph in $\mathcal{I}_c$ is isomorphic to an induced subgraph of R) and that it is a homogeneous graph (since every isomorphism between two finite induced subgraphs of R extends to an automorphism of R).

In this paper we construct a graph $U(H)$ which is H-universal in $\rightarrow H_c$, the induced-hereditary hom-property of H-colourable graphs consisting of all (countable) graphs which have a homomorphism into a given (countable) graph H. If H is the (finite) complete graph K_k, then $\rightarrow H_c$ is the property of k-colourable graphs. The universal graph $U(H)$ is characterised by showing that it is, up to isomorphism, the unique denumerable, H-universal graph in $\rightarrow H_c$ which is H-homogeneous in $\rightarrow H_c$. The graphs H for which $U(H) \cong R$ are also characterised.

With small changes to the definitions, our results translate effortlessly to hold for digraphs too. Another slight adaptation of our work yields related results for (k,l)-split graphs.

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1 Introduction

All graphs considered here are simple, undirected (except later only in Corollaries 1 and 2), unlabelled and have countable vertex sets. When the vertex set is taken to be the set, or some subset, of the positive integers $\mathbf{N} = \{1, 2, \ldots\}$, number-theoretic properties of the integers may be employed in constructions and proofs. Otherwise, the vertex set of a graph may be indexed by $\mathbf{N}$ or one of its subsets.

Let $\mathcal{P}$ be a class of countable graphs. Following [6], we define a graph U to be a **universal graph for** $\mathcal{P}$ if every graph in $\mathcal{P}$ is (isomorphic to) an induced subgraph of U; it is a **universal graph in** $\mathcal{P}$ if $U \in \mathcal{P}$ too. We shall often have occasion to refer to two graphs which are isomorphic; in that case we shall refer to (any) one of them as a **clone** of the other. For any graph property $\mathcal{P}$ (i.e., an isomorphically closed class of graphs) we use the symbols $\mathcal{P}_c$, $\mathcal{P}_d$, $\mathcal{P}_f$ to denote, respectively, the classes of countable, denumerable, and finite graphs of $\mathcal{P}$.

Rado [11] constructed the denumerable graph R on $\mathbf{N}$ with the following edges: For given m and n with $m < n$, m is adjacent to n in R (i.e., we have an edge mn in R) if n has a 1 in the m'th position of its binary expansion. It is well known that R is a universal graph in the set $\mathcal{I}_c$ of all countable graphs (since $R \in \mathcal{I}_c$ and every graph in $\mathcal{I}_c$ is isomorphic to an induced subgraph of R). Some known constructions of clones of this graph, together with many new constructions, are discussed in [2]. Important properties of R (sometimes called the "random graph") are discussed in [4].

A **homomorphism** from a graph G to a graph H is an edge-preserving map from the vertex set $V(G)$ of G into the vertex set $V(H)$ of H. If such a map exists, we say that G is **homomorphic to** H and we write $G \to H$. Given any (countable) graph H, the **hom-property** $\to H_c$ is the class of H**-colourable graphs**, i.e., it consists of all (countable) graphs which have a homomorphism into the given graph H. In symbols: $\to H_c = \{G \mid G \in \mathcal{I}_c, G \to H\}$. Using the terminology of [1], we note that every hom-property is an additive, induced-hereditary graph property. In [10] the authors describe the construction of a universal graph in the hom-property $\to H_c$ for any given finite graph H. Our aim in this paper is to construct and characterise a graph $U(H)$ which is universal in $\to H_c$ for any countable graph H. If H is the (finite) complete graph K_k, then $\to H_c$ is the property of k-colourable graphs denoted (following [1]) by $\mathcal{O}_c^k$, while $(\to K_{\aleph_0})_c = \mathcal{I}_c$. We also characterise those graphs H for which $U(H) \cong R$.

A slight adaptation of our work yields related results for (k, l)-split graphs.

2 Constructing universal H-colourable graphs

Throughout this section, H is any graph with vertex set $\{w_1, w_2, \ldots\}$ (in the denumerable case) or $\{w_1, w_2, \ldots, w_k\}$ (in the finite case). We introduce the notation $V(H) = \{w_1, w_2, \ldots \lhd w_k \rhd\}$ to cover both cases at once. We are going to construct a graph $U(H)$, universal in the property $\to H_c$, and then investigate its properties in sections 3 and 4.

Let $p_1, p_2, \ldots$ be any enumeration of the denumerable set of prime numbers. For any integer $n \geq 2$ there is a unique sequence $n_1, n_2, \ldots$ for which $n = \prod_{i \geq 1} p_i^{n_i}$. Here each power n_i is a non-negative integer, and at least one (but only finitely many) $n_i \geq 1$.

First we define countably many pairwise disjoint denumerable proper subsets $\mathbf{N}_1, \mathbf{N}_2, \ldots$ of $\mathbf{N}' = \{2, 3, \ldots\}$ by specifying $\mathbf{N}_j$, $1 \leq j$, to be the set of all integers $n \geq 2$ for which the power of any prime in its prime factorization is either 0 or j:

$$\mathbf{N}_j := \{n \in \mathbf{N}' \mid \text{ every } n_i \in \{0, j\}\}.$$

We now define the graph $U(H)$, co-determined by H, as follows: The vertex set of $U(H)$ is

$$V(U(H)) := \mathbf{N}_1 \cup \mathbf{N}_2 \cup \ldots \lhd \cup \mathbf{N}_k \rhd .$$

In $U(H)$ there are no edges between vertices from the same $\mathbf{N}_j$. If $m \in \mathbf{N}_h$ and $n \in \mathbf{N}_j$, $h \neq j$, then there is an edge in $U(H)$ between m and n if and only if

(i) there is an edge in H between w_h and w_j;

(ii) $m < n$; and

(iii) $n_m = j$.

For a graph G we now want to define the notion "G is H-universal in $\to H_c$" – where this is stronger than "G is universal in $\to H_c$" – and then prove that the $U(H)$ that we have just defined is H-universal in $\to H_c$. We need some preliminary definitions.

Consider two graphs $F, G \in \to H_c$ and two homomorphisms $\lambda : F \to H$ and $\zeta : G \to H$, together with a third homomorphism $\nu : F \to G$. Then we say that ν is (λ, ζ)**-respecting** when, for every $w_j \in V(H)$, $\nu(\lambda^{-1}(w_j)) \subseteq \zeta^{-1}(w_j)$. A graph $G \in \to H_c$ is now called H**-universal in**

$\to H_c$ if there exists a *surjective* homomorphism (H-colouring) $\zeta : G \to H$ such that for every $F \in \to H_c$ and every $\lambda : F \to H$ there exists a (λ, ζ)-respecting isomorphic embedding $\nu : F \to G$, i.e., $\nu : F \cong G[\nu(V(F))]$. This is a stronger property than universality in $\to H_c$, which requires only isomorphic embeddings into G. It says that there is some surjective H-colouring ζ of G such that for every H-colouring λ of any $F \in \to H_c$ there exists an isomorphic embedding ν of F into G which preserves the colours (elements of $V(H)$) assigned by λ to the vertices of F: for every $v \in V(F)$, $\zeta(\nu(v)) = \lambda(v)$. We say then that G is H-universal in $\to H_c$ *with respect to* ζ.

Theorem 1 *For any countable graph H the graph $U(H)$ is H-universal in $\to H_c$.*

Proof:
By the construction of $U(H)$ it is clear that $U(H) \in \to H_c$, since $n \mapsto w_j$ for every $n \in \mathbf{N}_j$ induces a homomorphism $\mu : U(H) \to H$, which is even a surjection. In the sequel we shall call this μ the *canonical* homomorphism from $U(H)$ onto H, or the *canonical H-colouring* of $U(H)$. Note that, for every j, $\mu^{-1}(w_j) = \mathbf{N}_j$. We shall establish H-universality of $U(H)$ in $\to H_c$ with respect to μ.

Let F with $V(F) = \{v_1, v_2, \ldots \lhd v_\ell \rhd\}$ be any countable graph for which $F \to H$, indeed, let λ be a homomorphism from F into H. We recursively construct an injection $\nu : V(F) \to V(U(H))$ such that $\nu(V(F))$ induces a subgraph of $U(H)$ which, under ν, is isomorphic to F. By its construction, ν will be (λ, μ)-respecting.

We begin by defining subsets V_j of $V(F)$ by $V_j := \lambda^{-1}(w_j)$, $1 \leq j \lhd \leq k \rhd$, and note that $V_1 \cup V_2 \cup \ldots \lhd \cup V_k \rhd$ is a partition of $V(F)$. (For some indices j, V_j may be empty, since λ need not be surjective onto H.) Furthermore, since λ is a homomorphism from F into the (simple) graph H, there is no edge in F between any two vertices from the same V_j. Note also that, for every edge uv of F, $\lambda(u)\lambda(v)$ is an edge of H.

We now define ν by recursion on the indices $1, 2, \ldots \lhd \ell \rhd$ of the vertices of F. Let us suppose that $v_1 \in V_s$; then we can choose $\nu(v_1)$ to be any element of $\mathbf{N}_s$. Next assume that $\nu(v_1), \nu(v_2), \ldots, \nu(v_{p-1})$ have already been specified so that for every q and r with $1 \leq q, r \leq p-1$

- if $q \neq r$, then $\nu(v_q) \neq \nu(v_r)$;
- if $v_q \in V_j$, then $\nu(v_q) \in \mathbf{N}_j$; and
- $F[\{v_1, v_2, \ldots, v_{p-1}\}] \cong U(H)[\{\nu(v_1), \nu(v_2), \ldots, \nu(v_{p-1})\}]$ under $\nu|\{v_1, v_2, \ldots, v_{p-1}\}$.

Note that, for every q and r with $1 \leq q, r \leq p-1$, an edge $v_q v_r$ of F corresponds to an edge $\nu(v_q)\nu(v_r)$ of $U(H)$ and an edge $\lambda(v_q)\lambda(v_r)$ of H, and that $\mu(\nu(v_q)) = \lambda(v_q)$.

Now we consider v_p to decide on $\nu(v_p)$. Suppose that $v_p \in V_t$. We shall construct (by specifying its prime factorization) an $n \in \mathbf{N}_t$ which is a suitable choice for $\nu(v_p)$. Let $\{u_1, u_2, \ldots, u_m\}$ be the subset of $\{v_1, v_2, \ldots, v_{p-1}\}$ of those vertices which are adjacent to v_p in F, (none of them is with v_p in V_t, of course). Now, in the prime factorization of n, the primes with indices $\nu(u_1), \nu(u_2), \ldots, \nu(u_m)$ all occur to the power t, and so does one extra prime with a value so large as to ensure that n is larger than each of $\nu(u_1), \nu(u_2), \ldots, \nu(u_m)$; all other primes have power zero in the factorization of n. Then, by specifying $\nu(v_p) = n$, we have that ν establishes the isomorphism $F[\{v_1, v_2, \ldots, v_p\}] \cong U(H)[\{\nu(v_1), \nu(v_2), \ldots, \nu(v_p)\}]$. Thus a (λ, μ)-respecting isomorphism ν from F onto an induced subgraph of $U(H)$ (i.e., $\lambda = \mu \circ \nu$) is constructed in countably many recursive steps. $\square$

Let us here digress for a moment and devote some thought to the case when H is a directed graph and, correspondingly, consider H-colourings of directed graphs, i.e., homomorphisms into H which not only preserve edges, but also the directions of those edges. The obvious way for the directions of the edges of H to induce directions on the edges of $U(H)$ is as follows: where we say (in the definition of $U(H)$) that "there is an edge in $U(H)$ between m and n if and only if (i) there is an edge in H between w_h and w_j; . . . ; (iii) . . . ", we just add
"and the direction of the edge between m and n corresponds to the direction of the edge between

w_h and w_j in H".
In this way $U(H)$ becomes directed ("H-directed", if you like). Similarly, the directions of the edges on H determine unambiguously the direction of every single edge of *any* H-colourable graph – irrespective of the choice of a particular H-colouring of that graph. Colour F by $\lambda : F \to H$ and consider an edge in F between vertices v_q and v_r. If in H the edge $(\lambda(v_q), \lambda(v_r))$ is directed from $\lambda(v_q)$ to $\lambda(v_r)$, then $(v_q, v_r) \in E(F)$ is directed from v_q to v_r, since λ preserves directions. And then any other H-colouring $\lambda' : F \to H$ would of course yield $(\lambda'(v_q), \lambda'(v_r))$, directed from $\lambda'(v_q)$ to $\lambda'(v_r)$ in H. So there is no possibility whatsoever for any edge in F to be directed in the opposite direction to the one consistent with an arbitrary H-colouring of F.

The definitions of "$\to H_c$" and "H-universal in $\to H_c$" when H is directed are also obvious. Thus the proof of Theorem 1 can be rewritten for the directed case with minimal changes. To the crucial "an edge $v_q v_r$ of F corresponds to an edge $\nu(v_q)\nu(v_r)$ of $U(H)$ and an edge $\lambda(v_q)\lambda(v_r)$ of H", we can then add "with corresponding directions". To summarise, we obtain

Corollary 1 *For any countable digraph H the digraph $U(H)$ is H-universal in $\to H_c$.*

3 Two characterisations of $U(H)$

The Rado graph R, universal in $\mathcal{I}_c$, has interesting properties, some of which characterise it [4]. It is, up to isomorphism, the unique countable graph with the "extension property". It is also characterised by being universal in $\mathcal{I}_c$ and "homogeneous". (These properties are defined in the sequel.) The relative simplicity of these properties relates to the extreme symmetry of the structure of R. H-colouring complicates issues somewhat for $U(H)$. We have, however, already relativised the notion of universality in Section 2 and shall now do so for the extension property and homogeneity to incorporate H-colourings. This will facilitate characterisations of $U(H)$ analogous to those of R.

Suppose, again throughout this section, that $V(H) = \{w_1, w_2, \ldots \lhd w_k \rhd\}$ is the vertex set of a countable graph H.

Consider any $G \in \to H_c$ and let $\lambda : G \to H$ be a surjective homomorphism with $V_1 \cup V_2 \cup \ldots \lhd \cup V_k \rhd$ of $V(G)$ the partition associated with it, i.e., for every j, $V_j = \lambda^{-1}(w_j)$. Then $V_1 \cup V_2 \cup \ldots \lhd \cup V_k \rhd$ is called the **λ-induced partition** of $V(G)$. Note that every $G[V_j]$ is edgeless.

We say that a countable graph G **has the H-extension property** if there exists a surjective homomorphism $\lambda : G \to H$ such that, with respect to the λ-induced partition $V_1 \cup V_2 \cup \ldots \lhd \cup V_k \rhd$ of $V(G)$, for every possible choice of all of the following:
(i) any index j with $1 \le j \lhd \le k \rhd$;
(ii) any finite subset Z_j of V_j;
(iii) any finite subset Q of the set of indices $\{1, 2, \ldots \lhd k \rhd\}$ such that, for every $h \in Q$, $w_h w_j \in E(H)$;
(iv) any two finite disjoint subsets X_h and Y_h of V_h for each $h \in Q$,
the following holds: there exists a vertex in V_j which is not in Z_j and which is adjacent in G to every vertex in every X_h and to no vertex in any Y_h.
If the graph G has this property, we also say that G has the H-extension property *with respect to λ*.

Consider a graph $G \in \to H_c$ and the partition $V(G) = V_1 \cup V_2 \cup \ldots \lhd \cup V_k \rhd$ induced by a homomorphism $\lambda : G \to H$. Let $\alpha : C \to D$ be any isomorphism between two induced subgraphs C and D of G. Then we say that α is **λ-isochromatic** if, for every j, $\alpha(V(C) \cap V_j) \subseteq V_j$. We also say that such a graph G is **λ-homogeneous** if every λ-isochromatic isomorphism α between two finite induced subgraphs of G extends to a λ-isochromatic automorphism α^+ of G. Note that, for every j, $\alpha^+|V_j$ is then a permutation of V_j. When $G \in \rightarrow H_c$ is λ-homogeneous for some *surjective* $\lambda : G \to H$, then we say that G is H**-homogeneous in** $\rightarrow H_c$ *with respect to* λ. This is in general a property different from homogeneity (as defined in [8]), which requires the extendability to an *ordinary* automorphism of *any* isomorphism between two finite induced subgraphs. (It is known that every homogeneous graph belongs to one of only three types, one of which is "clone of R" [9].)

Theorem 2 *Let G be any denumerable graph. Then the following three conditions on G are equivalent:*
(a) G is a clone of $U(H)$
(b) G has the H-extension property
(c) There exists a surjective H-colouring of G with respect to which G is both H-universal and H-homogeneous in $\rightarrow H_c$.

Proof:
(a) implies (b): Since an isomorphism preserves all graph theoretical properties, it suffices to prove that $U(H)$ has the properties ascribed to G in (b).
We shall prove that $U(H)$ has the H-extension property with respect to the partition $\mathbf{N}_1 \cup \mathbf{N}_2 \cup \ldots \triangleleft \cup \mathbf{N}_k \triangleright$ of $V(U(H))$ which is induced by the canonical surjective homomorphism $\mu : U(H) \to H$ defined by $n \mapsto w_j$ for every $n \in \mathbf{N}_j$.

Consider for any index j with $1 \leq j \triangleleft \leq k \triangleright$ any finite subset Z_j of $\mathbf{N}_j$, a finite $Q \subseteq \{1, 2, \ldots \triangleleft k \triangleright\}$ with $w_h w_j \in E(H)$ for every $h \in Q$, and any two finite disjoint subsets X_h and Y_h of V_h for each $h \in Q$. Construct $n \in \mathbf{N}_j$ by having in its prime factorization a factor p_u^j for every u in any X_h; a factor $p_u^0 = 1$ for every u in any Y_h; a factor p^j for some prime p large enough to ensure that n is larger than all the numbers in $Z_j \cup \bigcup\{X_h \cup Y_h \mid h \in Q\}$; and all other factors $p_r^0 = 1$.

It is easy to see that this vertex n has the required properties to ensure that $U(H)$ has the H-extension property with respect to μ.

(b) implies (a): Assume that G has the H-extension property with respect to the surjective homomorphism $\lambda : G \to H$ with concomitant λ-induced partition $V_1 \cup V_2 \cup \ldots \triangleleft \cup V_k \triangleright$ of $V(G) = \{v_1, v_2, \ldots\}$. For $V(U(H)) := \mathbf{N}_1 \cup \mathbf{N}_2 \cup \ldots \triangleleft \cup \mathbf{N}_k \triangleright$ we consider any enumeration $V(U(H)) = \{u_1, u_2, \ldots\}$. By a construction going back and forth between G and $U(H)$, recursive on both the indices of $v_1, v_2, \ldots$ as well as those of $u_1, u_2, \ldots$, we shall now build an isomorphism $\alpha : G \to U(H)$ with the property that, for every j, $\alpha \mid V_j$ is a bijection from V_j to $\mathbf{N}_j$, i.e., α is (λ, μ)-respecting.

Suppose $v_1 \in V_i$. Define $\alpha(v_1)$ to be any element of $\mathbf{N}_i$. Let u_r be the vertex of $U(H)$ with smallest index r such that $u_r \notin \{\alpha(v_1)\}$, and suppose that $u_r \in \mathbf{N}_j$. Now define the finite subset $Z_j := V_j \cap \{v_1\}$ of V_j, $Q = \{i\}$ if $w_i w_j \in E(H)$ and $Q = \emptyset$ otherwise, and, for each $h \in Q$, two finite disjoint subsets X_h and Y_h of V_h as follows:

$$X_h := \{x \in V_h \mid \alpha(x) \in \{\alpha(v_1)\} \text{ and } \alpha(x)u_r \in E(U(H))\};$$

$$Y_h := \{y \in V_h \mid \alpha(y) \in \{\alpha(v_1)\} \text{ and } \alpha(y)u_r \notin E(U(H))\}.$$

(An explanatory remark: By writing, say $\alpha(x)$, we of course mean "α has already been defined on x at this stage and . . . ".)

Employing the H-extension property of G with respect to λ, we now have a vertex, say v_s, in V_j which is not in Z_j, i.e., $v_s \notin \{v_1\}$, and v_s is adjacent in G to v_1 if and only if $\alpha(v_1)$ is adjacent in $U(H)$ to u_r. By defining $\alpha(v_s) = u_r$, we have that
• $v_1 \in V_i$ and $\alpha(v_1) \in \mathbf{N}_i$; $v_s \in V_j$ and $\alpha(v_s) \in \mathbf{N}_j$; and

• $v_1v_s \in E(G)$ if and only if $\alpha(v_1)\alpha(v_s) \in E(U(H))$; so
• $G[\{v_1, v_s\}] \cong U(H)[\{\alpha(v_1), \alpha(v_s)\}]$ under $\alpha \mid \{v_1, v_s\}$.

The next step in the back and forth construction of α has to start again in G with vertex v_t, say, with lowest index t of those outside $\{v_1, v_s\}$. But let us rather describe the recursive step in general, distinguishing the two cases, starting from either vertex $v_t \in V(G)$ with smallest index t of some sort, or starting from vertex $u_r \in V(U(H))$ with smallest index r of some sort. In both cases the starting situation before the recursive step is the same:

I is a finite subset of the index set $\{1, 2, \ldots\}$ of $V(G) = \{v_1, v_2, \ldots\}$, and $V_I = \{v_i \in V(G) \mid i \in I\}$. For each $i \in I$ we have now already defined $\alpha(v_i) \in V(U(H))$, with, for each j, elements of V_j mapping into $\mathbf{N}_j$ and $G[V_I] \cong U(H)[\alpha(V_I)]$ under $\alpha \mid V_I$.

Case 1, starting from $v_t \in V(G)$: Let $v_t \in V(G)$ be the vertex with the lowest index of all those outside V_I. We need to define $\alpha(v_t) \in V(U(H))$. Suppose $v_t \in V_\ell$; we want $\alpha(v_t) \in \mathbf{N}_\ell$. Define the finite subset $Z_\ell := \mathbf{N}_\ell \cap \alpha(V_I)$ of $\mathbf{N}_\ell$. Suppose that $\alpha(V_I) \subseteq \mathbf{N}_{j_1} \cup \mathbf{N}_{j_2} \cup \ldots \cup \mathbf{N}_{j_a}$, and define $Q = \{h \in \{j_1, j_2, \ldots, j_a\} \mid w_h w_\ell \in E(H)\}$. Now define, for every $h \in Q$, two finite disjoint subsets X_h and Y_h of $\mathbf{N}_h$ as follows:

$$X_h := \{x \in \mathbf{N}_h \mid \text{ for some } i \in I,\ x = \alpha(v_i) \text{ and } v_iv_t \in E(G)\};$$

$$Y_h := \{y \in \mathbf{N}_h \mid \text{ for some } i \in I,\ y = \alpha(v_i),\ \text{ but } v_iv_t \notin E(G)\}.$$

By the H-extension property of $U(H)$ with respect to μ, there exists a vertex, say $u_s \in \mathbf{N}_\ell$, which is not in $\alpha(V_I)$, which is adjacent in $U(H)$ to every $\alpha(v_i)$, $i \in I$, for which $v_iv_t \in E(G)$, while u_s is adjacent in $U(H)$ to no $\alpha(v_i)$ for which $v_iv_t \notin E(G)$. By defining $\alpha(v_t) = u_s$, we now have extended α to establish $G[V_I \cup \{v_t\}] \cong U(H)[\alpha(V_I) \cup \{u_s\}]$.

Case 2, starting from $u_r \in V(U(H))$: Let u_r be that vertex of $U(H)$ with the least index r among those vertices not already in $\alpha(V_I)$. Suppose that $u_r \in \mathbf{N}_j$. We need to find a $v_s \in V_j$ suitable for defining $\alpha(v_s) = u_r$. Define the finite subset $Z_j := V_j \cap V_I$ of V_j. Suppose that $V_I \subseteq V_{g_1} \cup V_{g_2} \cup \ldots \cup V_{g_b}$, and define $Q = \{h \in \{g_1, g_2, \ldots, g_b\} \mid w_h w_j \in E(H)\}$. Now define, for each $h \in Q$, the finite disjoint subsets X_h and Y_h of V_h as follows:

$$X_h := \{x \in V_h \mid x \in V_I \text{ and } \alpha(x)u_r \in E(U(H))\};$$

$$Y_h := \{y \in V_h \mid y \in V_I \text{ and } \alpha(y)u_r \notin E(U(H))\}.$$

Employing the H-extension property of G with respect to λ, we have a vertex, say $v_s \in V_j \setminus V_I$, which is adjacent in G to every v_i, $i \in I$, for which $\alpha(v_i)$ is adjacent in $U(H)$ to u_r and to no v_i for which $\alpha(v_i)$ is not adjacent in $U(H)$ to u_r. By defining $\alpha(v_s) = u_r$ we extend α to ensure that $G[V_I \cup \{v_s\}] \cong U(H)[\alpha(V_I) \cup \{u_r\}]$.

By alternating the two cases in a back and forth manner through denumerably many recursive steps we construct the isomorphism $\alpha : G \cong U(H)$.

(b) implies (c): Suppose G satisfies (b) with respect to the surjective homomorphism $\zeta : G \to H$. Suppose (again) that $V(G) = \{v_1, v_2, \ldots\}$, with $V_1 \cup V_2 \cup \ldots \lhd \cup V_k \rhd$ the partition of $V(G)$ induced by ζ.

To demonstrate H-universality of G in $\to H_c$ with respect to ζ, let C be any graph in $\to H_c$ and suppose $V(C) = \{u_1, u_2, \ldots \lhd u_\ell \rhd\}$ with $U_1 \cup U_2 \cup \ldots \lhd \cup U_k \rhd$ the induced partition of its vertex set with respect to a homomorphism $\lambda : C \to H$. By recursion on the indices of the vertices of C we now construct a (λ, ζ)-respecting embedding $\alpha : C \to G$ of C into G.

If $u_1 \in U_i$, let $\alpha(u_1)$ be any element of V_i. Next suppose that r is the least index of a vertex of C for which $\alpha(u_r)$ has not yet been defined and suppose that $u_r \in U_j$. Let $R = \{\alpha(u_1), \alpha(u_2), \ldots, \alpha(u_{r-1})\}$ and suppose that $R \subseteq V_{g_1} \cup V_{g_2} \cup \ldots \cup V_{g_q}$. Now define the finite subset $Z_j := V_j \cap R$ of V_j, $Q := \{h \in \{g_1, g_2, \ldots, g_q\} \mid w_h w_j \in E(H)\}$, and, for each $h \in Q$, two finite disjoint subsets X_h and Y_h of $V_h \cap R$ as follows:

$$X_h := \{x \in V_h \cap R \mid \alpha^{-1}(x)u_r \in E(C)\} \text{ and } Y_h := \{y \in V_h \cap R \mid \alpha^{-1}(y)u_r \notin E(C)\}.$$

Employing the H-extension property of G with respect to ζ, we have a vertex, say $v_s \in V_j$, which is not in Z_j and which is, for every $h \in Q$, adjacent in G to every $x \in X_h$ and not adjacent in G to any $y \in Y_h$. By defining $\alpha(u_r) = v_s$ we extend α and ensure in the process that $C[\{u_1, u_2, \ldots, u_r\}] \cong G[\{\alpha(u_1), \alpha(u_2), \ldots, \alpha(u_r)\}]$. Thus a (λ, ζ)-respecting isomorphism from C onto an induced subgraph of G is constructed in countably many recursive steps. So G is H-universal in $\rightarrow H_c$ with respect to ζ.

Next we prove that G is H-homogeneous in $\rightarrow H_c$ with respect to the *same* ζ with respect to which it is H-universal in $\rightarrow H_c$. Consider any two isomorphic finite induced subgraphs C and D of G with a ζ-isochromatic isomorphism α from C onto D. We prove that α can be extended to a ζ-isochromatic automorphism α^+ of G by a recursive construction on the indices of the vertices in $V(G)$. Hence suppose that V is a finite subset of $V(G)$ containing $V(C)$, and that a ζ-isochromatic isomorphism α^+, which extends α, has already been defined from V into $V(G)$: $\alpha^+ : G[V] \cong G[\alpha^+(V)]$. Let r be the least index of a vertex of G for which $v_r \notin V$; we want to define $\alpha^+(v_r)$.

Suppose that, with respect to the ζ-induced partition of $V(G)$, $v_r \in V_j$. Define $Z_j := \alpha^+(V) \cap V_j$. Suppose $\alpha^+(V) \subseteq V_{g_1} \cup V_{g_2} \cup \ldots \cup V_{g_q}$ and let $Q := \{h \in \{g_1, g_2, \ldots, g_q\} \mid w_h w_j \in E(H)\}$. For every $h \in Q$, define

$$X_h := \{\alpha^+(x) \in V_h \mid x \in V \cap V_h \text{ and } xv_r \in E(G)\};$$

$$Y_h := \{\alpha^+(y) \in V_h \mid y \in V \cap V_h \text{ and } yv_r \notin E(G)\}.$$

By the H-extension property with respect to the ζ-induced partition of $V(G)$, there exists a vertex $z \in V_j$ which is not in $\alpha^+(V)$ and which is adjacent to every vertex $\alpha^+(x)$ in $\alpha^+(V)$ for which x is adjacent to v_r in G and to no vertex $\alpha^+(y)$ in $\alpha^+(V)$ for which y is not adjacent to v_r in G. Hence, by defining $\alpha^+(v_r) = z$, we have extended α^+ to a ζ-isochromatic isomorphism on $V \cup \{v_r\}$ into G.

By symmetry (between C and D), the same proof can now be used to extend it starting with $V(D) \cup \{z\}$. Hence, in denumerably many such recursive back and forth steps, a ζ-isochromatic automorphism of G which extends α is built, establishing the H-homogeneity of G with respect to ζ.

(c) implies (b): Suppose G is H-universal in $\rightarrow H_c$ and H-homogeneous in $\rightarrow H_c$, both with respect to the same surjective homomorphism $\zeta : G \rightarrow H$ and corresponding partition

$$V(G) = V_1 \cup V_2 \cup \ldots \lhd \cup V_k \rhd$$

in which each $V_j = \zeta^{-1}(w_j)$. With respect to this ζ-induced partition we show that G has the H-extension property. Pick a finite subset Z_j of V_j, a finite set Q of indices $h \in \{1, 2, \ldots \lhd k \rhd\}$ for which $w_h w_j \in E(H)$, and, for each $h \in Q$, two finite disjoint subsets X_h and Y_h of V_h.

First we construct a finite graph A (considered as if outside G) from the subgraph $F = G[Z_j \cup \bigcup\{X_h \cup Y_h \mid h \in Q\}]$ of G by adding to F a vertex $m \notin V(G)$ and, for each $q \in \bigcup\{X_h \mid h \in Q\}$ an edge between m and q. From the partitioning

$$V(A) = \bigcup\{(X_h \cup Y_h) \mid h \in Q\} \cup (\{m\} \cup Z_j)$$

it is clear that $A \in \rightarrow H_c$: This is ensured by the homomorphism $\lambda : A \rightarrow H$ which, for $h \in Q$, maps each element of $X_h \cup Y_h$ to w_h, and each element of $\{m\} \cup Z_j$ to w_j.

By the H-universality of G in $\rightarrow H_c$ with respect to ζ, the graph A can (λ, ζ)-respectingly be isomorphically embedded into G by, say, $\alpha : A \to B$, where $B = \alpha(A)$ is an induced subgraph of G. Suppose $\alpha(m) = n \in V_j$. (That $\alpha(m) \in V_j$ follows from the fact that $\zeta(\alpha(m)) = \lambda(m) = w_j$.) It is clear that the subgraph F of G and the subgraph $B - n$ of G are ζ-isochromatically isomorphic. By the H-homogeneity of G in $\rightarrow H_c$ with respect to ζ, this isomorphism can be extended to a ζ-isochromatic automorphism β of G. Clearly the vertex $\beta^{-1}(n)$, corresponding to n, has the properties required to ensure that G has the H-extension property with respect to the ζ-induced partition of $V(G)$ with which we started. $\Box$

Properties of R, universality, extension, and homogeneity, have now been H-"relativised". R is also self-complementary, i.e., isomorphic to its complement. Does $U(H)$ have what we could call its "H-complement" to which it is isomorphic? The answer is "yes".

We shall now define a graph $U^*(H)$, which will turn out to be a clone of $U(H)$. Its vertex set is the same as that of $U(H)$:

$$V(U^*(H)) := \mathbf{N}_1 \cup \mathbf{N}_2 \cup \ldots \lhd \cup \mathbf{N}_k \rhd .$$

In $U^*(H)$ there are no edges between vertices from the same $\mathbf{N}_j$. If $m \in \mathbf{N}_h$ and $n \in \mathbf{N}_j$, $h \neq j$, and $w_h w_j \notin E(H)$, then there is no edge in $U^*(H)$ between m and n. If, however, $w_h w_j \in E(H)$, then $mn \in E(U^*(H))$ if and only if $mn \notin E(U(H))$:

$$E(U^*(H)) := \{mn \mid \mu(m)\mu(n) \in E(H) \text{ but } mn \notin E(U(H))\}.$$

Theorem 3 $U^*(H) \cong U(H)$.

Proof:
By Theorem 2 it is sufficient to prove that $U^*(H)$ has the H-extension property. The same function $\mu : V(U^*(H) \to V(H))$ that we had before as the canonical $\mu : U(H) \to H$ is also a surjective H-colouring of $U^*(H)$, inducing the same partition of $V(U^*(H)$ into $\mathbf{N}_j$'s. We show that $U^*(H)$ has the H-extension property with respect to this μ.

Choose a j; a finite subset Z_j of $\mathbf{N}_j$; a finite set Q of indices h such that $w_h w_j \in E(H)$; and, for each $h \in Q$, two finite disjoint subsets X_h^* and Y_h^* of $\mathbf{N}_h$. Now define, for each $h \in Q$, $X_h = Y_h^*$ and $Y_h = X_h^*$. By the H-extension property of $U(H)$ there is a vertex $z \in \mathbf{N}_j \setminus Z_j$ which is adjacent in $U(H)$ to every vertex in every X_h and to no vertex in any Y_h. By the definition of $E(U^*(H))$, z is adjacent in $U^*(H)$ to every vertex in every X_h^* and to no vertex in any Y_h^*. Hence $U^*(H)$ has the H-extension property with respect to μ, and so is a clone of $U(H)$. $\Box$

At the end of Section 1 in Corollary 1 we formulated how, with apt small changes to definitions, the H-universality of $U(H)$ in $\rightarrow H_c$ (as established in Theorem 1) holds also in the case of digraphs. Similarly now, modulo slight adaptations of the definitions of the "H-extension property" and being "H-homogeneous in $\rightarrow H_c$", no major conceptual struggle is required to see the truth of

Corollary 2 *Given the obvious definitions of the concepts involved, Theorems 2 and 3 hold for digraphs too.*

4 Special cases: $H \cong K_k$, and $U(H) \cong R$

We now investigate the special case of the definitions and results of the previous two sections if we choose $H = K_k$. Note that the existence of a homomorphism $\lambda : G \to K_k$ is equivalent to the existence of a partition $V_1 \cup V_2 \cup \ldots \cup V_k$ of the vertex set $V(G)$ into subsets inducing edgeless subgraphs of G, i.e., the existence of a (classical) k-colouring of G. Hence the hom-property $\rightarrow(K_k)_c$ is the property of countable k-colourable graphs.

Taking these remarks into account, the concepts defined above simplify to:

• G is K_k-universal in $\rightarrow(K_k)_c$ if there is a k-colouring ζ of G which uses all k colours, such that for every k-colourable graph F (in $\rightarrow(K_k)_c$) and every K_k-colouring $\lambda : F \rightarrow K_k$ there is a (λ, ζ)-respecting isomorphic embedding $\nu : F \rightarrow G$; hence for every k-colouring $\lambda : V(F) \rightarrow \{1, 2, \ldots, k\}$ of F the k-colouring $\zeta : V(G) \rightarrow \{1, 2, \ldots, k\}$ of G is such that the isomorphism ν respects the colours assigned to vertices, i.e., if a vertex x of F is assigned the colour j by λ, then the vertex $\nu(x)$ of G is assigned the same colour j by ζ.

• In this strong sense of universality, $U(K_k)$ is K_k-universal in $\mathcal{O}_c^k$ by Theorem 1.

We can also simplify the characterisation of $U(H)$ when taking $H = K_k$. Again we start with the simplified definitions:

• Consider (again) a k-colourable graph G and consider any k-colouring $\lambda : G \rightarrow K_k$ of it with $V_1 \cup V_2 \cup \ldots \cup V_k$ its colour classes. Let $\alpha : C \rightarrow D$ be any isomorphism between induced subgraphs C and D of G. Then α is λ-isochromatic if it preserves the colours assigned to the vertices of C. Furthermore, G is λ-homogeneous if every λ-isochromatic isomorphism between finite induced subgraphs C and D of G extends to a λ-isochromatic automorphism of G. Furthermore, a k-colourable graph $G \in \mathcal{O}_c^k$ is K_k-homogeneous in $\mathcal{O}_c^k$ if it is λ-homogeneous for some surjective k-colouring $\lambda : G \rightarrow K_k$.

• These concepts can now be used to characterise the clones of $U(K_k)$ by simply formulating a special case of Theorem 2.

The concept of a universal graph in $\mathcal{O}_c^k$ is revisited in the next section where a second construction of such a graph is given.

We now turn our attention to the special case $H = K_{\aleph_0}$. In this case, it is immediate to see that $U(H)$ is universal in $\mathcal{I}_c$ since $(\rightarrow K_{\aleph_0})_c = \mathcal{I}_c$. Indeed, we shall see that $U(H)$ is a clone of the Rado graph R, the classical universal graph in $\mathcal{I}_c$. Hence our construction of $U(H)$, using the prime factorizations of positive integers, supplements the many constructions of R discussed in [2].

Our result uses the fact that the Rado graph is characterised by the (classical) extension property (for which see [4]): A graph G is said to have the **extension property** if for every two finite disjoint sets of vertices X and Y of G there is a vertex of G outside $X \cup Y$ which is adjacent in G to every vertex of X and to no vertex of Y. Furthermore, a graph G is here said to have the **weak extension property** if for every finite set of vertices X of G there is a vertex of G outside X which is adjacent in G to every vertex of X.

Theorem 4 *The graph $U(H)$ has the extension property (i.e., $U(H) \cong R$) if and only if H has the weak extension property.*

Proof:

$\Rightarrow$: Suppose $U(H)$ has the extension property. Let $X = \{w_{i_1}, w_{i_2}, \ldots, w_{i_p}\}$ be any finite subset of $V(H)$. Pick a finite set $X' = \{u_1, u_2, \ldots, u_p\}$ of vertices in $V(U(H))$ with $u_j \in \mathbf{N}_{i_j}$ for $1 \leq j \leq p$ – and $Y' = \emptyset$ if you insist. By the extension property of $U(H)$ there is a vertex $u \in V(U(H))$, say $u \in \mathbf{N}_\ell$, with $u_j u \in E(U(H))$ for every j. By the definition of adjacency in $U(H)$, $\mathbf{N}_\ell \neq \mathbf{N}_j$ for every j. Since the canonical $\mu : U(H) \rightarrow H$ preserves adjacency, it follows that $w_{i_j} w_\ell \in E(H)$ for every j, establishing the weak extension property of H.

$\Leftarrow$: Suppose H has the weak extension property. Let $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$ be any two finite disjoint subsets of $V(U(H))$. Assume that $X \subseteq \mathbf{N}_{i_1} \cup \mathbf{N}_{i_2} \cup \ldots \cup \mathbf{N}_{i_p}$ where the sets in this union need not all be different. Consider the finite set $\{w_{i_1}, w_{i_2}, \ldots, w_{i_p}\} \subseteq V(H)$ (not necessarily all different vertices). By the weak extension property of H, there is a vertex $w_\ell \in V(H)$ with $w_{i_j} w_\ell \in E(H)$ for every j. By specifying its prime factorization, we define a vertex $n \in \mathbf{N}_\ell$ of $U(H)$ which is adjacent in $U(H)$ to every element of X and to no element of Y:

• $n_m = \ell$ if $m \in X$;

• $n_s = \ell$ for some $s \notin Y$ for which the associated prime p_s is larger than every element in $X \cup Y$; and
• $n_t = 0$ for each other t, including any $t \in Y$.
This n has the properties required for the extension property of $U(H)$. $\square$

We note that, while $U(H)$ is always H-homogeneous in $\rightarrow H_c$ (Theorem 2), when $U(H)$ has the extension property it also has the different property of being homogeneous, like its clone R.

Since $K_{\aleph_0}$ obviously has the weak extension property, we immediately have that $U(K_{\aleph_0}) \cong R$. Clearly, $K_{\aleph_0}$ does not have the extension property. We conclude this section with an example of a non-complete (denumerable) graph G which also has the weak extension property but not the extension property: Let $V(G)$ be the set of all finite subsets of $\mathbf{N}$ and let $AB \in E(G)$ if and only if A and B are different comparable subsets of $\mathbf{N}$, i.e., subsets satisfying $A \subset B$ or $B \subset A$. The weak extension property easily follows by considering the union of finitely many finite sets (together with an extra element, if needed); it is the vertex with the required properties. The fact that G does not have the extension property follows by considering, for example, for X any singleton $\{F\}$ consisting of any finite set F with at least two elements and for Y the finite set of its proper subsets $\{H \mid H \subset F\}$. Clearly, every (finite) subset of $\mathbf{N}$ comparable to F is then comparable to some element of Y.

5 Another universal graph in k-colourable countable graphs

A graph G is called **k-colourable** ($k \geq 1$) if its vertex set can be partitioned into k subsets such that every edge has its endpoints in two of these (different) sets. The graph property of countable k-colourable graphs is $\mathcal{O}_c^k = \{G \mid G \text{ is a countable } k\text{-colourable graph}\}$; it is also an induced-hereditary graph property (of finite character, by the Compactness Theorem – see [5]). The existence of a universal graph in $\mathcal{O}_c^k$ is already guaranteed by the results of [10] (and reassured by Theorem 1 since $\mathcal{O}_c^k = (\rightarrow K_k)_c$).

Every positive integer n has a unique **$(k+1)$-ary expansion** with entries from $\{0, 1, \ldots, k\}$, i.e., a (finite) power series $n = \sum_{i=0}^{\infty} n_i (k+1)^i$ with $k \geq 1$ and $0 \leq n_i \leq k$. We shall refer to n_{i-1} ($i \geq 1$) as the **entry in the i'th position** of the expansion. We use this fact to define k denumerable, pairwise disjoint, proper subsets $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_k$ of $\mathbf{N} = \{1, 2, \ldots\}$: $\mathbf{M}_i$ ($1 \leq i \leq k$) is the set of all those positive integers whose $(k+1)$-ary expansion has all entries from only $\{0, i\}$. We now define a graph U_k as follows: The vertex set of U_k is

$$V(U_k) := \mathbf{M}_1 \cup \mathbf{M}_2 \cup \ldots \cup \mathbf{M}_k.$$

In U_k there are no edges between vertices from the same $\mathbf{M}_\ell$. If $m \in \mathbf{M}_i$ and $n \in \mathbf{M}_j$ ($i \neq j$), then there is an edge in U_k between m and n if and only if $m < n$ and n has the entry j in position number m of its $(k+1)$-ary expansion: $n_{m-1} = j$. For this graph we shall now prove universality in the set $\mathcal{O}_c^k$.

Theorem 5 *For any positive integer k the graph U_k is universal in $\mathcal{O}_c^k$.*

Proof:
By the construction of U_k, it is clear that $U_k \in \mathcal{O}_c^k$, in fact $U_k \in \mathcal{O}_d^k$.

Let G with $V(G) = \{v_1, v_2, \ldots\}$ be any countable k-colourable graph. We recursively construct an injection $\alpha : V(G) \to V(U_k)$ such that $\alpha(V(G))$ induces a subgraph of U_k which, under α, is isomorphic to G. Since G is k-colourable, there is a partition $V(G) = V_1 \cup V_2 \cup \ldots \cup V_k$ such that G has no edge between any two vertices from the same V_ℓ. Suppose $v_1 \in V_i$ and let $\alpha(v_1)$ be any element of $\mathbf{M}_i$. Next assume that $\alpha(v_1), \alpha(v_2), \ldots, \alpha(v_{p-1})$ have already been specified so that $G[\{v_1, v_2, \ldots, v_{p-1}\}] \cong U_k[\{\alpha(v_1), \alpha(v_2), \ldots, \alpha(v_{p-1})\}]$. Suppose that $v_p \in V_j$.

We shall construct an $n \in \mathbf{M}_j$ which is a suitable choice for $\alpha(v_p)$ by prescribing its $(k+1)$-ary expansion. Let $\{u_1, u_2, \ldots, u_\ell\}$ be the subset of $\{v_1, v_2, \ldots, v_{p-1}\}$ of those vertices which are adjacent to v_p in G, (none of them is with v_p in V_j, of course). Now the $(k+1)$-ary expansion of n has entry j in the positions with numbers $\{\alpha(u_1), \alpha(u_2), \ldots, \alpha(u_\ell)\}$, as well as in one position with a number so large as to ensure that n is larger than each of $\alpha(u_1), \alpha(u_2), \ldots, \alpha(u_\ell)$ – and entry 0 in all other positions. It is clear that by specifying $\alpha(v_p) = n$ we have that $G[\{v_1, v_2, \ldots, v_p\}] \cong U_k[\{\alpha(v_1), \alpha(v_2), \ldots, \alpha(v_p)\}]$. Thus an isomorphism from G onto an induced subgraph of U_k is constructed in countably many steps. $\square$

If you now wonder whether U_k is even K_k-universal in $\mathcal{O}_c^k$, the answer is "yes", as will be seen in Theorem 6. Note that both constructions we provide of universal graphs in $\mathcal{O}_c^k$, namely $U(K_k)$ in Section 2 and U_k above, have the desirable properties of what is called an A-type universal graph in [3].

We now proceed to show that the graph U_k also has, like $U(K_k)$, the K_k-extension property with respect to the surjective homomorphism $\zeta : U_k \to K_k$ which maps every vertex in $\mathbf{M}_j$ to the vertex j of K_k, $1 \leq j \leq k$.

Lemma 1 *The graph U_k has the K_k-extension property with respect to ζ.*

Proof:
Let $\mathbf{M}_1 \cup \mathbf{M}_2 \cup \ldots \cup \mathbf{M}_k$ be the partition of $V(U_k)$ induced by ζ. Pick a j, a finite subset Z_j of $\mathbf{M}_j$, any subset Q of $\{1, 2, \ldots, k\} \setminus \{j\}$, and, for every $h \in Q$, two finite disjoint subsets X_h and Y_h of $\mathbf{M}_h$. We define $v \in \mathbf{M}_j$ through its $(k+1)$-ary expansion by putting an entry j in every position u for which $u \in \bigcup\{X_h \mid h \in Q\}$ and in some position which is larger than all of the numbers in $Z_j \cup \bigcup\{(X_h \cup Y_h) \mid h \in Q\}$ and a 0 in every position u for which $u \in \bigcup\{Y_h \mid h \in Q\}$; the remaining positions can be filled with 0's and (finitely many) j's at will. It is easy to see that this vertex v has the required properties. $\square$

From Lemma 1, together with Theorem 2, it now follows that U_k and $U(K_k)$, although constructed differently, are clones of each other, and hence share the same graph theoretical properties. From Theorem 2 we then have as a special case

Theorem 6 *Let G be any denumerable graph. Then the following four conditions on G are equivalent:*
(a) G is a clone of U_k
(b) G is a clone of $U(K_k)$
(c) G has the K_k-extension property
(d) There exists a surjective K_k-colouring of G with respect to which G is both K_k-universal and K_k-homogeneous in $\to(K_k)_c$.

6 Split graphs

In logic, a contradiction and a tautology harbour no employable information: no possibility, or every possibility, is allowed. Somewhat analogously, in graph theory, an edgeless graph and a complete graph have similarly inane internal structure. This has the effect that if in U_{k+l} we replace l of the edgeless induced subgraphs by complete subgraphs, we easily obtain another universal graph. We now describe this briefly.

Following [7], a graph G is called a (k,l)**-split graph** ($k \geq 0$, $l \geq 0$ and $k + l \geq 2$) if its vertex set can be partitioned into $k + l$ (possibly empty) subsets

$$V_1, V_2, \ldots, V_k, V_{k+1}, V_{k+2}, \ldots, V_{k+l}$$

such that each induced subgraph $G[V_i], i = 1, 2, \ldots, k$ is edgeless, while each induced subgraph $G[V_j], j = k+1, k+2, \ldots, k+l$ is a complete subgraph of G. The graph property of countable (k,l)-split graphs is denoted by $\mathcal{S}_c^{k,l} = \{G \mid G$ is a countable (k,l)-split graph$\}$; it is also an induced-hereditary graph property. Note that the $(k,0)$-split graphs are exactly the k-colourable graphs, i.e., $\mathcal{S}_c^{k,0} = \mathcal{O}_c^k$.

We now define the graph $U_{k,l}$ with the following adaptation of our previous graph U_{k+l}. It has vertex set

$$V(U_{k,l}) = V(U_{k+l}) = \mathbf{M}_1 \cup \ldots \cup \mathbf{M}_k \cup \mathbf{M}_{k+1} \cup \ldots \cup \mathbf{M}_{k+l}$$

but now with no edges between vertices from the same $\mathbf{M}_i$ when $1 \leq i \leq k$; and all possible edges between vertices from the same $\mathbf{M}_j$ when $k+1 \leq j \leq k+l$. Edges between vertices from $\mathbf{M}_g$ and $\mathbf{M}_h$ $(g \neq h)$ are exactly as in U_{k+l}. Then the proof of Theorem 5 can be rewritten with slight adaptations to yield

Theorem 7 *For any integers k and l, where $k \geq 0, l \geq 0$ and $k+l \geq 2$, the graph $U_{k,l}$ is universal in $\mathcal{S}_c^{k,l}$.*

We have not (yet) been able to prove or disprove that $U_{k,k}$ is self-complementary, but formulate the open

Conjecture 1 $U_{k,k} \cong \overline{U_{k,k}}$.

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