# A note on large rainbow matchings in edge-coloured graphs

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June 13, 2018

Submitted to Graphs and Combinatorics on April 19, 2012.

#### Abstract

A rainbow subgraph in an edge-coloured graph is a subgraph such that its edges have distinct colours. The minimum colour degree of a graph is the smallest number of distinct colours on the edges incident with a vertex over all vertices. Kostochka, Pfender, and Yancey showed that every edge-coloured graph on n vertices with minimum colour degree at least k contains a rainbow matching of size at least k, provided  $n \ge \frac{17}{4}k^2$ . In this paper, we show that  $n \ge 4k - 4$  is sufficient for  $k \ge 4$ .

## 1 Introduction

Let G be a simple graph, that is, no loops or multiple edges. We write V(G) for the vertex set of G and  $\delta(G)$  for the minimum degree of G. An *edge-coloured graph* is a graph in which each edge is assigned a colour. We say such an edge-coloured G is *proper* if no two adjacent edges have the same colour. A subgraph H of G is *rainbow* if all its edges have distinct colours. Rainbow subgraphs are also called totally multicoloured, polychromatic, or heterochromatic subgraphs.

For a vertex v of an edge-coloured graph G, the *colour degree* of v is the number of distinct colours on the edges incident with v. The smallest colour degree of all vertices in G is the *minimum colour degree of* G and is denoted by  $\delta^c(G)$ . Note that a properly edge-coloured graph G with  $\delta(G) \geq k$  has  $\delta^c(G) \geq k$ .

In this paper, we are interested in rainbow matchings in edge-coloured graphs. The study of rainbow matchings began with a conjecture of Ryser [11], which states that every Latin square of odd order contains a Latin transversal. Equivalently, for n odd, every properly n-edge-colouring

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of  $K_{n,n}$ , the complete bipartite graph with n vertices on each part, contains a rainbow copy of perfect matching. In a more general setting, given a graph H, we wish to know if an edgecoloured graph G contains a rainbow copy of H. A survey on rainbow matchings and other rainbow subgraphs in edge-coloured subgraph can be found in [4]. From now onwards, we often refer to G for an edge-coloured graph G (not necessarily proper) of order n.

Li and Wang [9] showed that if  $\delta^c(G) = k$ , then G contains a rainbow matching of size  $\left\lceil \frac{5k-3}{12} \right\rceil$ . They further conjectured that if  $k \ge 4$ , then G contains a rainbow matching of size  $\left\lceil \frac{k}{2} \right\rceil$ . This bound is tight for properly edge-coloured complete graphs. LeSaulnier et al. [8] proved that if  $\delta^c(G) = k$ , then G contains a rainbow matching of size  $\lfloor \frac{k}{2} \rfloor$ . Furthermore, if G is properly edge-coloured with  $G \ne K_4$  or  $|V(G)| \ne \delta(G) + 2$ , then there is a rainbow matching of size  $\lceil \frac{k}{2} \rceil$ . The conjecture was later proved in full by Kostochka and Yancey [7].

What happens if we have a larger graph? Wang [12] proved that every properly edge-coloured graph G with  $\delta(G) = k$  and  $|V(G)| \geq \frac{8k}{5}$  contains a rainbow matching of size at least  $\lfloor \frac{3k}{5} \rfloor$ . He then asked if there is a function, f(k), such that every properly edge-coloured graph G with  $\delta(G) \geq k$  and  $|V(G)| \geq f(k)$  contains a rainbow matching of size k. The bound on the size of rainbow matching is sharp, as shown for example by any k-edge-coloured k-regular graph. If f(k) exists, then we trivially have  $f(k) \geq 2k$ . In fact, f(k) > 2k for even k as there exists  $k \times k$  Latin square without any Latin transversal (see [1, 13]). Diemunsch et at. [2] gave an affirmative answer to Wang's question and showed that  $f(k) \leq \frac{93}{5}k$ . The bound was then improved to  $f(k) \leq \frac{9}{2}k$  in [10], and shortly thereafter, to  $f(k) \leq \frac{98}{23}k$  in [3].

Kostochka, Pfender and Yancey [6] considered a similar problem with  $\delta^c(G)$  instead of properly edge-coloured graphs. They showed that if G is such that  $\delta^c(G) \ge k$  and  $n > \frac{17}{4}k^2$ , then G contains a rainbow matching of size k. Kostochka [5] then asked: can n be improved to a linear bound in k? In this paper, we show that  $n \ge 4k - 4$  is sufficient for  $k \ge 4$ . Furthermore, this implies that  $f(k) \le 4k - 4$  for  $k \ge 4$ .

**Theorem 1.1.** If G is an edge-coloured graph on n vertices with  $\delta^c(G) \ge k$ , then G contains a rainbow matching of size k, provided  $n \ge 4k - 4$  for  $k \ge 4$  and  $n \ge 4k - 3$  for  $k \le 3$ .

### 2 Main Result

We write [k] for  $\{1, 2, ..., k\}$ . For an edge uv in G, we denote by c(uv) the colour of uv and let the set of colours be  $\mathbb{N}$ , the set of natural numbers.

The idea of the proof is as follows. By induction, G contains a rainbow matching M of size k-1. Suppose that G does not contain a rainbow matching of size k. We are going to show that there exists another rainbow matching M' of size k-1 in  $V(G) \setminus V(M)$ . Clearly, the colours of M equal to the colours of M'. If  $n \ge 4k-3$ , then there exists a vertex z not in  $M \cup M'$ . Since  $\delta^c(G) \ge k, z$  has a neighbour w such that zw does not use any colour of M. Hence, it is easy to deduce that G contains a rainbow matching of size k.

Proof of Theorem 1.1. We proceed by induction on k. The theorem is trivially true for k = 1. So fix k > 1 and assume that the theorem is true for k - 1. Let G be an edge-coloured graph with  $\delta^c(G) \ge k$  and  $n = |V(G)| \ge 4k - 4$  if  $k \ge 4$  and  $n \ge 4k - 3$  otherwise. Suppose that the theorem is false and so G does not contain a rainbow matching of size k.

By induction, there exists a rainbow matching  $M = \{x_i y_i : i \in [k-1]\}$  in G, say with  $c(x_i y_i) = i$ for each  $i \in [k-1]$ . Let M' be another rainbow matching of size s (which could be empty) in G vertex-disjoint from M. Clearly  $s \leq k-1$  and the colours on M' is a subset of [k-1], as otherwise G contains a rainbow matching of size k. Without loss of generality, we may assume that  $M' = \{z_i w_i : i \in [s]\}$  with  $c(z_i w_i) = i$  for each  $i \in [s]$ . We further assume that M and M' are chosen such that s is maximal. Now, let  $W = V(G) \setminus V(M \cup M')$  and  $S = \{x_i, y_i, z_i, w_i : i \in [s]\}$ . Clearly, if there is an edge in W, it must have colour in [s], otherwise G contains a rainbow matching of size k, or s is not maximal.

**Fact A** If uw is an edge in W, then  $c(uw) \in [s]$ .

Furthermore, if uv is an edge with  $u \in S$  and  $v \in W$ , then  $c(uv) \in [k-1]$ , otherwise G contains a rainbow matching of size k. First, we are going to show that s = k - 1. Suppose the contrary, s < k - 1. We then claim the following.

**Claim** By relabeling the indices of i (in the interval  $\{s+1, s+2, \ldots, k-1\}$ ) and swapping the roles of  $x_i$  and  $y_i$  if necessary, there exist distinct vertices  $z_{k-1}, z_{k-2}, \ldots, z_{s+1}$  in W such that for  $s+1 \leq i \leq k-1$  the following holds for  $s+1 \leq i \leq k-1$ :

- (a)  $y_i z_i$  is an edge and  $c(y_i z_i) \notin [i]$ .
- (b) Let  $T_i$  be the vertex set  $\{x_j, y_j, z_j : i \le j \le k-1\}$ . For any colour j, there exists a rainbow matching of size k i on  $T_i$  which does not use any colour in  $[i 1] \cup \{j\}$ .
- (c) Let  $W_i = W \setminus \{z_i, z_{i+1}, \dots, z_{k-1}\}$ . If  $x_i w$  is an edge with  $w \in W_i$ , then  $c(x_i w) \in [s]$ .
- (d) If uw is an edge with  $u \in S$  and  $w \in W_i$ , then  $c(uw) \in [i-1]$ .
- (e) If uw is an edge with  $u \in S \cup T_i \cup W$  and  $w \in W_i$ , then  $c(uw) \in [i-1]$  or  $u \in \{y_i, \dots, y_{k-1}\}$ .

Proof of Claim. Let  $W_k = W$  and  $T_k = \emptyset$ . Observe that part (d) and (e) of the claim hold for i = k. For each i = k - 1, k - 2, ..., s + 1 in terms, we are going to find  $z_i$  satisfying (a) – (e). Suppose that we have already found  $z_{k-1}, z_{k-1}, ..., z_{i+1}$ .

Note that  $|W_{i+1}| \ge n - 2(k-1) - 2s - (k-i-1) \ge 1$ , so  $W_{i+1} \ne \emptyset$ . Let z be a vertex in  $W_{i+1}$ . By the colour degree condition, z must incident with at least k edges of distinct colours, and in particular, at least k - i distinct coloured edges not using colours in [i]. Then, there exists a vertex  $u \in \{x_j, y_j : s + 1 \le j \le i\}$  such that uz is an edge with  $c(uz) \notin [i]$  by part (e) of the claim for the case i + 1. Without loss of generality,  $u = y_i$  and we set  $z_i = z$ .

Part (b) of the claim is true for colour  $j \neq i$ , simply by taking the edge  $x_i y_i$  together with a rainbow matching of size k - i - 1 on  $T_{i+1}$  which does not use any colour in  $[i] \cup \{j\}$ . For colour

j = i, we take the edge  $y_i z_i$  together with a rainbow matching of size k - i - 1 on  $T_{i+1}$  which does not use any colour in  $[i] \cup \{c(y_i z_i)\}$ .

To show part (c) of the claim, let  $x_i w$  be an edge for some  $w \in W_i$ . By part (b) of the claim for the case i + 1, there exists a rainbow matching M'' of size k - i - 1 on  $T_{i+1}$  which does not use any colour in  $[i] \cup \{c(y_i z_i)\}$ . Set  $M_0 = \{x_j y_j : j \in [i-1]\} \cup M'' \cup \{y_i z_i\}$ . Then,  $M_0$  is a rainbow matching of size k - 1 vertex-disjoint from M'. Now, by considering the pair  $(M_0, M')$  instead of (M, M'), we must have  $c(x_i w) \in [s]$ . Otherwise, G contains a rainbow matching of size k or s is not maximal.

Let uw be an edge with  $u \in S$ ,  $w \in W_i$  and  $c(uw) \notin [i-1]$ . Pick a rainbow matching  $M_u$  of size s on  $S \setminus \{u\}$  with colours [s], and a rainbow matching  $M'_u$  of size k-i on  $T_i$  which does not contain any colour in  $[i-1] \cup \{c(uw)\}$ . Then,  $\{uw\} \cup M_u \cup M'_u \cup \{x_jy_j : s+1 \leq j \leq i-1\}$  is a rainbow matching of size k in G, a contradiction. So  $c(uw) \in [i-1]$  for any  $u \in S$  and  $w \in W$ , showing part (d) of the claim.

Part (e) of the claim follows easily from Fact A, (c) and (d). This completes the proof of the claim.  $\hfill \Box$ 

Recall that s < k-1. So we have  $|W_{s+1}| = n - 2(k-1) - 2s - (k-1-s) \ge k-1-s \ge 1$ . Pick a vertex  $w \in W_{s+1}$ . By part (e) of the claim, w adjacent to vertices in  $\{y_{s+1}, y_{s+2}, \ldots, y_{k-1}\}$ or w incident with edges of colours in [s]. Hence, w has colour degree at most k-1, which contradicts  $\delta^c(G) \ge k$ . Thus, we must have s = k - 1 as claimed. In summary, we have  $M = \{x_i y_i : i \in [k-1]\}$  and  $M' = \{z_i w_i : i \in [k-1]\}$  with  $c(x_i y_i) = i = c(z_i w_i)$  for  $i \in [k-1]$ .

Now, if  $n \ge 4k - 3$ , then  $V(G) \ne V(M \cup M')$ . Pick a vertex  $w \notin V(M \cup M')$  and since w has colour degree at least k, there exists a vertex u such that uw is an edge and  $c(uw) \notin [k-1]$ . It is easy to see that G contains a rainbow matching of size k, contradicting our assumption. Therefore, we may assume n = 4k - 4 and  $k \ge 4$ .

Since  $\delta^c(G) \geq k$ , any vertex  $u \in \{x_1, y_1, z_1, w_1\}$  must have a neighbour v such that  $c(uv) \notin [k-1]$ . If  $v \notin \{x_1, y_1, z_1, w_1\}$ , then G contains a rainbow matching of size k. So, without loss of generality,  $x_1z_1$  and  $y_1w_1$  are edges in G with  $c(x_1z_1), c(y_1w_1) \notin [k-1]$ . By symmetry, we may assume that for each  $i \in [k-1]$ ,  $x_iz_i$  and  $y_iw_i$  are edges in G with  $c(x_iz_i), c(y_iw_i) \notin [k-1]$ . As  $\delta^c(G) \geq k \geq 4$ ,  $x_1$  must have a neighbour  $v \notin \{y_1, z_1, w_1\}$  with  $c(x_1v) \neq 1$ . Without loss of generality, we may assume  $v = z_j$  for some j and  $c(x_1z_j) = 2$ . Now,  $\{x_1z_j, z_1w_1, y_2w_2, \} \cup \{x_iy_i : i \in \{3, 4, \ldots, k-1\}$  is a rainbow matching of size k in G, which again is a contradiction. This completes the proof of the theorem.

#### 3 Remarks

In Theorem 1.1, the bound on n, the number of vertices, is sharp for k = 2, 3 (and trivially for k = 1), as shown by properly 3-edge-coloured  $K_4$  for k = 2 and by properly 3-edge-coloured two disjoint copies of  $K_4$  for k = 3. However, we do not know if the bound is sharp for  $k \ge 4$ .

**Question.** Given k, what is the minimum n such that every edge-coloured graph G of order n with  $\delta^{c}(G) = k$  contains a rainbow matching of size k?

#### Acknowledgment

The authors thank Alexandr Kostochka for suggesting the problem during 'Probabilistic Methods in Graph Theory' workshop at University of Birmingham, United Kingdom. We would also like to thank Daniela Kühn, Richard Mycroft and Deryk Osthus for organizing this nice event.

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