# A note on large rainbow matchings in edge-coloured graphs 

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#### Abstract

A rainbow subgraph in an edge-coloured graph is a subgraph such that its edges have distinct colours. The minimum colour degree of a graph is the smallest number of distinct colours on the edges incident with a vertex over all vertices. Kostochka, Pfender, and Yancey showed that every edge-coloured graph on $n$ vertices with minimum colour degree at least $k$ contains a rainbow matching of size at least $k$, provided $n \geq \frac{17}{4} k^{2}$. In this paper, we show that $n \geq 4 k-4$ is sufficient for $k \geq 4$.


## 1 Introduction

Let $G$ be a simple graph, that is, no loops or multiple edges. We write $V(G)$ for the vertex set of $G$ and $\delta(G)$ for the minimum degree of $G$. An edge-coloured graph is a graph in which each edge is assigned a colour. We say such an edge-coloured $G$ is proper if no two adjacent edges have the same colour. A subgraph $H$ of $G$ is rainbow if all its edges have distinct colours. Rainbow subgraphs are also called totally multicoloured, polychromatic, or heterochromatic subgraphs.

For a vertex $v$ of an edge-coloured graph $G$, the colour degree of $v$ is the number of distinct colours on the edges incident with $v$. The smallest colour degree of all vertices in $G$ is the minimum colour degree of $G$ and is denoted by $\delta^{c}(G)$. Note that a properly edge-coloured graph $G$ with $\delta(G) \geq k$ has $\delta^{c}(G) \geq k$.

In this paper, we are interested in rainbow matchings in edge-coloured graphs. The study of rainbow matchings began with a conjecture of Ryser [11], which states that every Latin square of odd order contains a Latin transversal. Equivalently, for $n$ odd, every properly $n$-edge-colouring

[^0]of $K_{n, n}$, the complete bipartite graph with $n$ vertices on each part, contains a rainbow copy of perfect matching. In a more general setting, given a graph $H$, we wish to know if an edgecoloured graph $G$ contains a rainbow copy of $H$. A survey on rainbow matchings and other rainbow subgraphs in edge-coloured subgraph can be found in 4]. From now onwards, we often refer to $G$ for an edge-coloured graph $G$ (not necessarily proper) of order $n$.

Li and Wang [9] showed that if $\delta^{c}(G)=k$, then $G$ contains a rainbow matching of size $\left\lceil\frac{5 k-3}{12}\right\rceil$. They further conjectured that if $k \geq 4$, then $G$ contains a rainbow matching of size $\left\lceil\frac{k}{2}\right\rceil$. This bound is tight for properly edge-coloured complete graphs. LeSaulnier et al. 8] proved that if $\delta^{c}(G)=k$, then $G$ contains a rainbow matching of size $\left\lfloor\frac{k}{2}\right\rfloor$. Furthermore, if $G$ is properly edge-coloured with $G \neq K_{4}$ or $|V(G)| \neq \delta(G)+2$, then there is a rainbow matching of size $\left\lceil\frac{k}{2}\right\rceil$. The conjecture was later proved in full by Kostochka and Yancey [7].

What happens if we have a larger graph? Wang [12] proved that every properly edge-coloured graph $G$ with $\delta(G)=k$ and $|V(G)| \geq \frac{8 k}{5}$ contains a rainbow matching of size at least $\left\lfloor\frac{3 k}{5}\right\rfloor$. He then asked if there is a function, $f(k)$, such that every properly edge-coloured graph $G$ with $\delta(G) \geq k$ and $|V(G)| \geq f(k)$ contains a rainbow matching of size $k$. The bound on the size of rainbow matching is sharp, as shown for example by any $k$-edge-coloured $k$-regular graph. If $f(k)$ exists, then we trivially have $f(k) \geq 2 k$. In fact, $f(k)>2 k$ for even $k$ as there exists $k \times k$ Latin square without any Latin transversal (see [1, 13]). Diemunsch et at. [2] gave an affirmative answer to Wang's question and showed that $f(k) \leq \frac{13}{5} k$. The bound was then improved to $f(k) \leq \frac{9}{2} k$ in [10], and shortly thereafter, to $f(k) \leq \frac{98}{23} k$ in [3].

Kostochka, Pfender and Yancey [6] considered a similar problem with $\delta^{c}(G)$ instead of properly edge-coloured graphs. They showed that if $G$ is such that $\delta^{c}(G) \geq k$ and $n>\frac{17}{4} k^{2}$, then $G$ contains a rainbow matching of size $k$. Kostochka 5 then asked: can $n$ be improved to a linear bound in $k$ ? In this paper, we show that $n \geq 4 k-4$ is sufficient for $k \geq 4$. Furthermore, this implies that $f(k) \leq 4 k-4$ for $k \geq 4$.

Theorem 1.1. If $G$ is an edge-coloured graph on $n$ vertices with $\delta^{c}(G) \geq k$, then $G$ contains a rainbow matching of size $k$, provided $n \geq 4 k-4$ for $k \geq 4$ and $n \geq 4 k-3$ for $k \leq 3$.

## 2 Main Result

We write $[k]$ for $\{1,2, \ldots, k\}$. For an edge $u v$ in $G$, we denote by $c(u v)$ the colour of $u v$ and let the set of colours be $\mathbb{N}$, the set of natural numbers.

The idea of the proof is as follows. By induction, $G$ contains a rainbow matching $M$ of size $k-1$. Suppose that $G$ does not contain a rainbow matching of size $k$. We are going to show that there exists another rainbow matching $M^{\prime}$ of size $k-1$ in $V(G) \backslash V(M)$. Clearly, the colours of $M$ equal to the colours of $M^{\prime}$. If $n \geq 4 k-3$, then there exists a vertex $z$ not in $M \cup M^{\prime}$. Since $\delta^{c}(G) \geq k, z$ has a neighbour $w$ such that $z w$ does not use any colour of $M$. Hence, it is easy to deduce that $G$ contains a rainbow matching of size $k$.

Proof of Theorem 1.1. We proceed by induction on $k$. The theorem is trivially true for $k=1$. So fix $k>1$ and assume that the theorem is true for $k-1$. Let $G$ be an edge-coloured graph with $\delta^{c}(G) \geq k$ and $n=|V(G)| \geq 4 k-4$ if $k \geq 4$ and $n \geq 4 k-3$ otherwise. Suppose that the theorem is false and so $G$ does not contain a rainbow matching of size $k$.

By induction, there exists a rainbow matching $M=\left\{x_{i} y_{i}: i \in[k-1]\right\}$ in $G$, say with $c\left(x_{i} y_{i}\right)=i$ for each $i \in[k-1]$. Let $M^{\prime}$ be another rainbow matching of size $s$ (which could be empty) in $G$ vertex-disjoint from $M$. Clearly $s \leq k-1$ and the colours on $M^{\prime}$ is a subset of [k-1], as otherwise $G$ contains a rainbow matching of size $k$. Without loss of generality, we may assume that $M^{\prime}=\left\{z_{i} w_{i}: i \in[s]\right\}$ with $c\left(z_{i} w_{i}\right)=i$ for each $i \in[s]$. We further assume that $M$ and $M^{\prime}$ are chosen such that $s$ is maximal. Now, let $W=V(G) \backslash V\left(M \cup M^{\prime}\right)$ and $S=\left\{x_{i}, y_{i}, z_{i}, w_{i}: i \in[s]\right\}$. Clearly, if there is an edge in $W$, it must have colour in $[s]$, otherwise $G$ contains a rainbow matching of size $k$, or $s$ is not maximal.

Fact A If $u w$ is an edge in $W$, then $c(u w) \in[s]$.
Furthermore, if $u v$ is an edge with $u \in S$ and $v \in W$, then $c(u v) \in[k-1]$, otherwise $G$ contains a rainbow matching of size $k$. First, we are going to show that $s=k-1$. Suppose the contrary, $s<k-1$. We then claim the following.

Claim By relabeling the indices of $i$ (in the interval $\{s+1, s+2, \ldots, k-1\}$ ) and swapping the roles of $x_{i}$ and $y_{i}$ if necessary, there exist distinct vertices $z_{k-1}, z_{k-2}, \ldots, z_{s+1}$ in $W$ such that for $s+1 \leq i \leq k-1$ the following holds for $s+1 \leq i \leq k-1$ :
(a) $y_{i} z_{i}$ is an edge and $c\left(y_{i} z_{i}\right) \notin[i]$.
(b) Let $T_{i}$ be the vertex set $\left\{x_{j}, y_{j}, z_{j}: i \leq j \leq k-1\right\}$. For any colour $j$, there exists a rainbow matching of size $k-i$ on $T_{i}$ which does not use any colour in $[i-1] \cup\{j\}$.
(c) Let $W_{i}=W \backslash\left\{z_{i}, z_{i+1}, \ldots, z_{k-1}\right\}$. If $x_{i} w$ is an edge with $w \in W_{i}$, then $c\left(x_{i} w\right) \in[s]$.
(d) If $u w$ is an edge with $u \in S$ and $w \in W_{i}$, then $c(u w) \in[i-1]$.
(e) If $u w$ is an edge with $u \in S \cup T_{i} \cup W$ and $w \in W_{i}$, then $c(u w) \in[i-1]$ or $u \in\left\{y_{i}, \ldots, y_{k-1}\right\}$.

Proof of Claim. Let $W_{k}=W$ and $T_{k}=\emptyset$. Observe that part (d) and (e) of the claim hold for $i=k$. For each $i=k-1, k-2, \ldots, s+1$ in terms, we are going to find $z_{i}$ satisfying (a) - (e). Suppose that we have already found $z_{k-1}, z_{k-1}, \ldots, z_{i+1}$.

Note that $\left|W_{i+1}\right| \geq n-2(k-1)-2 s-(k-i-1) \geq 1$, so $W_{i+1} \neq \emptyset$. Let $z$ be a vertex in $W_{i+1}$. By the colour degree condition, $z$ must incident with at least $k$ edges of distinct colours, and in particular, at least $k-i$ distinct coloured edges not using colours in $[i]$. Then, there exists a vertex $u \in\left\{x_{j}, y_{j}: s+1 \leq j \leq i\right\}$ such that $u z$ is an edge with $c(u z) \notin[i]$ by part (e) of the claim for the case $i+1$. Without loss of generality, $u=y_{i}$ and we set $z_{i}=z$.

Part (b) of the claim is true for colour $j \neq i$, simply by taking the edge $x_{i} y_{i}$ together with a rainbow matching of size $k-i-1$ on $T_{i+1}$ which does not use any colour in $[i] \cup\{j\}$. For colour
$j=i$, we take the edge $y_{i} z_{i}$ together with a rainbow matching of size $k-i-1$ on $T_{i+1}$ which does not use any colour in $[i] \cup\left\{c\left(y_{i} z_{i}\right)\right\}$.

To show part (c) of the claim, let $x_{i} w$ be an edge for some $w \in W_{i}$. By part (b) of the claim for the case $i+1$, there exists a rainbow matching $M^{\prime \prime}$ of size $k-i-1$ on $T_{i+1}$ which does not use any colour in $[i] \cup\left\{c\left(y_{i} z_{i}\right)\right\}$. Set $M_{0}=\left\{x_{j} y_{j}: j \in[i-1]\right\} \cup M^{\prime \prime} \cup\left\{y_{i} z_{i}\right\}$. Then, $M_{0}$ is a rainbow matching of size $k-1$ vertex-disjoint from $M^{\prime}$. Now, by considering the pair ( $M_{0}, M^{\prime}$ ) instead of $\left(M, M^{\prime}\right)$, we must have $c\left(x_{i} w\right) \in[s]$. Otherwise, $G$ contains a rainbow matching of size $k$ or $s$ is not maximal.

Let $u w$ be an edge with $u \in S, w \in W_{i}$ and $c(u w) \notin[i-1]$. Pick a rainbow matching $M_{u}$ of size $s$ on $S \backslash\{u\}$ with colours [s], and a rainbow matching $M_{u}^{\prime}$ of size $k-i$ on $T_{i}$ which does not contain any colour in $[i-1] \cup\{c(u w)\}$. Then, $\{u w\} \cup M_{u} \cup M_{u}^{\prime} \cup\left\{x_{j} y_{j}: s+1 \leq j \leq i-1\right\}$ is a rainbow matching of size $k$ in $G$, a contradiction. So $c(u w) \in[i-1]$ for any $u \in S$ and $w \in W$, showing part (d) of the claim.

Part (e) of the claim follows easily from Fact A, (c) and (d). This completes the proof of the claim.

Recall that $s<k-1$. So we have $\left|W_{s+1}\right|=n-2(k-1)-2 s-(k-1-s) \geq k-1-s \geq 1$. Pick a vertex $w \in W_{s+1}$. By part (e) of the claim, $w$ adjacent to vertices in $\left\{y_{s+1}, y_{s+2}, \ldots, y_{k-1}\right\}$ or $w$ incident with edges of colours in $[s]$. Hence, $w$ has colour degree at most $k-1$, which contradicts $\delta^{c}(G) \geq k$. Thus, we must have $s=k-1$ as claimed. In summary, we have $M=\left\{x_{i} y_{i}: i \in[k-1]\right\}$ and $M^{\prime}=\left\{z_{i} w_{i}: i \in[k-1]\right\}$ with $c\left(x_{i} y_{i}\right)=i=c\left(z_{i} w_{i}\right)$ for $i \in[k-1]$.

Now, if $n \geq 4 k-3$, then $V(G) \neq V\left(M \cup M^{\prime}\right)$. Pick a vertex $w \notin V\left(M \cup M^{\prime}\right)$ and since $w$ has colour degree at least $k$, there exists a vertex $u$ such that $u w$ is an edge and $c(u w) \notin[k-1]$. It is easy to see that $G$ contains a rainbow matching of size $k$, contradicting our assumption. Therefore, we may assume $n=4 k-4$ and $k \geq 4$.

Since $\delta^{c}(G) \geq k$, any vertex $u \in\left\{x_{1}, y_{1}, z_{1}, w_{1}\right\}$ must have a neighbour $v$ such that $c(u v) \notin[k-1]$. If $v \notin\left\{x_{1}, y_{1}, z_{1}, w_{1}\right\}$, then $G$ contains a rainbow matching of size $k$. So, without loss of generality, $x_{1} z_{1}$ and $y_{1} w_{1}$ are edges in $G$ with $c\left(x_{1} z_{1}\right), c\left(y_{1} w_{1}\right) \notin[k-1]$. By symmetry, we may assume that for each $i \in[k-1], x_{i} z_{i}$ and $y_{i} w_{i}$ are edges in $G$ with $c\left(x_{i} z_{i}\right), c\left(y_{i} w_{i}\right) \notin[k-1]$. As $\delta^{c}(G) \geq k \geq 4, x_{1}$ must have a neighbour $v \notin\left\{y_{1}, z_{1}, w_{1}\right\}$ with $c\left(x_{1} v\right) \neq 1$. Without loss of generality, we may assume $v=z_{j}$ for some $j$ and $c\left(x_{1} z_{j}\right)=2$. Now, $\left\{x_{1} z_{j}, z_{1} w_{1}, y_{2} w_{2},\right\} \cup\left\{x_{i} y_{i}\right.$ : $i \in\{3,4, \ldots, k-1\}$ is a rainbow matching of size $k$ in $G$, which again is a contradiction. This completes the proof of the theorem.

## 3 Remarks

In Theorem [1.1, the bound on $n$, the number of vertices, is sharp for $k=2,3$ (and trivially for $k=1$ ), as shown by properly 3 -edge-coloured $K_{4}$ for $k=2$ and by properly 3 -edge-coloured two disjoint copies of $K_{4}$ for $k=3$. However, we do not know if the bound is sharp for $k \geq 4$.

Question. Given $k$, what is the minimum $n$ such that every edge-coloured graph $G$ of order $n$ with $\delta^{c}(G)=k$ contains a rainbow matching of size $k$ ?

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