# Maximum hitting for $n$ sufficiently large 

Ben Barber*

November 5, 2018


#### Abstract

For a left-compressed intersecting family $\mathcal{A} \subseteq[n]^{(r)}$ and a set $X \subseteq[n]$, let $\mathcal{A}(X)=\{A \in \mathcal{A}: A \cap X \neq \emptyset\}$. Borg asked: for which $X$ is $|\mathcal{A}(X)|$ maximised by taking $\mathcal{A}$ to be all $r$-sets containing the element 1 ? We determine exactly which $X$ have this property, for $n$ sufficiently large depending on $r$.


## 1 Introduction

Write $[n]=\{1,2, \ldots, n\}$ and $[m, n]=\{m, m+1, \ldots, n\}$. Denote the set of $r$-sets from a set $S$ by $S^{(r)}$. A family of sets is a subset of $[n]^{(r)}$ for some $n$ and $r$. We think of a set $A$ as an increasing sequence of elements $a_{1} a_{2} \ldots a_{r}$. The compression order on $[n]^{(r)}$ has $A \leq B$ if and only if $a_{i} \leq b_{i}$ for $1 \leq i \leq r$. A family $\mathcal{A}$ is left-compressed if $A \in \mathcal{A}$ whenever $A \leq B$ for some $B \in \mathcal{A}$. The corresponding notion of left-compression is described in Section 2

We call a family intersecting if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$. (If $n<2 r$ then every family is intersecting.) The most basic result about intersecting families is the Erdős-Ko-Rado Theorem. For any $n$ and $r$, write $\mathcal{S}=\left\{A \in[n]^{(r)}: 1 \in A\right\}$ for the star at 1.

Theorem 1 (Erdős-Ko-Rado [3]). If $n \geq 2 r$ and $\mathcal{A} \subseteq[n]^{(r)}$ is intersecting, then $|\mathcal{A}| \leq|\mathcal{S}|$.

Borg considered a variant problem where we only count members that meet some fixed set $X$. For a family $\mathcal{A}$ and a non-empty set $X$, write

$$
\mathcal{A}(X)=\{A \in \mathcal{A}: A \cap X \neq \emptyset\} .
$$

Theorem 1 tells us that we can maximise $|\mathcal{A}(X)|$ by taking $\mathcal{A}$ to consist of all $r$ sets containing some fixed element of $X$. To avoid this trivial case we insist that $\mathcal{A}$ be left-compressed, which rules out stars centred anywhere but 1 . The star at 1 remains the optimal family if $1 \in X$, so we assume further that $X \subseteq[2, n]$.

Question 2. For which $X$ do we have $|\mathcal{A}(X)| \leq|\mathcal{S}(X)|$ for all left-compressed intersecting families $\mathcal{A}$ ?

Borg asked this question in [2], giving a complete answer for the case $|X| \geq r$ and a partial answer for the case $|X|<r$. Call $X \operatorname{good}$ (for $n$ and $r$ ) if for every left-compressed intersecting family $\mathcal{A} \subseteq[n]^{(r)}$ we have $|\mathcal{A}(X)| \leq|\mathcal{S}(X)|$.

[^0]Theorem 3 (Borg [2]). Let $r \geq 2, n \geq 2 r$ and $X \subseteq[2, n]$.
(a) If $|X|>r$, then $X$ is good.
(b) If $X$ is good and $X \leq X^{\prime}$, then $X^{\prime}$ is good.
(c) For any $k \leq r,\{2 k, 2 k+2, \ldots, 2 r\}$ is good.
(d) If $n=2 r$ and $|X|=r$, then $X$ is good if and only if $\{2,4, \ldots, 2 r\} \leq X$.
(e) If $n>2 r,|X|=r$ and either
(i) $r \geq 4$ and $X \neq[2, r+1]$,
(ii) $r=3$ and $\{2,3\} \nsubseteq X$, or
(iii) $r=2$ and $\{2,3\} \neq X$,
then $X$ is good. Otherwise, $X$ is not good.
It is not true that all $X$ are good. For example, consider the Hilton-Milner family $\mathcal{T}=\mathcal{S}([2, r+1]) \cup\{[2, r+1]\}$. The family $\mathcal{T}$ is left-compressed and for any $X \subseteq[2, r+1],|\mathcal{T}(X)|=|\mathcal{S}(X)|+1$, so $X$ is not good.

Our main result is that, surprisingly, for large $n$ and $|X| \geq 4$ this turns out to be the only obstruction.

Theorem 4. Let $r \geq 3, n \geq 2 r$ and $X \subseteq[2, n]$ with $|X| \leq r$. If $X \nsubseteq[2, r+1]$ and either
(i) $|X| \geq 4$,
(ii) $|X|=3$ and $\{2,3\} \nsubseteq X$,
(iii) $|X|=2$ and $2,3 \notin X$, or
(iv) $|X|=1$,
then, for $n$ sufficiently large, $X$ is good. Otherwise, $X$ is not good.
For $r=2$, condition (iii) needs to be replaced by $X \neq\{2,3\}$. The result can then be checked easily by hand or read out of Theorem 3 in conjunction with the Hilton-Milner example, so we assume $r \geq 3$ for simplicity.

Our proof uses Ahlswede and Khachatrian's notion of generating sets to express the sizes of maximal left-compressed intersecting families, and their restrictions under $X$, as polynomials in $n$. It turns out to be sufficient to consider only leading terms, reducing a question about intersecting families of $r$-sets to a question about intersecting families of 2 -sets, which have a very simple structure.

Section 2 sets out the basic properties of compressions and generating sets that we shall use. Section 3 describes a way of thinking about maximal leftcompressed intersecting families and proves the lemma that allows us to compare coefficients of polynomials instead of set sizes. Section 4 completes the proof of Theorem 4 Section 5 discusses possible improvements and generalisations.

## 2 Compressions and generating sets

In this section we describe the notion of left-compression corresponding to $\leq$ on $[n]^{(r)}$ and the use of generating sets.

### 2.1 Compressions

For a set $A$, and $i<j$, the $i j$-compression of $A$ is

$$
C_{i j}(A)= \begin{cases}A-j+i & \text { if } j \in A, i \notin A \\ A & \text { otherwise }\end{cases}
$$

that is, replace $j$ by $i$ if possible. Observe that $A \leq B$ if and only if $A$ can be obtained from $B$ by a sequence of $i j$-compressions.

For a set family $\mathcal{A}$, define

$$
C_{i j}(\mathcal{A})=\left\{C_{i j}(A): A \in \mathcal{A} \text { and } C_{i j}(A) \notin \mathcal{A}\right\} \cup\left\{A: A \in \mathcal{A} \text { and } C_{i j}(A) \in \mathcal{A}\right\} ;
$$

that is, compress $A$ if possible. Observe that $\mathcal{A}$ is left-compressed if and only if $C_{i j}(\mathcal{A})=\mathcal{A}$ for all $i<j$. We will use the following basic result.

Lemma 5. If $\mathcal{A}$ is intersecting then $C_{i j}(\mathcal{A})$ is intersecting.
Proof. The proof is an easy case check. Details, and a further introduction to compressions, can be found in Frankl's survey article 4].

Lemma 5 means that we can always compress an intersecting family to a left-compressed intersecting family of the same size by repeatedly applying $i j$ compressions. We eventually reach a left-compressed family as $\sum_{A \in \mathcal{A}} \sum_{i=1}^{r} a_{i}$ is positive and strictly decreases with each successful compression.

### 2.2 Generating sets

For any $r$ and $n$, and a collection $\mathcal{G}$ of sets, the family generated by $\mathcal{G}$ is

$$
\mathcal{F}(r, n, \mathcal{G})=\left\{A \in[n]^{(r)}: A \supseteq G \text { for some } G \in \mathcal{G}\right\}
$$

Generating sets were introduced by Ahlswede and Khachatrian [1], and are useful for the study of intersecting families because they give a restricted number of sets on which all the intersecting actually happens.

Lemma 6 (1]). For $n \geq 2 r, \mathcal{F}(r, n, \mathcal{G})$ is intersecting if and only if $\mathcal{G}$ is.
Proof. If $\mathcal{G}$ is intersecting then certainly $\mathcal{F}(r, n, \mathcal{G})$ is. Conversely, if $\mathcal{G}$ contains two disjoint sets then (since $n \geq 2 r$ ) they can be completed to disjoint $r$-sets in $\mathcal{F}(r, n, \mathcal{G})$.

If $\mathcal{G}$ generates a left-compressed intersecting family then

$$
\mathcal{G}^{\prime}=\left\{G^{\prime}: G^{\prime} \leq G \text { for some } G \in \mathcal{G}\right\}
$$

generates the same family, so we may assume that $\mathcal{G}$ is 'left-compressed' (overlooking non-uniformity) and can therefore be described by listing its maximal elements. It is convenient to take

$$
\mathcal{F}(r, n, \mathcal{G})=\left\{A \in[n]^{(r)}: A \prec G \text { for some } G \in \mathcal{G}\right\}
$$

where $A \prec G$ (' $A$ is generated by $G$ ') if and only if $|G| \leq|A|$ and $a_{i} \leq g_{i}$ for $1 \leq i \leq|G|$. We can think of $\prec$ as an extension of $\leq$ to the non-uniform case, where 'missing' elements are assumed to take the value $\infty$. Thus

$$
\begin{aligned}
123 & \prec 12(=12 \infty) ; \\
(12 \infty=) 12 & \nprec 123 .
\end{aligned}
$$

The following weaker form of Lemma 6 is better suited to our new definition and is sufficient for our purposes.

Corollary 7. Let $n \geq 2 r$ and $\mathcal{G}$ be a collection of subsets of $[2 s]$ of size at most s. If $\mathcal{F}(s, 2 s, \mathcal{G})$ is intersecting, then so is $\mathcal{F}(r, n, \mathcal{G})$.

## 3 Maximal left-compressed intersecting families

We say an intersecting family $\mathcal{A} \subseteq[n]^{(r)}$ is maximal if no other set can be added to $\mathcal{A}$ while preserving the intersecting property. The maximal objects in the set of left-compressed intersecting families are maximal intersecting families (otherwise an extension could be compressed to a left-compressed extension), so the ordering of 'maximal' and 'left-compressed' is unimportant.

The maximal left-compressed intersecting subfamilies of $[n]^{(2)}$ are $\{12,13, \ldots, 1 n\}$ and $\{12,13,23\}$, and we can already distinguish between these families when $n=4$. In fact, the same phenomenon occurs for all $r$.
Lemma 8. Let $\mathcal{A} \subseteq[2 r]^{(r)}$ be a maximal left-compressed intersecting family and $n \geq 2 r$. Then $\mathcal{A}$ extends uniquely to a maximal left-compressed intersecting subfamily of $[n]^{(r)}$. Moreover, every maximal left-compressed intersecting subfamily of $[n]^{(r)}$ arises in this way.

Proof. Since $\mathcal{A}$ is left-compressed, it can be completely described by listing its $\leq$-maximal elements $A_{1}, \ldots, A_{k}$. Some of these sets might contain final segments of [2r]. The idea is that the elements of these final segments would take larger values if they were allowed to, so we obtain a generating set by 'replacing them by $\infty^{\prime}$.

For $A=A_{i}$, take $s$ greatest with $a_{s}<r+s(s$ exists since $[r+1,2 r]$ is not a member of any left-compressed intersecting family), and let $A^{\prime}=a_{1} \ldots a_{s}$. Then $\mathcal{G}=\left\{A_{1}^{\prime}, \ldots, A_{k}^{\prime}\right\}$ generates $\mathcal{A}$, as the sets generated by $A_{i}^{\prime}$ are precisely those lying below $A_{i}$. Since $\mathcal{G}$ is a collection of subsets of $[2 r]$ of size at most $r$ and $\mathcal{A}=\mathcal{F}(r, 2 r, \mathcal{G})$ is intersecting, Corollary 7 tells us that $\mathcal{F}(r, n, \mathcal{G})$ is a left-compressed intersecting family for every $n$.

Now let $\mathcal{B}$ be any extension of $\mathcal{A}$ to a left-compressed intersecting subfamily of $[n]^{(r)}$. We will show that $\mathcal{B} \subseteq \mathcal{F}(r, n, \mathcal{G})$. Indeed, if $\mathcal{B} \nsubseteq \mathcal{F}(r, n, \mathcal{G})$ then there is a $B \in \mathcal{B} \backslash \mathcal{F}(r, n, \mathcal{G})$. We claim that there is a $B^{\prime} \in[2 r]^{(r)}$ with $B^{\prime} \leq B$ and $B^{\prime} \notin \mathcal{F}(r, 2 r, \mathcal{G})$, contradicting the maximality of $\mathcal{A}$.

We obtain $B^{\prime}$ from $B$ by compressing as little as possible to get $B^{\prime} \subseteq[2 r]$; that is, we take $B^{\prime}=(B \cap[2 r]) \cup[q, 2 r]$ with $q$ chosen such that $\left|B^{\prime}\right|=r$. Explicitly, $b_{i}^{\prime}=\min \left(b_{i}, r+i\right)$. Now take $G \in \mathcal{G}$. Since $B \notin \mathcal{F}(r, n, \mathcal{G})$, there is an $i$ with $b_{i}>g_{i}$. By construction, $r+i>g_{i}$. So $b_{i}^{\prime}=\min \left(b_{i}, r+i\right)>g_{i}$, and $G$ does not generate $B^{\prime}$. Hence $\mathcal{A}$ extends uniquely to a maximal left-compressed intersecting subfamily of $[n]^{(r)}$.

It remains to show that every maximal left-compressed intersecting subfamily of $[n]^{(r)}$ arises in this way. So suppose $\mathcal{C} \subseteq[n]^{(r)}$ is a maximal left-compressed intersecting family with $\mathcal{C} \cap[2 r]^{(r)}$ not maximal. Let $\mathcal{D}_{0}$ be an extension of $\mathcal{C} \cap[2 r]^{(r)}$ to a maximal left-compressed intersecting subfamily of $[2 r]^{(r)}$, and let $\mathcal{D}$ be the unique maximal extension of $\mathcal{D}_{0}$ to $[n]^{(r)}$. Since $\mathcal{C}$ is maximal and $\mathcal{D} \backslash \mathcal{C} \neq \emptyset$, there is a $C \in \mathcal{C} \backslash \mathcal{D}$. As above, we obtain $C^{\prime} \in[2 r]^{(r)}$ with $C^{\prime} \leq C$ and $C^{\prime} \notin \mathcal{D}_{0}$. But then $C^{\prime} \notin \mathcal{C}$, contradicting the assumption that $\mathcal{C}$ is left-compressed.

Lemma 8 allows a compact description of maximal left-compressed intersecting families. For example, $\{1\}$ generates the star and $\{1(r+1),[2, r+1]\}$ generates the Hilton-Milner family. Enumerating the generating sets using a computer is feasible for small $r$; for $r=3$ they are $\{1\},\{23\},\{345\},\{14,234\},\{13,235,145\}$ and $\{12,245\}$.

In view of Lemma 8, our key tool is the following.
Lemma 9. Let $n \geq 2, X \subseteq[2,2 r]$. Then

$$
|\mathcal{F}(r, n, \mathcal{G})(X)|=\sum_{i=1}^{r}|\mathcal{F}(i, 2 r, \mathcal{G})(X)|\binom{n-2 r}{r-i}
$$

Proof. How do we construct a member of $\mathcal{F}(r, n, \mathcal{G})(X)$ ? We first choose an initial segment for our set that is contained in $[2 r]$ and witnesses the membership of $\mathcal{F}(r, n, \mathcal{G})(X)$ (i.e. meets $X$ and is $\prec$ some $G \in \mathcal{G})$. We then complete our set by taking as many elements as we need from outside $[2 r]$. This gives rise to the size claimed.

## 4 Proof of Theorem 4

We first show that $X$ is not good if the given conditions do not hold. We have already seen that for $X \subseteq[2, r+1]$ the Hilton-Milner family shows that $X$ is not good for any $n$. In each of the remaining cases we claim that the family generated by $\{23\}$ shows that $X$ is not good for any $n$.

So take $X=23 k$ with $k \geq r+2$. We have

$$
|\mathcal{F}(r, n,\{1\})(23 k)|=\binom{n-2}{r-2}+\binom{n-3}{r-2}+\binom{n-4}{r-2}
$$

where the first term counts the sets containing 1 and 2 , the second term the sets containing 1 and 3 but not 2 , and the third term the sets containing 1 and $k$ but neither 2 nor 3 . Similarly,

$$
|\mathcal{F}(r, n,\{23\})(23 k)|=\binom{n-2}{r-2}+\binom{n-3}{r-2}+\binom{n-3}{r-2}
$$

where the terms count the sets containing 1 and 2 , the sets containing 1 and 3 but not 2 , and the sets containing 2 and 3 but not 1 respectively. Since $r \geq 3$, $|\mathcal{F}(r, n,\{23\})(23 k)|>|\mathcal{F}(r, n,\{1\})(23 k)|$ and $23 k$ is not good.

Next take $X=3 j$ with $j \geq r+2$. We have

$$
|\mathcal{F}(r, n,\{1\})(3 j)|=\binom{n-2}{r-2}+\binom{n-3}{r-2}
$$

where the terms count the sets containing 1 and 3 , and the sets containing 1 and $j$ but not 3 respectively. Similarly,

$$
|\mathcal{F}(r, n,\{23\})(3 j)|=\binom{n-2}{r-2}+\binom{n-3}{r-2}+\binom{n-4}{r-3}
$$

where the terms count the sets containing 1 and 3 , the sets containing 2 and 3 but not 1, and the sets containing 1,2 and $j$ but not 3 respectively. Again, since $r \geq 3,|\mathcal{F}(r, n,\{23\})(3 j)|>|\mathcal{F}(r, n,\{1\})(3 j)|$ and $3 j$ is not good. It follows from Theorem 3(b) that $2 j$ is not good either.

Now we take $X$ satisfying the conditions of the theorem and show that $X$ is good for $n$ sufficiently large. We will show that, for any $\mathcal{G} \neq\{1\}$, $|\mathcal{F}(2,2 r, \mathcal{G})(X)|<|\mathcal{F}(2,2 r,\{1\})(X)|=|X|$. Note that, for any $\mathcal{G},|\mathcal{F}(1,2 r, \mathcal{G})(X)|=$ 0 as the only possible singleton generator is 1 , which does not meet $X$. So by Lemma $9, \mathcal{F}(2, n, \mathcal{G})(X)$ has size polynomial in $n$ with leading coefficient $|\mathcal{F}(2,2 r, \mathcal{G})(X)|$, from which the result will follow.

There are two maximal left-compressed intersecting families of 2-sets, and $\mathcal{F}(2,2 r, \mathcal{G})(X)$ must be contained in one of them. We handle each case separately.

Suppose first that $\mathcal{F}(2,2 r, \mathcal{G})(X) \subseteq\{12,13,23\}$. Then it is enough to show that

$$
|\{12,13,23\}(X)|<|X| .
$$

This is clearly true for $|X| \geq 4$. If $|X|=3$, then it is true because one of 2 or 3 is missing from $X$ so that $|\{12,13,23\}(X)| \leq 2$. If $|X|=2$, then it is true because both 2 and 3 are missing from $X$, so that $|\{12,13,23\}(X)|=0$. Finally, if $|X|=1$, then it is true because $X=\{i\}$ with $i \geq r+2 \geq 4$.

Next suppose that $\mathcal{F}(2,2 r, \mathcal{G})(X) \subseteq\{12,13, \ldots, 1(2 r)\}$. Since $\mathcal{F}(r, 2 r, \mathcal{G})$ is left-compressed and has a member not containing the element 1 , it has $[2, r+1]$ as a member. Hence by the intersecting property of the generators, $\mathcal{F}(2,2 r, \mathcal{G})(X)$ cannot contain $1 j$ for any $j \geq r+2$. But $X \nsubseteq[2, r+1]$, so there is such a $j \in X \backslash[2, r+1]$ and $|\mathcal{F}(2,2 r, \mathcal{G})(X)|<|X|$.

## 5 Improvements and generalisations

What happens for small $n$ ? Theorem [3(c) tells us that our characterisation cannot be correct for all $n \geq 2 r$.

Question 10. How large is 'sufficiently large' for $n$ in Theorem 4??
For $2 \leq r \leq 5$, computational results suggest that $n \geq 2 r+2$ is large enough for our characterisation to be correct. It would be particularly nice to show that $n \geq 2 r+c$ is sufficient for some constant $c$ independent of $r$.

A natural conjecture is that for $n=2 r,[2 k, 2 k+2, \ldots, 2 r]$ is the unique minimal good set of its size. However, this is false; computational results give that $\{7,10\}$ and $\{5,8,10\}$ are unique minimal good sets of their size when $r=5$.

Question 11. Is there a 'nice' characterisation of the good sets for $n=2 r$ when $r$ is sufficiently large?

It seems unlikely that a good explicit description exists for intermediate values of $r$ and $n$. The following may be easier.

Question 12. Is there a short list of families, one of which maximises $|\mathcal{A}(X)|$ for any $X$ ?

Versions of Lemma 8 hold for any property that is preserved under leftcompression and can be detected on generating sets. The most obvious candidate is that of being $t$-intersecting (a family $\mathcal{A}$ is $t$-intersecting if $|A \cap B| \geq t$ for all $A, B \in \mathcal{A}$ ). Indeed, an identical argument gives the corresponding result that, for large $n$, a set $X \subseteq[t+1, n]$ with $|X| \geq t+3$ is good if and only if $X \nsubseteq[t+1, r+1]$. (For smaller $X$ the form of good $X$ is again decided by the need to prevent problems caused when $\mathcal{F}(t+1,2 r-t+1, \mathcal{G})(X) \subseteq[t+2]^{(t+1)}$.)

In the context of $t$-intersecting families it may be more natural to consider

$$
\mathcal{A}(s, X)=\{A \in \mathcal{A}:|A \cap X| \geq s\} .
$$

For $s=1$ the argument relies on the fact that maximal left-compressed $t$ intersecting families of $(t+1)$-sets have one of two very simple forms. For $s=2$, even the $t=1$ case is complicated by the larger number of structures of intersecting families of 3 -sets (more generally, $(t+s)$-sets); this problem seems likely to get worse for larger $s$ and $t$.

Acknowledgements. I would like to thank the anonymous referees for carefully reading an earlier draft of this paper and making a number of helpful comments.

## References

[1] Rudolf Ahlswede and Levon H. Khachatrian. The complete intersection theorem for systems of finite sets. European J. Combin., 18(2):125-136, 1997.
[2] Peter Borg. Maximum hitting of a set by compressed intersecting families. Graphs and Combinatorics, 27(6):785-797, 2011.
[3] P. Erdős, Chao Ko, and R. Rado. Intersection theorems for systems of finite sets. Quart. J. Math. Oxford Ser. (2), 12:313-320, 1961.
[4] P. Frankl. The shifting technique in extremal set theory. In C. Whitehead, editor, Surveys in Combinatorics, volume 123 of London Math. Soc. Lecture Notes Series, pages 81-110. Cambridge University Press, Cambridge, 1987.


[^0]:    *Department of Pure Mathematics and Mathematical Statistics, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WB, UK. b.a.barber@dpmms.cam.ac.uk

