ON A FAMILY OF CONJECTURES OF JOEL LEWIS ON ALTERNATING PERMUTATIONS

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ABSTRACT. We prove generalized versions of some conjectures of Joel Lewis on the number of alternating permutations avoiding certain patterns. Our main tool is the perhaps surprising observation that a classic bijection on pattern avoiding permutations often preserves the alternating property.

1. INTRODUCTION

The theory of pattern avoiding permutations has seen tremendous progress during the last two decades. The key definition is the following. Let $k \leq n$, let $p = p_1p_2\cdots p_n$ be a permutation of length n, and let $q = q_1q_2\cdots q_k$ be a permutation of length k. We say that p avoids q if there are no k indices $i_1 < i_2 < \cdots < i_k$ so that for all a and b, the inequality $p_{i_a} < p_{i_b}$ holds if and only if the inequality $q_a < q_b$ holds. For instance, $p = 2537164$ avoids $q = 1234$ because p does not contain an increasing subsequence of length four. See [1] for an overview of the main results on pattern avoiding permutations.

Recently, there has been an interest to extend the study of pattern avoiding permutations to alternating permutations. A permutation $p = p_1p_2\cdots p_n$ is called *alternating* if $p_1 < p_2 > p_3 < p_4 > \cdots$, that is, if $p_i < p_{i+1}$ if and only if i is odd. In [2], Joel Brewster Lewis has made a number of interesting conjectures on the numbers $A_n(q)$ of alternating permutations of length n that avoid a given pattern q. In particular, he conjectured that for all positive integers n, the equalities

$$A_{2n}(1234) = A_{2n}(1243), \tag{1}$$

and

$$A_{2n}(12345) = A_{2n}(12354) \tag{2}$$

hold. In this paper, we prove a general version of these conjectures, showing that for all n and for all k, the equality

$$A_{2n}(12\cdots k) = A_{2n}(12\cdots k(k-1)) \tag{3}$$

holds. We also explain why the same equality does not hold if $2n$ is replaced by $2n+1$. On the other hand, a slight modification of our method will show that for all n and for all k, the equality

$$A_n(12\cdots k) = A_n(21\cdots (k-1)k) \tag{4}$$

holds. The special case of $k = 4$, that is, the equality $A_n(1234) = A_n(2134)$ was conjectured by Joel Lewis in [2].

2. A classic bijection

In this section, we review a classic bijection of Julian West that will be useful for us. We point out that in this section, our permutations do *not* have to be alternating. One crucial definition is the following.

Definition 2.1. *The* rank *of an entry of a permutation is the length of the longest increasing subsequence that ends in that entry.*

For instance, in $p = 3526174$, entries 3, 2, and 1 are of rank one, entries 5 and 4 are of rank two, entry 6 is of rank three, and entry 7 is of rank 4. It is straightforward to prove that entries of the same rank always form a decreasing subsequence. It is also easy to see that if a permutation p avoids the increasing pattern $12\cdots k$, then all entries of p have rank $k-1$ or less.

For any permutation pattern q, let $S_n(q)$ denote the number of permutations of length n (or, in what follows, n-permutations) that avoid the pattern q.

Lemma 2.2. [4] *Let $k \geq 3$ be an integer. Then for all positive integers n, the equality $S_n(12\cdots k) = S_n(12\cdots k(k-1))$ holds.*

Note that in the special case of $k = 3$, the equality of the lemma reduces to $S_n(123) = S_n(132)$, and the proof we are going to present below reduces to the classic Simion-Schmidt bijection [3].

Proof. We construct a bijection f from the set X_n of all $12\cdots k$-avoiding n-permutations to the set Y_n of $12\cdots k(k-1)$-avoiding n-permutations.

Let $p \in X_n$. In order to obtain $f(p)$, leave all entries of p that are of rank $k-2$ or less in their place. Rearrange the entries of rank $k-1$ of p as follows. Let P be the set of positions of p in which an entry of rank $k-1$ is located, and let R be the set of entries of p that are of rank $k-1$. Now fill the positions of P with the entries in R from left to right, so that each position $i \in P$ is filled with the *smallest* entry r of R that has not been placed yet and that is larger than the closest entry of rank $k-2$ on the left of position i. Let $f(p)$ be the obtained permutation. Note that $f(p)$ avoids $12\cdots k(k-1)$ since the existence of such a pattern in $f(p)$ would mean that the last two entries of that pattern were not placed according to the rule specified above.

It is easy to see that this definition always enables us to create $f(p)$. Indeed, the very existence of p shows that there is at least one way to assign the entries of R to the positions in P so that each of these entries will have rank $k-1$ or higher. Putting the smallest eligible entry in the leftmost available position can only push other entries of R back, which will not decrease their rank.

Note that if entry p_i of p was of rank $k-2$ or less, then p_i did not move in the above procedure, and the rank of p_i did not change. If p_i was of rank

$k-1$, then p_i may have moved, and the rank of p_i as an entry of $f(p)$ is $k-1$ or higher.

In order to see that f is a bijection, we show that it has an inverse. Let $q \in Y_n$. The unique preimage $f^{-1}(q)$ can then be obtained by keeping all entries of q that are of rank $k-2$ or less fixed, and placing all the remaining entries (whose set is R) in the remaining slots in decreasing order. It is easy to see that this can always be done, and that in their new positions, each element of R will have rank $k-1$. Indeed, fill the available slots from left to right with available elements of R as follows. In each step, move the largest element of R that has not been placed yet into the leftmost available slot j, and move each element of R that has been weakly on the right of j one notch to the right. Then in each step, each slot i either contains an entry that is larger than what was in i before, or an entry that was on the left of i before. Both of these steps result in the new entry in position i having rank $k-1$. □

Example 2.3. *Let $k=4$. Then $f(893624751) = 893624571$. Indeed, the only entries of rank three in 893624751 are 7 and 5, so f rearranges them so that each spot is filled with the smallest entry larger than the closest entry of rank two on the left of that spot (in this case, the entry 4).*

3. Alternating Permutations

Now we turn our attention to alternating permutations, and prove the results announced in the introduction.

Theorem 3.1. *Let $k \geq 3$ be an integer, and let n be an* even *positive integer. Then we have $A_n(12\cdots k) = A_n(12\cdots k(k-1))$.*

Proof. We claim that the fact that n is even implies that the bijection f of Lemma 2.2 preserves the alternating property. In other words, if p is an alternating permutation of length n that avoids $12\cdots k$, then $f(p)$ is an alternating permutation of length n that avoids $12\cdots k(k-1)$, and vice versa, that is, if q is an alternating permutation of length n that avoids the pattern $12\cdots k(k-1)$, then $f^{-1}(q)$ is an alternating n-permutation that avoids $12\cdots k$.

Let $p = p_1p_2\cdots p_n$ be an alternating, $12\cdots k$-avoiding n-permutation, where n is an even positive integer. Call the entries of p that are larger than both their neighbors *peaks* and call the entries of p that are smaller than both of their neighbors *valleys*. Let us also say that p_1 is a valley and p_n is a peak. It is clear that all entries of p that are of rank $k-1$ must be peaks. Indeed, *because n is even*, all valleys in p are followed by a larger entry, so a valley of rank $k-1$ would have to be followed by a peak of rank k or higher, which is a contradiction.

Now let us apply the map f of Lemma 2.2 to our permutation p. As we have seen, that map keeps entries of rank $k-2$ or less fixed; it only moves entries of rank $k-1$, which are all peaks. Therefore, in order to prove that

$f(p)$ is an alternating permutation, it suffices to show that if f displaces entry p_i of p, that entry p_i will be a peak in $f(p)$.

So let us assume that f moves the peak entry p_i of p to a new position. As all valleys are fixed by f, that new position is necessarily between two valleys, say a and b. We need to prove that $p_i > a$ and $p_i > b$. (If the new position of p_i is at the very end of $f(p)$, then we only need to prove that $p_i > a$.)

By definition, p_i is *larger* than the closest entry y of rank $k-2$ on the left of its new position. If $a = y$, then this means that $p_i > a$. Otherwise, a is of rank $j \leq k-3$. This implies that $a < y$, otherwise the rank of a would be higher than the rank of y. So $a < y < p_i$, and therefore, $p_i > a$ again.

Similarly, $b < y$, otherwise b would be of rank at least $k-1$ since y is of rank $k-2$. As $p_i > y$, it follows that $p_i > b$, proving our claim that p_i is a peak in $f(p)$. This implies that $f(p)$ is an alternating permutation, since its entries in even positions are all peaks.

Now let q be an alternating, $12\cdots k(k-1)$-avoiding n-permutation. Consider $f^{-1}(q)$, where f^{-1} is the inverse of the bijection $f : X_n \to Y_n$, as defined in the proof of Lemma 2.2. As we saw in the proof of that lemma, $f^{-1}(q)$ is obtained from q by rearranging the entries of q that are of rank $k-1$ or higher in decreasing order. It is easy to see that all these entries are peaks in q. Indeed, if w were a valley of rank $k-1$ or higher in q, then there would be a increasing subsequence $w_1w_2\cdots w_{k-2}w$ in q. If the entry immediately preceding w is v, then this would mean that $w_1w_2\cdots w_{k-2}vw$ is a $12\cdots k(k-1)$-pattern in q, which is a contradiction.

So f^{-1} simply permutes some peaks of q among themselves. Therefore, in order to prove that $f^{-1}(q)$ is alternating, it is again sufficient to prove that each peak q_i that is displaced by f^{-1} is a peak in its new position. Let us say that f^{-1} moves q_i into a new position, where its new neighbors will be s and t. If q_i is larger than the old peak entry Q that was between s and t before, then of course q_i is a peak in $f^{-1}(q)$. If not, that means that f^{-1} moved q_i to the *right*. However, that means that q_i must be larger than both s and t, otherwise one of s and t would have rank k or more in q. Indeed, q_i, which is an entry of rank $k-1$, would be on their left in q. That would be a contradiction, since s and t are not peaks of q, so they are of rank less than $k-1$. □

Example 3.2. *Let $n = 8$, let $k = 4$, and let $p = 47581623$. Then the only entries of rank three in p are 8, 6 and 3. Rearranging them as described above yields the alternating permutation $f(p) = 47561823$.*

A careful look at the above proof reveals which parts of the argument will carry over to the case of *odd* n, and which parts will not. The proof of the fact that if q_i is a peak in q, then f^{-1} moves q_i into a position where q_i will be a peak again, did not use the fact that n was even. So f^{-1} maps alternating permutations to alternating permutations, even if n is odd. However, if n is odd, and p is alternating, and $12\cdots k$-avoiding, then $f(p)$ will not be alternating if and only if there is a entry of rank $k-1$ in p that is not a peak. It is easy to see that that happens precisely when the *last* entry of p is of rank $k-1$, like in $p = 23154$. So we have proved the following Corollary.

Corollary 3.3. *Let n be an odd positive integer. Then the inequality*

$$A_n(12\cdots k) \geq A_n(12\cdots k(k-1))$$

holds. Furthermore, $A_n(12\cdots k) - A_n(12\cdots k(k-1))$ is equal to the number of alternating, $12\cdots k$-avoiding n-permutations whose last entry is of rank $k-1$.

Finally, we use a slightly modified version of our argument to prove the following theorem.

Theorem 3.4. *Let n be any positive integer. Then for all k, we have*

$$A_n(12\cdots k) = A_n(21\cdots k).$$

Proof. The proof is similar to the proof of Theorem 3.1. First, we construct a bijection g that proves the equality

$$S_n(12\cdots k) = S_n(213\cdots k)$$

for every n. To this end, let us say that an entry of a permutation is of *co-rank* i if the longest increasing subsequence *starting* in that entry has length i. If $p \in X_n$, then we define $g(p)$ as follows. Let g keep all entries of p that are of co-rank $k-2$ or less fixed. Fill the remaining slots with the remaining entries from right to left, so that in position j, we always put the *largest* remaining entry that is smaller than the closest entry of co-rank $k-2$ on the right of j. Then g is a bijection from X_n to the set Z_n of $213\cdots k$-avoiding permutations of length n as can be proved in a way analogous to the proof of Lemma 2.2.

Next, we claim that g preserves the alternating property. In order to prove this, we point out that if $p \in X_n$, then entries of p of co-rank $k-1$ are necessarily valleys, since if they were peaks, they would be immediately preceded by a valley, which would have co-rank at least k, a contradiction. *Note that unlike in the proof of Theorem 3.1, there is no need for parity restrictions here.* The rest of the proof is analogous to that of Theorem 3.1. □

4. Further directions

As we mentioned, there is a large collection of conjectures in [2] claiming that $A_n(q) = A_n(q')$ for some patterns q and q'. Some of these conjectures are for all integers n, some others for integers of a given parity. In many cases, the corresponding equalities $S_n(q) = S_n(q')$ are known to be true. As the results of this paper show, sometimes the bijection that proves an equality for *all* pattern avoiding permutations preserves the alternating property, and hence can be used to prove the corresponding equality for alternating permutations. The question is, of course, exactly when we can do this.

References

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