On heterochromatic out-directed spanning trees in tournaments^{*}

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Abstract

Given a tournament T, let h(T) be the smallest integer k such that every arc-coloring of T with k or more colors produces at least one out-directed spanning tree of T with no pair of arcs with the same color. In this paper we give the exact value of h(T).

Keywords: Out-directed Tree. Tournament. Heterochromatic

1 Introduction

Given a graph G and an edge-coloring of G, a subgraph H of G is said to be *hete*rochromatic if no pair of edges of H have the same color. Problems concerning the existence of heterochromatic subgraphs with a specific property in edge-colorings of a host graph are known as *anti-Ramsey problems* (see, for instance, [1, 4, 5, 7, 9, 11]). Typically, the host graph G is a complete graph or some graph with a particular

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structure, and the property which defines the set of heterochromatic subgraphs in consideration is that they are *isomorphic to a given graph* H or that they are subgraphs of G with a general property like, for example, being edge-cuts or spanning trees of G (see [2, 3, 6, 8, 10]).

A tournament is a digraph D = (V(D), A(D)) such that for every pair $\{x, y\} \subseteq V(D)$, either $xy \in A(D)$ or $yx \in A(D)$ but not both. A spanning tree S of a tournament T is an *out-directed spanning tree* of T if there is a root vertex r of S such that for each vertex $u \in V(S)$, the unique r - u path in S is directed from r to u.

In this paper, the host graphs are tournaments, and the property that defines the set of heterochromatic subgraphs in consideration is that of being an out-directed spanning tree of the corresponding tournament.

Let T = (V(T), A(T)) be a tournament. An arc-coloring of T is a function $\Gamma : A(T) \to C$, where C is a set of "colors"; if $|\Gamma[A(T)]| = k$ we say that Γ is a k-arc-coloring of T. A subdigraph H of T is said to be *heterochromatic* if no pair of arcs of H have the same color. We define h(T) as the smallest integer k such that every k-arc-coloring of T produces at least one heterochromatic-out directed spanning tree of T. Our main result is the following theorem:

Theorem 1. Let T be a tournament of order $n \ge 3$. Then $h(T) = \binom{n}{2} - \delta_3^-(T) + 2$, where $\delta_3^-(T) = \min\{d_T^-(x) + d_T^-(y) + d_T^-(w) : \{x, y, w\} \subseteq V(T)\}$. Moreover, if the arcs of T are colored with h(T) - 1 colors, and there is no heterochromatic out-directed spanning tree of T, then there is a triple $\{x, y, w\} \subseteq V(T)$ such that $\delta_3^-(T) = d_T^-(x) + d_T^-(y) + d_T^-(w)$, all the in-arcs of x, y, and w receive the same color and each of the remaining arcs of T receives a new different color.

2 Notation and Preliminary Results

Let D = (V(D), A(D)) be a digraph and x be a vertex of D. We denote by $N_D^+(x) = \{v \in V(D) : xv \in A(D)\}$ and $N_D^-(x) = \{v \in V(D) : vx \in A(D)\}$ the sets of *out-neighbors* and of *in-neighbors* of x in D, respectively. Likewise, we denote by $d_D^+(x) = |N_D^+(x)|$ and $d_D^-(x) = |N_D^-(x)|$ the *ex-degree* and the *in-degree* of x in D, respectively.

For every $Q \subseteq V(D)$, let $F_D^+(Q) = \{zw \in A(D) : z \in Q \text{ and } w \in V(D) \setminus Q\}$, $F_D^-(Q) = \{wz : z \in Q \text{ and } w \in V(D) \setminus Q\}$ and $F_D(Q) = F_D^+(Q) \cup F_D^-(Q)$. Given $x \in V(D)$ the sets $F_D^+(\{x\}), F_D^-(\{x\})$ and $F_D(\{x\})$ are called the set of ex-arcs, the set of in-arcs and the set of arcs of x, respectively. For $Q, R \subseteq V(D)$, we denote by $(Q \to R)$ the set $\{xy \in A(D) : x \in Q \text{ and } y \in R\}$. Let $\Gamma : A(D) \to C$ be an arc-coloring of D. We denote by C(x) the set of colors that appear only on arcs of D incident to x, and by c(x) the number of colors in C(x). A color $i \in C$ is a Γ_D -singular color if $|\Gamma^{-1}(i)| = 1$.

For any vertex $x \in V(D)$ and any arc $wy \in A(D)$, we denote by D-x and D-wythe digraphs obtained from D by deleting the vertex x and the arc wy, respectively. For an arc $zy \notin A(D)$, D + zy is the digraph obtained from D by adding the arc zy.

We say that a vertex $z \in V(D)$ is reachable from a vertex x in D if there is a directed path in D from x to z.

Let $\delta_3^-(D) = \min\{d_D^-(x) + d_D^-(y) + d_D^-(w) : \{x, y, w\} \subseteq V(D)\}.$

Lemma 1. Let T be a tournament of order $n \geq 3$. Then

$$h(T) \ge \binom{n}{2} - \delta_3^-(T) + 2.$$

Proof. Let $\{x, y, w\} \subseteq V(T)$ such that $d_T^-(x) + d_T^-(y) + d_T^-(w) = \delta_3^-(T)$ and color the arcs of T with $\binom{n}{2} - \delta_3^-(T) + 1$ colors in the following way: all the in-arcs of x, y and w receive the same color, say color black, and the remaining $\binom{n}{2} - \delta_3^-(T)$ arcs receive $\binom{n}{2} - \delta_3^-(T)$ new colors.

Given an out-directed spanning tree S of T we can assume, without loss of generality, that neither x nor y is the root of S, and therefore $d_S^-(x) = d_S^-(y) = 1$. From here we see that S has at least two black arcs, thus S is not heterochromatic and the lemma follows.

3 Proof of Theorem 1

Lemma 1 gives the lower bound for h(T) in Theorem 1. The proof of the upper bound and of the remainder of the theorem is by induction on n. For better readability, we break down the proof into several lemmas.

It is not hard to see that if T is a tournament of order 3, and Γ is and arccoloring of T with no heterochromatic out-directed spanning tree, then Γ uses $1 = \binom{3}{2} - \delta_3^-(T) + 1$ color. It is also clear that $V(T) = \{x, y, z\}$ is such that $d_T^-(x) + d_T^-(y) + d_T^-(z) = \delta_3^-(T) = 3$ and that the three in-arcs of x, y and z receive the same color. This shows that Theorem 1 holds for tournaments of order 3.

Let T be a tournament of order $n \ge 4$. For the rest of the proof we assume as inductive hypothesis that Theorem 1 holds for every tournament of order m, with $3 \le m < n$.

Let Γ be an arc-coloring of T which uses h(T) - 1 colors and produces no heterochromatic out-directed spanning trees of T. Observe that by Lemma 1, $h(T) \geq$ $\binom{n}{2} - \delta_3^-(T) + 2$ and therefore the number of colors in $\Gamma[A(T)]$ (from now on $\Gamma[T]$ for short) is at least $\binom{n}{2} - \delta_3^-(T) + 1$.

A vertex x of T is of type 1 if there is an in-arc e of x such that $\Gamma(e) \in C(x)$; of type 2 if none of the in-arcs of x receive a color in C(x) and there are at least two in-arcs of x which receive different colors; and of type 3 if none of the in-arcs of x receive a color in C(x) and all the in-arcs of x receive the same color.

The next three lemmas will show some properties of the vertices of type 1 and 2, and that there are at most n-2 vertices of type 1. With these at hand, we will return to the proof of Theorem 1.

Lemma 2. If x is a vertex of T of type 1, then $c(x) \ge n - 4$.

Proof. Since x is of type 1, there is an arc $yx \in A(T)$ such that $\Gamma(yx) \in C(x)$. Since $\Gamma(yx) \notin \Gamma[T-x]$, the tournament T-x has no heterochromatic out-directed spanning tree S, otherwise S + yx would be a heterochromatic out-directed spanning tree of T, which is not possible. Therefore, by our induction hypothesis, the number of colors appearing in $\Gamma[T-x]$ is at most $\binom{n-1}{2} - \delta_3^-(T-x) + 1$. Thus

$$c(x) \ge \binom{n}{2} - \delta_3^-(T) + 1 - \left(\binom{n-1}{2} - \delta_3^-(T-x) + 1\right) = n - 1 + \delta_3^-(T-x) - \delta_3^-(T).$$

Now just observe that $\delta_3^-(T) - \delta_3^-(T-x) \le 3$ and therefore $c(x) \ge n-4$.

Lemma 3. If x is a vertex of T of type 2, then $d_T^+(x) \ge c(x) = n - 4$.

Proof. By definition of type 2, none of the colors of the in-arcs of x is in C(x), so all the colors from C(x) appear on the out-arcs of x and therefore $d_T^+(x) \ge c(x)$. Also by definition, there are vertices $y_1, y_2 \in N_T^-(x)$ such that $c_1 = \Gamma(y_1 x) \ne \Gamma(y_2 x) = c_2$ with $c_1, c_2 \notin C(x)$.

Let Γ' be an arc-coloring of T - x obtained from Γ by recoloring the arcs of color c_2 with color c_1 .

Suppose T-x has an out-directed spanning tree S which is heterochromatic with respect to Γ' . Clearly S is also heterochromatic with respect to Γ and it is such that either color c_1 or color c_2 does not appear in $\Gamma[S]$. Thus, either $S + y_1 x$ or $S + y_2 x$ is a heterochromatic out-directed spanning tree of T with respect to Γ , which is not possible. Therefore T - x has no heterochromatic out-directed spanning tree with respect to Γ' . By our induction hypothesis, there are at most $\binom{n-1}{2} - \delta_3^-(T-x) + 1$ colors in $\Gamma'[T-x]$. It follows at most $\binom{n-1}{2} - \delta_3^-(T-x) + 2$ colors of Γ are used in T - x which implies

$$c(x) \ge \binom{n}{2} - \delta_3^-(T) + 1 - \left(\binom{n-1}{2} - \delta_3^-(T-x) + 2\right) = n - 2 - \delta_3^-(T) + \delta_3^-(T-x) \ge n - 5$$

If c(x) = n - 5, each of the following must happen: i) $\delta_3^-(T) - \delta_3^-(T-x) = 3$; ii) $|\Gamma[T-x]| = \binom{n-1}{2} - \delta_3^-(T-x) + 2$; iii) $|\Gamma'[T-x]| = \binom{n-1}{2} - \delta_3^-(T-x) + 1$ and iv) T-x has no heterochromatic out-directed spanning tree with respect to Γ' .

By induction $h(T-x) = \binom{n-1}{2} - \delta_3^-(T-x) + 2$ and therefore, according to *iii*), Γ' is an arc-coloring of T-x with h(T-x) - 1 colors. Also by induction, there is a triple $\{x_1, x_2, x_3\} \subseteq V(T-x)$ such that $\delta_3^-(T-x) = d_{T-x}^-(x_1) + d_{T-x}^-(x_2) + d_{T-x}^-(x_3)$, all the in-arcs of x_1, x_2 , and x_3 have the same color in Γ' and each of the remaining arcs of T-x has a singular color in Γ' .

Recall that there are arcs in T - x with colors c_1 and c_2 , since $c_1, c_2 \notin C(x)$. Therefore c_1 is the non-singular color in Γ' and all the in-arcs of x_1, x_2 , and x_3 have color c_1 in Γ' . This implies that all the in-arcs of x_1, x_2 , and x_3 have color c_1 or color c_2 in Γ ; and each of the remaining arcs of T - x has a singular color in Γ .

By i, $\delta_3^-(T) - \delta_3^-(T-x) = 3$ and this implies $\{x_1, x_2, x_3\} \subseteq N_T^+(x)$. Therefore $\{y_1, y_2\} \subseteq V(T) \setminus \{x, x_1, x_2, x_3\}$ and $N_T^+(x) \subseteq V(T) \setminus \{x, y_1, y_2\}$. Since c(x) = n - 5, it follows that there is at least one vertex $z \in \{x_1, x_2, x_3\}$ such that $\Gamma(xz) \in C(x)$. Without loss of generality assume $z = x_1$.

Case 1. $\{\Gamma(xx_1), \Gamma(xx_2), \Gamma(xx_3)\} \cap C(x) = \Gamma(xx_1).$

The ex-arcs of x with the other (n-6) colors of C(x) appear in $(x \to [V(T) \setminus \{x, x_1, x_2, x_3\}])$. Thus $N_T^-(x) = \{y_1, y_2\}$ and $\delta_T^-(x) = 2$. Since $\delta_3^-(T) - \delta_3^-(T-x) = 3$, it follows that $\delta_3^-(T) = d_T^-(x_1) + d_T^-(x_2) + d_T^-(x_3)$ and therefore $\delta_T^-(x_i) \leq 2$ for i = 1, 2, 3. Since $\{x_1, x_2, x_3\} \subseteq N_T^+(x)$, it follows that $\{x_1, x_2, x_3\}$ induces a directed cycle with length 3 in T (with colors c_1 and c_2), and $V(T) \setminus \{x, x_1, x_2, x_3\} \subseteq N^+(x_i)$ for i = 1, 2, 3, where each of the arcs in $(\{x_1, x_2, x_3\} \to [V(T) \setminus \{x, x_1, x_2, x_3\}])$ receives a Γ_{T-x} -singular color (none of them a color in C(x)). Therefore, the tournament H induced by $V(T) \setminus \{x, x_1\}$ is a heterochromatic tournament in which either c_1 or c_2 appear, but not both. Thus, in H there is a hamiltonian heterochromatic path P where, without loss of generality, color c_1 does not appear. Therefore $E(P) \cup \{y_2x\} \cup \{xx_1\}$ induces a heterochromatic out-directed spanning tree of T which is not possible.

Case 2. $|\{\Gamma(xx_1), \Gamma(xx_2), \Gamma(xx_3)\} \cap C(x)| \ge 2.$

Suppose $\Gamma(xx_2) \in C(x)$ and $\Gamma(xx_1) \neq \Gamma(xx_2)$. Consider the tournament H induced by $V(T) \setminus \{x, x_1, x_2\}$ and let P be a hamiltonian path in H. Except for the in-arcs of x_3 , which receive color c_1 or c_2 , all the other arcs in H receive Γ_{T-x} -singular colors. Thus P is a heterochromatic path in which either color c_1 or color c_2 appear, but not both. Without loss of generality, suppose color c_1 does not appear in P. In this case $E(P) \cup \{y_2x\} \cup \{xx_1, xx_2\}$ induces a heterochromatic out-directed spanning tree of T which again is not possible.

From Case 1 and Case 2, it follows that $c(x) \ge n-4$. Suppose $c(x) \ge n-3$.

Since $d_T^+(x) \ge c(x)$ and $\Gamma(y_1x), \Gamma(y_2x) \notin C(x)$, all the ex-arcs of x receive different colors and all of them lie in C(x). Since $\Gamma(y_1x) = c_1 \neq c_2 = \Gamma(y_2x)$ and the color of the arc with endpoints y_1 and y_2 is not in C(x), it is not hard to see that either $F_T^+(\{x\}) \cup \{y_1x, y_1y_2\}$ or $F_T^+(\{x\}) \cup \{y_1y_2, y_2x\}$ induces a heterochromatic outdirected spanning tree of T which is not possible. Therefore c(x) = n-4 and Lemma 3 follows.

Lemma 4. There are at most n - 2 vertices of T of type 1.

Proof. Suppose there are at least n-1 vertices of type 1. Let D be a spanning subdigraph of T with the minimum number of connected components whose arc set is obtained as follows: choose a set A with n-1 vertices of type 1, and for each vertex $x \in A$, choose one in-arc of x with a color in C(x).

Clearly D is heterochromatic. Since there are no heterochromatic out-directed spanning trees of T, D is not connected. Let D_1, D_2, \ldots, D_r be the connected components of D. Since D has n vertices and n-1 arcs and the maximum in-degree of D is 1, it is not hard to see that one connected component, say D_1 , is an out-directed tree, while, for $i = 2, 3, \ldots, r$, component D_i contains exactly one directed cycle C_i such that D - e is an outdirected tree for each edge e of C_i . Let z_1 be the root of D_1 and notice that $A = V(T) \setminus \{z_1\}$.

Claim 1. Let $x \in V(C_2)$, $y \in \bigcup_{j \neq 2} V(C_j) \cup \{z_1\}$ and e be the arc with endpoints $\{x, y\}$. If $\Gamma(e) \in C(x)$ then e is an exarc of x and $\Gamma(e)$ is not a Γ_T -singular color.

Suppose $\Gamma(e) \in C(x)$. If e is an in-arc of x, the digraph (D - wx) + e, with $wx \in A(C_2) \subseteq A(D)$, has fewer connected components than D and can be obtained in the same way as D by choosing in C(x) the edge e instead of wx, which is a contradiction. Hence e is an ex-arc of x, and therefore an in-arc of y. Let us suppose $\Gamma(e)$ is a Γ_T -singular color. Thus $\Gamma(e) \in C(y)$ and y is of type 1. On the one hand, if $y \in V(C_j)$ for some $j \neq 2$, in an analogous way as with the vertex x, we reach a contradiction. On the other hand, if $y = z_1$ the digraph (D - wx) + e (which has fewer connected components than D) can be obtained in the same way as D by choosing the set $A' = (A \setminus \{x\}) \cup \{z\}$ as the set of n - 1 vertices of type 1 and choosing the edge e in $C(z_1)$ instead of the edge wx in C(x), which is a contradiction. From here, Claim 1 follows.

Let $x \in V(C_2)$. Since c(x) = n - 4 it follows there are at least n - 7 arcs incident to x with Γ_T -singular colors. Thus, by Claim 1 it follows that $|\{z_1\} \cup \bigcup_{j \neq 2} V(C_j)| \leq 6$ and therefore $r \leq 3$. Let us suppose r = 2 and let e be the arc with endpoints $\{z_1, x\}$. The color $\Gamma(e)$ must appear in D, otherwise D + e is a heterochromatic digraph containing an out-directed spanning tree of T which is a contradiction. By the choice of the arcs of D, $\Gamma(e) \in C(x)$ and there is an arc $wx \in A(C_2)$ with color $\Gamma(e)$, but then (D - wx) + e is a heterochromatic out-directed spanning tree of Twhich is a contradiction. Thus r = 3. Since c(x) = n - 4 and $|\{z_1\} \cup V(C_3)| \ge 4$, there is a color $c \in C(x)$ which only appears in arcs incident to x and with the other endpoint in $V(C_3) \cup \{z_1\}$. By Claim 1, these arcs are ex-arcs of x and there are at least two of them, since c is not a Γ_T -singular color. Thus there is $y \in V(C_3)$ such that $\Gamma(xy) = c$. Let $w \in V(C_2) \setminus \{x\}$ and let e be the arc with endpoints $\{z_1, w\}$. The color $\Gamma(e)$ must appear in D + xy, otherwise $D + \{xy, e\}$ is a heterochromatic digraph containing an out-directed spanning tree of T which is a contradiction. Thus, by the choice of the arcs of D and since $\Gamma(xy) \in C(x)$, $\Gamma(e) \in C(w)$ and there is an arc $ww' \in A(C_2)$ with color $\Gamma(e)$, but then $(D - ww') + \{xy, e\}$ is a heterochromatic digraph containing an out-directed spanning tree of T which is a contradiction. Thus, by the choice of the arcs of D and since $\Gamma(xy) \in C(x)$, $\Gamma(e) \in C(w)$ and there is an arc $ww' \in A(C_2)$ with color $\Gamma(e)$, but then $(D - ww') + \{xy, e\}$ is a heterochromatic digraph containing an out-directed spanning tree of T which is a contradiction. This ends the proof of Lemma 4.

Now we return to the proof of Theorem 1. First we will show that there is an arc $x_1x_2 \in A(T)$ and a vertex $x_3 \in V(T) \setminus \{x_1, x_2\}$ such that the spanning subdigraph D of T with set of arcs

$$A(D) = \left(A(T) \setminus \bigcup_{i=1}^{3} F_{T}^{-}(\{x_{i}\})\right) \cup \{x_{1}x_{2}\}$$

is an heterochromatic spanning subdigraph of T with h(T) - 1 arcs. Observe that these will imply that

$$h(T) - 1 = |A(D)| = \binom{n}{2} - \left(d_T^-(x_1) + d_T^-(x_2) + d_T^-(x_3)\right) + 1 \le \binom{n}{2} - \delta_3^-(T) + 1$$

which will prove the first part of the theorem.

Recall that if v is a vertex of T of type 3, then all the in-arcs of v recieve the same color. For each such vertex v we denote by c_v the color assigned to every in-arc of v.

Now we will choose a pair of vertices $\{x, y\}$ in the following way: By Lemma 4 there are at least two vertices that are not of type 1. If there are at least two vertices of type 3, choose x and y to be vertices of type 3 such that $c_x = c_y$ if possible, otherwise chose any two vertices of type 3. If there is exactly one vertex of type 3, choose it together with any vertex of type 2. Otherwise choose x and y to be vertices of type 2.

Without loss of generality assume $xy \in A(T)$ and let $c_0 = \Gamma(xy)$. Let D be a maximal heterochromatic spanning subdigraph of $A(T) \setminus F_T^-[\{x, y\}] \cup xy$ that contains

xy. Observe that the number of arcs in D is

$$|A(D)| = \Gamma[T] - k(x, y), \tag{1}$$

where k(x, y) is the number of colors that only appear in the set of arcs $F_T^-[\{x, y\}]$.

Claim 2. k(x, y) = 0.

Suppose $k(x, y) \geq 1$ and let c_1 be a color that only appears in the set of arcs $F_T^-(\{x, y\})$. Since neither x nor y are of type 1, $c_1 \notin C(x) \cup C(y)$, there is a pair of arcs $\{z_x x, z_y y\} \subseteq F_T^-[\{x, y\}]$ (where z_x and z_y are not necessarily different) such that $\Gamma(z_x x) = \Gamma(z_y y) = c_1$. Since y is not of type 1 and $\Gamma(xy) = c_0 \neq c_1 = \Gamma(z_y y)$, it follows that y is of type 2.

Let $A = \{yx_1, yx_2, \ldots, yx_{c(y)}\}$ be a set of ex-arcs of y, all of them with different colors in C(y), contained in A(D). By Lemma 3, c(y) = n - 4, since y is of type 2. Thus $\{xy\} \cup A$ induces a heterochromatic out-directed tree of order n - 2, with root x and with colors in $\{c_0\} \cup C(y)$.

Let $\{w_1, w_2\} = V(T) \setminus (\{x, y\} \cup \{x_i : yx_i \in A\})$. Observe that $z_y \in \{w_1, w_2\}$ and, without loss of generality, assume $z_y = w_1$. Since y is of type 2, by the way x and y were chosen, it follows that neither w_1 nor w_2 is of type 3. For i = 1, 2, observe that if w_i is of type 1, then there is an in-arc of w_i with a color in $C(w_i)$, which does not appear in $xy \cup A$. Also notice that if w_1 is of type 2, then there are two in-arcs of w_1 , with different colors such that those colors are not in C(y) and that if w_2 is of type 2, then there are two in-arcs of w_2 , also with different colors, such that at least one of those colors is not in C(y) (maybe $yw_2 \in A(T)$ and $\Gamma(yw_2) \in C(y)$).

In any case, there exist in-arcs e_1 of $w_1 = z_y$ and e_2 of w_2 with different colors, none of them with color in C(y), none of them with color c_1 (recall that all the arcs of color c_1 are in-arcs of x and y), and maybe one of them with color c_0 . Since $\Gamma(z_x x) = c_1$, it follows that $A \cup \{xy, z_x x, e_1, e_2\}$ contains a heterochromatic outdirected spanning tree of T which is not possible and therefore, Claim 2 holds.

Since k(x, y) = 0 and $|A(D)| = \Gamma[T] - k(x, y)$, we see that the number of arcs in D is $\Gamma[T] = h(T) - 1 \ge {n \choose 2} - \delta_3^-(T) + 1$. Notice that none of the in-arcs of x are in A(D) and, except for xy, none of the in-arcs of y are in A(D). Let $H \subseteq V(T)$ be the set of vertices which are reachable from x by directed paths in D. Since T has no heterochromatic out-directed spanning tree with respect to Γ , it follows that $W = V(T) \setminus H \neq \emptyset$. Thus, none of the arcs in $F_T^-(W)$ are present in D. Therefore,

$$|A(D)| = \binom{n}{2} - d_T^-(x) - d_T^-(y) - |F_T^-(W)| + 1 - \alpha$$
(2)

with $\alpha \geq 0$ (maybe other arcs in $A(T) \setminus (F_T^-(W) \cup F_T^-(\{x, y\}))$ do not appear in D).

Since

$$|A(D)| \ge \binom{n}{2} - \delta_3^-(T) + 1,$$

it follows from (2) that

$$\delta_3^-(T) \ge d_T^-(x) + d_T^-(y) + |F_T^-(W)| + \alpha.$$
(3)

It is not hard to see that $|F_T^-(W)| = \sum_{z \in W} d_T^-(z) - {|W| \choose 2}$ and therefore

$$d_T^-(x) + d_T^-(y) + |F_T^-(W)| + \alpha = \sum_{z \in W \cup \{x,y\}} d_T^-(z) - \binom{|W|}{2} + \alpha.$$
(4)

On the other hand, by an averaging argument we see that

$$\left(\frac{3}{|W|+2}\right)\sum_{z\in W\cup\{x,y\}}d_T^-(z)\geq \delta_3^-(T)$$

and then, by (3) and (4),

$$\left(\frac{3}{|W|+2}\right)\sum_{z\in W\cup\{x,y\}}d_T^{-}(z) \ge \sum_{z\in W\cup\{x,y\}}d_T^{-}(z) - \binom{|W|}{2} + \alpha$$

Therefore

$$\binom{|W|}{2} \ge \left(\frac{|W|-1}{|W|+2}\right) \sum_{z \in W \cup \{x,y\}} d_T^-(z) + \alpha,$$

but since $\sum_{z \in W \cup \{x,y\}} d_T^-(z) \ge {|W|+2 \choose 2}$, we see that

$$\binom{|W|}{2} \ge \frac{(|W|+1)(|W|-1)}{2} + \alpha$$

and hence

$$0 \ge \frac{|W| - 1}{2} + \alpha. \tag{5}$$

Since $W \neq \emptyset$, $\frac{|W|-1}{2} \ge 0$ and then, from (5) it follows that |W| = 1 and $\alpha = 0$. Let $\{w\} = W$. Clearly $|F_T^-(W)| = d_T^-(w)$, and by (3) we see that

$$\delta_3^-(T) \ge d_T^-(x) + d_T^-(y) + |F_T^-(W)| + \alpha = d_T^-(x) + d_T^-(y) + d_T^-(w)$$

which, by definition of $\delta_3^-(T)$ implies that

$$\delta_3^-(T) = d_T^-(x) + d_T^-(y) + d_T^-(w).$$
(6)

Since $\alpha = 0$, it follows that all the arcs of T are present in D except for the in-arcs of x, the in-arcs of w and, besides the arc xy, all the in-arcs of y. Thus

$$A(D) = \left(A(T) \setminus \bigcup_{z \in \{x, y, w\}} F_T^-(\{z\})\right) \cup \{xy\}$$

and

$$|A(D)| = \binom{n}{2} - (d^{-}(x) + d^{-}(y) + d^{-}(w)) + 1 = \binom{n}{2} - \delta_{3}^{-}(T) + 1$$

and since |A(D)| = h(T) - 1 it follows that

$$h(T) = {n \choose 2} - \delta_3^-(T) + 2.$$
 (7)

From here, to end the proof of Theorem 1 just remain to show that all the in-arcs of x, y, and w receive the same color. For this, first we will prove that all the in-arcs of w receive color c_0 . Let suppose there is an arc $zw \in A(T)$ such that $\Gamma(zw) = c_3 \neq c_0$. Since all the colors in $\Gamma[T]$ are present in A(D), there is an arc $z'w' \in A(D)$ such that $\Gamma(z'w') = c_3$. Notice that $w' \notin \{x, y, w\}$, since no in-arcs of x nor w are present in Dand the only in-arc of y in D has color c_0 . Let $D' = (D \setminus z'w') \cup zw$. Observe that both vertices z and z' are reachable from x in both digraphs D and D'. Also notice that D' is a maximal heterochromatic spanning subdigraph of $A(T) \setminus F_T^-[\{x, y\}] \cup xy$ that contains xy. Thus, by an analogous procedure as for D, we find that in D'there is a vertex v such that all the arcs of T are present in D' with exception of the in-arcs of x, the in-arcs of v and, besides the arc xy, all the in-arcs of y. Since w'has an in-arc missing in D' and $v \notin \{x, y\}$, it follows that v = w'.

Since $w \neq w'$, either $ww' \in A(T)$ or $w'w \in A(T)$. If $w'w \in A(T)$, $w'w \notin A(D)$ but $w'w \in A(D')$, and since $D' = (D \setminus z'w') \cup zw$ it follows that zw = w'w and w' = z which is not possible since z is reachable from x in D' and w' is not reachable from x in D'. In an analogous way, if $ww' \in A(T)$, $ww' \notin A(D')$ but $ww' \in A(D)$ and then z'w' = ww' and w = z', which is not possible since z' is reachable from x in D and w is not.

Therefore all the in-arcs of w receive color c_0 . Thus w is a vertex of type 3, and, by the way the pair $\{x, y\}$ were chosen, this implies that $\{w, x, y\}$ is a triple of vertices of type 3, and since $c_0 = c_x = c_w$, again, by the way the pair $\{x, y\}$ were chosen, $c_y = c_x$. Therefore all the in-arcs of the triple $\{x, y, w\}$ receive the same color c_0 and this ends the proof of Theorem 1.

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