# On heterochromatic out-directed spanning trees in tournaments* 

Juan José Montellano-Ballesteros<br>Instituto de Matemáticas, Universidad Nacional Autónoma de México<br>juancho@math.unam.mx<br>Eduardo Rivera-Campo<br>Departamento de Matemáticas<br>Universidad Autónoma Metropolitana-Iztapalapa<br>erc@xanum.uam.mx


#### Abstract

Given a tournament $T$, let $h(T)$ be the smallest integer $k$ such that every arc-coloring of $T$ with $k$ or more colors produces at least one out-directed spanning tree of $T$ with no pair of arcs with the same color. In this paper we give the exact value of $h(T)$.


Keywords: Out-directed Tree. Tournament. Heterochromatic

## 1 Introduction

Given a graph $G$ and an edge-coloring of $G$, a subgraph $H$ of $G$ is said to be heterochromatic if no pair of edges of $H$ have the same color. Problems concerning the existence of heterochromatic subgraphs with a specific property in edge-colorings of a host graph are known as anti-Ramsey problems (see, for instance, [1, 4, 5, 7, 9, 11). Typically, the host graph $G$ is a complete graph or some graph with a particular

[^0]structure, and the property which defines the set of heterochromatic subgraphs in consideration is that they are isomorphic to a given graph $H$ or that they are subgraphs of $G$ with a general property like, for example, being edge-cuts or spanning trees of $G$ (see [2, 3, 6, 8, 10]).

A tournament is a digraph $D=(V(D), A(D))$ such that for every pair $\{x, y\} \subseteq$ $V(D)$, either $x y \in A(D)$ or $y x \in A(D)$ but not both. A spanning tree $S$ of a tournament $T$ is an out-directed spanning tree of $T$ if there is a root vertex $r$ of $S$ such that for each vertex $u \in V(S)$, the unique $r-u$ path in $S$ is directed from $r$ to $u$.

In this paper, the host graphs are tournaments, and the property that defines the set of heterochromatic subgraphs in consideration is that of being an out-directed spanning tree of the corresponding tournament.

Let $T=(V(T), A(T))$ be a tournament. An arc-coloring of $T$ is a function $\Gamma: A(T) \rightarrow C$, where $C$ is a set of "colors"; if $|\Gamma[A(T)]|=k$ we say that $\Gamma$ is a $k$-arc-coloring of $T$. A subdigraph $H$ of $T$ is said to be heterochromatic if no pair of arcs of $H$ have the same color. We define $h(T)$ as the smallest integer $k$ such that every $k$-arc-coloring of $T$ produces at least one heterochromatic-out directed spanning tree of $T$. Our main result is the following theorem:

Theorem 1. Let $T$ be a tournament of order $n \geq 3$. Then $h(T)=\binom{n}{2}-\delta_{3}^{-}(T)+2$, where $\delta_{3}^{-}(T)=\min \left\{d_{T}^{-}(x)+d_{T}^{-}(y)+d_{T}^{-}(w):\{x, y, w\} \subseteq V(T)\right\}$. Moreover, if the arcs of $T$ are colored with $h(T)-1$ colors, and there is no heterochromatic out-directed spanning tree of $T$, then there is a triple $\{x, y, w\} \subseteq V(T)$ such that $\delta_{3}^{-}(T)=d_{T}^{-}(x)+d_{T}^{-}(y)+d_{T}^{-}(w)$, all the in-arcs of $x, y$, and $w$ receive the same color and each of the remaining arcs of $T$ receives a new different color.

## 2 Notation and Preliminary Results

Let $D=(V(D), A(D))$ be a digraph and $x$ be a vertex of $D$. We denote by $N_{D}^{+}(x)=$ $\{v \in V(D): x v \in A(D)\}$ and $N_{D}^{-}(x)=\{v \in V(D): v x \in A(D)\}$ the sets of out-neighbors and of in-neighbors of $x$ in $D$, respectively. Likewise, we denote by $d_{D}^{+}(x)=\left|N_{D}^{+}(x)\right|$ and $d_{D}^{-}(x)=\left|N_{D}^{-}(x)\right|$ the ex-degree and the in-degree of $x$ in $D$, respectively.

For every $Q \subseteq V(D)$, let $F_{D}^{+}(Q)=\{z w \in A(D): z \in Q$ and $w \in V(D) \backslash Q\}$, $F_{D}^{-}(Q)=\{w z: z \in Q$ and $w \in V(D) \backslash Q\}$ and $F_{D}(Q)=F_{D}^{+}(Q) \cup F_{D}^{-}(Q)$. Given $x \in V(D)$ the sets $F_{D}^{+}(\{x\}), F_{D}^{-}(\{x\})$ and $F_{D}(\{x\})$ are called the set of ex-arcs, the set of in-arcs and the set of arcs of $x$, respectively. For $Q, R \subseteq V(D)$, we denote by $(Q \rightarrow R)$ the set $\{x y \in A(D): x \in Q$ and $y \in R\}$.

Let $\Gamma: A(D) \rightarrow C$ be an arc-coloring of $D$. We denote by $C(x)$ the set of colors that appear only on arcs of $D$ incident to $x$, and by $c(x)$ the number of colors in $C(x)$. A color $i \in C$ is a $\Gamma_{D}$-singular color if $\left|\Gamma^{-1}(i)\right|=1$.

For any vertex $x \in V(D)$ and any arc $w y \in A(D)$, we denote by $D-x$ and $D-w y$ the digraphs obtained from $D$ by deleting the vertex $x$ and the arc $w y$, respectively. For an arc $z y \notin A(D), D+z y$ is the digraph obtained from $D$ by adding the arc $z y$.

We say that a vertex $z \in V(D)$ is reachable from a vertex $x$ in $D$ if there is a directed path in $D$ from $x$ to $z$.

Let $\delta_{3}^{-}(D)=\min \left\{d_{D}^{-}(x)+d_{D}^{-}(y)+d_{D}^{-}(w):\{x, y, w\} \subseteq V(D)\right\}$.
Lemma 1. Let $T$ be a tournament of order $n \geq 3$. Then

$$
h(T) \geq\binom{ n}{2}-\delta_{3}^{-}(T)+2
$$

Proof. Let $\{x, y, w\} \subseteq V(T)$ such that $d_{T}^{-}(x)+d_{T}^{-}(y)+d_{T}^{-}(w)=\delta_{3}^{-}(T)$ and color the arcs of $T$ with $\binom{n}{2}-\delta_{3}^{-}(T)+1$ colors in the following way: all the in-arcs of $x, y$ and $w$ receive the same color, say color black, and the remaining $\binom{n}{2}-\delta_{3}^{-}(T) \operatorname{arcs}$ receive $\binom{n}{2}-\delta_{3}^{-}(T)$ new colors.

Given an out-directed spanning tree $S$ of $T$ we can assume, without loss of generality, that neither $x$ nor $y$ is the root of $S$, and therefore $d_{S}^{-}(x)=d_{S}^{-}(y)=1$. From here we see that $S$ has at least two black arcs, thus $S$ is not heterochromatic and the lemma follows.

## 3 Proof of Theorem 1

Lemma 1 gives the lower bound for $h(T)$ in Theorem 1 . The proof of the upper bound and of the remainder of the theorem is by induction on $n$. For better readability, we break down the proof into several lemmas.

It is not hard to see that if $T$ is a tournament of order 3 , and $\Gamma$ is and arccoloring of $T$ with no heterochromatic out-directed spanning tree, then $\Gamma$ uses $1=$ $\binom{3}{2}-\delta_{3}^{-}(T)+1$ color. It is also clear that $V(T)=\{x, y, z\}$ is such that $d_{T}^{-}(x)+$ $d_{T}^{-}(y)+d_{T}^{-}(z)=\delta_{3}^{-}(T)=3$ and that the three in-arcs of $x, y$ and $z$ receive the same color. This shows that Theorem 1 holds for tournaments of order 3.

Let $T$ be a tournament of order $n \geq 4$. For the rest of the proof we assume as inductive hypothesis that Theorem 1 holds for every tournament of order $m$, with $3 \leq m<n$.

Let $\Gamma$ be an arc-coloring of $T$ which uses $h(T)-1$ colors and produces no heterochromatic out-directed spanning trees of $T$. Observe that by Lemma [1, $h(T) \geq$
$\binom{n}{2}-\delta_{3}^{-}(T)+2$ and therefore the number of colors in $\Gamma[A(T)]$ (from now on $\Gamma[T]$ for short) is at least $\binom{n}{2}-\delta_{3}^{-}(T)+1$.

A vertex $x$ of $T$ is of type 1 if there is an in-arc $e$ of $x$ such that $\Gamma(e) \in C(x)$; of type 2 if none of the in-arcs of $x$ receive a color in $C(x)$ and there are at least two in-arcs of $x$ which receive different colors; and of type 3 if none of the in-arcs of $x$ receive a color in $C(x)$ and all the in-arcs of $x$ receive the same color.

The next three lemmas will show some properties of the vertices of type 1 and 2 , and that there are at most $n-2$ vertices of type 1 . With these at hand, we will return to the proof of Theorem [1.

Lemma 2. If $x$ is a vertex of $T$ of type 1 , then $c(x) \geq n-4$.
Proof. Since $x$ is of type 1, there is an arc $y x \in A(T)$ such that $\Gamma(y x) \in C(x)$. Since $\Gamma(y x) \notin \Gamma[T-x]$, the tournament $T-x$ has no heterochromatic out-directed spanning tree $S$, otherwise $S+y x$ would be a heterochromatic out-directed spanning tree of $T$, which is not possible. Therefore, by our induction hypothesis, the number of colors appearing in $\Gamma[T-x]$ is at most $\binom{n-1}{2}-\delta_{3}^{-}(T-x)+1$. Thus
$c(x) \geq\binom{ n}{2}-\delta_{3}^{-}(T)+1-\left(\binom{n-1}{2}-\delta_{3}^{-}(T-x)+1\right)=n-1+\delta_{3}^{-}(T-x)-\delta_{3}^{-}(T)$.
Now just observe that $\delta_{3}^{-}(T)-\delta_{3}^{-}(T-x) \leq 3$ and therefore $c(x) \geq n-4$.
Lemma 3. If $x$ is a vertex of $T$ of type 2, then $d_{T}^{+}(x) \geq c(x)=n-4$.
Proof. By definition of type 2, none of the colors of the in-arcs of $x$ is in $C(x)$, so all the colors from $C(x)$ appear on the out-arcs of $x$ and therefore $d_{T}^{+}(x) \geq c(x)$. Also by definition, there are vertices $y_{1}, y_{2} \in N_{T}^{-}(x)$ such that $c_{1}=\Gamma\left(y_{1} x\right) \neq \Gamma\left(y_{2} x\right)=c_{2}$ with $c_{1}, c_{2} \notin C(x)$.

Let $\Gamma^{\prime}$ be an arc-coloring of $T-x$ obtained from $\Gamma$ by recoloring the arcs of color $c_{2}$ with color $c_{1}$.

Suppose $T-x$ has an out-directed spanning tree $S$ which is heterochromatic with respect to $\Gamma^{\prime}$. Clearly $S$ is also heterochromatic with respect to $\Gamma$ and it is such that either color $c_{1}$ or color $c_{2}$ does not appear in $\Gamma[S]$. Thus, either $S+y_{1} x$ or $S+y_{2} x$ is a heterochromatic out-directed spanning tree of $T$ with respect to $\Gamma$, which is not possible. Therefore $T-x$ has no heterochromatic out-directed spanning tree with respect to $\Gamma^{\prime}$. By our induction hypothesis, there are at most $\binom{n-1}{2}-\delta_{3}^{-}(T-x)+1$ colors in $\Gamma^{\prime}[T-x]$. It follows at most $\binom{n-1}{2}-\delta_{3}^{-}(T-x)+2$ colors of $\Gamma$ are used in $T-x$ which implies
$c(x) \geq\binom{ n}{2}-\delta_{3}^{-}(T)+1-\left(\binom{n-1}{2}-\delta_{3}^{-}(T-x)+2\right)=n-2-\delta_{3}^{-}(T)+\delta_{3}^{-}(T-x) \geq n-5$.

If $c(x)=n-5$, each of the following must happen: i) $\delta_{3}^{-}(T)-\delta_{3}^{-}(T-x)=3$; ii) $|\Gamma[T-x]|=\binom{n-1}{2}-\delta_{3}^{-}(T-x)+2$; iii) $\left|\Gamma^{\prime}[T-x]\right|=\binom{n-1}{2}-\delta_{3}^{-}(T-x)+1$ and iv) $T-x$ has no heterochromatic out-directed spanning tree with respect to $\Gamma^{\prime}$.

By induction $h(T-x)=\binom{n-1}{2}-\delta_{3}^{-}(T-x)+2$ and therefore, according to $\left.i i i\right)$, $\Gamma^{\prime}$ is an arc-coloring of $T-x$ with $h(T-x)-1$ colors. Also by induction, there is a triple $\left\{x_{1}, x_{2}, x_{3}\right\} \subseteq V(T-x)$ such that $\delta_{3}^{-}(T-x)=d_{T-x}^{-}\left(x_{1}\right)+d_{T-x}^{-}\left(x_{2}\right)+d_{T-x}^{-}\left(x_{3}\right)$, all the in-arcs of $x_{1}, x_{2}$, and $x_{3}$ have the same color in $\Gamma^{\prime}$ and each of the remaining arcs of $T-x$ has a singular color in $\Gamma^{\prime}$.

Recall that there are arcs in $T-x$ with colors $c_{1}$ and $c_{2}$, since $c_{1}, c_{2} \notin C(x)$. Therefore $c_{1}$ is the non-singular color in $\Gamma^{\prime}$ and all the in-arcs of $x_{1}, x_{2}$, and $x_{3}$ have color $c_{1}$ in $\Gamma^{\prime}$. This implies that all the in-arcs of $x_{1}, x_{2}$, and $x_{3}$ have color $c_{1}$ or color $c_{2}$ in $\Gamma$; and each of the remaining arcs of $T-x$ has a singular color in $\Gamma$.

By $i), \delta_{3}^{-}(T)-\delta_{3}^{-}(T-x)=3$ and this implies $\left\{x_{1}, x_{2}, x_{3}\right\} \subseteq N_{T}^{+}(x)$. Therefore $\left\{y_{1}, y_{2}\right\} \subseteq V(T) \backslash\left\{x, x_{1}, x_{2}, x_{3}\right\}$ and $N_{T}^{+}(x) \subseteq V(T) \backslash\left\{x, y_{1}, y_{2}\right\}$. Since $c(x)=n-5$, it follows that there is at least one vertex $z \in\left\{x_{1}, x_{2}, x_{3}\right\}$ such that $\Gamma(x z) \in C(x)$. Without loss of generality assume $z=x_{1}$.
Case 1. $\left\{\Gamma\left(x x_{1}\right), \Gamma\left(x x_{2}\right), \Gamma\left(x x_{3}\right)\right\} \cap C(x)=\Gamma\left(x x_{1}\right)$.
The ex-arcs of $x$ with the other $(n-6)$ colors of $C(x)$ appear in $(x \rightarrow[V(T) \backslash$ $\left.\left.\left\{x, x_{1}, x_{2}, x_{3}\right\}\right]\right)$. Thus $N_{T}^{-}(x)=\left\{y_{1}, y_{2}\right\}$ and $\delta_{T}^{-}(x)=2$. Since $\delta_{3}^{-}(T)-\delta_{3}^{-}(T-x)=3$, it follows that $\delta_{3}^{-}(T)=d_{T}^{-}\left(x_{1}\right)+d_{T}^{-}\left(x_{2}\right)+d_{T}^{-}\left(x_{3}\right)$ and therefore $\delta_{T}^{-}\left(x_{i}\right) \leq 2$ for $i=1,2,3$. Since $\left\{x_{1}, x_{2}, x_{3}\right\} \subseteq N_{T}^{+}(x)$, it follows that $\left\{x_{1}, x_{2}, x_{3}\right\}$ induces a directed cycle with length 3 in $T$ (with colors $c_{1}$ and $c_{2}$ ), and $V(T) \backslash\left\{x, x_{1}, x_{2}, x_{3}\right\} \subseteq N^{+}\left(x_{i}\right)$ for $i=1,2,3$, where each of the $\operatorname{arcs}$ in $\left(\left\{x_{1}, x_{2}, x_{3}\right\} \rightarrow\left[V(T) \backslash\left\{x, x_{1}, x_{2}, x_{3}\right\}\right]\right)$ receives a $\Gamma_{T-x}$-singular color (none of them a color in $C(x)$ ). Therefore, the tournament $H$ induced by $V(T) \backslash\left\{x, x_{1}\right\}$ is a heterochromatic tournament in which either $c_{1}$ or $c_{2}$ appear, but not both. Thus, in $H$ there is a hamiltonian heterochromatic path $P$ where, without loss of generality, color $c_{1}$ does not appear. Therefore $E(P) \cup$ $\left\{y_{2} x\right\} \cup\left\{x x_{1}\right\}$ induces a heterochromatic out-directed spanning tree of $T$ which is not possible.
Case 2. $\quad\left|\left\{\Gamma\left(x x_{1}\right), \Gamma\left(x x_{2}\right), \Gamma\left(x x_{3}\right)\right\} \cap C(x)\right| \geq 2$.
Suppose $\Gamma\left(x x_{2}\right) \in C(x)$ and $\Gamma\left(x x_{1}\right) \neq \Gamma\left(x x_{2}\right)$. Consider the tournament $H$ induced by $V(T) \backslash\left\{x, x_{1}, x_{2}\right\}$ and let $P$ be a hamiltonian path in $H$. Except for the in-arcs of $x_{3}$, which receive color $c_{1}$ or $c_{2}$, all the other arcs in $H$ receive $\Gamma_{T-x}$-singular colors. Thus $P$ is a heterochromatic path in which either color $c_{1}$ or color $c_{2}$ appear, but not both. Without loss of generality, suppose color $c_{1}$ does not appear in $P$. In this case $E(P) \cup\left\{y_{2} x\right\} \cup\left\{x x_{1}, x x_{2}\right\}$ induces a heterochromatic out-directed spanning tree of $T$ which again is not possible.

From Case 1 and Case 2, it follows that $c(x) \geq n-4$. Suppose $c(x) \geq n-3$.

Since $d_{T}^{+}(x) \geq c(x)$ and $\Gamma\left(y_{1} x\right), \Gamma\left(y_{2} x\right) \notin C(x)$, all the ex-arcs of $x$ receive different colors and all of them lie in $C(x)$. Since $\Gamma\left(y_{1} x\right)=c_{1} \neq c_{2}=\Gamma\left(y_{2} x\right)$ and the color of the arc with endpoints $y_{1}$ and $y_{2}$ is not in $C(x)$, it is not hard to see that either $F_{T}^{+}(\{x\}) \cup\left\{y_{1} x, y_{1} y_{2}\right\}$ or $F_{T}^{+}(\{x\}) \cup\left\{y_{1} y_{2}, y_{2} x\right\}$ induces a heterochromatic outdirected spanning tree of $T$ which is not possible. Therefore $c(x)=n-4$ and Lemma 3 follows.

Lemma 4. There are at most $n-2$ vertices of $T$ of type 1.
Proof. Suppose there are at least $n-1$ vertices of type 1 . Let $D$ be a spanning subdigraph of $T$ with the minimum number of connected components whose arc set is obtained as follows: choose a set $A$ with $n-1$ vertices of type 1 , and for each vertex $x \in A$, choose one in-arc of $x$ with a color in $C(x)$.

Clearly $D$ is heterochromatic. Since there are no heterochromatic out-directed spanning trees of $T, D$ is not connected. Let $D_{1}, D_{2}, \ldots, D_{r}$ be the connected components of $D$. Since $D$ has $n$ vertices and $n-1$ arcs and the maximum in-degree of $D$ is 1 , it is not hard to see that one connected component, say $D_{1}$, is an out-directed tree, while, for $i=2,3, \ldots, r$, component $D_{i}$ contains exactly one directed cycle $C_{i}$ such that $D-e$ is an outdirected tree for each edge $e$ of $C_{i}$. Let $z_{1}$ be the root of $D_{1}$ and notice that $A=V(T) \backslash\left\{z_{1}\right\}$.
Claim 1. Let $x \in V\left(C_{2}\right), y \in \bigcup_{j \neq 2} V\left(C_{j}\right) \cup\left\{z_{1}\right\}$ and $e$ be the arc with endpoints $\{x, y\}$. If $\Gamma(e) \in C(x)$ then $e$ is an ex-arc of $x$ and $\Gamma(e)$ is not a $\Gamma_{T}$-singular color.

Suppose $\Gamma(e) \in C(x)$. If $e$ is an in-arc of $x$, the digraph $(D-w x)+e$, with $w x \in A\left(C_{2}\right) \subseteq A(D)$, has fewer connected components than $D$ and can be obtained in the same way as $D$ by choosing in $C(x)$ the edge $e$ instead of $w x$, which is a contradiction. Hence $e$ is an ex-arc of $x$, and therefore an in-arc of $y$. Let us suppose $\Gamma(e)$ is a $\Gamma_{T}$-singular color. Thus $\Gamma(e) \in C(y)$ and $y$ is of type 1. On the one hand, if $y \in V\left(C_{j}\right)$ for some $j \neq 2$, in an analogous way as with the vertex $x$, we reach a contradiction. On the other hand, if $y=z_{1}$ the digraph $(D-w x)+e$ (which has fewer connected components than $D$ ) can be obtained in the same way as $D$ by choosing the set $A^{\prime}=(A \backslash\{x\}) \cup\{z\}$ as the set of $n-1$ vertices of type 1 and choosing the edge $e$ in $C\left(z_{1}\right)$ instead of the edge $w x$ in $C(x)$, which is a contradiction. From here, Claim 1 follows.

Let $x \in V\left(C_{2}\right)$. Since $c(x)=n-4$ it follows there are at least $n-7$ arcs incident to $x$ with $\Gamma_{T}$-singular colors. Thus, by Claim 1 it follows that $\left|\left\{z_{1}\right\} \cup \bigcup_{j \neq 2} V\left(C_{j}\right)\right| \leq 6$ and therefore $r \leq 3$. Let us suppose $r=2$ and let $e$ be the arc with endpoints $\left\{z_{1}, x\right\}$. The color $\Gamma(e)$ must appear in $D$, otherwise $D+e$ is a heterochromatic
digraph containing an out-directed spanning tree of $T$ which is a contradiction. By the choice of the arcs of $D, \Gamma(e) \in C(x)$ and there is an arc $w x \in A\left(C_{2}\right)$ with color $\Gamma(e)$, but then $(D-w x)+e$ is a heterochromatic out-directed spanning tree of $T$ which is a contradiction. Thus $r=3$. Since $c(x)=n-4$ and $\left|\left\{z_{1}\right\} \cup V\left(C_{3}\right)\right| \geq 4$, there is a color $c \in C(x)$ which only appears in arcs incident to $x$ and with the other endpoint in $V\left(C_{3}\right) \cup\left\{z_{1}\right\}$. By Claim 1, these arcs are ex-arcs of $x$ and there are at least two of them, since $c$ is not a $\Gamma_{T}$-singular color. Thus there is $y \in V\left(C_{3}\right)$ such that $\Gamma(x y)=c$. Let $w \in V\left(C_{2}\right) \backslash\{x\}$ and let $e$ be the arc with endpoints $\left\{z_{1}, w\right\}$. The color $\Gamma(e)$ must appear in $D+x y$, otherwise $D+\{x y, e\}$ is a heterochromatic digraph containing an out-directed spanning tree of $T$ which is a contradiction. Thus, by the choice of the arcs of $D$ and since $\Gamma(x y) \in C(x), \Gamma(e) \in C(w)$ and there is an arc $w w^{\prime} \in A\left(C_{2}\right)$ with color $\Gamma(e)$, but then $\left(D-w w^{\prime}\right)+\{x y, e\}$ is a heterochromatic digraph containing an out-directed spanning tree of $T$ which is a contradiction. This ends the proof of Lemma 4.

Now we return to the proof of Theorem 1. First we will show that there is an arc $x_{1} x_{2} \in A(T)$ and a vertex $x_{3} \in V(T) \backslash\left\{x_{1}, x_{2}\right\}$ such that the spanning subdigraph $D$ of $T$ with set of arcs

$$
A(D)=\left(A(T) \backslash \bigcup_{i=1}^{3} F_{T}^{-}\left(\left\{x_{i}\right\}\right)\right) \cup\left\{x_{1} x_{2}\right\}
$$

is an heterochromatic spanning subdigraph of $T$ with $h(T)-1$ arcs. Observe that these will imply that

$$
h(T)-1=|A(D)|=\binom{n}{2}-\left(d_{T}^{-}\left(x_{1}\right)+d_{T}^{-}\left(x_{2}\right)+d_{T}^{-}\left(x_{3}\right)\right)+1 \leq\binom{ n}{2}-\delta_{3}^{-}(T)+1
$$

which will prove the first part of the theorem.
Recall that if $v$ is a vertex of $T$ of type 3 , then all the in-arcs of $v$ recieve the same color. For each such vertex $v$ we denote by $c_{v}$ the color assigned to every in-arc of $v$.

Now we will choose a pair of vertices $\{x, y\}$ in the following way: By Lemma 4 there are at least two vertices that are not of type 1. If there are at least two vertices of type 3, choose $x$ and $y$ to be vertices of type 3 such that $c_{x}=c_{y}$ if possible, otherwise chose any two vertices of type 3 . If there is exactly one vertex of type 3 , choose it together with any vertex of type 2 . Otherwise choose $x$ and $y$ to be vertices of type 2 .

Without loss of generality assume $x y \in A(T)$ and let $c_{0}=\Gamma(x y)$. Let $D$ be a maximal heterochromatic spanning subdigraph of $A(T) \backslash F_{T}^{-}[\{x, y\}] \cup x y$ that contains
$x y$. Observe that the number of $\operatorname{arcs}$ in $D$ is

$$
\begin{equation*}
|A(D)|=\Gamma[T]-k(x, y) \tag{1}
\end{equation*}
$$

where $k(x, y)$ is the number of colors that only appear in the set of $\operatorname{arcs} F_{T}^{-}[\{x, y\}]$.
Claim 2. $k(x, y)=0$.
Suppose $k(x, y) \geq 1$ and let $c_{1}$ be a color that only appears in the set of arcs $F_{T}^{-}(\{x, y\})$. Since neither $x$ nor $y$ are of type $1, c_{1} \notin C(x) \cup C(y)$, there is a pair of $\operatorname{arcs}\left\{z_{x} x, z_{y} y\right\} \subseteq F_{T}^{-}[\{x, y\}]$ (where $z_{x}$ and $z_{y}$ are not necessarily different) such that $\Gamma\left(z_{x} x\right)=\Gamma\left(z_{y} y\right)=c_{1}$. Since $y$ is not of type 1 and $\Gamma(x y)=c_{0} \neq c_{1}=\Gamma\left(z_{y} y\right)$, it follows that $y$ is of type 2 .

Let $A=\left\{y x_{1}, y x_{2}, \ldots, y x_{c(y)}\right\}$ be a set of ex-arcs of $y$, all of them with different colors in $C(y)$, contained in $A(D)$. By Lemma 3, $c(y)=n-4$, since $y$ is of type 2 . Thus $\{x y\} \cup A$ induces a heterochromatic out-directed tree of order $n-2$, with root $x$ and with colors in $\left\{c_{0}\right\} \cup C(y)$.

Let $\left\{w_{1}, w_{2}\right\}=V(T) \backslash\left(\{x, y\} \cup\left\{x_{i}: y x_{i} \in A\right\}\right)$. Observe that $z_{y} \in\left\{w_{1}, w_{2}\right\}$ and, without loss of generality, assume $z_{y}=w_{1}$. Since $y$ is of type 2 , by the way $x$ and $y$ were chosen, it follows that neither $w_{1}$ nor $w_{2}$ is of type 3 . For $i=1,2$, observe that if $w_{i}$ is of type 1 , then there is an in-arc of $w_{i}$ with a color in $C\left(w_{i}\right)$, which does not appear in $x y \cup A$. Also notice that if $w_{1}$ is of type 2, then there are two in-arcs of $w_{1}$, with different colors such that those colors are not in $C(y)$ and that if $w_{2}$ is of type 2 , then there are two in-arcs of $w_{2}$, also with different colors, such that at least one of those colors is not in $C(y)$ (maybe $y w_{2} \in A(T)$ and $\Gamma\left(y w_{2}\right) \in C(y)$ ).

In any case, there exist in-arcs $e_{1}$ of $w_{1}=z_{y}$ and $e_{2}$ of $w_{2}$ with different colors, none of them with color in $C(y)$, none of them with color $c_{1}$ (recall that all the arcs of color $c_{1}$ are in-arcs of $x$ and $y$ ), and maybe one of them with color $c_{0}$. Since $\Gamma\left(z_{x} x\right)=c_{1}$, it follows that $A \cup\left\{x y, z_{x} x, e_{1}, e_{2}\right\}$ contains a heterochromatic outdirected spanning tree of $T$ which is not possible and therefore, Claim 2 holds.

Since $k(x, y)=0$ and $|A(D)|=\Gamma[T]-k(x, y)$, we see that the number of arcs in $D$ is $\Gamma[T]=h(T)-1 \geq\binom{ n}{2}-\delta_{3}^{-}(T)+1$. Notice that none of the in-arcs of $x$ are in $A(D)$ and, except for $x y$, none of the in-arcs of $y$ are in $A(D)$. Let $H \subseteq V(T)$ be the set of vertices which are reachable from $x$ by directed paths in $D$. Since $T$ has no heterochromatic out-directed spanning tree with respect to $\Gamma$, it follows that $W=V(T) \backslash H \neq \emptyset$. Thus, none of the $\operatorname{arcs}$ in $F_{T}^{-}(W)$ are present in $D$. Therefore,

$$
\begin{equation*}
|A(D)|=\binom{n}{2}-d_{T}^{-}(x)-d_{T}^{-}(y)-\left|F_{T}^{-}(W)\right|+1-\alpha \tag{2}
\end{equation*}
$$

with $\alpha \geq 0$ (maybe other arcs in $A(T) \backslash\left(F_{T}^{-}(W) \cup F_{T}^{-}(\{x, y\})\right)$ do not appear in D).

Since

$$
|A(D)| \geq\binom{ n}{2}-\delta_{3}^{-}(T)+1
$$

it follows from (22) that

$$
\begin{equation*}
\delta_{3}^{-}(T) \geq d_{T}^{-}(x)+d_{T}^{-}(y)+\left|F_{T}^{-}(W)\right|+\alpha \tag{3}
\end{equation*}
$$

It is not hard to see that $\left|F_{T}^{-}(W)\right|=\sum_{z \in W} d_{T}^{-}(z)-\binom{|W|}{2}$ and therefore

$$
\begin{equation*}
d_{T}^{-}(x)+d_{T}^{-}(y)+\left|F_{T}^{-}(W)\right|+\alpha=\sum_{z \in W \cup\{x, y\}} d_{T}^{-}(z)-\binom{|W|}{2}+\alpha . \tag{4}
\end{equation*}
$$

On the other hand, by an averaging argument we see that

$$
\left(\frac{3}{|W|+2}\right) \sum_{z \in W \cup\{x, y\}} d_{T}^{-}(z) \geq \delta_{3}^{-}(T)
$$

and then, by (3) and (4),

$$
\left(\frac{3}{|W|+2}\right) \sum_{z \in W \cup\{x, y\}} d_{T}^{-}(z) \geq \sum_{z \in W \cup\{x, y\}} d_{T}^{-}(z)-\binom{|W|}{2}+\alpha
$$

Therefore

$$
\binom{|W|}{2} \geq\left(\frac{|W|-1}{|W|+2}\right) \sum_{z \in W \cup\{x, y\}} d_{T}^{-}(z)+\alpha
$$

but since $\sum_{z \in W \cup\{x, y\}} d_{T}^{-}(z) \geq\binom{|W|+2}{2}$, we see that

$$
\binom{|W|}{2} \geq \frac{(|W|+1)(|W|-1)}{2}+\alpha
$$

and hence

$$
\begin{equation*}
0 \geq \frac{|W|-1}{2}+\alpha \tag{5}
\end{equation*}
$$

Since $W \neq \emptyset, \frac{|W|-1}{2} \geq 0$ and then, from (15) it follows that $|W|=1$ and $\alpha=0$. Let $\{w\}=W$. Clearly $\left|F_{T}^{-}(W)\right|=d_{T}^{-}(w)$, and by (3) we see that

$$
\delta_{3}^{-}(T) \geq d_{T}^{-}(x)+d_{T}^{-}(y)+\left|F_{T}^{-}(W)\right|+\alpha=d_{T}^{-}(x)+d_{T}^{-}(y)+d_{T}^{-}(w)
$$

which, by definition of $\delta_{3}^{-}(T)$ implies that

$$
\begin{equation*}
\delta_{3}^{-}(T)=d_{T}^{-}(x)+d_{T}^{-}(y)+d_{T}^{-}(w) \tag{6}
\end{equation*}
$$

Since $\alpha=0$, it follows that all the arcs of $T$ are present in $D$ except for the in-arcs of $x$, the in-arcs of $w$ and, besides the arc $x y$, all the in-arcs of $y$. Thus

$$
A(D)=\left(A(T) \backslash \bigcup_{z \in\{x, y, w\}} F_{T}^{-}(\{z\})\right) \cup\{x y\}
$$

and

$$
|A(D)|=\binom{n}{2}-\left(d^{-}(x)+d^{-}(y)+d^{-}(w)\right)+1=\binom{n}{2}-\delta_{3}^{-}(T)+1
$$

and since $|A(D)|=h(T)-1$ it follows that

$$
\begin{equation*}
h(T)=\binom{n}{2}-\delta_{3}^{-}(T)+2 \tag{7}
\end{equation*}
$$

From here, to end the proof of Theorem 1 just remain to show that all the in-arcs of $x, y$, and $w$ receive the same color. For this, first we will prove that all the in-arcs of $w$ receive color $c_{0}$. Let suppose there is an arc $z w \in A(T)$ such that $\Gamma(z w)=c_{3} \neq c_{0}$. Since all the colors in $\Gamma[T]$ are present in $A(D)$, there is an $\operatorname{arc} z^{\prime} w^{\prime} \in A(D)$ such that $\Gamma\left(z^{\prime} w^{\prime}\right)=c_{3}$. Notice that $w^{\prime} \notin\{x, y, w\}$, since no in-arcs of $x$ nor $w$ are present in $D$ and the only in-arc of $y$ in $D$ has color $c_{0}$. Let $D^{\prime}=\left(D \backslash z^{\prime} w^{\prime}\right) \cup z w$. Observe that both vertices $z$ and $z^{\prime}$ are reachable from $x$ in both digraphs $D$ and $D^{\prime}$. Also notice that $D^{\prime}$ is a maximal heterochromatic spanning subdigraph of $A(T) \backslash F_{T}^{-}[\{x, y\}] \cup x y$ that contains $x y$. Thus, by an analogous procedure as for $D$, we find that in $D^{\prime}$ there is a vertex $v$ such that all the arcs of $T$ are present in $D^{\prime}$ with exception of the in-arcs of $x$, the in-arcs of $v$ and, besides the arc $x y$, all the in-arcs of $y$. Since $w^{\prime}$ has an in-arc missing in $D^{\prime}$ and $v \notin\{x, y\}$, it follows that $v=w^{\prime}$.

Since $w \neq w^{\prime}$, either $w w^{\prime} \in A(T)$ or $w^{\prime} w \in A(T)$. If $w^{\prime} w \in A(T), w^{\prime} w \notin A(D)$ but $w^{\prime} w \in A\left(D^{\prime}\right)$, and since $D^{\prime}=\left(D \backslash z^{\prime} w^{\prime}\right) \cup z w$ it follows that $z w=w^{\prime} w$ and $w^{\prime}=z$ which is not possible since $z$ is reachable from $x$ in $D^{\prime}$ and $w^{\prime}$ is not reachable from $x$ in $D^{\prime}$. In an analogous way, if $w w^{\prime} \in A(T), w w^{\prime} \notin A\left(D^{\prime}\right)$ but $w w^{\prime} \in A(D)$ and then $z^{\prime} w^{\prime}=w w^{\prime}$ and $w=z^{\prime}$, which is not possible since $z^{\prime}$ is reachable from $x$ in $D$ and $w$ is not.

Therefore all the in-arcs of $w$ receive color $c_{0}$. Thus $w$ is a vertex of type 3, and, by the way the pair $\{x, y\}$ were chosen, this implies that $\{w, x, y\}$ is a triple of vertices of type 3 , and since $c_{0}=c_{x}=c_{w}$, again, by the way the pair $\{x, y\}$ were chosen, $c_{y}=c_{x}$. Therefore all the in-arcs of the triple $\{x, y, w\}$ receive the same color $c_{0}$ and this ends the proof of Theorem [1.

## References

[1] Axenovich, M.; Harborth, H.; Kemnitz, A.; Mller, M.; Schiermeyer, I.: Rainbows in the hypercube. Graphs Combin. 23, no. 2, (2007) 123-133 .
[2] Bialostocki, A.; Voxman, W.: On the anti-Ramsey numbers for spanning trees. Bull. Inst. Combin. Appl. 32 (2001), 23-26.
[3] Carraher, J.; Hartkey, S.; Hornz, P.: Edge-disjoint rainbow spanning trees in complete graphs. Submitted.
[4] Erdős, P.; Simonovits, M.; Sós, V. T.; (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II, 633-643. Colloq. Math. Soc. Janos Bolyai, Vol. 10, North-Holland, Amsterdam, (1975).
[5] Haas, R.; Young, M.: The anti-Ramsey number of perfect matching. Discrete Math. 312, no. 5, (2012) 933-937.
[6] Jahanbekam, S.; West, D. B. West: Rainbow spanning subgraphs of edge-colored complete graphs. Submitted
[7] Kano; M. Li, X. : Monochromatic and heterochromatic subgraphs in edge-colored graphs - A survey. Graphs Combin. 24, no. 4, (2008) 237-263.
[8] Montágh, B.: Anti-Ramsey numbers of spanning double stars. Acta Univ. Sapientiae Math. 1, no. 1, (2009) 21-34.
[9] Montellano-Ballesteros, J. J.; Neumann-Lara, V.: An Anti-Ramsey Theorem, Combinatorica 22 (3), (2002) 445-449.
[10] Montellano-Ballesteros, J. J.; An anti-Ramsey theorem on edge-cuts. Discuss. Math. Graph Theory 26, no. 1, (2006) 19-21.
[11] Simonovits, M.; Sós, V. T.: On Restricted Colourings of $K_{n}$, Combinatorica 4 (1), (1984) 101-110.


[^0]:    *Research partially supported by Conacyt, México and by PAPIIT-México project IN101912.

