# CHROMATIC POLYNOMIALS OF SIMPLICIAL COMPLEXES 

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#### Abstract

In this note we consider $s$-chromatic polynomials of finite simplicial complexes. The $s$-chromatic polynomials of simplicial complexes are higher dimensional analogues of chromatic polynomials for graphs.


## 1. Introduction

Let $K$ be a finite simplicial complex with vertex set $V(K) \neq \emptyset$ and let $r \geq 1$ and $s \geq 1$ be two natural numbers. A map col: $V(K) \rightarrow\{1,2, \ldots, r\}$ is an $(r, s)$-coloring of $K$ if there are no monochrome $s$-simplices in $K$ [5]. We write $\chi^{s}(K, r)$ for the number of $(r, s)$-colorings of $K$.
Definition 1.1. The s-chromatic polynomial of $K$ is the function $\chi^{s}(K, r)$ of $r$. The $s$-chromatic number of $K$, $\operatorname{chr}^{s}(K)$, is the minimal $r \geq 1$ with $\chi^{s}(K, r)>0$.

The theorem below shows that $\chi^{s}(K, r)$ is indeed polynomial in $r$ for fixed $K$ and $s$. (By notational convention, $[r]_{i}=r(r-1) \cdots(r-i+1)$ is the $i$ th falling factorial in $r$.)

Theorem 1.2. The s-chromatic polynomial of $K$ is

$$
\chi^{s}(K, r)=\sum_{i=\operatorname{chr}^{s}(K)}^{|V(K)|} S(K, i, s)[r]_{i}
$$

where $S(K, i, s)$ is the number of partitions of $V(K)$ into $i$ blocks containing no s-simplex of $K$.
For $s=1$, an $(r, 1)$-coloring of $K$ is a usual graph coloring, $\chi^{1}(K, r)$ is the usual chromatic polynomial, and $\operatorname{chr}^{1}(K)$ the usual chromatic number of the 1 -skeleton of $K$. In general, $\chi^{s}(K, r)$ depends only on the $s$-skeleton of $K$. Although the higher $s$-chromatic polynomials for simplicial complexes are analogues of 1-chromatic polynomials for graphs we shall shortly see that there are structural differences between the cases $s=$ and $s>1$.

Figure 1 shows a triangulation MB of the Möbius band. To the left is a ( 5,1 )- and to the right a ( 2,2 )-coloring of MB. The chromatic polynomials and chromatic numbers ${ }^{1}$ of MB are

$$
\chi^{s}(\mathrm{MB}, r)=\left\{\begin{array}{ll}
r^{5}-10 r^{4}+35 r^{3}-50 r^{2}+24 r & s=1 \\
r^{5}-5 r^{3}+5 r^{2}-r & s=2 \\
r^{5} & s \geq 3
\end{array} \quad \operatorname{chr}^{s}(\mathrm{MB})= \begin{cases}5 & s=1 \\
2 & s=2 \\
1 & s \geq 3\end{cases}\right.
$$



Figure 1. A (5, 1)-coloring and a (2,2)-coloring of a 5 -vertex triangulated Möbius band MB

[^0]1.1. Notation. We shall use the following notation throughout the paper:
$K$ : a finite simplicial complex
$K^{s}$ : the $s$-skeleton of $K$
$F^{s}(K)$ : the set of $s$-simplices $K$
$\# V$ or $|V|$ : the number of elements in the finite set $V$
$V(K)$ : the vertex set $\bigcup K$ of $K$ and $m(K)=\mid V(K)$ is the number of vertices in $K$
$D[V]$ : the complete simplicial complex of all subsets of the finite set $V$
$[m]$ : the finite set $\{1, \ldots, m\}$ of cardinality $m$
$[r]_{i}$ : the $i$ the falling factorial polynomial $[r]_{i}=i!\binom{r}{i}$ in $r$
$P(a, b)$ : the open interval $(a, b)$ in the poset $P$

## 2. Three ways to the $s$-Chromatic polynomial of a simplicial complex

In this section we present three different to approaches to the $s$-chromatic polynomial $\chi^{s}(K, r)$ :

- Theorem 2.5 via 1-chromatic polynomials of graphs;
- Theorem 2.25 via the Möbius function for the $s$-chromatic lattice;
- Theorem 1.2 via the simplicial $s$-Stirling numbers of the second kind.
2.1. Block-connected $s$-independent vertex partitions. Let $s \geq 1$ be a natural number.

Definition 2.1. Let $B \subset V(K)$ be a set of vertices of $K$. Then

- $B$ is s-independent if $B$ contains no $s$-simplex of $K$;
- $B$ is connected if $K \cap D[B]$ is a connected simplicial complex;
- the connected components of $B$ are the maximal connected subsets of $B$.

Definition 2.2. Let $P$ be a partition of $V(K)$.

- The graph $G_{0}(P)$ of $P$ is the simple graph whose vertices are the blocks of $P$ and with two blocks connected by and edge if their union is connected;
- The block-connected refinement $P_{0}$ of $P$ is the refinement whose blocks are the connected components of the blocks of $P$;
- $P$ is block-connected if the blocks of $P$ are connected (ie if $P=P_{0}$ ).

Lemma 2.3. Let $P$ be a partition of $V(K)$. If two different blocks of the block-connected refinement $P_{0}$ are connected by an edge in the graph $G_{0}\left(P_{0}\right)$ of $P$ then they lie in different blocks of $P$.

Proof. The connected components of the blocks of $P$ are maximal with respect to connectedness.
Definition 2.4. $\mathrm{BCP}^{s}(K)$ is the set of all block-connected s-independent partitions of $V(K)$.
Recall that $\chi^{1}\left(G_{0}(P), r\right)$ is the 1-chromatic polynomial of the simple graph $G_{0}(P)$ of the partition $P$.
Theorem 2.5. The s-chromatic polynomial for $K$ is the sum

$$
\chi^{s}(K, r)=\sum_{P \in \mathrm{BCP}^{s}(K)} \chi^{1}\left(G_{0}(P), r\right)
$$

of the 1-chromatic polynomials and the s-chromatic number of $K$ is the minimum

$$
\operatorname{chr}^{s}(K)=\min _{P \in \mathrm{BCP}^{s}(K)} \operatorname{chr}^{1}\left(G_{0}(P)\right)
$$

of the 1-chromatic numbers for the graphs of all the block-connected s-independent partitions of $V(K)$.
Proof. Let col: $V(K) \rightarrow[r]$ be an $(r, s)$-coloring of $K$. The monochrome partition $P(\operatorname{col})$ of $V(K)$ is the $s$ independent partition whose blocks are the nonempty monochrome sets of vertices $\{\operatorname{col}=i\}$ for $i \in[r]$. The block-connected refinement $P(\mathrm{col})_{0}$ of the monochrome partition is a block-connected $s$-independent partition of $K$. The original coloring col of $K$ is also a coloring of the graph $G_{0}\left(P(\mathrm{col})_{0}\right)$ of $P(\mathrm{col})_{0}$ for, by Lemma 2.3, distinct vertices of 1 -simplices of this graph have distinct colors. We have shown that any $(r, s)$-coloring col of $K$ induces an $(r, 1)$-coloring $\mathrm{col}_{0}$ of the graph $G_{0}\left(P(\mathrm{col})_{0}\right)$ of the block-connected refinement of the monochrome partition.

Let $P \in \mathrm{BCP}^{s}(K)$ be a block-connected $s$-independent partition of $V(K)$ and $\operatorname{col}_{0}: P \rightarrow\{1, \ldots, r\}$ an $(r, 1)$ coloring of its graph $G_{0}(P)$. Then $\mathrm{col}_{0}$ determines a map col: $V(K) \rightarrow[r]$ that is constant on the blocks of $P$. An $s$-simplex of $K$ can not be monochrome under col as it intersects at least two different blocks of $P$ connected by an edge of $G_{0}(P)$. Thus col is an $(r, s)$-coloring of $K$.

These two constructions are inverses of each other.

Remark 2.6 (The minimal block-connected $s$-independent partition). Let $C_{0}=\{\{v\} \mid v \in V(K)\}$ be the blockconnected $s$-independent partition of $V(K)$ whose blocks are singletons. The graph $G_{0}\left(C_{0}\right)=K^{1}$ is the 1-skeleton of $K$. Thus the 1 -chromatic polynomial of the 1 -skeleton of $K$ is always one of the polynomials in the sum of Theorem 2.5. If $K$ is 1 -dimensional, $\mathrm{BCP}^{1}(K)$ consists only of the partition $C_{0}$ and Theorem 2.5 simply says that the 1-chromatic polynomial of a simplicial complex is the 1-chromatic polynomial of its 1-skeleton.

Example 2.7 (The block-connected 2-independent partitions for $D[3]$ ). The 2-simplex $D[3]$ has 4 block-connected 2-independent partitions $C_{0},\{\{1\},\{2,3\}\},\{\{2\},\{1,3\}\}$, and $\{\{3\},\{1,3\}\}$. The graph of $C_{0}$ is the complete graph $K_{3}$, the 1 -skeleton of $D[3]$. The graphs of the other three partitions are all the complete graph $K_{2}$. Thus the 2-chromatic polynomial of $D[3]$ is $\chi^{2}(D[3], r)=\chi^{1}\left(K_{3}, r\right)+3 \chi^{1}\left(K_{2}, r\right)=[r]_{3}+3[r]_{2}=[r]_{2}(r+1)=r^{3}-r$ and the 2 -chromatic number is $\operatorname{chr}^{2}(D[3])=2$.
Example 2.8 (A (2,2)-coloring and the graph of the block-connected refinement of its monochrome partition). The picture below illustrates a $(2,2)$-coloring of a 9 -vertex triangulation of the torus

and its corresponding graph. There are 6937 block-connected partitions of the vertex set, and 3 of them has the graph shown above. The 2 -chromatic polynomial is $21[r]_{2}+742[r]_{3}+3747[r]_{4}+4908[r]_{5}+2295[r]_{6}+444[r]_{7}+36[r]_{8}+[r]_{9}=$ $[r]_{2}\left(r^{7}+r^{6}-17 r^{5}+10 r^{4}+82 r^{3}-116 r^{2}-23 r+67\right)$ and the 2 -chromatic number is 2 .
Example 2.9 (The $(r, 2)$-colorings of a simplicial complex $K$ ). Let $K$ be the pure 2-dimensional complex with facets $F^{2}(K)=\{\{1,2,3\},\{2,3,4\},\{4,5,6\}\}$.


The picture shows a $(2,2)$-coloring of $K$ and the corresponding $(2,1)$-coloring of the associated graph, $G_{0}\left(P_{0}\right)$, the block connected refinement of the monochrome partition $P=\{\{1,2,5,6\},\{3,4\}\}$. Table 1 shows the graphs $G_{0}(P)$ for all block connected partitions $P \in \mathrm{BCP}^{2}(K)$. For each graph, the table records its 1-chromatic polynomial and its 1-chromatic number. The 2-chromatic polynomial of $K$ is $\chi^{2}(K, 2)=15[r]_{2}+73[r]_{3}+62[r]_{4}+15[r]_{5}+[r]_{6}=$ $[r]_{2}(r-1)(r+1)\left(r^{2}+r-1\right)$ and the 2-chromatic number is $\operatorname{chr}^{2}(K)=2$.
Example 2.10 (The ( $r, 2$ )-colorings of the Möbius band). The set $\mathrm{BCP}^{2}(\mathrm{MB})$ of block-connected 2-independent partitions of the triangulated Möbius band MB (Figure 1) has 36 elements. There are $5,5,15,10,1$ partitions in $\mathrm{BCP}^{2}(\mathrm{MB})$ realizing the partitions $[3,2],[3,1,1],[2,2,1],[2,1,1,1],[1,1,1,1,1]$ of the integer $|V(\mathrm{MB})|=5$. All associated graphs are complete graphs. This yields the 2 -chromatic polynomial $\chi^{2}(\mathrm{MB}, r)=5[r]_{2}+20[r]_{3}+10[r]_{4}+$ $[r]_{5}=[r]_{2}\left(r^{3}+r^{2}-4 r+1\right)=r^{5}-5 r^{3}+5 r^{2}-r$ and the 2 -chromatic number is $\operatorname{chr}^{2}(\mathrm{MB})=2$.
Remark 2.11 (The $\mathcal{S}$-chromatic polynomial of $K$ ). Let $\mathcal{S}$ be a set of connected subcomplexes of $K$. A set $B \subset V(K)$ of vertices is $\mathcal{S}$-independent if $B$ is not a superset of any member of $\mathcal{S}$. Let $\mathrm{BCP}^{\mathcal{S}}(K)$ be the set of
\# in $\mathrm{BCP}^{2}(K) \quad \chi^{1}\left(G_{0}(P), r\right) \quad \operatorname{chr}^{1}\left(G_{0}(P)\right)$

Table 1. The graphs for the block-connected partitions in $\operatorname{BCP}^{2}(K)$
$\mathcal{S}$-independent partitions of $V(K)$. An $(r, \mathcal{S})$-coloring is a map $V(K) \rightarrow\{1, \ldots, r\}$ such that $\# \operatorname{col}(S)>1$ for all $S \in \mathcal{S}$. The number of $(r, \mathcal{S})$-colorings of $K$ is

$$
\chi^{\mathcal{S}}(K, r)=\sum_{P \in \operatorname{BCP}^{\mathcal{S}}(K)} \chi^{1}\left(G_{0}(P), r\right)
$$

as one sees by an obvious generalization of Theorem 2.5. An $(r, s)$-coloring of $K$ is an $(r, \mathcal{S})$-coloring of $K$ where $\mathcal{S}=F^{s}(K)$ is the set of $s$-simplices.
2.2. The $s$-chromatic linear program. Read $[9, \S 10]$ explains how to construct a linear program with minimal value equal to the $s$-chromatic number $\operatorname{chr}^{s}(K)$ of $K$.
Definition 2.12. $M^{s}(K)$ is the set of all maximal s-independent subsets of $V(K)$.
Let $A$ be the $\left(m(K) \times\left|M^{s}(K)\right|\right)$-matrix

$$
A(v, M)= \begin{cases}1 & v \in M \\ 0 & v \notin M\end{cases}
$$

recording which vertices $v \in V(K)$ belong to which maximal $s$-independent sets $M \in M^{s}(K)$. Now the $s$-chromatic number

$$
\operatorname{chr}^{s}(K)=\min \left\{\sum_{M \in M^{s}(K)} x(M) \mid x: M^{s}(K) \rightarrow\{0,1\}, \forall v \in V(K): \sum_{M \in M^{s}(K)} A(v, M) x(M) \geq 1\right\}
$$

is the minimal value of the objective function $\sum_{M \in M^{s}(K)} x(M)$ in $\left|M^{s}(K)\right|$ variables $x: M^{s}(K) \rightarrow\{0,1\}$, taking values 0 or 1 , and $m(K)$ constraints $\sum_{M \in M^{s}(K)} A(v, M) x(M) \geq 1, v \in V(K)$.
2.3. The $s$-chromatic lattice. Our approach here simply follows Rota's classical method for computing chromatic polynomials from Möbius functions of lattices $[10, \S 9]$. We need some terminology in order to characterize the monochrome loci for colorings of $K$. Recall that $F^{s}(K)$ is the set of $s$-simplices of $K$.
Definition 2.13. Let $S \subset F^{s}(K)$ be a set of $s$-simplices of $K$.

- The equivalence relation $\sim$ is the smallest equivalence relation in $S$ such that $s_{1} \cap s_{2} \neq \emptyset \Longrightarrow s_{1} \sim s_{2}$ for all $s_{1}, s_{2} \in S$;
- the connected components of $S$ are the equivalence classes under ~;
- $\pi_{0}(S)$ is the set of connected components of $S$;
- $S$ is connected if it has at most one component;
- $V(S)=\bigcup S$ is the vertex set of $S$
- $\pi(S)$ is the partition of $V(K)$ whose blocks are the vertex sets of the connected components of $S$ together with the singleton blocks $\{v\}, v \in V(K)-V(S)$, of vertices not in any simplex in $S$;
- $S$ is closed if $S$ contains any s-simplex in $K$ contained in the vertex set of $S$, ie if

$$
\left\{\sigma \in F^{s}(K) \mid \sigma \subset V(S)\right\}=S
$$

- the closure of $S$ is the smallest closed set of s-simplices containing $S$.

For instance, the empty set $S=\emptyset$ of $0 s$-simplices is connected with 0 connected components. If $K=D[4]$, the set $\{\{1,2\},\{2,4\}\}$ of 1 -simplices is connected while $\{\{1,2\},\{3,4\}\}$ has the two components $\{\{1,2\}\}$ and $\{\{3,4\}\}$.

A set of $s$-simplices is closed if and only if it equals its closure. For instance in $F^{2}(D[5])$, the set $\{\{1,2,3\},\{3,4,5\}\}$ is not closed because its closure is the set of all 2 -simplices in $D[5]$. The empty set of $s$-simplices, any set of just one $s$-simplex, and any set of disjoint $s$-simplices are closed.

In this picture the green set of 2 -simplices is

connected and not closed, closed and not connected, closed and connected, respectively.
The partition $\pi(S)$ has $|\pi(S)|=\left|\pi_{0}(S)\right|+m(K)-|V(S)|$ blocks.
Lemma 2.14. Let $S$ be a set of s-simplices in $K$ and $S_{0}$ a connected component of $S$. Then $S_{0}$ is closed if and only if

$$
\left\{\sigma \in F^{s}(K) \mid \sigma \subset V\left(S_{0}\right)\right\} \subset S
$$

Proof. Since the condition is certainly necessary we only need to see that it is sufficient. Let $\sigma$ be an $s$-simplex in $K$ with all its vertices in $V\left(S_{0}\right)$. Then $\sigma$ lies in $S$ by assumption. But $\sigma$ is equivalent to all elements of the equivalence class $S_{0}$. Thus $\sigma \in S_{0}$.

Lemma 2.15. Let $S$ and $T$ be sets of $s$-simplices in $K$.
(1) If $S$ and $T$ are closed, so is $S \cap T$.
(2) If $S$ and $T$ have closed connected components, so does $S \cap T$

Proof. (1) Let $\sigma$ be an $s$-simplex of $K$ and suppose that $\sigma \subset V(S \cap T)$. Then $\sigma \subset V(S)$ an $\sigma \subset V(T)$ so that $\sigma \in S$ and $\sigma \in T$ as $S$ and $T$ are closed.
(2) Let $R$ be a connected component of $S \cap T$. Let $S_{0}$ be the connected component of $S$ containing $R$ and $T_{0}$ be the connected component of $T$ containing $R$. Then $R \subset S_{0} \cap T_{0}$. Suppose that $\sigma \in F^{s}(K)$ is an $s$-simplex with $\sigma \subset V(R)$. Then $\sigma \subset V\left(S_{0} \cap T_{0}\right)$ so $\sigma \in S_{0} \cap T_{0}$ by (1) as the connected components $S_{0}$ and $T_{0}$ are assumed to be closed. In particular, $\sigma \in S \cap T$. According to Lemma 2.14, the connected component $R$ is closed.
Definition 2.16. The s-chromatic lattice of $K$ is the set $L^{s}(K)$ of all subsets of $F^{s}(K)$ with closed connected components. $L^{s}(K)$ is a partially ordered by set inclusion.

The set $L^{s}(K)$ contains the empty set $\emptyset$ of $s$-simplices and the set $F^{s}(K)$ of all $s$-simplices. These two elements of $L^{s}(K)$ are distinct when $K$ has dimension at least $s$.

Corollary 2.17. $L^{s}(K)$ is a finite lattice with $\widehat{0}=\emptyset, \widehat{1}=F^{s}(K)$, and meet $S \wedge T=S \cap T$.
Proof. If $S, T \in L^{s}(K)$ then $S \cap T$ is also in $L^{s}(K)$ by Lemma 2.15 and this is clearly the greatest lower bound of $S$ and $T$. It is now a standard result that $L^{s}(K)$ is a finite lattice [12, Proposition 3.3.1]. The join $S \vee T$ of $S, T \in L^{s}(K)$ is the intersection of all supersets $U \in L^{s}(K)$ of $S \cup T$.
Example 2.18 (The $s$-chromatic lattice $L^{s}(D[m])$ ). The closed and connected elements of the $s$-chromatic lattice $L^{s}(D[m])$ of the complete simplex $D[m]$ on $m>s$ vertices are $\emptyset$ and the $\binom{m}{k}$ sets $F^{s}(D[k])$ of all $s$-simplices in the subcomplexes $D[k]$ for $s<k \leq m$. The map $S \rightarrow \pi(S)$ is an isomorphism between the lattice $L^{s}(D[m])$ and the lattice, ordered by refinement, of all partitions of the set $[m]$ into blocks of size $>s$ or 1 . The least element, $\widehat{0}=(1) \cdots(m)$, is the partition with $m$ blocks and the greatest element, $\widehat{1}=(1 \cdots m)$, the partition with 1 block. $L^{s}(D[m])$ is not a graded lattice [12, p 99] in general when $s \geq 2$. To see this, observe that the 2-chromatic lattices $L^{2}(D[3]), L^{2}(D[4])$, and $L^{2}(D[4])$ are graded but the lattice $L^{2}(D[6])$ is not graded as it contains two maximal chains

$$
\begin{gathered}
\widehat{0}=(1)(2)(3)(4)(5)(6)<(123)(4)(5)(6)<(1234)(5)(6)<(12345)(6)<(123456)=\widehat{1} \\
\widehat{0}=(1)(2)(3)(4)(5)(6)<(123)(4)(5)(6)<(123)(456)<(123456)=\widehat{1}
\end{gathered}
$$

of unequal length. In contrast, the 1-chromatic lattice of any finite simplicial complex is always graded and even geometric $[10, \S 9$, Lemma 1].

Remark 2.19 (The Möbius function for the $s$-chromatic lattices $L^{s}(D[m])$ ). Our discussion of the Möbius function for the lattice $L^{s}(D[m])$ echoes the exposition of the Möbius function for the geometric lattice $L^{1}(D[m])$ of all partitions from [12, Example 3.10.4].

Let $w:[m] \rightarrow \mathbf{N}$ be a function that to every element of $[m]$ associates a natural number, thought of as a weight function. We write $w=1^{i_{1}} 2^{i_{2}} \cdots r^{i_{r}}$, or something similar, for the weight function $w$ defined on the set $[m$ ] of cardinality $m=\sum_{j} i_{j}$ and mapping $i_{j}$ elements to $j$ for $1 \leq j \leq r$. The map $w$ extends to a map, also called $w$, defined on the set of all nonempty subsets $X$ of $[m]$ given by $w(X)=\sum_{x \in X} w(x)$. Let $L_{m}^{s}(w)$ be the lattice of all partitions of the set $[m$ ] into blocks $X$ that are singletons or have weight $w(X)>s$. The non-singleton blocks of the meet $\sigma \wedge \tau$ of two partitions $\sigma, \tau \in L_{m}^{s}(w)$ are the subsets of weight $>s$ of the form $S \cap T$ where $S$ is a block in $\sigma$ and $T$ a block in $\tau$. Write $\mu_{m}^{s}(w)$ for the Möbius function of $L_{m}^{s}(w)$.

In particular, $L_{m}^{s}\left(1^{m}\right)$ is a synonym for $L^{s}(D[m])$ and we are primarily interested in the Möbius function $\mu_{m}^{s}\left(1^{m}\right)$ of the uniform weight $w=1^{m}$. However, the computation of this Möbius function will involve the Möbius functions of other weights as well. We shall therefore discuss the Möbius functions $\mu_{m}^{s}(w)$ for general weight functions $w$.

Suppose that $\sigma \in L_{m}^{s}(w), \sigma<\widehat{1}$, is a partition of $[m]$ into singleton blocks or blocks of weight $>s$. Let $w(\sigma)$ be the restriction of $w$ to the set of blocks of $\sigma$. Thus $w(\sigma)(X)=\sum_{x \in X} w(x)$ for any block $X$ of $\sigma$. Then the interval

$$
L_{m}^{s}(w) \supset[\sigma, \widehat{1}]=L_{|\sigma|}^{s}(w(\sigma))
$$

so that $\mu_{m}^{s}(w)(\sigma, \widehat{1})=\mu_{|\sigma|}^{s}(w(\sigma))(\widehat{0}, \widehat{1})$. More generally, suppose that $\sigma<\tau$ for some $\tau \in L_{m}^{s}(w)$. Assume that the partition $\tau$ has blocks $\tau_{j}$. Let $\sigma_{j}$ be the set of those blocks of $\sigma$ that intersect the block $\tau_{j}$ of $\tau$. Let $w\left(\sigma_{j}\right)$ be the restriction of $w(\sigma)$ to $\sigma_{j}$. Then the interval

$$
L_{m}^{s}(w) \supset[\sigma, \tau]=\prod_{j} L_{\left|\sigma_{j}\right|}^{s}\left(w\left(\sigma_{j}\right)\right)
$$

and therefore the value of the Möbius function on the pair $(\sigma, \tau)$

$$
\mu_{m}^{s}(w)(\sigma, \tau)=\prod_{j} \mu_{\left|\sigma_{j}\right|}^{s}\left(w\left(\sigma_{j}\right)\right)(\widehat{0}, \widehat{1})
$$

by the product theorem for Möbius functions [12, Proposition 3.8.2]. We conclude that the complete Möbius functions on all the lattices $L_{m}^{s}(w)$, are actually determined by the values $\mu_{m}^{s}(w)(\widehat{0}, \widehat{1})$ of these Möbius functions on just $(\widehat{0}, \widehat{1})$. See Equation (2.36) for more information about these Euler characteristics.

For the following it is convenient to name the elements of the domain $[m]$ of $w$ so that the element $m$ carries minimal weight. Assume that $a_{m}=(1 \cdots m-1)(m)$ is an element of $L_{m}^{s}(w)$, ie that $w(1)+\cdots+w(m-1)>s$. We shall determine the set of lattice elements $x$ with $x \wedge a_{m}=\widehat{0}$. There is only one solution to this equation with $x \leq a_{m}$ and that is $x=\widehat{0}$. As the other solutions satisfy $x \not \leq a_{m}$, they must have a block that contains $m$ and at least one other element. It follows that the solutions $x \neq \widehat{0}$ are all elements of the form

$$
x=\left(x_{1} \cdots x_{t} m\right)(\cdot) \cdots(\cdot) \text { with } \begin{cases}w\left(x_{1}\right)>s-w(m) & t=1 \\ s \geq w\left(x_{1}\right)+\cdots+w\left(x_{t}\right)>s-w(m) & t>1\end{cases}
$$

where all blocks but the unique block containing $m$ are singletons. There are $t+1$ elements in the block containing $m$ where $t$ is some number in the range $1 \leq t \leq s$. (All the solutions $x \neq \widehat{0}$ are atoms in the lattice $L_{m}^{s}(w)$.) Since we are in a lattice, the Möbius function satisfies the equation [12, Corollary 3.9.3]

$$
\mu_{m}^{s}(w)(\widehat{0}, \widehat{1})=-\sum_{\substack{x \wedge a_{m}=\widehat{0} \\ x \neq 0}} \mu_{m}^{s}(w)(x, \widehat{1})
$$

which translates to

$$
\begin{align*}
& \text { (2.20) } \mu_{m}^{s}(w)(\widehat{0}, \widehat{1})=-\sum_{\substack{x \wedge a_{m}=\widehat{0} \\
x \neq 0}} \mu_{|x|}^{s}(w(x))(\widehat{0}, \widehat{1})=  \tag{2.20}\\
& \left.-\sum_{\substack{1 \leq x_{1} \leq m-1 \\
w\left(x_{1}\right)>s-w(m)}} \mu_{m-1}^{s}\left(w\left(x_{1} m\right) w(\cdot) \cdots w(\cdot)\right)(\widehat{0}, \widehat{1})-\sum_{1<t \leq s} \sum_{\substack{1 \leq x_{1}, \ldots, x_{t} \leq m-1 \\
s \geq w\left(x_{1}\right)+\cdots+w\left(x_{t}\right)>s-w(m)}} \mu_{m-t}^{s}\left(w\left(x_{1} \cdots x_{t} m\right)\right) w(\cdot) \cdots w(\cdot)\right)(\widehat{0}, \widehat{1})
\end{align*}
$$

This describes a recursive procedure for computing all values of the Möbius function on the weight lattices $L_{m}^{s}(w)$.

As an illustration we compute $\mu_{6}^{2}\left(1^{6}\right)(\widehat{0}, \widehat{1})$. Using (2.20) twice gives

$$
\mu_{6}^{2}\left(1^{6}\right)(\widehat{0}, \widehat{1})=-10 \mu_{4}^{2}(3111)(\widehat{0}, \widehat{1})=10\left(\mu_{3}^{2}(411)(\widehat{0}, \widehat{1})+\mu_{2}^{2}(33)(\widehat{0}, \widehat{1})\right)
$$

The lattices $L_{4}^{2}(411)$ and $L_{2}^{2}(33)$ have 4 and 2 elements, respectively, and they look like

so that $\mu_{3}^{2}(411)(\widehat{0}, \widehat{1})=1$ and $\mu_{2}^{2}(33)(\widehat{0}, \widehat{1})=-1$. Therefore $\mu_{6}^{2}\left(1^{6}\right)(\widehat{0}, \widehat{1})=0$.
We remind the reader of the well-known fact that $\mu_{m}^{s}(w)(\widehat{0}, \widehat{1})$ is the reduced Euler characteristic of the open interval $L_{m}^{s}(w)(\widehat{0}, \widehat{1})$ between $\widehat{0}$ and $\widehat{1}$ in the lattice $L_{m}^{s}(w)$.
Proposition 2.21. [10, §6] [12, Proposition 3.8.5] Let $x<y$ be two elements in a finite poset. The value of the Möbius function on the pair $(x, y)$ is the reduced Euler characteristic of the open interval $(x, y)$.
Proof. Write $\mu$ be the Möbius function of $P$ and E for Euler characteristic. The closed interval from $x$ to $y$ has Euler characteristic 1 since it has a smallest element. Thus

$$
\begin{aligned}
1=\mathrm{E}([x, y])=\sum_{a, b \in[x, y]} \mu(a, b)=\sum_{a, b \in(x, y)} \mu(a, b)+\sum_{a \in[x, y]} \mu(a, y) & +\sum_{b \in[x, y]} \mu(x, b)-\mu(x, y) \\
& =\mathrm{E}((x, y))+0+0-\mu(x, y)=\mathrm{E}((x, y))-\mu(x, y)
\end{aligned}
$$

or $\mu(x, y)=\widetilde{\mathrm{E}}((x, y))$.
For $1 \leq s \leq m+1$ let $B(m, s)$ be the graded poset of nonempty subsets of $[m]$ of cardinality less than $s$.
Lemma 2.22. The reduced Euler characteristic of $B(m, s)$ is

$$
\widetilde{E}(B(m, s))=(-1)^{s}\binom{m-1}{s-1}, \quad 1 \leq s \leq m+1
$$

Proof. It is rather easy to get the recurrence relation

$$
\begin{aligned}
& E(B(m, 2))=m \\
& E(B(m, s))=E(B(m, s-1))+\binom{m}{s-1} \sum_{j=1}^{s-1}(-1)^{s-1-j}\binom{s-1}{j}, \quad 2<s<2+m
\end{aligned}
$$

Since the sum of binomial coefficients has value $(-1)^{s}$, we get the recurrence relation

$$
\begin{aligned}
& \widetilde{E}(B(m, 2))=m-1 \\
& \widetilde{E}(B(m, s))=\widetilde{E}(B(m, s-1))+(-1)^{s}\binom{m}{s-1}, \quad 2<s<2+m
\end{aligned}
$$

for the reduced Euler characteristic. The claim of the lemma follows immediately.
Example 2.23 (Reduced Euler characteristics of the $s$-chromatic lattice intervals $\left.L_{m}^{s}(w)(\widehat{0}, \widehat{1})\right)$. The reduced Euler characteristics $\mu_{m}^{s}\left(1^{m}\right)(\widehat{0}, \widehat{1})=\widetilde{E}\left(L_{m}^{s}\left(1^{m}\right)(\widehat{0}, \widehat{1})\right), m \geq s+2$, for $s=1,2, \ldots, 8$ are

$$
\begin{aligned}
& 2,-6,24,-120,720,-5040,40320,-362880,3628800,-39916800,479001600,-6227020800,87178291200, \ldots \\
& 3,-6,0,90,-630,2520,0,-113400,1247400,-7484400,0,681080400,-10216206000,81729648000, \ldots \\
& 4,-10,20,-70,560,-4200,25200,-138600,924000,-8408400,84084000,-798798000,7399392000, \ldots \\
& 5,-15,35,-70,0,2100,-23100,173250,-1051050,5255250,-15765750,-105105000,2858856000, \ldots \\
& 6,-21,56,-126,252,-924,11088,-126126,1093092,-7693686,46414368,-254438184,1492322832, \ldots \\
& 7,-28,84,-210,462,-924,0,42042,-630630,6390384,-51459408,351639288,-2118412296,11406835440 \ldots \\
& 8,-36,120,-330,792,-1716,3432,-12870,205920,-3150576,35706528,-322583976,2460949920 \ldots \\
& 9,-45,165,-495,1287,-3003,6435,-12870,0,787644,-14965236,191222460,-1920538620 \ldots
\end{aligned}
$$

The first sequence, $\mu_{m}^{1}\left(1^{m}\right)(\widehat{0}, \widehat{1}), m \geq 2$, is the sequence $(-1)^{m-1}(m-1)$ ! of reduced Euler characteristics of the lattice of partitions of $[m]\left[12\right.$, Example 3.10.4]. The second sequence, $\mu_{m}^{2}\left(1^{m}\right)(\widehat{0}, \widehat{1}), m \geq 3$, seems to coincide with first terms of the sequence A009014 from The On-Line Encyclopedia of Integer Sequences (OES). The remaining 6 sequences apparently do not match any sequences of the OES.

The first $s$ terms of these sequences are signed binomial coefficients. This is because the interval ( $\widehat{0}, \widehat{1}$ ) in $L^{s}(D[m])$ is isomorphic to the opposite of the poset $B(m, m-s)$ when $s+2 \leq m \leq 2 s+1$. Thus the reduced Euler characteristic

$$
\mu_{m}^{s}\left(1^{m}\right)(\widehat{0}, \widehat{1})=\widetilde{E}(B(m, m-s))=(-1)^{m-s}\binom{m-1}{s}, \quad s+2 \leq m \leq 2 s+1
$$

according to Lemma 2.22.
The first terms of the sequence $\mu_{m}^{2}\left(3^{1} 1^{m-1}\right)(\widehat{0}, \widehat{1}), m \geq 3$, of reduced Euler characteristics of the weighted lattice intervals $L_{m}^{2}\left(3^{1} 1^{m-1}\right)(\widehat{0}, \widehat{1})$,

$$
1,0,-6,30,-90,0,2520,-22680,113400,0,-7484400,97297200,-681080400,0,81729648000,-1389404016000, \ldots
$$

seem to coincide up to sign with first terms of the sequence A009775 from OES. The sequence of reduced Euler characteristics $\mu_{m}^{2}\left(3^{2} 1^{m-2}\right)(\widehat{0}, \widehat{1}), m \geq 3$, of the lattice interval $L_{m}^{2}\left(3^{2} 1^{m-2}\right)(\widehat{0}, \widehat{1})$,

$$
\begin{aligned}
& 2,-4,6,6,-120,720,-2520,-2520,136080,-1360800,7484400,7484400, \\
&-778377600,10897286400,-81729648000,-81729648000,13894040160000, \ldots
\end{aligned}
$$

apparently does not match any sequence in the OES.
Define the $s$-monochrome set of a map col: $V(K) \rightarrow[r]=\{1, \ldots, r\}$ to be the set

$$
M^{s}(\operatorname{col})=\left\{\sigma \in F^{s}(K)| | \operatorname{col}(\sigma) \mid=1\right\}
$$

of all monochrome $s$-simplices in $K$. The map col is an $(r, s)$-coloring of $K$ if and only if $M^{s}(\operatorname{col})=\emptyset$.
Lemma 2.24. The s-monochrome set $M^{s}(\mathrm{col})$ of any map col: $V(K) \rightarrow[r]$ is an element of the $s$-chromatic lattice $L^{s}(K)$.
Proof. Let $S$ be a connected component of $M^{s}(\operatorname{col})$. Since $S$ is connected, all vertices in $S$ have the same color. Let $\sigma \in F^{s}(K)$ be an $s$-simplex of $K$ such that $\sigma \subset V(S)$. The $\sigma$ is monochrome: $\sigma \in M^{s}(\operatorname{col})$. By Lemma 2.14, $S$ is closed.

Theorem 2.25. The number of $(r, s)$-colorings of $K$ is

$$
\chi^{s}(K, r)=\sum_{T \in L^{s}(K)} \mu(\widehat{0}, T) r^{|\pi(T)|}
$$

where $\mu$ the Möbius function for the s-chromatic lattice $L^{s}(K)$.
Proof. For any $B \in L^{s}(K)$, let $\chi(K, r, s, B)$ be the number of maps col: $V(K) \rightarrow[r]$ with $M^{s}(\operatorname{col})=B$. We want to determine $\chi(K, r, s, \emptyset)=\chi^{r}(s, K)$. For any $A \in L^{s}(K)$,

$$
r^{|\pi(A)|}=\sum_{A \leq B} \chi(K, r, s, B)
$$

because there are $r^{\left|\pi_{0}(A)\right|} r^{m(K)-|V(A)|}=r^{|\pi(A)|}$ maps col: $V(K) \rightarrow[r]$ with $A \leq M^{s}(\mathrm{col})$. Equivalently,

$$
\sum_{A \leq B} \mu(A, B) r^{|\pi(B)|}=\chi(K, r, s, A)
$$

by Möbius inversion [12, Proposition 3.7.1]. The statement of the theorem is the particular case of this formula where $A=\widehat{0}$.

The defining rules for the Möbius function of the poset $L^{s}(K)[12,3.7]$

- $\mu(S, S)=1$ for all $S \in L^{s}(K)$
- $\sum_{R \leq S \leq T} \mu(R, S)=0$ when $R \nsupseteq T$
- $\mu(R, S)=0$ when $R \not \leq S$
imply that $\mu(\widehat{0}, \widehat{0})=1$ and $\mu(\widehat{0},\{\sigma\})=-1$ for every $s$-simplex $\sigma \in F^{s}(K)$.
Corollary 2.26. The highest degree terms of the s-chromatic polynomial are

$$
\chi^{s}(K, r)=r^{m(K)}-f_{s}(K) r^{m(K)-s}+\cdots
$$

Thus the s-chromatic polynomial determines $f_{0}(K)$ and $f_{s}(K)$.

Proof. The $s$-chromatic polynomial is

$$
\chi^{s}(K, r)=\mu(\widehat{0}, \widehat{0}) r^{f_{0}(K)}+\sum_{\sigma \in F^{s}(K)} \mu(\widehat{0},\{\sigma\}) r^{f_{0}(K)-s}+\cdots
$$

where $\mu(\widehat{0}, \widehat{0})=1$ and $\mu(\widehat{0},\{\sigma\})=-1$ for all $s$-simplices $\sigma$ of $K$.
Example 2.27. Consider the 2-dimensional complex $K$ from Example 2.9. The 2-chromatic lattice $L^{2}(K)$ of $K$

consists of all subsets of $F^{2}(K)$. The 2-chromatic polynomial is

$$
\chi^{2}(K, r)=r^{6}-r^{4}-r^{4}-r^{4}+r^{2}+r^{3}+r^{2}-r=r^{6}-3 r^{4}+r^{3}+2 r^{2}-r
$$

$K$ has $\chi^{2}(K, 2)=30(2,2)$-colorings and $\chi^{2}(K, 3)=528(3,2)$-colorings.
Example 2.28. The triangulation $M B$ of the Möbius band with $f$-vector $f(\mathrm{MB})=(5,10,5)$ shown in Figure 1 has the following (reduced) 2-chromatic lattice $L^{2}(\mathrm{MB})-\{\widehat{0}, \widehat{1}\}$

and 2 -chromatic polynomial

$$
\chi^{2}(\mathrm{MB}, r)=r^{5}-5 r^{3}+5 r^{2}-r
$$

The lattice $L^{2}(\mathrm{MB})$ is graded but it is still not semi-modular [12, Proposition 3.3.2]: The meet and join of $a=$ $\{\{2,3,5\}\}$ and $b=\{\{1,3,4\}\}$ are $a \wedge b=\widehat{0}$ and $a \vee b=\widehat{1}$. Thus $a$ and $b$ cover $a \wedge b$ but $a \vee b$ covers neither $a$ nor $b$.

Example 2.29. Let MT be Möbius's minimal triangulation of the torus with $f$-vector $f(\mathrm{MT})=(7,21,14)$ and P 2 the triangulation of the projective plane with $f$-vector $f(\mathrm{P} 2)=(1,6,15,10)$ shown in Figure 2 (decorated with $(3,2)$-colorings). The chromatic polynomials of these two simplicial complexes are

$$
\begin{array}{ll}
\chi^{1}(\mathrm{MT}, r)=[r]_{7}, & \chi^{2}(\mathrm{MT}, r)=r^{7}-14 r^{5}+21 r^{4}+7 r^{3}-21 r^{2}+6 r \\
\chi^{1}(\mathrm{P} 2, r)=[r]_{6}, & \chi^{2}(\mathrm{P} 2, r)=r^{6}-10 r^{4}+15 r^{3}-6 r^{2}
\end{array}
$$

In both cases, the 1 -skeleton is the complete graph on the vertex set. The chromatic numbers are chr ${ }^{1}(\mathrm{MT})=7$, $\operatorname{chr}^{1}(\mathrm{P} 2)=6$, and $\operatorname{chr}^{2}(\mathrm{MT})=3=\operatorname{chr}^{2}(\mathrm{P} 2)$.

The chromatic polynomials of simple graphs (the 1-chromatic polynomials of simplicial complexes) are known to have these properties:

- The coefficients are sign-alternating [10, $\S 7$, Corollary]
- The coefficients are log-concave (Definition 2.43) in absolute value [7]
- There are no negative roots and no roots between 0 and 1 [14]


Figure 2. (3, 2)-colorings of P2 and MT

In contrast, the coefficients of the 2-chromatic polynomial

$$
\chi^{2}(\mathrm{MT}, r)=r^{7}-14 r^{5}+21 r^{4}+7 r^{3}-21 r^{2}+6 r=[r]_{3}(r+1)\left(r^{3}+2 r^{2}-9 r+3\right)
$$

are not sign-alternating, not log-concave in absolute value, and the polynomial has a negative root and a root between 0 and 1.
2.4. The $s$-chromatic polynomial in falling factorial form. Theorem 1.2 provides an interpretation of the coefficients of the falling factorial $[r]_{i}$ in the $s$-chromatic polynomial of the simplicial complex $K$.

Definition 2.30. $S(K, r, s)$ is the number of partitions of $V(K)$ into $r$-independent blocks.
We think of $S(K, r, s)$ as an $s$-Stirling number of the second kind for the simplicial complex $K$. If $s>\operatorname{dim}(K)$, then there are no $s$-simplices in $K$ and all partitions of $V(K)$ are $s$-independent, so that $S(K, r, s)$ is the Stirling number of the second kind $S(m(K), r)$ [12, p 33]. We now explain the general relation between these simplicial Stirling numbers $S(K, r, s)$ and the usual Stirling numbers of the second kind.

Define the s-monochrome set of a partition $P$ of $V(K)$ to be the set

$$
M^{s}(P)=\left\{\sigma \in F^{s}(K) \mid \sigma \text { is contained in a block of } P\right\}
$$

of all $s$-simplices entirely contained in one of the blocks of $P$. The set $M^{s}(P)$ is an element of the $s$-chromatic lattice $L^{s}(K)$ by Lemma 2.24.
Theorem 2.31. The number of partitions of $V(K)$ into $r$ s-independent blocks is

$$
S(K, r, s)=\sum_{T \in L^{s}(K)} \mu(\widehat{0}, T) S(|\pi(T)|, r)
$$

where $\mu$ the Möbius function for the s-chromatic lattice $L^{s}(K)$.
Proof. For any $B \in L^{s}(K)$, let $S(K, r, s, B)$ be the number of partitions $P$ of $V(K)$ into $r$ blocks with monochrome set $M^{s}(P)=B$. We want to determine $S(K, r, s, \emptyset)=S(K, r, s)$. For any $A \in L^{s}(K)$,

$$
S(|\pi(A)|, r)=\sum_{A \leq B} S(K, r, s, B)
$$

because there are $S(|\pi(A)|, r)$ partitions $P$ of $V(K)$ into $r$ blocks with $A \leq M^{s}(P)$. Equivalently,

$$
\sum_{A \leq B} \mu(A, B) S(|\pi(B)|, r)=S(K, r, s, A)
$$

by Möbius inversion [12, Proposition 3.7.1]. The statement of the theorem is the particular case of this formula where $A=\widehat{0}$.

Proof of Theorem 1.2. We simply follow the proof of the similar statement for chromatic polynomials for graphs [9, Theorem 15]. When $r \geq i$ we can get an $(r, s)$-coloring out of one of the $S(K, i, s)$ partitions of $V(K)$ into $i$ $s$-independent blocks by choosing $i$ out of the $r$ colors and assigning them to the $i$ blocks. There are $\binom{r}{i}$ ways of
choosing the $i$ out of $r$ colors and $i$ ! ways of coloring $i$ blocks in $i$ colors. The number of $(r, s)$-colorings of $K$ in exactly $i$ colors is thus

$$
S(K, i, s)\binom{r}{i} i!=S(K, i, s)[r]_{i}
$$

so that

$$
\chi^{s}(K, r)=\sum_{i=1}^{m(K)} S(K, i, s)[r]_{i}
$$

is the total number of $(r, s)$-colorings of $K$.
Corollary 2.32. The reduced Euler characteristic of the open interval $(\hat{0}, \widehat{1})$ in $s$-chromatic lattice $L^{s}(K)$ is

$$
\mu\left(L^{s}(K)\right)(\widehat{0}, \widehat{1})=\sum_{i=\operatorname{chr}^{s}(K)}^{m(K)}(-1)^{i-1}(i-1)!S(K, i, s)
$$

Proof. Equate the terms of degree 1 of the two expressions

$$
\begin{equation*}
\sum_{T \in L^{s}(K)} \mu(\widehat{0}, T) r^{|\pi(T)|}=\sum_{i=\operatorname{chr}^{s}(K)}^{m(K)} S(K, i, s)[r]_{i} \tag{2.33}
\end{equation*}
$$

from Theorem 2.25 and Theorem 1.2 for the $s$-chromatic polynomial of $K$.
We observe that

$$
\sum_{i} S(K, i, s)[r]_{i}=\sum_{i} \sum_{T} \mu(\widehat{0}, T) S(|\pi(T)|, i)[r]_{i}=\sum_{T} \mu(\widehat{0}, T) \sum_{i} S(|\pi(T)|, i)[r]_{i}=\sum_{T} \mu(\widehat{0}, T) r^{|\pi(T)|}
$$

so that Theorem 2.31 implies Theorem 1.2.
The $s$-chromatic number of $K$ is immediately visible with the $s$-chromatic polynomial in factorial form because

$$
\operatorname{chr}^{s}(K)=\min \{i \mid S(K, i, s) \neq 0\}
$$

is the lowest degree of the nonzero terms. The positive integer sequence

$$
\chi^{s}\left(K, \operatorname{chr}^{s}(K)\right), \ldots, \chi^{s}(K, m(K))=1
$$

has no internal zeros. (If there is a partition of $V(K)$ into $r$ blocks not containing any $s$-simplex of $K$ and $r<m(K)$, then split one of the blocks with more than one vertex into two sub-blocks to get a partition of $V(K)$ into $r+1$ blocks containing no $s$-simplices of $K$.)

The simplicial Stirling numbers satisfy the recurrence relations

$$
S(K, r, s)=\sum_{\substack{\emptyset \subseteq U \subseteq V(K)-\left\{v_{0}\right\} \\ V(K) \\(K)}} S(K \cap D[U], r-1, s), \quad S(K, 1, s)= \begin{cases}1 & s>\operatorname{dim}(K) \geq 0 \\ 0 & \text { otherwise }\end{cases}
$$

To see this, fix a vertex $v_{0}$ of $K$. Let $P$ be partition of $V(K)$ into $r s$-independent subsets. Let $U_{0}$ be the block containing $v_{0}$. The other blocks in $P$ form a partition $P_{0}$ of $K \cap D\left[V(K)-U_{0}\right.$ ] into $r-1 s$-independent subsets. The map $P \leftrightarrow\left(P_{0}, U_{0}\right)$ is a bijection.

The familiar recurrence relation $S(m, r)=S(m-1, r-1)+r S(m-1, r)$ for Stirling numbers of the second kind does not readily apply to simplicial Stirling numbers. The closest analogue may be
$S(K, r, s)=S\left(K \cap D\left[V(K)-\left\{v_{0}\right\}\right], r-1, s\right)+\sum_{P \in \mathcal{S}\left(K \cap D\left[V(K)-\left\{v_{0}\right\}\right], r, s\right)} \mid\left\{B \in P \mid B \cup\left\{v_{0}\right\}\right.$ is $s$-independent in $\left.K\right\} \mid$
where $v_{0}$ is a vertex of $K$ and $\mathcal{S}\left(K \cap D\left[V(K)-\left\{v_{0}\right\}, r, s\right)\right.$ is the set of partitions $P$ of the vertex set of $K \cap D[V(K)-$ $\left\{v_{0}\right\}$ into $r s$-independent subsets.
Proposition 2.34. Let $K$ be a subcomplex of $L$ and assume that $V(K)=V(L)$.
(1) $S(K, r, s) \geq S(L, r, s)$ for all $r$.
(2) If $S(K, r, s)=S(L, r, s)$ for some $r$ with $\frac{1}{s}(|V|-1) \leq r \leq|V|-s$, then $K^{s}=L^{s}$.

$$
\begin{gathered}
\left(\begin{array}{llll}
0 & 1
\end{array}\right)\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 3 & 1
\end{array}\right) \\
\left(\begin{array}{ccccc}
0 & 0 & 0 & 1 \\
0 & 3 & 6 & 1 \\
0 & 7 & 6 & 1
\end{array}\right) \\
\left.\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 15 & 45 & 15 & 1 \\
0 & 10 & 75 & 65 & 15 & 1 \\
0 & 25 & 90 & 65 & 15 & 1 \\
0 & 31 & 90 & 65 & 15 & 1
\end{array}\right) \quad\left(\begin{array}{cccccc}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 15 & 10 & 1 \\
0 & 10 & 25 & 10 & 1 \\
0 & 15 & 25 & 10 & 1
\end{array}\right)
\end{gathered}
$$

TABLE 2. Chromatic tables for complete simplices $D[m]$ for $m=2, \ldots, 7$

Proof. (1) Let $V$ be the vertex set of $K$ and $L$. Write $\mathcal{S}(K, r, s)$ and $\mathcal{S}(L, r, s)$ for the set of partitions of $V$ into $r$ blocks containing no $s$-simplex of $K$ or $L$, respectively. Then $\mathcal{S}(L, r, s) \subseteq \mathcal{S}(K, r, s)$ for all $r$ and $s$. Thus $S(L, r, s) \leq S(K, r, s)$.
(2) Suppose that $\sigma \in F^{s}(L)-F^{s}(K)$ is an $s$-simplex of $L$ that is not an $s$-simplex of $K$. Any partition of the form

$$
\{\sigma\} \cup \tau, \quad \tau \in \mathcal{S}(D[V-\sigma], r-1, s)
$$

in $\mathcal{S}(K, r, s)-\mathcal{S}(L, r, s)$. The set $\mathcal{S}(D[V-\sigma], r-1, s)$ is nonempty when

$$
\operatorname{chr}^{s}(D[V-\sigma])=\left\lceil\frac{|V|-s-1}{s}\right\rceil \leq r-1 \leq|V|-s-1
$$

and thus $S(K, r, s)$ is strictly greater than $S(L, r, s)$ when $\frac{|V|-1}{s} \leq r \leq|V|-s$.
Remark $2.35(S(K, r, s)$ for the complete simplex $K=D[m])$. For any finite set $M$, let $S(M, r, s)$ stand for $S(D[M], r, s)$ (Definition 2.30), the number of partitions of the set $M$ into $r$ blocks containing at most $s$ elements. Let us even write $S(m, r, s)$ in case $M=[m], m \geq 1, r, s \geq 0$. Clearly, $S(m, r, s)$ is nonzero only when $m / s \leq r \leq m$. Also, $S(m, r, s)=S(m, r)$ when $r$ is among the $s$ numbers $m-s+1, \ldots, m$. The recurrence relation

$$
S(m, r, s)=\sum_{j=m-s}^{m-1}\binom{m-1}{j} S(j, r-1, s)
$$

can be used to compute these numbers. Table 2 shows $S(m, r, s)$ for small $m$; the number $S(m, r, s)$ is in row $s$ and column $r$ in the chromatic table (Definition 2.39) for $D[m]$. All the red numbers are usual Stirling numbers of the second kind.

According to Theorem 1.2, the numbers $S(m, r, s)$ determine the $s$-chromatic polynomial in falling factorial form of the complete simplex on $m$ vertices

$$
\chi^{s}(D[m], r)=\sum_{i=\lceil m / s\rceil}^{m} S(m, i, s)[r]_{i}
$$

and, according to Corollary 2.32, they also determine the reduced Euler characteristic

$$
\mu_{m}^{s}\left(1^{m}\right)(\widehat{0}, \widehat{1})=\sum_{i=\lceil m / s\rceil}^{m}(-1)^{i-1}(i-1)!S(m, i, s)
$$

of the $s$-chromatic lattice $L^{s}(D[m])$.
More generally, if $w: M \rightarrow \mathbf{N}$ is a function on $M$ with natural numbers as values, let $S(M, w, r, s)$ be the number of partitions of $M$ into admissible blocks, where we declare a block admissible if it is a singleton or it has weight at most $s$. (Then $S(m, r, s)=S\left([m], 1^{m}, r, s\right)$ occur when $M=[m]$ and $w=1^{m}$ places weight 1 on all elements.) Any such partition is a partition of $M$ into blocks of weight at most $s$, and therefore $S(M, w, r, s) \leq S(\# M, r, s)$. In particular, $S(M, w, r, s)$ is nonzero only when $\# M / s \leq r \leq \# M$. The recurrence relation

$$
S(M, w, r, s)=\sum_{\substack{\emptyset \neq J \subset M-\{\max (M)\} \\ M-J \text { admissible }}} S(J, w \mid J, r-1, s)
$$

provides a means to compute these numbers.

The weighted version of Equation (2.33) for $K=D[m]$,

$$
\sum_{\sigma \in L_{m}^{s}(w)} \mu_{m}^{s}(w)(\widehat{0}, \sigma) r^{|\sigma|}=\sum_{i=\lceil m / s\rceil}^{m} S([m], w, i, s)[r]_{i}
$$

implies, by equating coefficients of first degree terms, the expression

$$
\begin{equation*}
\mu_{m}^{s}(w)(\widehat{0}, \widehat{1})=\sum_{i=\lceil m / s\rceil}^{m}(-1)^{i-1}(i-1)!S([m], w, i, s) \tag{2.36}
\end{equation*}
$$

for the Euler characteristic of the weighted lattice $L_{m}^{s}(w)$ from Remark 2.19.
Because any simplicial complex $K$ is a subcomplex of the complete simplex $D[m(K)]$ on its vertex set, we have

$$
\begin{equation*}
S(m(K), r) \geq S(K, r, s) \geq S(m(K), r, s), \quad 1 \leq r \leq m(K) \tag{2.37}
\end{equation*}
$$

Moreover, these inequalities are equalities for the $s$ highest values $m(K)-s+1, \ldots, m(K)$ of $r$. Thus the $s$ terms of highest falling factorial degree in the $s$-chromatic polynomial of $K$

$$
\chi^{s}(K, r)=\sum_{i=0}^{m(K)-s} S(K, i, s)[r]_{i}+\sum_{i=m(K)-s+1}^{m(K)} S(m(K), i)[r]_{i}
$$

are given by the $s$ Stirling numbers $S(m(K), m(K)-s+1), \ldots, S(m(K), m(K))$ of the second kind. These coefficients depend only on the size of the vertex set of $K$. We shall next show that the coefficient number $s+1$ counted from above, $S(K, m(K)-s, s)$, informs about the number $f_{s}(K)$ of $s$-simplices in $K$.

Proposition 2.38. $S(K, m(K)-s, s)=S(m(K), m(K)-s)-f_{s}(K)$. If $S(K, m(K)-s, s)=S(m(K), m(K)-s, s)$ then $K^{s}=D[m(K)]^{s}$.

Proof. The only partitions of the $S(m, m-s)$ partitions of $V(K)$ into $m-s$ blocks that are not $s$-independent are those consisting of one $s$-simplex of $K$ together with singleton blocks. If $S(K, m(K)-s, s)=S(D[m(K)], m(K)-$ $s, s)$ then $f_{s}(K)=f_{s}(D[m(K)])$ so $K^{s}=D[m(K)]^{s}$. (This is a special case of Proposition 2.34.(2).)

Definition 2.39. The chromatic table, $\chi(K)$, of $K$ is the $(\operatorname{dim}(K) \times m(K))$-table with $S(K, r, s)$ in row $s$ and column $r$.

This means that row $s$ in the chromatic table lists the coefficients of the $s$-chromatic polynomial. The chromatic table of a 3-dimensional simplicial complex $K$, for instance, looks like this

|  | $r=1$ | $r=2$ | $\cdots$ | $r=m-3$ | $r=m-2$ | $r=m-1$ | $r=m$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $S(K, \cdot 1)$ | $S(K, 1,1)$ | $S(K, 2,1)$ | $\cdots$ | $S(K, m-3,1)$ | $S(K, m-2,1)$ | $S(m, m-1)-f_{1}$ | $S(m, m)=1$ |
| $S(K, \cdot, 2)$ | $S(K, 1,2)$ | $S(K, 2,2)$ | $\cdots$ | $S(K, m-3,2)$ | $S(m, m-2)-f_{2}$ | $S(m, m-1)$ | $S(m, m)=1$ |
| $S(K, \cdot, 3)$ | $S(K, 1,3)$ | $S(K, 2,3)$ | $\cdots$ | $S(m, m-3)-f_{3}$ | $S(m, m-2)$ | $S(m, m-1)$ | $S(m, m)=1$ |

where the red entries in row $s$ are Stirling numbers of the second kind $S(m, r)$ for $m-s+1 \leq r \leq m$, and the blue entry in row $s$ is $S(m(K), m(K)-s)-f_{s}(K)$.

Example 2.40. The chromatic tables of the 2-dimensional simplicial complexes from Examples 2.9, 2.28, and 2.29 are

$$
\begin{array}{rlrl}
\chi(K) & =\left(\begin{array}{cccccc}
0 & 0 & 2 & 10 & 7 & 1 \\
0 & 15 & 73 & 62 & 15 & 1
\end{array}\right) & \chi(\mathrm{MB}) & =\left(\begin{array}{ccccc}
0 & 0 & 0 & 0 & 1 \\
0 & 5 & 20 & 10 & 1
\end{array}\right) \\
\chi(\mathrm{MT}) & =\left(\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 84 & 231 & 126 & 21 & 1
\end{array}\right) & \chi(\mathrm{P} 2)=\left(\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 45 & 55 & 15 \\
0 & 1
\end{array}\right)
\end{array}
$$

The red entries in column $r$ are Stirling numbers $S(m, r)$ and they are independent of the row index. The blue entry in row $s$ and column $m-s$, which equals $S(m-s, s)-f_{s}(K)$, detects if $K$ has maximal $s$-skeleton by Proposition 3 .
Example 2.41. Let $K=$ AS3 be Altshuler's peculiar triangulation of the 3 -sphere with $f$-vector $f=(10,45,70,35)$ [1]. The 1-chromatic polynomial is $\chi^{1}(\mathrm{AS} 3, r)=[r]_{10}$ as $K^{1}$ is the complete graph on 10 vertices. The chromatic table is

$$
\chi(\mathrm{AS} 3)=\left(\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1360 & 8475 & 10355 & 4200 & 680 & 45 & 1 \\
0 & 26 & 4320 & 25915 & 38550 & 22152 & 5845 & 750 & 45 & 1
\end{array}\right)
$$

The blue numbers determine the $f$-vector

$$
f(\mathrm{AS} 3)=\left(10, S(10,9)-\chi(\mathrm{AS} 3)_{19}, S(10,8)-\chi(\mathrm{AS} 3)_{28}, S(10,7)-\chi(\mathrm{AS} 3)_{37}\right)
$$



Figure 3. The simplicial Stirling numbers for $S_{17,74}^{3}$

The row numbers of the first nonzero term in each row tell us that $\operatorname{chr}^{1}(\mathrm{AS} 3)=10, \operatorname{chr}^{2}(\mathrm{AS} 3)=4, \operatorname{and}^{\operatorname{chr}}{ }^{3}(\mathrm{AS} 3)=$ 2.

Example 2.42. The nonconstructible, nonshellable 3-sphere $S_{17,74}^{3}, f=(17,91,148,74)$, found by Lutz [8], has

|  | $r=1$ | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ | $r=7$ | $r=8$ | $r=9$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $s=1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 88 | 3089 |
| $s=2$ | 0 | 0 | 36 | 702475 | 82949364 | 1075420155 | 3827766587 | 5493687086 | 3876597169 |
| $s=3$ | 0 | 422 | 4319865 | 338438489 | 3903094622 | 14292381565 | 22946854806 | 19158310796 | 9202775199 |


|  | $r=10$ | $r=11$ | $r=12$ | $r=13$ | $r=14$ | $r=15$ | $r=16$ | $r=17$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $s=1$ | 23017 | 55285 | 54973 | 25941 | 6210 | 762 | 45 | 1 |
| $s=2$ | 1507939074 | 346346664 | 48855523 | 4302470 | 235026 | 7672 | 136 | 1 |
| $s=3$ | 2708454744 | 507528561 | 61784524 | 4903589 | 249826 | 7820 | 136 | 1 |

as its chromatic table. Figure 3 shows a semi-logarithmic plot of the simplicial Stirling numbers $S\left(S_{17,74}^{3}, r, s\right)$. The triangulation $\Sigma_{16}^{3}, f=(16,106,180,90)$, of the Poincaré homology 3 -sphere constructed by Björner and Lutz [2, Theorem 5] has

|  | $r=1$ | $r=2$ | $r=3$ | $r=4$ | $r=5$ | $r=6$ | $r=7$ | $r=8$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $s=1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |  |
| $s=2$ | 0 | 0 | 0 | 4589 | 2974411 | 69671411 | 300475213 | 442354547 |  |
| $s=3$ | 0 | 3 | 845561 | 70005500 | 701299653 | 2158716508 | 2888730959 | 2000811501 |  |
|  |  | $r=9$ | $r=10$ | $r=11$ | $r=12$ | $r=13$ | $r=14$ | $r=15$ | $r=16$ |
| $s=1$ | 0 | 0 | 0 | 0 | 28 | 44 | 14 | 1 |  |
| $s=2$ | 292864435 | 100793551 | 19546606 | 2225261 | 150095 | 5840 | 120 | 1 |  |
| $s=3$ | 792553648 | 190527025 | 28730056 | 2750278 | 165530 | 6020 | 120 | 1 |  |

as its chromatic table.
Observe that all the above chromatic tables have strictly log-concave rows.
Definition 2.43. [11] A finite sequence $a_{1}, a_{2}, \ldots, a_{N}$ of $N \geq 3$ nonnegative integers is strictly log-concave if $a_{i-1} a_{i+1}<a_{i}^{2}$ for $1<i<N$ (and log-concave if $a_{i-1} a_{i+1} \leq a_{i}^{2}$ ).

It has been conjectured that the sequence of coefficients of the 1-chromatic polynomial of a simple graph in falling factorial form, $r \rightarrow S(K, 1, r), \operatorname{chr}^{1}(K) \leq r \leq m(K)$, is log-concave [4, Conjecture 3.11]. More generally, one may ask

Question 2.44. Is the finite sequence of simplicial Stirling numbers

$$
r \rightarrow S(K, r, s), \quad \operatorname{chr}^{s}(K) \leq r \leq m(K)
$$

log-concave for fixed $K$ and $s$ ?
This seems to be the right question to ask as it may be true for all the chromatic polynomials of a simplicial complex and we have seen that the absolute value of the coefficients of the $s$-chromatic polynomial are simply not log-concave for $s>1$.

Note that the Stirling numbers of the second kind, which are upper bounds for the simplicial Stirling numbers $S(K, r, s)$ by the inequalities (2.37), are log-concave in $r$ [11, Corollary 2].

We shall now examine Question 2.44 on two spherical boundary complexes of cyclic $n$-polytopes.
Definition 2.45. $\partial \mathrm{CP}(m, n), m>n$, is the $(n-1)$-dimensional simplicial complex on the ordered set $[m]$ with the following facets: An n-subset $\sigma$ of $[m]$ is a facet if and only if between any two elements of $[m]-\sigma$ there is an even number of vertices in $\sigma$.

By Gale's Evenness Theorem [6], the simplicial complex $\partial \mathrm{CP}(m, n)$ triangulates the boundary of the cyclic $n$ polytope on $m$ vertices. Thus $\partial \mathrm{CP}(m, n)$ is a simplicial $(n-1)$-sphere on $m$ vertices and it is $\lfloor n / 2\rfloor$-neighborly in the sense that $\partial \mathrm{CP}(m, n)$ has the same $s$-skeleton as the full simplex on its vertex set when $s<\lfloor n / 2\rfloor$.
Example 2.46 (Cyclic polytopes with log-concave simplicial Stirling numbers of the second kind). Let $\partial \mathrm{CP}(m, n)$ be the triangulated boundary of the cyclic polytope on $m$ vertices in $\mathbf{R}^{n}$. The simplicial complex $\partial \mathrm{CP}(m, n)$ is an $m$-vertex triangulation of $S^{n-1}$. The chromatic tables of the simplicial 3 -spheres $\partial \mathrm{CP}(m, 4)$ on $m=6,7,8,9,10$ vertices are

$$
\begin{gathered}
\left(\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 21 & 47 & 15 & 1 \\
0 & 16 & 81 & 65 & 15 & 1
\end{array}\right)\left(\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 28 & 147 & 112 & 21 & 1 \\
0 & 21 & 238 & 336 & 140 & 21 & 1
\end{array}\right)\left(\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 50 & 393 & 582 & 226 & 28 & 1 \\
0 & 29 & 654 & 1533 & 1030 & 266 & 28 & 1
\end{array}\right) \\
\left(\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 94 & 1062 & 2523 & 1719 & 408 & 36 \\
0 & 1 \\
0 & 36 & 1729 & 6471 & 6591 & 2619 & 462 & 36 \\
1
\end{array}\right)\left(\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 180 & 2980 & 10200 & 10777 & 4225 & 680 & 45 \\
0 & 46 & 4445 & 25960 & 38550 & 22152 & 5845 & 750 & 45 \\
0 & 4
\end{array}\right)
\end{gathered}
$$

All rows are strictly log-concave. As $\partial \mathrm{CP}(m, 4)^{1}=D[m]^{1}$, the 1-chromatic number $\operatorname{chr}^{1}(\partial \mathrm{CP}(m, 4))=m$, and it is not difficult to see that the 2 -chromatic number $\operatorname{chr}^{2}(\partial \mathrm{CP}(m, 4))$ is 2 if $m$ is even and 3 if $m$ is odd [5].

Right multiplication with the upper triangular matrix $\left([j]_{i}\right)_{1 \leq i, j \leq m(K)}$ with $[j]_{i}=\binom{j}{i} i!=\frac{j!}{(i-j)!}$ in row $i$ and column $j$ transforms, by Theorem 1.2, the chromatic table into the $(\operatorname{dim}(K) \times m(K))$-matrix

$$
\chi(K)\left([j]_{i}\right)_{1 \leq i, j \leq m(K)}=\left(\chi^{s}(K, i)\right)_{\substack{1 \leq s \leq \operatorname{dim}(K) \\ 1 \leq i \leq m(K)}}
$$

with the $m(K)$ values $\chi^{s}(K, i), 1 \leq i \leq m(K)$, of the $s$-chromatic polynomial in row $s$. This matrix of chromatic polynomial values appears also to have log-concave rows.

## 3. Chromatic uniqueness

In this section we briefly discuss to what extent simplicial complexes are determined by their chromatic polynomials. Proposition shows that the chromatic table of a simplicial complex determines its $f$-vector.

Definition 3.1. $K$ is chromatically unique if it is determined up to isomorphism by its chromatic table.
In Lemma 3.2 below, $K \amalg L$ is the disjoint union and $K \vee L$ the one-point union of $K$ and $L$. The proof is identical to the one for the similar statements about chromatic polynomials for simple graphs.

Lemma 3.2. If $K$ and $L$ are finite simplicial complexes then

$$
\chi^{s}(K \amalg L, r)=\chi^{s}(K, r) \chi^{s}(L, r), \quad \chi^{s}(K \vee L, r)=\frac{\chi^{s}(K, r) \chi^{s}(L, r)}{r}
$$

for all $r$ and all $s \geq 0$.
The two nonisomorphic simplicial complexes

are not chromatically unique as they have identical chromatic tables

$$
\left(\begin{array}{cccccc}
0 & 0 & 2 & 10 & 7 & 1 \\
0 & 15 & 73 & 62 & 15 & 1
\end{array}\right)
$$

by Lemma 3.2. (These two complexes are, however, PL-isomorphic.)
On the other hand, Proposition 2.34.(2) immediately implies that the $s$-skeleton of a full simplex is chromatically unique (in a very strong sense).

Proposition 3.3. If $K$ has the same s-chromatic polynomial as a full simplex $D[N]$, then $K$ and $D[N]$ have isomorphic s-skeleta.

Proof. If $K$ and $D[N]$ have the same $s$-chromatic polynomial for some $s \geq 1$, then $K$ has $N$ vertices (Corollary 2.26), and, since $\chi^{s}(K, N-s)=\chi^{s}(D[N], N-s)$, the $s$-skeleton of $K$ is isomorphic to the $s$-skeleton of the full simplex on $N$ vertices (Proposition 2.34.(2)).

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