CHROMATIC POLYNOMIALS OF SIMPLICIAL COMPLEXES

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ABSTRACT. In this note we consider *s*-chromatic polynomials of finite simplicial complexes. The *s*-chromatic polynomials of simplicial complexes are higher dimensional analogues of chromatic polynomials for graphs.

1. INTRODUCTION

Let K be a finite simplicial complex with vertex set $V(K) \neq \emptyset$ and let $r \ge 1$ and $s \ge 1$ be two natural numbers. A map col: $V(K) \rightarrow \{1, 2, ..., r\}$ is an (r, s)-coloring of K if there are no monochrome s-simplices in K [5]. We write $\chi^s(K, r)$ for the number of (r, s)-colorings of K.

Definition 1.1. The s-chromatic polynomial of K is the function $\chi^s(K, r)$ of r. The s-chromatic number of K, $\operatorname{chr}^s(K)$, is the minimal $r \ge 1$ with $\chi^s(K, r) > 0$.

The theorem below shows that $\chi^s(K, r)$ is indeed polynomial in r for fixed K and s. (By notational convention, $[r]_i = r(r-1)\cdots(r-i+1)$ is the *i*th falling factorial in r.)

Theorem 1.2. The s-chromatic polynomial of K is

$$\chi^{s}(K,r) = \sum_{i=chr^{s}(K)}^{|V(K)|} S(K,i,s)[r]_{i}$$

where S(K, i, s) is the number of partitions of V(K) into i blocks containing no s-simplex of K.

For s = 1, an (r, 1)-coloring of K is a usual graph coloring, $\chi^1(K, r)$ is the usual chromatic polynomial, and $chr^1(K)$ the usual chromatic number of the 1-skeleton of K. In general, $\chi^s(K, r)$ depends only on the s-skeleton of K. Although the higher s-chromatic polynomials for simplicial complexes are analogues of 1-chromatic polynomials for graphs we shall shortly see that there are structural differences between the cases s = and s > 1.

Figure 1 shows a triangulation MB of the Möbius band. To the left is a (5, 1)- and to the right a (2, 2)-coloring of MB. The chromatic polynomials and chromatic numbers ¹ of MB are

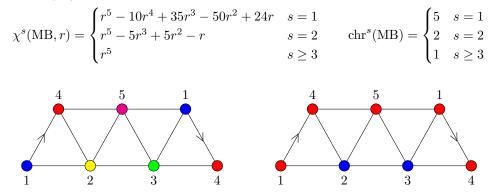


FIGURE 1. A (5,1)-coloring and a (2,2)-coloring of a 5-vertex triangulated Möbius band MB

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¹The computations behind the examples of this note were carried out in the computer algebra system Magma [3].

- 1.1. Notation. We shall use the following notation throughout the paper:
 - K: a finite simplicial complex
 - K^s : the *s*-skeleton of *K*
 - $F^{s}(K)$: the set of s-simplices K
 - #V or |V|: the number of elements in the finite set V
 - V(K): the vertex set $\bigcup K$ of K and m(K) = |V(K)| is the number of vertices in K
 - D[V]: the complete simplicial complex of all subsets of the finite set V
 - [m]: the finite set $\{1, \ldots, m\}$ of cardinality m
 - $[r]_i$: the *i*the falling factorial polynomial $[r]_i = i! \binom{r}{i}$ in r
 - P(a, b): the open interval (a, b) in the poset P

2. Three ways to the s-chromatic polynomial of a simplicial complex

In this section we present three different to approaches to the s-chromatic polynomial $\chi^s(K,r)$:

- Theorem 2.5 via 1-chromatic polynomials of graphs;
- Theorem 2.25 via the Möbius function for the *s*-chromatic lattice;
- Theorem 1.2 via the simplicial s-Stirling numbers of the second kind.

2.1. Block-connected s-independent vertex partitions. Let $s \ge 1$ be a natural number.

Definition 2.1. Let $B \subset V(K)$ be a set of vertices of K. Then

- B is s-independent if B contains no s-simplex of K;
- B is connected if $K \cap D[B]$ is a connected simplicial complex;
- the connected components of B are the maximal connected subsets of B.

Definition 2.2. Let P be a partition of V(K).

- The graph $G_0(P)$ of P is the simple graph whose vertices are the blocks of P and with two blocks connected by and edge if their union is connected;
- The block-connected refinement P_0 of P is the refinement whose blocks are the connected components of the blocks of P;
- P is block-connected if the blocks of P are connected (ie if $P = P_0$).

Lemma 2.3. Let P be a partition of V(K). If two different blocks of the block-connected refinement P_0 are connected by an edge in the graph $G_0(P_0)$ of P then they lie in different blocks of P.

Proof. The connected components of the blocks of P are maximal with respect to connectedness.

Definition 2.4. BCP^s(K) is the set of all block-connected s-independent partitions of V(K).

Recall that $\chi^1(G_0(P), r)$ is the 1-chromatic polynomial of the simple graph $G_0(P)$ of the partition P.

Theorem 2.5. The s-chromatic polynomial for K is the sum

$$\chi^s(K,r) = \sum_{P \in \mathrm{BCP}^s(K)} \chi^1(G_0(P),r)$$

of the 1-chromatic polynomials and the s-chromatic number of K is the minimum

$$\operatorname{chr}^{s}(K) = \min_{P \in \operatorname{BCP}^{s}(K)} \operatorname{chr}^{1}(G_{0}(P))$$

of the 1-chromatic numbers for the graphs of all the block-connected s-independent partitions of V(K).

Proof. Let col: $V(K) \to [r]$ be an (r, s)-coloring of K. The monochrome partition P(col) of V(K) is the sindependent partition whose blocks are the nonempty monochrome sets of vertices $\{\text{col} = i\}$ for $i \in [r]$. The block-connected refinement $P(\text{col})_0$ of the monochrome partition is a block-connected s-independent partition of K. The original coloring col of K is also a coloring of the graph $G_0(P(\text{col})_0)$ of $P(\text{col})_0$ for, by Lemma 2.3, distinct vertices of 1-simplices of this graph have distinct colors. We have shown that any (r, s)-coloring col of K induces an (r, 1)-coloring col₀ of the graph $G_0(P(\text{col})_0)$ of the block-connected refinement of the monochrome partition.

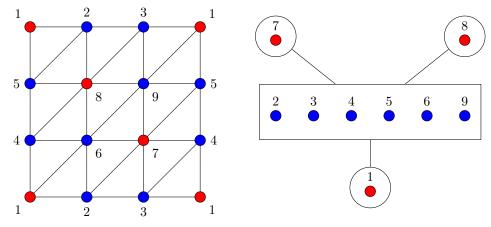
Let $P \in BCP^s(K)$ be a block-connected s-independent partition of V(K) and $col_0: P \to \{1, \ldots, r\}$ an (r, 1)coloring of its graph $G_0(P)$. Then col_0 determines a map $col: V(K) \to [r]$ that is constant on the blocks of P. An s-simplex of K can not be monochrome under col as it intersects at least two different blocks of P connected by an edge of $G_0(P)$. Thus col is an (r, s)-coloring of K.

These two constructions are inverses of each other.

Remark 2.6 (The minimal block-connected s-independent partition). Let $C_0 = \{\{v\} \mid v \in V(K)\}$ be the blockconnected s-independent partition of V(K) whose blocks are singletons. The graph $G_0(C_0) = K^1$ is the 1-skeleton of K. Thus the 1-chromatic polynomial of the 1-skeleton of K is always one of the polynomials in the sum of Theorem 2.5. If K is 1-dimensional, BCP¹(K) consists only of the partition C_0 and Theorem 2.5 simply says that the 1-chromatic polynomial of a simplicial complex is the 1-chromatic polynomial of its 1-skeleton.

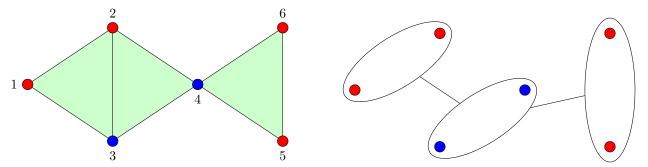
Example 2.7 (The block-connected 2-independent partitions for D[3]). The 2-simplex D[3] has 4 block-connected 2-independent partitions C_0 , {{1}, {2,3}}, {{2}, {1,3}}, and {{3}, {1,3}}. The graph of C_0 is the complete graph K_3 , the 1-skeleton of D[3]. The graphs of the other three partitions are all the complete graph K_2 . Thus the 2-chromatic polynomial of D[3] is $\chi^2(D[3], r) = \chi^1(K_3, r) + 3\chi^1(K_2, r) = [r]_3 + 3[r]_2 = [r]_2(r+1) = r^3 - r$ and the 2-chromatic number is $\operatorname{chr}^2(D[3]) = 2$.

Example 2.8 (A (2, 2)-coloring and the graph of the block-connected refinement of its monochrome partition). The picture below illustrates a (2, 2)-coloring of a 9-vertex triangulation of the torus



and its corresponding graph. There are 6937 block-connected partitions of the vertex set, and 3 of them has the graph shown above. The 2-chromatic polynomial is $21[r]_2 + 742[r]_3 + 3747[r]_4 + 4908[r]_5 + 2295[r]_6 + 444[r]_7 + 36[r]_8 + [r]_9 = [r]_2(r^7 + r^6 - 17r^5 + 10r^4 + 82r^3 - 116r^2 - 23r + 67)$ and the 2-chromatic number is 2.

Example 2.9 (The (r, 2)-colorings of a simplicial complex K). Let K be the pure 2-dimensional complex with facets $F^2(K) = \{\{1, 2, 3\}, \{2, 3, 4\}, \{4, 5, 6\}\}$.



The picture shows a (2, 2)-coloring of K and the corresponding (2, 1)-coloring of the associated graph, $G_0(P_0)$, the block connected refinement of the monochrome partition $P = \{\{1, 2, 5, 6\}, \{3, 4\}\}$. Table 1 shows the graphs $G_0(P)$ for all block connected partitions $P \in BCP^2(K)$. For each graph, the table records its 1-chromatic polynomial and its 1-chromatic number. The 2-chromatic polynomial of K is $\chi^2(K, 2) = 15[r]_2 + 73[r]_3 + 62[r]_4 + 15[r]_5 + [r]_6 = [r]_2(r-1)(r+1)(r^2+r-1)$ and the 2-chromatic number is $chr^2(K) = 2$.

Example 2.10 (The (r, 2)-colorings of the Möbius band). The set BCP²(MB) of block-connected 2-independent partitions of the triangulated Möbius band MB (Figure 1) has 36 elements. There are 5, 5, 15, 10, 1 partitions in BCP²(MB) realizing the partitions [3, 2], [3, 1, 1], [2, 2, 1], [2, 1, 1, 1], [1, 1, 1, 1, 1] of the integer |V(MB)| = 5. All associated graphs are complete graphs. This yields the 2-chromatic polynomial $\chi^2(MB, r) = 5[r]_2 + 20[r]_3 + 10[r]_4 + [r]_5 = [r]_2(r^3 + r^2 - 4r + 1) = r^5 - 5r^3 + 5r^2 - r$ and the 2-chromatic number is $chr^2(MB) = 2$.

Remark 2.11 (The S-chromatic polynomial of K). Let S be a set of connected subcomplexes of K. A set $B \subset V(K)$ of vertices is S-independent if B is not a superset of any member of S. Let $BCP^{S}(K)$ be the set of

TABLE 1. The graphs for the block-connected partitions in $BCP^2(K)$

S-independent partitions of V(K). An (r, S)-coloring is a map $V(K) \to \{1, \ldots, r\}$ such that #col(S) > 1 for all $S \in S$. The number of (r, S)-colorings of K is

$$\chi^{\mathcal{S}}(K,r) = \sum_{P \in \mathrm{BCP}^{\mathcal{S}}(K)} \chi^1(G_0(P),r)$$

as one sees by an obvious generalization of Theorem 2.5. An (r, s)-coloring of K is an (r, S)-coloring of K where $S = F^s(K)$ is the set of s-simplices.

2.2. The s-chromatic linear program. Read [9, §10] explains how to construct a linear program with minimal value equal to the s-chromatic number $chr^{s}(K)$ of K.

Definition 2.12. $M^{s}(K)$ is the set of all maximal s-independent subsets of V(K).

Let A be the $(m(K) \times |M^s(K)|)$ -matrix

$$A(v,M) = \begin{cases} 1 & v \in M \\ 0 & v \notin M \end{cases}$$

recording which vertices $v \in V(K)$ belong to which maximal s-independent sets $M \in M^{s}(K)$. Now the s-chromatic number

$$\mathrm{chr}^{s}(K) = \min\{\sum_{M \in M^{s}(K)} x(M) \mid x \colon M^{s}(K) \to \{0,1\}, \forall v \in V(K) \colon \sum_{M \in M^{s}(K)} A(v,M)x(M) \ge 1\}$$

is the minimal value of the objective function $\sum_{M \in M^s(K)} x(M)$ in $|M^s(K)|$ variables $x \colon M^s(K) \to \{0, 1\}$, taking values 0 or 1, and m(K) constraints $\sum_{M \in M^s(K)} A(v, M)x(M) \ge 1, v \in V(K)$.

2.3. The s-chromatic lattice. Our approach here simply follows Rota's classical method for computing chromatic polynomials from Möbius functions of lattices [10, §9]. We need some terminology in order to characterize the monochrome loci for colorings of K. Recall that $F^{s}(K)$ is the set of s-simplices of K.

Definition 2.13. Let $S \subset F^s(K)$ be a set of s-simplices of K.

- The equivalence relation \sim is the smallest equivalence relation in S such that $s_1 \cap s_2 \neq \emptyset \Longrightarrow s_1 \sim s_2$ for all $s_1, s_2 \in S$;
- the connected components of S are the equivalence classes under \sim ;
- $\pi_0(S)$ is the set of connected components of S;
- S is connected if it has at most one component;
- $V(S) = \bigcup S$ is the vertex set of S
- $\pi(S)$ is the partition of V(K) whose blocks are the vertex sets of the connected components of S together with the singleton blocks $\{v\}, v \in V(K) - V(S)$, of vertices not in any simplex in S;

• S is closed if S contains any s-simplex in K contained in the vertex set of S, ie if

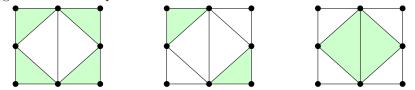
$$\{\sigma \in F^s(K) \mid \sigma \subset V(S)\} = S$$

• the closure of S is the smallest closed set of s-simplices containing S.

For instance, the empty set $S = \emptyset$ of 0 s-simplices is connected with 0 connected components. If K = D[4], the set $\{\{1,2\},\{2,4\}\}$ of 1-simplices is connected while $\{\{1,2\},\{3,4\}\}$ has the two components $\{\{1,2\}\}$ and $\{\{3,4\}\}$.

A set of s-simplices is closed if and only if it equals its closure. For instance in $F^2(D[5])$, the set $\{\{1, 2, 3\}, \{3, 4, 5\}\}$ is not closed because its closure is the set of all 2-simplices in D[5]. The empty set of s-simplices, any set of just one s-simplex, and any set of disjoint s-simplices are closed.

In this picture the green set of 2-simplices is



connected and not closed, closed and not connected, closed and connected, respectively.

The partition $\pi(S)$ has $|\pi(S)| = |\pi_0(S)| + m(K) - |V(S)|$ blocks.

Lemma 2.14. Let S be a set of s-simplices in K and S_0 a connected component of S. Then S_0 is closed if and only if

$$[\sigma \in F^s(K) \mid \sigma \subset V(S_0)\} \subset S$$

Proof. Since the condition is certainly necessary we only need to see that it is sufficient. Let σ be an *s*-simplex in K with all its vertices in $V(S_0)$. Then σ lies in S by assumption. But σ is equivalent to all elements of the equivalence class S_0 . Thus $\sigma \in S_0$.

Lemma 2.15. Let S and T be sets of s-simplices in K.

- (1) If S and T are closed, so is $S \cap T$.
- (2) If S and T have closed connected components, so does $S \cap T$

Proof. (1) Let σ be an s-simplex of K and suppose that $\sigma \subset V(S \cap T)$. Then $\sigma \subset V(S)$ an $\sigma \subset V(T)$ so that $\sigma \in S$ and $\sigma \in T$ as S and T are closed.

(2) Let R be a connected component of $S \cap T$. Let S_0 be the connected component of S containing R and T_0 be the connected component of T containing R. Then $R \subset S_0 \cap T_0$. Suppose that $\sigma \in F^s(K)$ is an s-simplex with $\sigma \subset V(R)$. Then $\sigma \subset V(S_0 \cap T_0)$ so $\sigma \in S_0 \cap T_0$ by (1) as the connected components S_0 and T_0 are assumed to be closed. In particular, $\sigma \in S \cap T$. According to Lemma 2.14, the connected component R is closed.

Definition 2.16. The s-chromatic lattice of K is the set $L^{s}(K)$ of all subsets of $F^{s}(K)$ with closed connected components. $L^{s}(K)$ is a partially ordered by set inclusion.

The set $L^{s}(K)$ contains the empty set \emptyset of s-simplices and the set $F^{s}(K)$ of all s-simplices. These two elements of $L^{s}(K)$ are distinct when K has dimension at least s.

Corollary 2.17. $L^{s}(K)$ is a finite lattice with $\widehat{0} = \emptyset$, $\widehat{1} = F^{s}(K)$, and meet $S \wedge T = S \cap T$.

Proof. If $S, T \in L^s(K)$ then $S \cap T$ is also in $L^s(K)$ by Lemma 2.15 and this is clearly the greatest lower bound of S and T. It is now a standard result that $L^s(K)$ is a finite lattice [12, Proposition 3.3.1]. The join $S \vee T$ of $S, T \in L^s(K)$ is the intersection of all supersets $U \in L^s(K)$ of $S \cup T$.

Example 2.18 (The s-chromatic lattice $L^s(D[m])$). The closed and connected elements of the s-chromatic lattice $L^s(D[m])$ of the complete simplex D[m] on m > s vertices are \emptyset and the $\binom{m}{k}$ sets $F^s(D[k])$ of all s-simplices in the subcomplexes D[k] for $s < k \le m$. The map $S \to \pi(S)$ is an isomorphism between the lattice $L^s(D[m])$ and the lattice, ordered by refinement, of all partitions of the set [m] into blocks of size > s or 1. The least element, $\widehat{0} = (1) \cdots (m)$, is the partition with m blocks and the greatest element, $\widehat{1} = (1 \cdots m)$, the partition with 1 block. $L^s(D[m])$ is not a graded lattice $[12, p \ 99]$ in general when $s \ge 2$. To see this, observe that the 2-chromatic lattices $L^2(D[3])$, $L^2(D[4])$, and $L^2(D[4])$ are graded but the lattice $L^2(D[6])$ is not graded as it contains two maximal chains

$$\widehat{0} = (1)(2)(3)(4)(5)(6) < (123)(4)(5)(6) < (1234)(5)(6) < (12345)(6) < (123456) = \widehat{1}$$

$$\widehat{0} = (1)(2)(3)(4)(5)(6) < (123)(4)(5)(6) < (123)(456) < (123456) = \widehat{1}$$

of unequal length. In contrast, the 1-chromatic lattice of any finite simplicial complex is always graded and even geometric [10, §9, Lemma 1].

Remark 2.19 (The Möbius function for the *s*-chromatic lattices $L^{s}(D[m])$). Our discussion of the Möbius function for the lattice $L^{s}(D[m])$ echoes the exposition of the Möbius function for the geometric lattice $L^{1}(D[m])$ of all partitions from [12, Example 3.10.4].

Let $w \colon [m] \to \mathbf{N}$ be a function that to every element of [m] associates a natural number, thought of as a weight function. We write $w = 1^{i_1} 2^{i_2} \cdots r^{i_r}$, or something similar, for the weight function w defined on the set [m] of cardinality $m = \sum_j i_j$ and mapping i_j elements to j for $1 \le j \le r$. The map w extends to a map, also called w, defined on the set of all nonempty subsets X of [m] given by $w(X) = \sum_{x \in X} w(x)$. Let $L_m^s(w)$ be the lattice of all partitions of the set [m] into blocks X that are singletons or have weight w(X) > s. The non-singleton blocks of the meet $\sigma \land \tau$ of two partitions $\sigma, \tau \in L_m^s(w)$ are the subsets of weight > s of the form $S \cap T$ where S is a block in σ and T a block in τ . Write $\mu_m^s(w)$ for the Möbius function of $L_m^s(w)$.

In particular, $L_m^s(1^m)$ is a synonym for $L^s(D[m])$ and we are primarily interested in the Möbius function $\mu_m^s(1^m)$ of the uniform weight $w = 1^m$. However, the computation of this Möbius function will involve the Möbius functions of other weights as well. We shall therefore discuss the Möbius functions $\mu_m^s(w)$ for general weight functions w.

Suppose that $\sigma \in L^s_m(w)$, $\sigma < 1$, is a partition of [m] into singleton blocks or blocks of weight > s. Let $w(\sigma)$ be the restriction of w to the set of blocks of σ . Thus $w(\sigma)(X) = \sum_{x \in X} w(x)$ for any block X of σ . Then the interval

$$L_m^s(w) \supset [\sigma, 1] = L_{|\sigma|}^s(w(\sigma))$$

so that $\mu_m^s(w)(\sigma, \widehat{1}) = \mu_{|\sigma|}^s(w(\sigma))(\widehat{0}, \widehat{1})$. More generally, suppose that $\sigma < \tau$ for some $\tau \in L_m^s(w)$. Assume that the partition τ has blocks τ_j . Let σ_j be the set of those blocks of σ that intersect the block τ_j of τ . Let $w(\sigma_j)$ be the restriction of $w(\sigma)$ to σ_j . Then the interval

$$L_m^s(w) \supset [\sigma, \tau] = \prod_j L_{|\sigma_j|}^s(w(\sigma_j))$$

and therefore the value of the Möbius function on the pair (σ, τ)

$$\mu_m^s(w)(\sigma,\tau) = \prod_j \mu_{|\sigma_j|}^s(w(\sigma_j))(\widehat{0},\widehat{1})$$

by the product theorem for Möbius functions [12, Proposition 3.8.2]. We conclude that the complete Möbius functions on all the lattices $L_m^s(w)$, are actually determined by the values $\mu_m^s(w)(\hat{0},\hat{1})$ of these Möbius functions on just $(\hat{0},\hat{1})$. See Equation (2.36) for more information about these Euler characteristics.

For the following it is convenient to name the elements of the domain [m] of w so that the element m carries minimal weight. Assume that $a_m = (1 \cdots m - 1)(m)$ is an element of $L_m^s(w)$, it that $w(1) + \cdots + w(m - 1) > s$. We shall determine the set of lattice elements x with $x \wedge a_m = \hat{0}$. There is only one solution to this equation with $x \leq a_m$ and that is $x = \hat{0}$. As the other solutions satisfy $x \leq a_m$, they must have a block that contains m and at least one other element. It follows that the solutions $x \neq \hat{0}$ are all elements of the form

$$x = (x_1 \cdots x_t m)(\cdot) \cdots (\cdot) \text{ with } \begin{cases} w(x_1) > s - w(m) & t = 1\\ s \ge w(x_1) + \cdots + w(x_t) > s - w(m) & t > 1 \end{cases}$$

where all blocks but the unique block containing m are singletons. There are t+1 elements in the block containing m where t is some number in the range $1 \le t \le s$. (All the solutions $x \ne \hat{0}$ are atoms in the lattice $L_m^s(w)$.) Since we are in a lattice, the Möbius function satisfies the equation [12, Corollary 3.9.3]

$$\mu_m^s(w)(\widehat{0},\widehat{1}) = -\sum_{\substack{x \wedge a_m = \widehat{0} \\ x \neq \widehat{0}}} \mu_m^s(w)(x,\widehat{1})$$

which translates to

$$(2.20) \quad \mu_m^s(w)(\widehat{0},\widehat{1}) = -\sum_{\substack{x \wedge a_m = \widehat{0} \\ x \neq \widehat{0}}} \mu_{|x|}^s(w(x))(\widehat{0},\widehat{1}) = -\sum_{\substack{x \wedge a_m = \widehat{0} \\ x \neq \widehat{0}}} \mu_{m-1}^s(w(x_1m)w(\cdot) \cdots w(\cdot))(\widehat{0},\widehat{1}) - \sum_{\substack{1 < t \le s \\ s \ge w(x_1) + \cdots + w(x_t) > s - w(m)}} \sum_{\substack{t \le x_1, \dots, x_t \le m - 1 \\ w(x_1) > s - w(m)}} \mu_{m-t}^s(w(x_1 \cdots x_tm))w(\cdot) \cdots w(\cdot))(\widehat{0},\widehat{1}) = -\sum_{\substack{t \le x_1, \dots, x_t \le m - 1 \\ w(x_1) > s - w(m)}} \mu_{m-t}^s(w(x_1 \cdots x_tm))w(\cdot) \cdots w(\cdot))(\widehat{0},\widehat{1}) = -\sum_{\substack{t \le x_1, \dots, x_t \le m - 1 \\ w(x_1) > s - w(m)}} \mu_{m-t}^s(w(x_1 \cdots x_tm))w(\cdot) \cdots w(\cdot))(\widehat{0},\widehat{1}) = -\sum_{\substack{t \le x_1, \dots, x_t \le m - 1 \\ w(x_1) > s - w(m)}} \mu_{m-t}^s(w(x_1 \cdots x_tm))w(\cdot) \cdots w(\cdot))(\widehat{0},\widehat{1}) = -\sum_{\substack{t \le x_1, \dots, x_t \le m - 1 \\ w(x_1) > s - w(m)}} \mu_{m-t}^s(w(x_1 \cdots x_tm))w(\cdot) \cdots w(\cdot))(\widehat{0},\widehat{1}) = -\sum_{\substack{t \le x_1, \dots, x_t \le m - 1 \\ w(x_1) > s - w(m)}} \mu_{m-t}^s(w(x_1 \cdots x_tm))w(\cdot) \cdots w(\cdot))(\widehat{0},\widehat{1}) = -\sum_{\substack{t \le x_1, \dots, x_t \le m - 1 \\ w(x_1) > s - w(m)}} \mu_{m-t}^s(w(x_1 \cdots x_tm))w(\cdot) \cdots w(\cdot))(\widehat{0},\widehat{1}) = -\sum_{\substack{t \le x_1, \dots, x_t \le m - 1 \\ w(x_1) > s - w(m)}} \mu_{m-t}^s(w(x_1 \cdots x_tm))w(\cdot) \cdots w(\cdot))(\widehat{0},\widehat{1}) = -\sum_{\substack{t \le x_1, \dots, x_t \le m - 1 \\ w(x_1) > s - w(m)}} \mu_{m-t}^s(w(x_1 \cdots x_tm))w(\cdot) \cdots w(\cdot))(\widehat{0},\widehat{1}) = -\sum_{\substack{t \le x_1, \dots, x_t \le m - 1 \\ w(x_1) > s - w(m)}} \mu_{m-t}^s(w(x_1 \cdots x_tm))w(\cdot) \cdots w(\cdot))(\widehat{0},\widehat{1}) = -\sum_{\substack{t \le x_1, \dots, x_t \le m - 1 \\ w(x_1) > s - w(m)}} \mu_{m-t}^s(w(x_1 \cdots x_tm))w(\cdot) \cdots w(\cdot))(\widehat{0},\widehat{1}) = -\sum_{\substack{t \le x_1, \dots, x_t \le m - 1 \\ w(x_1) > w(x_1 \cdots x_tm)} \mu_{m-t}^s(w(x_1 \cdots x_tm))w(\cdot) \cdots w(\cdot))(\widehat{0},\widehat{1}) = -\sum_{\substack{t \ge x_1, \dots, x_t \le m - 1 \\ w(x_1) > w(x_1 \cdots x_tm)} \mu_{m-t}^s(w(x_1 \cdots x_tm))w(\cdot) \cdots w(\cdot))(\widehat{0},\widehat{1}) = -\sum_{\substack{t \ge x_1, \dots, x_t \ge w_tm} \mu_{m-t}^s(w(x_1 \cdots x_tm))w(\cdot) \cdots w(\cdot))(\widehat{0},\widehat{1}) = -\sum_{\substack{t \ge x_1, \dots, x_t \ge w_tm} \mu_{m-t}^s(w(x_1 \cdots x_tm))w(\cdot) \cdots w(\cdot))(\widehat{0},\widehat{1})$$

This describes a recursive procedure for computing all values of the Möbius function on the weight lattices $L_m^s(w)$.

As an illustration we compute $\mu_6^2(1^6)(\widehat{0},\widehat{1})$. Using (2.20) twice gives

$$\mu_6^2(1^6)(\widehat{0},\widehat{1}) = -10\mu_4^2(3111)(\widehat{0},\widehat{1}) = 10(\mu_3^2(411)(\widehat{0},\widehat{1}) + \mu_2^2(33)(\widehat{0},\widehat{1}))$$

The lattices $L_4^2(411)$ and $L_2^2(33)$ have 4 and 2 elements, respectively, and they look like

so that $\mu_3^2(411)(\hat{0},\hat{1}) = 1$ and $\mu_2^2(33)(\hat{0},\hat{1}) = -1$. Therefore $\mu_6^2(1^6)(\hat{0},\hat{1}) = 0$.

We remind the reader of the well-known fact that $\mu_m^s(w)(\widehat{0},\widehat{1})$ is the reduced Euler characteristic of the open interval $L_m^s(w)(\widehat{0},\widehat{1})$ between $\widehat{0}$ and $\widehat{1}$ in the lattice $L_m^s(w)$.

Proposition 2.21. [10, §6] [12, Proposition 3.8.5] Let x < y be two elements in a finite poset. The value of the Möbius function on the pair (x, y) is the reduced Euler characteristic of the open interval (x, y).

Proof. Write μ be the Möbius function of P and E for Euler characteristic. The closed interval from x to y has Euler characteristic 1 since it has a smallest element. Thus

$$1 = \mathcal{E}([x,y]) = \sum_{a,b\in[x,y]} \mu(a,b) = \sum_{a,b\in(x,y)} \mu(a,b) + \sum_{a\in[x,y]} \mu(a,y) + \sum_{b\in[x,y]} \mu(x,b) - \mu(x,y) = \mathcal{E}((x,y)) + 0 + 0 - \mu(x,y) = \mathcal{E}((x,y)) - \mu(x,y)$$
$$= \mathcal{E}((x,y)) + 0 + 0 - \mu(x,y) = \mathcal{E}((x,y)) - \mu(x,y)$$

or $\mu(x, y) = E((x, y)).$

For $1 \le s \le m+1$ let B(m,s) be the graded poset of nonempty subsets of [m] of cardinality less than s. **Lemma 2.22.** The reduced Euler characteristic of B(m, s) is

$$\widetilde{E}(B(m,s)) = (-1)^s \binom{m-1}{s-1}, \qquad 1 \le s \le m+1$$

Proof. It is rather easy to get the recurrence relation

 $E(B(m \ 2)) - m$

$$E(B(m,2)) = m$$

$$E(B(m,s)) = E(B(m,s-1)) + \binom{m}{s-1} \sum_{j=1}^{s-1} (-1)^{s-1-j} \binom{s-1}{j}, \qquad 2 < s < 2+m$$

Since the sum of binomial coefficients has value $(-1)^s$, we get the recurrence relation

$$\begin{split} \widetilde{E}(B(m,2)) &= m - 1 \\ \widetilde{E}(B(m,s)) &= \widetilde{E}(B(m,s-1)) + (-1)^s \binom{m}{s-1}, \qquad 2 < s < 2 + m \end{split}$$

for the reduced Euler characteristic. The claim of the lemma follows immediately.

Example 2.23 (Reduced Euler characteristics of the *s*-chromatic lattice intervals $L_m^s(w)(\widehat{0},\widehat{1})$). The reduced Euler characteristics $\mu_m^s(1^m)(\widehat{0},\widehat{1}) = \widehat{E}(L_m^s(1^m)(\widehat{0},\widehat{1})), m \ge s+2$, for $s = 1, 2, \dots, 8$ are

 $3, -6, 0, 90, -630, 2520, 0, -113400, 1247400, -7484400, 0, 681080400, -10216206000, 81729648000, \ldots$

 $6, -21, 56, -126, 252, -924, 11088, -126126, 1093092, -7693686, 46414368, -254438184, 1492322832, \ldots$

 $7, -28, 84, -210, 462, -924, 0, 42042, -630630, 6390384, -51459408, 351639288, -2118412296, 11406835440\ldots$

 $8, -36, 120, -330, 792, -1716, 3432, -12870, 205920, -3150576, 35706528, -322583976, 2460949920\ldots$

 $9, -45, 165, -495, 1287, -3003, 6435, -12870, 0, 787644, -14965236, 191222460, -1920538620 \dots$

The first sequence, $\mu_m^1(1^m)(\widehat{0},\widehat{1}), m \ge 2$, is the sequence $(-1)^{m-1}(m-1)!$ of reduced Euler characteristics of the lattice of partitions of [m] [12, Example 3.10.4]. The second sequence, $\mu_m^2(1^m)(\widehat{0},\widehat{1}), m \ge 3$, seems to coincide with first terms of the sequence A009014 from The On-Line Encyclopedia of Integer Sequences (OES). The remaining 6 sequences apparently do not match any sequences of the OES.

The first s terms of these sequences are signed binomial coefficients. This is because the interval (0, 1) in $L^{s}(D[m])$ is isomorphic to the opposite of the poset B(m, m-s) when $s+2 \le m \le 2s+1$. Thus the reduced Euler characteristic

$$\mu_m^s(1^m)(\widehat{0},\widehat{1}) = \widetilde{E}(B(m,m-s)) = (-1)^{m-s} \binom{m-1}{s}, \qquad s+2 \le m \le 2s+1,$$

according to Lemma 2.22.

The first terms of the sequence $\mu_m^2(3^{1}1^{m-1})(\widehat{0},\widehat{1}), m \geq 3$, of reduced Euler characteristics of the weighted lattice intervals $L^2_m(3^{1}1^{m-1})(\widehat{0},\widehat{1}),$

seem to coincide up to sign with first terms of the sequence A009775 from OES. The sequence of reduced Euler characteristics $\mu_m^2(3^{2}1^{m-2})(\widehat{0},\widehat{1}), m \geq 3$, of the lattice interval $L_m^2(3^{2}1^{m-2})(\widehat{0},\widehat{1}),$

 $-778377600, 10897286400, -81729648000, -81729648000, 13894040160000, \ldots$

apparently does not match any sequence in the OES.

Define the s-monochrome set of a map col: $V(K) \to [r] = \{1, \ldots, r\}$ to be the set

$$M^{s}(\operatorname{col}) = \{ \sigma \in F^{s}(K) \mid |\operatorname{col}(\sigma)| = 1 \}$$

of all monochrome s-simplices in K. The map col is an (r, s)-coloring of K if and only if $M^s(col) = \emptyset$.

Lemma 2.24. The s-monochrome set $M^{s}(col)$ of any map $col: V(K) \to [r]$ is an element of the s-chromatic lattice $L^{s}(K).$

Proof. Let S be a connected component of $M^{s}(col)$. Since S is connected, all vertices in S have the same color. Let $\sigma \in F^s(K)$ be an s-simplex of K such that $\sigma \subset V(S)$. The σ is monochrome: $\sigma \in M^s(\text{col})$. By Lemma 2.14, S is closed.

Theorem 2.25. The number of (r, s)-colorings of K is

$$\chi^s(K,r) = \sum_{T \in L^s(K)} \mu(\widehat{0},T) r^{|\pi(T)|}$$

where μ the Möbius function for the s-chromatic lattice $L^{s}(K)$.

Proof. For any $B \in L^s(K)$, let $\chi(K, r, s, B)$ be the number of maps col: $V(K) \to [r]$ with $M^s(col) = B$. We want to determine $\chi(K, r, s, \emptyset) = \chi^r(s, K)$. For any $A \in L^s(K)$,

$$r^{|\pi(A)|} = \sum_{A \le B} \chi(K, r, s, B)$$

because there are $r^{|\pi_0(A)|}r^{m(K)-|V(A)|} = r^{|\pi(A)|}$ maps col: $V(K) \to [r]$ with $A \leq M^s(\text{col})$. Equivalently,

$$\sum_{A \leq B} \mu(A, B) r^{|\pi(B)|} = \chi(K, r, s, A)$$

by Möbius inversion [12, Proposition 3.7.1]. The statement of the theorem is the particular case of this formula where $A = \hat{0}$.

The defining rules for the Möbius function of the poset $L^{s}(K)$ [12, 3.7]

- $\mu(S,S) = 1$ for all $S \in L^s(K)$
- $\sum_{R \leq S \leq T} \mu(R, S) = 0$ when $R \lneq T$ $\mu(R, S) = 0$ when $R \nleq S$

imply that $\mu(\widehat{0}, \widehat{0}) = 1$ and $\mu(\widehat{0}, \{\sigma\}) = -1$ for every s-simplex $\sigma \in F^s(K)$.

Corollary 2.26. The highest degree terms of the s-chromatic polynomial are

$$\chi^{s}(K,r) = r^{m(K)} - f_{s}(K)r^{m(K)-s} + \cdots$$

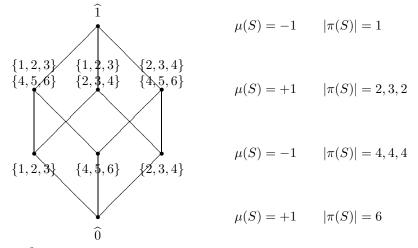
Thus the s-chromatic polynomial determines $f_0(K)$ and $f_s(K)$.

Proof. The s-chromatic polynomial is

$$\chi^{s}(K,r) = \mu(\widehat{0},\widehat{0})r^{f_{0}(K)} + \sum_{\sigma \in F^{s}(K)} \mu(\widehat{0}, \{\sigma\})r^{f_{0}(K)-s} + \cdots$$

where $\mu(\widehat{0}, \widehat{0}) = 1$ and $\mu(\widehat{0}, \{\sigma\}) = -1$ for all s-simplices σ of K.

Example 2.27. Consider the 2-dimensional complex K from Example 2.9. The 2-chromatic lattice $L^2(K)$ of K



consists of all subsets of $F^2(K)$. The 2-chromatic polynomial is

$$\chi^{2}(K,r) = r^{6} - r^{4} - r^{4} - r^{4} + r^{2} + r^{3} + r^{2} - r = r^{6} - 3r^{4} + r^{3} + 2r^{2} - r$$

K has $\chi^2(K,2) = 30$ (2,2)-colorings and $\chi^2(K,3) = 528$ (3,2)-colorings.

Example 2.28. The triangulation MB of the Möbius band with f-vector f(MB) = (5, 10, 5) shown in Figure 1 has the following (reduced) 2-chromatic lattice $L^2(MB) - \{\widehat{0}, \widehat{1}\}$

and 2-chromatic polynomial

$$\chi^2(MB, r) = r^5 - 5r^3 + 5r^2 - r$$

The lattice $L^2(MB)$ is graded but it is still not semi-modular [12, Proposition 3.3.2]: The meet and join of $a = \{\{2,3,5\}\}$ and $b = \{\{1,3,4\}\}$ are $a \wedge b = \widehat{0}$ and $a \vee b = \widehat{1}$. Thus a and b cover $a \wedge b$ but $a \vee b$ covers neither a nor b.

Example 2.29. Let MT be Möbius's minimal triangulation of the torus with f-vector f(MT) = (7, 21, 14) and P2 the triangulation of the projective plane with f-vector f(P2) = (1, 6, 15, 10) shown in Figure 2 (decorated with (3, 2)-colorings). The chromatic polynomials of these two simplicial complexes are

$$\begin{split} \chi^1(\mathrm{MT},r) &= [r]_7, \qquad \chi^2(\mathrm{MT},r) = r^7 - 14r^5 + 21r^4 + 7r^3 - 21r^2 + 6r\\ \chi^1(\mathrm{P2},r) &= [r]_6, \qquad \chi^2(\mathrm{P2},r) = r^6 - 10r^4 + 15r^3 - 6r^2 \end{split}$$

In both cases, the 1-skeleton is the complete graph on the vertex set. The chromatic numbers are $chr^{1}(MT) = 7$, $chr^{1}(P2) = 6$, and $chr^{2}(MT) = 3 = chr^{2}(P2)$.

The chromatic polynomials of simple graphs (the 1-chromatic polynomials of simplicial complexes) are known to have these properties:

- The coefficients are sign-alternating [10, §7, Corollary]
- The coefficients are log-concave (Definition 2.43) in absolute value [7]
- There are no negative roots and no roots between 0 and 1 [14]

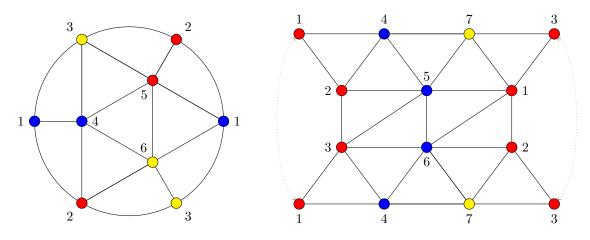


FIGURE 2. (3, 2)-colorings of P2 and MT

In contrast, the coefficients of the 2-chromatic polynomial

$$\chi^{2}(\mathrm{MT}, r) = r^{7} - 14r^{5} + 21r^{4} + 7r^{3} - 21r^{2} + 6r = [r]_{3}(r+1)(r^{3} + 2r^{2} - 9r + 3)$$

are not sign-alternating, not log-concave in absolute value, and the polynomial has a negative root and a root between 0 and 1.

2.4. The s-chromatic polynomial in falling factorial form. Theorem 1.2 provides an interpretation of the coefficients of the falling factorial $[r]_i$ in the s-chromatic polynomial of the simplicial complex K.

Definition 2.30. S(K,r,s) is the number of partitions of V(K) into r s-independent blocks.

We think of S(K, r, s) as an s-Stirling number of the second kind for the simplicial complex K. If $s > \dim(K)$, then there are no s-simplices in K and all partitions of V(K) are s-independent, so that S(K, r, s) is the Stirling number of the second kind S(m(K), r) [12, p 33]. We now explain the general relation between these simplicial Stirling numbers S(K, r, s) and the usual Stirling numbers of the second kind.

Define the s-monochrome set of a partition P of V(K) to be the set

 $M^{s}(P) = \{ \sigma \in F^{s}(K) \mid \sigma \text{ is contained in a block of } P \}$

of all s-simplices entirely contained in one of the blocks of P. The set $M^{s}(P)$ is an element of the s-chromatic lattice $L^{s}(K)$ by Lemma 2.24.

Theorem 2.31. The number of partitions of V(K) into r s-independent blocks is

$$S(K,r,s) = \sum_{T \in L^s(K)} \mu(\widehat{0},T) S(|\pi(T)|,r)$$

where μ the Möbius function for the s-chromatic lattice $L^{s}(K)$.

Proof. For any $B \in L^s(K)$, let S(K, r, s, B) be the number of partitions P of V(K) into r blocks with monochrome set $M^s(P) = B$. We want to determine $S(K, r, s, \emptyset) = S(K, r, s)$. For any $A \in L^s(K)$,

$$S(|\pi(A)|,r) = \sum_{A \leq B} S(K,r,s,B)$$

because there are $S(|\pi(A)|, r)$ partitions P of V(K) into r blocks with $A \leq M^{s}(P)$. Equivalently,

$$\sum_{A \leq B} \mu(A, B) S(|\pi(B)|, r) = S(K, r, s, A)$$

by Möbius inversion [12, Proposition 3.7.1]. The statement of the theorem is the particular case of this formula where $A = \hat{0}$.

Proof of Theorem 1.2. We simply follow the proof of the similar statement for chromatic polynomials for graphs [9, Theorem 15]. When $r \ge i$ we can get an (r, s)-coloring out of one of the S(K, i, s) partitions of V(K) into i s-independent blocks by choosing i out of the r colors and assigning them to the i blocks. There are $\binom{r}{i}$ ways of

choosing the *i* out of *r* colors and *i*! ways of coloring *i* blocks in *i* colors. The number of (r, s)-colorings of *K* in exactly *i* colors is thus

$$S(K, i, s) \binom{r}{i} i! = S(K, i, s)[r]_i$$

so that

$$\chi^{s}(K,r) = \sum_{i=1}^{m(K)} S(K,i,s)[r]_{i}$$

is the total number of (r, s)-colorings of K.

Corollary 2.32. The reduced Euler characteristic of the open interval $(\hat{0}, \hat{1})$ in s-chromatic lattice $L^{s}(K)$ is

$$\mu(L^{s}(K))(\widehat{0},\widehat{1}) = \sum_{i=\operatorname{chr}^{s}(K)}^{m(K)} (-1)^{i-1} (i-1)! S(K,i,s)$$

Proof. Equate the terms of degree 1 of the two expressions

(2.33)
$$\sum_{T \in L^s(K)} \mu(\widehat{0}, T) r^{|\pi(T)|} = \sum_{i=\operatorname{chr}^s(K)}^{m(K)} S(K, i, s)[r]_i$$

from Theorem 2.25 and Theorem 1.2 for the s-chromatic polynomial of K.

We observe that

$$\sum_{i} S(K, i, s)[r]_{i} = \sum_{i} \sum_{T} \mu(\widehat{0}, T) S(|\pi(T)|, i)[r]_{i} = \sum_{T} \mu(\widehat{0}, T) \sum_{i} S(|\pi(T)|, i)[r]_{i} = \sum_{T} \mu(\widehat{0}, T) r^{|\pi(T)|} S(|\pi(T)|, i)[r]_{i} = \sum_{T} \mu(\widehat{0}, T) r^{|\pi(T)|}$$

so that Theorem 2.31 implies Theorem 1.2.

The s-chromatic number of K is immediately visible with the s-chromatic polynomial in factorial form because

$$\operatorname{chr}^{s}(K) = \min\{i \mid S(K, i, s) \neq 0\}$$

is the lowest degree of the nonzero terms. The positive integer sequence

$$\chi^s(K, \operatorname{chr}^s(K)), \dots, \chi^s(K, m(K)) = 1$$

has no internal zeros. (If there is a partition of V(K) into r blocks not containing any s-simplex of K and r < m(K), then split one of the blocks with more than one vertex into two sub-blocks to get a partition of V(K) into r + 1blocks containing no s-simplices of K.)

The simplicial Stirling numbers satisfy the recurrence relations

$$S(K,r,s) = \sum_{\substack{\emptyset \subsetneq U \subseteq V(K) - \{v_0\}\\V(K) - U \text{ s-independent}}} S(K \cap D[U], r - 1, s), \qquad S(K, 1, s) = \begin{cases} 1 & s > \dim(K) \ge 0\\ 0 & \text{otherwise} \end{cases}$$

To see this, fix a vertex v_0 of K. Let P be partition of V(K) into r s-independent subsets. Let U_0 be the block containing v_0 . The other blocks in P form a partition P_0 of $K \cap D[V(K) - U_0]$ into r - 1 s-independent subsets. The map $P \leftrightarrow (P_0, U_0)$ is a bijection.

The familiar recurrence relation S(m,r) = S(m-1,r-1) + rS(m-1,r) for Stirling numbers of the second kind does not readily apply to simplicial Stirling numbers. The closest analogue may be

$$S(K, r, s) = S(K \cap D[V(K) - \{v_0\}], r - 1, s) + \sum_{P \in \mathcal{S}(K \cap D[V(K) - \{v_0\}], r, s)} |\{B \in P \mid B \cup \{v_0\} \text{ is s-independent in } K\}|$$

where v_0 is a vertex of K and $S(K \cap D[V(K) - \{v_0\}, r, s)$ is the set of partitions P of the vertex set of $K \cap D[V(K) - \{v_0\}]$ into r s-independent subsets.

Proposition 2.34. Let K be a subcomplex of L and assume that V(K) = V(L).

- (1) $S(K, r, s) \ge S(L, r, s)$ for all r.
- (2) If S(K, r, s) = S(L, r, s) for some r with $\frac{1}{s}(|V| 1) \le r \le |V| s$, then $K^s = L^s$.

TABLE 2. Chromatic tables for complete simplices D[m] for $m = 2, \ldots, 7$

Proof. (1) Let V be the vertex set of K and L. Write S(K, r, s) and S(L, r, s) for the set of partitions of V into r blocks containing no s-simplex of K or L, respectively. Then $S(L, r, s) \subseteq S(K, r, s)$ for all r and s. Thus $S(L, r, s) \leq S(K, r, s)$.

(2) Suppose that $\sigma \in F^s(L) - F^s(K)$ is an s-simplex of L that is not an s-simplex of K. Any partition of the form

$$\{\sigma\} \cup \tau, \qquad \tau \in \mathcal{S}(D[V-\sigma], r-1, s),$$

in $\mathcal{S}(K, r, s) - \mathcal{S}(L, r, s)$. The set $\mathcal{S}(D[V - \sigma], r - 1, s)$ is nonempty when

$$\operatorname{chr}^{s}(D[V-\sigma]) = \left\lceil \frac{|V|-s-1}{s} \right\rceil \le r-1 \le |V|-s-1$$

and thus S(K, r, s) is strictly greater than S(L, r, s) when $\frac{|V|-1}{s} \le r \le |V| - s$.

Remark 2.35 (S(K, r, s) for the complete simplex K = D[m]). For any finite set M, let S(M, r, s) stand for S(D[M], r, s) (Definition 2.30), the number of partitions of the set M into r blocks containing at most s elements. Let us even write S(m, r, s) in case $M = [m], m \ge 1, r, s \ge 0$. Clearly, S(m, r, s) is nonzero only when $m/s \le r \le m$. Also, S(m, r, s) = S(m, r) when r is among the s numbers $m - s + 1, \ldots, m$. The recurrence relation

$$S(m, r, s) = \sum_{j=m-s}^{m-1} {\binom{m-1}{j}} S(j, r-1, s)$$

can be used to compute these numbers. Table 2 shows S(m, r, s) for small m; the number S(m, r, s) is in row s and column r in the chromatic table (Definition 2.39) for D[m]. All the red numbers are usual Stirling numbers of the second kind.

According to Theorem 1.2, the numbers S(m, r, s) determine the s-chromatic polynomial in falling factorial form of the complete simplex on m vertices

$$\chi^{s}(D[m], r) = \sum_{i \in \lceil m/s \rceil}^{m} S(m, i, s)[r]_{i}$$

and, according to Corollary 2.32, they also determine the reduced Euler characteristic

$$\mu_m^s(1^m)(\widehat{0},\widehat{1}) = \sum_{i=\lceil m/s \rceil}^m (-1)^{i-1}(i-1)!S(m,i,s)$$

of the s-chromatic lattice $L^{s}(D[m])$.

More generally, if $w: M \to \mathbf{N}$ is a function on M with natural numbers as values, let S(M, w, r, s) be the number of partitions of M into admissible blocks, where we declare a block admissible if it is a singleton or it has weight at most s. (Then $S(m, r, s) = S([m], 1^m, r, s)$ occur when M = [m] and $w = 1^m$ places weight 1 on all elements.) Any such partition is a partition of M into blocks of weight at most s, and therefore $S(M, w, r, s) \leq S(\#M, r, s)$. In particular, S(M, w, r, s) is nonzero only when $\#M/s \leq r \leq \#M$. The recurrence relation

$$S(M, w, r, s) = \sum_{\substack{\emptyset \neq J \subset M - \{\max(M)\}\\M - J \text{ admissible}}} S(J, w | J, r - 1, s)$$

provides a means to compute these numbers.

The weighted version of Equation (2.33) for K = D[m],

$$\sum_{\sigma \in L^s_m(w)} \mu^s_m(w)(\widehat{0}, \sigma) r^{|\sigma|} = \sum_{i = \lceil m/s \rceil}^m S([m], w, i, s)[r]_i$$

implies, by equating coefficients of first degree terms, the expression

(2.36)
$$\mu_m^s(w)(\widehat{0},\widehat{1}) = \sum_{i=\lceil m/s \rceil}^m (-1)^{i-1} (i-1)! S([m], w, i, s)$$

for the Euler characteristic of the weighted lattice $L_m^s(w)$ from Remark 2.19.

Because any simplicial complex K is a subcomplex of the complete simplex D[m(K)] on its vertex set, we have (2.37) $S(m(K), r) \ge S(K, r, s) \ge S(m(K), r, s), \qquad 1 \le r \le m(K)$

Moreover, these inequalities are equalities for the s highest values $m(K) - s + 1, \ldots, m(K)$ of r. Thus the s terms of highest falling factorial degree in the s-chromatic polynomial of K

$$\chi^{s}(K,r) = \sum_{i=0}^{m(K)-s} S(K,i,s)[r]_{i} + \sum_{i=m(K)-s+1}^{m(K)} S(m(K),i)[r]_{i}$$

are given by the s Stirling numbers $S(m(K), m(K) - s + 1), \ldots, S(m(K), m(K))$ of the second kind. These coefficients depend only on the size of the vertex set of K. We shall next show that the coefficient number s + 1 counted from above, S(K, m(K) - s, s), informs about the number $f_s(K)$ of s-simplices in K.

Proposition 2.38. $S(K, m(K) - s, s) = S(m(K), m(K) - s) - f_s(K)$. If S(K, m(K) - s, s) = S(m(K), m(K) - s, s) then $K^s = D[m(K)]^s$.

Proof. The only partitions of the S(m, m-s) partitions of V(K) into m-s blocks that are not s-independent are those consisting of one s-simplex of K together with singleton blocks. If S(K, m(K) - s, s) = S(D[m(K)], m(K) - s, s) then $f_s(K) = f_s(D[m(K)])$ so $K^s = D[m(K)]^s$. (This is a special case of Proposition 2.34.(2).)

Definition 2.39. The chromatic table, $\chi(K)$, of K is the $(\dim(K) \times m(K))$ -table with S(K, r, s) in row s and column r.

This means that row s in the chromatic table lists the coefficients of the s-chromatic polynomial. The chromatic table of a 3-dimensional simplicial complex K, for instance, looks like this

	r = 1	r=2	• • •	r = m - 3	r = m - 2	r = m - 1	r = m
$S(K, \cdot, 1)$	S(K, 1, 1)	S(K, 2, 1)	• • •	S(K, m-3, 1)	S(K, m-2, 1)	$S(m,m-1) - f_1$	S(m,m) = 1
$S(K, \cdot, 2)$	S(K, 1, 2)	S(K, 2, 2)	• • •	S(K, m - 3, 2)	$S(m,m-2) - f_2$	S(m, m-1)	S(m,m) = 1
$S(K, \cdot, 3)$	S(K, 1, 3)	S(K,2,3)	• • •	$S(m,m-3) - f_3$	S(m,m-2)	S(m,m-1)	S(m,m) = 1

where the red entries in row s are Stirling numbers of the second kind S(m, r) for $m - s + 1 \le r \le m$, and the blue entry in row s is $S(m(K), m(K) - s) - f_s(K)$.

Example 2.40. The chromatic tables of the 2-dimensional simplicial complexes from Examples 2.9, 2.28, and 2.29 are

$$\chi(K) = \begin{pmatrix} 0 & 0 & 2 & 10 & 7 & 1 \\ 0 & 15 & 73 & 62 & 15 & 1 \end{pmatrix} \qquad \qquad \chi(MB) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 5 & 20 & 10 & 1 \end{pmatrix}$$
$$\chi(MT) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 84 & 231 & 126 & 21 & 1 \end{pmatrix} \qquad \qquad \chi(P2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 45 & 55 & 15 & 1 \end{pmatrix}$$

The red entries in column r are Stirling numbers S(m, r) and they are independent of the row index. The blue entry in row s and column m-s, which equals $S(m-s, s) - f_s(K)$, detects if K has maximal s-skeleton by Proposition 3.

Example 2.41. Let K = AS3 be Altshuler's peculiar triangulation of the 3-sphere with f-vector f = (10, 45, 70, 35)[1]. The 1-chromatic polynomial is $\chi^1(AS3, r) = [r]_{10}$ as K^1 is the complete graph on 10 vertices. The chromatic table is

$$\chi(\text{AS3}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1360 & 8475 & 10355 & 4200 & 680 & 45 & 1 \\ 0 & 26 & 4320 & 25915 & 38550 & 22152 & 5845 & 750 & 45 & 1 \end{pmatrix}$$

The blue numbers determine the f-vector

$$f(AS3) = (10, S(10, 9) - \chi(AS3)_{19}, S(10, 8) - \chi(AS3)_{28}, S(10, 7) - \chi(AS3)_{37})$$

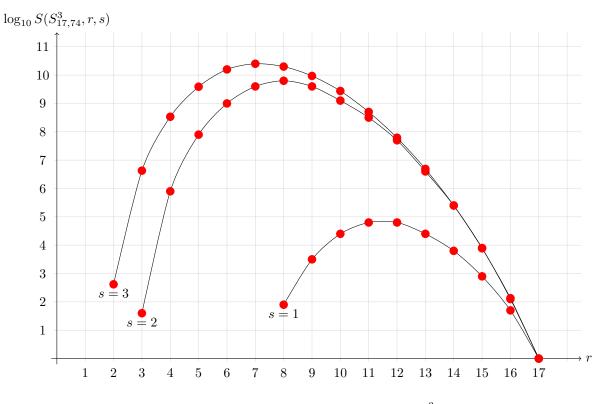


FIGURE 3. The simplicial Stirling numbers for $S^3_{17,74}$

The row numbers of the first nonzero term in each row tell us that $chr^1(AS3) = 10$, $chr^2(AS3) = 4$, and $chr^3(AS3) = 2$.

Example 2.42. The nonconstructible, nonshellable 3-sphere $S_{17,74}^3$, f = (17, 91, 148, 74), found by Lutz [8], has

	r = 1	r =	2 r = 3	r = 4	r = 5	r =	= 6	r = 7		r = 8	r = 9
			0								
s = 2	0	0	36	702475	82949364	107542	20155	38277665	87 549	93687086	3876597169
s = 3	0	42	2 4319865	338438489	3903094622	2 142923	81565	229468548	806 191	58310796	9202775199
			r = 10	r = 11	r = 12	r = 13	r = 14	r = 15	r = 16	r = 17	
	s	=1	23017	55285	54973	25941	6210	762	45	1	
	s	=2	1507939074	346346664	48855523	4302470	235026	5 7672	136	1	
	s	= 3	2708454744	507528561	61784524	4903589	249826	5 7820	136	1	

as its chromatic table. Figure 3 shows a semi-logarithmic plot of the simplicial Stirling numbers $S(S_{17,74}^3, r, s)$. The triangulation Σ_{16}^3 , f = (16, 106, 180, 90), of the Poincaré homology 3-sphere constructed by Björner and Lutz [2, Theorem 5] has

		r =	1 r = 2	r = 3	r = 4	<i>r</i> =	= 5	r = 6	r :	=7	r = 8	
s =	= 1	0	0	0	0	(0	0		0	0	
s =	= 2	0	0	0	4589	297_{-}	4411	69671411	3004	75213	44235454	17
s =	= 3	0	3	845561	70005500	70129	99653	215871650	8 28887	730959	20008115	01
		1	r = 9	r = 1	0 r =	11	r = 12	r = 13	r = 14	r = 15	r = 16	
		-	0	•	•		•					
	<i>s</i> =	= 2	292864435	100793	551 19546	606 2	2225261	150095	5840	120	1	
	<i>s</i> =	= 3	792553648	1905270	025 28730	0056 2	2750278	165530	6020	120	1	

as its chromatic table.

Observe that all the above chromatic tables have strictly log-concave rows.

Definition 2.43. [11] A finite sequence a_1, a_2, \ldots, a_N of $N \ge 3$ nonnegative integers is strictly log-concave if $a_{i-1}a_{i+1} < a_i^2$ for 1 < i < N (and log-concave if $a_{i-1}a_{i+1} \le a_i^2$).

It has been conjectured that the sequence of coefficients of the 1-chromatic polynomial of a simple graph in falling factorial form, $r \to S(K, 1, r)$, chr¹(K) $\leq r \leq m(K)$, is log-concave [4, Conjecture 3.11]. More generally, one may ask

Question 2.44. Is the finite sequence of simplicial Stirling numbers

 $r \to S(K, r, s), \qquad \operatorname{chr}^s(K) \le r \le m(K),$

 \log -concave for fixed K and s?

This seems to be the right question to ask as it may be true for *all* the chromatic polynomials of a simplicial complex and we have seen that the absolute value of the coefficients of the *s*-chromatic polynomial are simply not log-concave for s > 1.

Note that the Stirling numbers of the second kind, which are upper bounds for the simplicial Stirling numbers S(K, r, s) by the inequalities (2.37), are log-concave in r [11, Corollary 2].

We shall now examine Question 2.44 on two spherical boundary complexes of cyclic *n*-polytopes.

Definition 2.45. $\partial CP(m,n)$, m > n, is the (n-1)-dimensional simplicial complex on the ordered set [m] with the following facets: An n-subset σ of [m] is a facet if and only if between any two elements of $[m] - \sigma$ there is an even number of vertices in σ .

By Gale's Evenness Theorem [6], the simplicial complex $\partial CP(m, n)$ triangulates the boundary of the cyclic *n*-polytope on *m* vertices. Thus $\partial CP(m, n)$ is a simplicial (n - 1)-sphere on *m* vertices and it is $\lfloor n/2 \rfloor$ -neighborly in the sense that $\partial CP(m, n)$ has the same *s*-skeleton as the full simplex on its vertex set when $s < \lfloor n/2 \rfloor$.

Example 2.46 (Cyclic polytopes with log-concave simplicial Stirling numbers of the second kind). Let $\partial CP(m, n)$ be the triangulated boundary of the cyclic polytope on m vertices in \mathbb{R}^n . The simplicial complex $\partial CP(m, n)$ is an m-vertex triangulation of S^{n-1} . The chromatic tables of the simplicial 3-spheres $\partial CP(m, 4)$ on m = 6, 7, 8, 9, 10 vertices are

	(0	0	0	0	0	1	(0	0	0	0	0	0	1	(0	0	0	0	0	0	0	1	
	0	1	21	47	15	1	0	0	28	147	112	21	1	0	1	50	393	582	226	28	1	
	$\left(0 \right)$	16	81	65	15	1/	$\int 0$	21	238	336	140	21	1/	$\left(0 \right)$	29	654	0 393 1533	1030	266	28	1/	
(0	0	0)	0	()	0	0	0	1	(0	0	0		0	0	0		0	0	0	1
0	0	94	4	1062	25	23	1719	408	3 36	1	0	1	180	29	980	10200) 107	77 4	225	680	45	1
$\left(0 \right)$	36	172	29	6471	65	91	2619	462	2 36	1/	$\sqrt{0}$	46	4445	25	960	38550	$\begin{array}{c} 0 \\ 0 & 107' \\ 0 & 221 \end{array}$	52 <mark>5</mark>	845	750	45	1/

All rows are strictly log-concave. As $\partial CP(m, 4)^1 = D[m]^1$, the 1-chromatic number $chr^1(\partial CP(m, 4)) = m$, and it is not difficult to see that the 2-chromatic number $chr^2(\partial CP(m, 4))$ is 2 if m is even and 3 if m is odd [5].

Right multiplication with the upper triangular matrix $([j]_i)_{1 \le i,j \le m(K)}$ with $[j]_i = {j \choose i} i! = \frac{j!}{(i-j)!}$ in row *i* and column *j* transforms, by Theorem 1.2, the chromatic table into the $(\dim(K) \times m(K))$ -matrix

$$\chi(K)([j]_i)_{1 \le i,j \le m(K)} = (\chi^s(K,i))_{\substack{1 \le s \le \dim(K) \\ 1 \le i \le m(K)}}$$

with the m(K) values $\chi^s(K, i)$, $1 \le i \le m(K)$, of the s-chromatic polynomial in row s. This matrix of chromatic polynomial values appears also to have log-concave rows.

3. Chromatic uniqueness

In this section we briefly discuss to what extent simplicial complexes are determined by their chromatic polynomials. Proposition shows that the chromatic table of a simplicial complex determines its f-vector.

Definition 3.1. K is chromatically unique if it is determined up to isomorphism by its chromatic table.

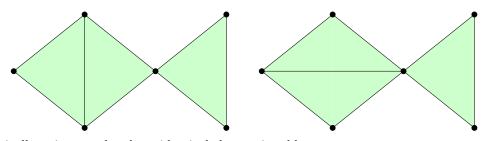
In Lemma 3.2 below, $K \amalg L$ is the disjoint union and $K \lor L$ the one-point union of K and L. The proof is identical to the one for the similar statements about chromatic polynomials for simple graphs.

Lemma 3.2. If K and L are finite simplicial complexes then

$$\chi^s(K\amalg L,r) = \chi^s(K,r)\chi^s(L,r), \qquad \chi^s(K\vee L,r) = \frac{\chi^s(K,r)\chi^s(L,r)}{r}$$

for all r and all $s \ge 0$.

The two nonisomorphic simplicial complexes



are not chromatically unique as they have identical chromatic tables

(0	0	2	10	7	1
$\left(0 \right)$	15	2 73	62	15	1)

by Lemma 3.2. (These two complexes are, however, PL-isomorphic.)

On the other hand, Proposition 2.34.(2) immediately implies that the *s*-skeleton of a full simplex is chromatically unique (in a very strong sense).

Proposition 3.3. If K has the same s-chromatic polynomial as a full simplex D[N], then K and D[N] have isomorphic s-skeleta.

Proof. If K and D[N] have the same s-chromatic polynomial for some $s \ge 1$, then K has N vertices (Corollary 2.26), and, since $\chi^s(K, N - s) = \chi^s(D[N], N - s)$, the s-skeleton of K is isomorphic to the s-skeleton of the full simplex on N vertices (Proposition 2.34.(2)).

References

- [1] A. Altshuler, A peculiar triangulation of the 3-sphere, Proc. Amer. Math. Soc. 54 (1976), 449–452. MR MR0397744 (53 #1602)
- [2] Anders Björner and Frank H. Lutz, Simplicial manifolds, bistellar flips and a 16-vertex triangulation of the Poincaré homology 3-sphere, Experiment. Math. 9 (2000), no. 2, 275–289. MR MR1780212 (2001h:57026)
- [3] Wieb Bosma, John Cannon, and Catherine Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput. 24 (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993). MR 1484478
- [4] Francesco Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update, Jerusalem combinatorics '93, Contemp. Math., vol. 178, Amer. Math. Soc., Providence, RI, 1994, pp. 71–89. MR 1310575 (95j:05026)
- [5] Natalia Dobrinskaya, Jesper M. Møller, and Dietrich Notbohm, Vertex colorings of simplicial complexes, arxive, 2010.
- [6] David Gale, Neighborly and cyclic polytopes, Proc. Sympos. Pure Math., Vol. VII, Amer. Math. Soc., Providence, R.I., 1963, pp. 225–232. MR MR0152944 (27 #2915)
- [7] June Huh, Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs, arXiv:1008.4749v3.
- [8] Frank H. Lutz, Small examples of nonconstructible simplicial balls and spheres, SIAM J. Discrete Math. 18 (2004), no. 1, 103–109 (electronic). MR MR2112491 (2005i:57028)
- [9] Ronald C. Read, An introduction to chromatic polynomials, J. Combinatorial Theory 4 (1968), 52-71. MR 0224505 (37 #104)
- [10] Gian-Carlo Rota, On the foundations of combinatorial theory. I. Theory of Möbius functions, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 2 (1964), 340–368 (1964). MR 0174487 (30 #4688)
- Bruce E. Sagan, Inductive and injective proofs of log concavity results, Discrete Math. 68 (1988), no. 2-3, 281–292. MR 926131 (89b:05009)
- [12] Richard P. Stanley, *Enumerative combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997, With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original. MR MR1442260 (98a:05001)
- [13] Yi Wang and Yeong-Nan Yeh, Log-concavity and LC-positivity, J. Combin. Theory Ser. A 114 (2007), no. 2, 195–210. MR 2293087 (2008g:11042)
- [14] D. R. Woodall, Zeros of chromatic polynomials, Combinatorial surveys (Proc. Sixth British Combinatorial Conf., Royal Holloway College, Egham, 1977), Academic Press, London, 1977, pp. 199–223. MR 0463010 (57 #2974)

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