# On star-wheel Ramsey numbers

Binlong Li\*

Ingo Schiermeyer<sup>†</sup>

Department of Applied Mathematics Northwestern Polytechnical University Xi'an, Shaanxi 710072, P.R. China European Centre of Excellence NTIS University of West Bohemia 306 14 Pilsen, Czech Republic; libinlong@mail.nwpu.edu.cn Institut für Diskrete Mathematik und Algebra Technische Universität Bergakademie Freiberg 09596 Freiberg, Germany Ingo.Schiermeyer@tu-freiberg.de

November 26, 2014

#### Abstract

For two given graphs  $G_1$  and  $G_2$ , the Ramsey number  $R(G_1, G_2)$  is the least integer r such that for every graph G on r vertices, either G contains a  $G_1$  or  $\overline{G}$  contains a  $G_2$ . In this note, we determined the Ramsey number  $R(K_{1,n}, W_m)$  for even m with  $n+2 \leq m \leq 2n-2$ , where  $W_m$  is the wheel on m+1 vertices, i.e., the graph obtained from a cycle  $C_m$  by adding a vertex v adjacent to all vertices of the  $C_m$ .

Keywords: Ramsey number; star; wheel

AMS Subject Classification: 05C55, 05D10

### 1 Introduction

Throughout this paper, all graphs are finite and simple. For a pair of graphs  $G_1$  and  $G_2$ , the Ramsey number  $R(G_1, G_2)$ , is defined as the smallest integer r such that for every graph G on r vertices, either G contains a  $G_1$  or  $\overline{G}$  contains a  $G_2$ , where  $\overline{G}$  is the complement of G. Note that  $R(G_1, G_2) = R(G_2, G_1)$ . We denote by  $P_n$   $(n \ge 1)$  and  $C_n$   $(n \ge 3)$  the path and cycle on n vertices, respectively. The bipartite graph  $K_{1,n}$   $(n \ge 2)$ 

<sup>\*</sup>The work is supported by NSFC (No. 11271300), the Doctorate Foundation of Northwestern Polytechnical University (No. cx201202) and the project NEXLIZ - CZ.1.07/2.3.00/30.0038, which is co-financed by the European Social Fund and the state budget of the Czech Republic.

<sup>&</sup>lt;sup>†</sup>Research was partly supported by the DAAD-PPP project "Rainbow connection and cycles in graphs" with project-id 56268242.

is called a *star*. The *wheel*  $W_n$   $(n \ge 3)$  is the graph obtained by joining a vertex and a cycle  $C_n$ .

In this note we consider the Ramsey numbers for stars versus wheels. There are many results on this area. Hasmawati [4] determined the Ramsey number  $R(K_{1,n}, W_m)$  for  $m \ge 2n$ .

**Theorem 1** (Hasmawati [4]). If  $n \ge 2$  and  $m \ge 2n$ , then

$$R(K_{1,n}, W_m) = \begin{cases} n+m-1, & \text{if both } n \text{ and } m \text{ are even;} \\ n+m, & \text{otherwise.} \end{cases}$$

So from now on we consider the case that  $m \leq 2n - 1$ . For odd m, Chen et al. [2] showed that if  $m \leq n + 2$ , then  $R(K_{1,n}, W_m) = 3n + 1$ . Hasmawati et al. [5] proved that the values remain the same even if  $m \leq 2n - 1$ .

**Theorem 2** (Hasmawati et al. [5]). If  $3 \le m \le 2n - 1$  and m is odd, then

$$R(K_{1,n}, W_m) = 3n + 1.$$

So it is remains the case when  $m \leq 2n - 2$  and m is even. Surahmat and Baskoro [7] determined the Ramsey numbers of stars versus  $W_4$ .

**Theorem 3** (Surahmat and Baskoro [7]). If  $n \ge 2$ , then

$$R(K_{1,n}, W_4) = \begin{cases} 2n+1, & \text{if } n \text{ is even;} \\ 2n+3, & \text{if } n \text{ is odd.} \end{cases}$$

Chen et al. [2] established  $R(K_{1,n}, W_6)$ , and Zhang et al. [8, 9] established  $R(K_{1,n}, W_8)$ .

In this note we first give a lower bound on  $R(K_{1,n}, W_m)$  for even  $m \leq 2n - 2$ . One can check that when m = 6, 8, the lower bound on  $R(K_{1,n}, W_m)$  in Theorem 4 is the exact value, see [2, 9, 8].

**Theorem 4.** If  $6 \le m \le 2n-2$  and m is even, then

$$R(K_{1,n}, W_m) \ge \begin{cases} 2n + m/2 - 1, & \text{if both } n \text{ and } m/2 \text{ are even}, \\ 2n + m/2, & \text{otherwise.} \end{cases}$$

Moreover, we establish the exact values when  $n + 2 \le m \le 2n - 2$ . We will show that the lower bound in Theorem 4 is the exact value if  $m \ge n + 2$ .

**Theorem 5.** If  $n + 2 \le m \le 2n - 2$  and m is even, then

$$R(K_{1,n}, W_m) = \begin{cases} 2n + m/2 - 1, & \text{if both } n \text{ and } m/2 \text{ are even}; \\ 2n + m/2, & \text{otherwise.} \end{cases}$$

### 2 Preliminaries

We denote by  $\nu(G)$  the order of G, by  $\delta(G)$  the minimum degree of G, c(G) the circumference of G, and g(G) the girth of G, respectively. The graph G is said to be *pancyclic* if G contains cycles of every length between 3 and  $\nu(G)$ , and *weakly pancyclic* if G contains cycles of every length between g(G) and c(G).

We will use the following results.

**Theorem 6** (Dirac [3]). Every 2-connected graph G has circumference  $c(G) \ge \min\{2\delta(G), \nu(G)\}$ .

**Theorem 7** (Brandt et al. [1]). Every non-bipartite graph G with  $\delta(G) \ge (\nu(G) + 2)/3$ is weakly pancyclic and has girth 3 or 4.

**Theorem 8** (Jackson [6]). Let G be a bipartite graph with partition sets X and Y,  $2 \le |X| \le |Y|$ . If for every vertex  $x \in X$ ,  $d(x) \ge \max\{|X|, |Y|/2 + 1\}$ , then G has a cycle containing all vertices in X, (i.e., of length 2|X|).

A graph G is said to be *k*-regular if every vertex of G has degree k.

**Lemma 1.** Let k and n be two integers with  $n \ge k+1$  and k or n is even. Then there is a k-regular graph of order n each component of which is of order at most 2k + 1.

Proof. We first assume that  $k + 1 \leq n \leq 2k + 1$ . If k is even, then let G be the graph with vertex set  $\{v_1, v_2, \ldots, v_n\}$  and every vertex  $v_i$  is adjacent to the k vertices in  $\{v_{i\pm 1}, v_{i\pm 2}, \ldots, v_{i\pm k/2}\}$ , where the subscripts are taken modulo n. Then G is a k-regular graph of order n. If k is odd, then n is even and n - 1 - k is even. Similarly as above we can get a (n - 1 - k)-regular graph H of order n. Then  $G = \overline{H}$  is a k-regular graph of order n. Since  $n \leq 2k + 1$ , every component of G has order at most 2k + 1.

Now we assume that  $n \ge 2k + 2$ .

If k is even, then let

$$n = q(2k+1) + r, \ 0 \le r \le 2k.$$

Note that  $q \ge 1$ . If r = 0, then the union of q copies of a k-regular graph of order 2k + 1 is a required graph. If  $k + 1 \le r \le 2k$ , then the union of q copies of a k-regular graph of order 2k + 1 and one copy of a k-regular graph of order r is a required graph. Now we assume that  $1 \le r \le k$ . Note that  $k + 1 \le k + r \le 2k$ . Then the union of q - 1 copies of a k-regular graph of order 2k + 1, one copy of a k-regular graph of order k + 1, and one copy of a k-regular graph of order k + r, is a required graph.

If k is odd, then n is even. Let

$$n = 2qk + r, \ 0 \le r < 2k.$$

Clearly r is even. If r = 0 then the union of q copies of a k-regular graph of order 2k is a required graph. If  $k + 1 \le r < 2k$ , then the union of q copies of a k-regular graph of order 2k and one copy of a k-regular graph of order r is a required graph. Now we assume that  $2 \le r \le k - 1$ . Note that  $k + 1 \le k + r - 1 \le 2k$ . Then the union of q - 1 copies of a k-regular graph of order 2k, one copy of a k-regular graph of order k + 1, and one copy of a k-regular graph of order  $k + r - 1 \le 2k$ . Then the union of q - 1 copies of a k-regular graph of order k + r - 1, is a required graph.

## 3 Proof of Theorem 4

For convenience we define a constant  $\theta$  such that  $\theta = 1$  if both n and m/2 are even, and  $\theta = 0$  otherwise. We will construct a graph G of order  $2n + m/2 - \theta - 1$  such that G contains no  $K_{1,n}$  and  $\overline{G}$  contains no  $W_m$ .

It is easy to check that m/2 - 1 or  $n + m/2 - \theta - 1$  is even. By Lemma 1, Let H be an (m/2 - 1)-regular graph of order  $n + m/2 - \theta - 1$  such that each component of which has order at most m - 1. Let  $G = \overline{H} \cup K_n$ . Then  $\nu(G) = 2n + m/2 - \theta - 1$ .

We first show that G contains no  $K_{1,n}$ . Clearly  $K_n$  contains no  $K_{1,n}$ . Note that every vertex in H has degree m/2-1, and then every vertex in  $\overline{H}$  has degree  $\nu(H)-1-m/2+1 = n-\theta - 1$ . Thus  $\overline{H}$  contains no  $K_{1,n}$ .

Second we show that  $\overline{G}$  contains no  $W_m$ . Suppose to contrary that  $\overline{G}$  contains a  $W_m$ . Let x be the hub of the  $W_m$ . If x is contained in  $K_n$ , then all vertices of the wheel other than x are in V(H). This implies that H has a cycle  $C_m$ . But every component of H has order less than m, a contradiction. So we assume that  $x \in V(H)$ . Note that x has m/2-1neighbors in H. At least m/2 + 1 vertices of the wheel are in the  $K_n$ . This implies that there are two vertices in the  $K_n$  such that they are adjacent in  $\overline{G}$ , a contradiction.

This implies that  $R(K_{1,n}, W_m) \ge 2n + m/2 - \theta$ .

### 4 Proof of Theorem 5

Note that by our assumption  $n \ge 4$  and  $m \ge 6$ . We already showed  $R(K_{1,n}, W_m) \ge 2n + m/2 - \theta$  in Theorem 4. Now we prove that  $R(K_{1,n}, W_m) \le 2n + m/2 - \theta$ . Let G be a graph of order

$$\nu(G) = 2n + m/2 - \theta.$$

Suppose that  $\overline{G}$  has no  $K_{1,n}$ , i.e.,

$$\delta(G) \ge n + m/2 - \theta. \tag{1}$$

We will prove that G has a  $W_m$ . We assume to the contrary that G contains no  $W_m$ . We choose such a G with minimum size.

Let u be a vertex of G with maximum degree. Set

$$H = G[N(u)]$$
 and  $I = V(G) \setminus (\{u\} \cup N(u)).$ 

Note that  $\nu(H) = d(u)$ .

Claim 1.  $d(u) \ge n + m/2$ ; and for every  $v \in V(H)$ ,  $d(v) = n + m/2 - \theta$ .

Proof. If  $\theta = 0$ , then by (1),  $d(u) \ge n + m/2$ . If  $\theta = 1$ , then n and m/2 are both even. Thus  $\nu(G) = 2n + m/2 - 1$  is odd. If every vertex of G has degree 2n + m/2 - 1, then G has an even order, a contradiction. This implies  $d(u) \ge n + m/2$ .

Let v be a vertex in H. Clearly  $d(v) \ge \delta(G) \ge n + m/2 - \theta$ . If  $d(v) \ge n + m/2 - \theta + 1$ , then  $d(u) \ge d(v) \ge n + m/2 - \theta + 1$ . Thus G' = G - uv has size less than G with  $\delta(G') \ge n + m/2 - \theta$ . Since G' is a subgraph of G, it contains no  $W_m$ , a contradiction.  $\Box$ 

By Claim 1, we assume that

$$\nu(H) = n + m/2 + \tau, \text{ where } \tau \ge 0.$$
(2)

Claim 2.  $\delta(H) \ge m/2 + \tau$ .

*Proof.* Let v be an arbitrary vertex of H. By Claim 1,  $d(v) = n + m/2 - \theta$ . Note that  $\nu(G - H) = (2n + m/2 - \theta) - (n + m/2 + \tau) = n - \theta - \tau$ . Thus

$$d_H(v) \ge d(v) - \nu(G - H) = (n + m/2 - \theta) - (n - \theta - \tau) = m/2 + \tau.$$

Thus the claim holds.

Claim 3. *H* is separable.

*Proof.* By (2),  $\nu(H) \ge m \ge 3$ . Suppose to contrary that H is 2-connected. By Claim 2 and Theorem 6,  $c(G) \ge m$ . Also note that

$$3\delta(H) \ge 3m/2 + 3\tau \ge n + m/2 + 3\tau + 2 \ge \nu(H) + 2,$$

i.e.,  $\delta(H) \ge (\nu(H) + 2)/3$ .

If H is non-bipartite, then by Theorem 3, H is weakly pancyclic and of girth 3 or 4. Thus H contains  $C_m$ . Note that u is adjacent to every vertex of the  $C_m$ , hence G contains a  $W_m$ , a contradiction.

If H is bipartite, say with partition sets X and Y, then  $|X| \ge m/2 + \tau$  and

$$|Y| = \nu(H) - |X| \le (n + m/2 + \tau) - (m/2 + \tau) = n_{0}$$

since  $\delta(H) \ge m/2 + \tau$ . Let X' be a subset of X with |X'| = m/2. Note that for every vertex x of X',

$$d_Y(x) = d_H(x) \ge m/2 \ge n/2 + 1 \ge |Y|/2 + 1.$$

By Theorem 8, the subgraph of H induced by  $X' \cup Y$  contains a  $C_m$ . Thus G contains a  $W_m$ , a contradiction.

If H is disconnected, then H has at least two components; if H is connected, then H has at least two end-blocks. Now let D be a component or an end-block of H such that  $\nu(D)$  is as small as possible. We define a constant  $\varepsilon$  such that  $\varepsilon = 1$  if D is an end-block of H, and  $\varepsilon = 0$  otherwise. Thus

$$\nu(D) \le (\nu(H) + \varepsilon)/2. \tag{3}$$

If D is an end-block of H, then let z be the cut-vertex of H contained in D.

Claim 4. For every two vertices  $v, w \in V(D)$  which are not cut-vertices of H,  $|N_I(v) \cap N_I(w)| \ge m/2 - 1$ .

*Proof.* Note that  $d_I(v) = d(v) - 1 - d_H(v) \ge d(v) - \nu(D)$ , and  $d_I(w) \ge d(w) - \nu(D)$ .

$$|N_I(v) \cap N_I(w)| \ge d_I(v) + d_I(w) - |I| \ge d(v) + d(w) - 2\nu(D) - |I|$$
$$\ge 2\delta(G) - (\nu(H) + \varepsilon) - |I| = 2\delta(G) - \nu(G) + 1 - \varepsilon$$
$$= 2(n + m/2 - \theta) - (2n + m/2 - \theta) + 1 - \varepsilon$$
$$= m/2 + 1 - \theta - \varepsilon \ge m/2 - 1.$$

Thus the claim holds.

Suppose that there is a vertex  $v \in V(D)$  which is not a cut-vertex of H such that v has m/2 neighbors in V(D) each of which is not a cut-vertex of H. Then let X be the set of such m/2 neighbors of v and  $Y = \{u\} \cup N_I(v)$ . Let B be the bipartite subgraph of G with partition sets X and Y, and for any two vertices  $x \in X$  and  $y \in Y$ ,  $xy \in E(B)$  if and only if  $xy \in E(G)$ .

Note that |X| = m/2. By Claim 4, every vertex of X has at least m/2 neighbors in Y. By Claims 1 and 2,  $d(v) = n + m/2 - \theta$  and  $d_H(v) \ge m/2 + \tau$ . Thus  $|Y| = d(v) - d_H(v) \le n - \theta - \tau$ . Since  $m \ge n + 2$ ,  $m/2 \ge |Y|/2 + 1$ . By Theorem 8, B contains a  $C_m$ . Note that v is adjacent to every vertex of the  $C_m$ , hence G has a  $W_m$ , a contradiction.

So we conclude that D is an end-block of H (i.e.,  $\varepsilon = 1$ ), and every vertex  $v \in V(D) \setminus \{z\}$ has at most m/2 - 1 neighbors in  $V(D) \setminus \{z\}$ . By Claim 2, we can see that z is adjacent to every vertex in  $V(D) \setminus \{z\}$  and every vertex in  $V(D) \setminus \{z\}$  has degree in H exactly m/2and  $\tau = 0$ . Claim 5. Every vertex in  $V(D) \setminus \{z\}$  is adjacent to every vertex in I.

*Proof.* Let v be a vertex in  $V(D) \setminus \{z\}$ . Since  $d(v) = n + m/2 - \theta$  and  $d_H(v) = m/2$ . we have

$$d_I(v) = d(v) - 1 - d_H(v) = n - 1 - \theta$$

Also note that

$$|I| = \nu(G) - 1 - \nu(H) = (2n + m/2 - \theta) - 1 - (n + m/2) = n - 1 - \theta.$$

This implies that v is adjacent to every vertex in I.

Case 1.  $N_I(z) \neq \emptyset$ .

Note that  $|I| = n - 1 - \theta \ge m/2 - 1$ . Let  $v \in V(D) \setminus \{z\}$  and  $u_1, u_2, \ldots, u_{m/2-1}$  be m/2 - 1 vertices in I such that  $zu_1 \in E(G)$ , and let  $v_1, v_2 \ldots, v_{m/2-1}$  be m/2 - 1 vertices in  $N_D(v) \setminus \{z\}$ . Then  $uzu_1v_1u_2v_2 \cdots u_{m/2-1}v_{m/2-1}u$  is a  $C_m$ . Since v is adjacent to every vertex of the  $C_m$ , G contains a  $C_m$ , a contradiction.

**Case 2.**  $N_I(z) = \emptyset$  and G[I] is not empty.

Let  $v \in V(D) \setminus \{z\}$  and  $u_1, u_2, \ldots, u_{m/2-1}$  be m/2 - 1 vertices in I such that  $u_1u_2 \in E(G)$ , and let  $v_1, v_2 \ldots, v_{m/2-1}$  be m/2-1 vertices in  $N_D(v) \setminus \{z\}$ . Then  $uzv_1u_1u_2v_2u_3v_3 \cdots u_{m/2-1}v_{m/2-1}u$  is a  $C_m$ . Since v is adjacent to every vertex of the  $C_m$ , G contains a  $C_m$ , a contradiction.

**Case 3.**  $N_I(z) = \emptyset$  and G[I] is empty.

Let w be an arbitrary vertex in I. Note that w is nonadjacent to every vertex in  $\{u, z\} \cup I$ . Hence

$$d(w) \le \nu(G) - 2 - |I| = (2n + m/2 - \theta) - 2 - (n - 1 - \theta) = n + m/2 - 1.$$

Since  $d(w) \ge \delta(G) = n + m/2 - \theta$ , we can see that  $\theta = 1$  and w is adjacent to every vertex of  $V(H) \setminus \{z\}$ . Moreover, every vertex in I is adjacent to every vertex in  $V(H) \setminus \{z\}$ .

Since  $\theta = 1$ , by Claim 1, d(u) = n + m/2 and d(z) = n + m/2 - 1. Thus there is a vertex  $x \in V(H) \setminus \{z\}$  such that  $xz \notin E(G)$ . By Claim 2, let  $v_1, v_2, \ldots, v_{m/2}$  be m/2 vertices in  $N_H(x)$  and  $u_1, u_2, \ldots, u_{m/2}$  be m/2 vertices in  $\{u\} \cup I$ . Then  $u_1v_1u_2v_2\cdots u_{m/2}v_{m/2}u_1$  is a  $C_m$ . Since x is adjacent to every vertex of the  $C_m$ , G contains a  $W_m$ , a contradiction.

The proof is complete.

# References

- S. Brandt, R.J. Faudree and W. Goddard, Weakly pancyclic graphs, J. Graph Theory 27 (1998) 141–176.
- [2] Y. Chen, Y. Zhang and K. Zhang, The Ramsey numbers of stars versus wheels, European J. Combin. 25 (2004) 1067–1075.
- [3] G.A. Dirac, Some theorems on abstract graphs, Proc. London. Math. Soc. 2 (1952) 69–81.
- [4] Hasmawati, Bilangan Ramsey untuk graf bintang terhadap graf roda, Tesis Magister, Departemen Matematika ITB, Indonesia, 2004.
- [5] Hasmawati, E.T. Baskoro, H. Assiyatun, Star-wheel Ramsey numbers, J. Combin. Math. Combin. Comput. 55 (2005), 123–128.
- [6] B. Jackson, Cycles in bipartite graphs, J. Comb. Theory, Ser. B 30 (3) (1981) 332–342.
- [7] Surahmat, E.T. Baskoro, On the Ramsey number of path or star versus W<sub>4</sub> or W<sub>5</sub>,
   in: Proceedings of the 12th Australasian Workshop on Combinatorial Algorithms (Bandung, Indonesia, 2001) 174–179.
- [8] Y. Zhang, Y. Chen and K. Zhang, The Ramsey numbers for stars of even order versus a wheel of order nine, European J. Combin. 29 (2008) 1744–1754.
- [9] Y. Zhang, T.C.E Cheng and Y. Chen, The Ramsey numbers for stars of odd order versus a wheel of order nine, Discrete Math., Alg. and Appl. 1 (3) (2009) 413–436.