# On star-wheel Ramsey numbers 

Binlong Li*<br>Department of Applied Mathematics<br>Northwestern Polytechnical University<br>Xi'an, Shaanxi 710072, P.R. China<br>European Centre of Excellence NTIS<br>University of West Bohemia 30614 Pilsen, Czech Republic;<br>libinlong@mail.nwpu.edu.cn

Ingo Schiermeyer ${ }^{\dagger}$<br>Institut für Diskrete Mathematik und Algebra<br>Technische Universität Bergakademie Freiberg<br>09596 Freiberg, Germany<br>Ingo.Schiermeyer@tu-freiberg.de

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#### Abstract

For two given graphs $G_{1}$ and $G_{2}$, the Ramsey number $R\left(G_{1}, G_{2}\right)$ is the least integer $r$ such that for every graph $G$ on $r$ vertices, either $G$ contains a $G_{1}$ or $\bar{G}$ contains a $G_{2}$. In this note, we determined the Ramsey number $R\left(K_{1, n}, W_{m}\right)$ for even $m$ with $n+2 \leq m \leq 2 n-2$, where $W_{m}$ is the wheel on $m+1$ vertices, i.e., the graph obtained from a cycle $C_{m}$ by adding a vertex $v$ adjacent to all vertices of the $C_{m}$.


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## 1 Introduction

Throughout this paper, all graphs are finite and simple. For a pair of graphs $G_{1}$ and $G_{2}$, the Ramsey number $R\left(G_{1}, G_{2}\right)$, is defined as the smallest integer $r$ such that for every graph $G$ on $r$ vertices, either $G$ contains a $G_{1}$ or $\bar{G}$ contains a $G_{2}$, where $\bar{G}$ is the complement of $G$. Note that $R\left(G_{1}, G_{2}\right)=R\left(G_{2}, G_{1}\right)$. We denote by $P_{n}(n \geq 1)$ and $C_{n}$ $(n \geq 3)$ the path and cycle on $n$ vertices, respectively. The bipartite graph $K_{1, n}(n \geq 2)$

[^0]is called a star. The wheel $W_{n}(n \geq 3)$ is the graph obtained by joining a vertex and a cycle $C_{n}$.

In this note we consider the Ramsey numbers for stars versus wheels. There are many results on this area. Hasmawati [4] determined the Ramsey number $R\left(K_{1, n}, W_{m}\right)$ for $m \geq 2 n$.

Theorem 1 (Hasmawati (4). If $n \geq 2$ and $m \geq 2 n$, then

$$
R\left(K_{1, n}, W_{m}\right)= \begin{cases}n+m-1, & \text { if both } n \text { and } m \text { are even } ; \\ n+m, & \text { otherwise } .\end{cases}
$$

So from now on we consider the case that $m \leq 2 n-1$. For odd $m$, Chen et al. [2] showed that if $m \leq n+2$, then $R\left(K_{1, n}, W_{m}\right)=3 n+1$. Hasmawati et al. 5 proved that the values remain the same even if $m \leq 2 n-1$.

Theorem 2 (Hasmawati et al. (5). If $3 \leq m \leq 2 n-1$ and $m$ is odd, then

$$
R\left(K_{1, n}, W_{m}\right)=3 n+1 .
$$

So it is remains the case when $m \leq 2 n-2$ and $m$ is even. Surahmat and Baskoro [7] determined the Ramsey numbers of stars versus $W_{4}$.

Theorem 3 (Surahmat and Baskoro (7). If $n \geq 2$, then

$$
R\left(K_{1, n}, W_{4}\right)= \begin{cases}2 n+1, & \text { if } n \text { is even; } \\ 2 n+3, & \text { if } n \text { is odd. }\end{cases}
$$

Chen et al. [2] established $R\left(K_{1, n}, W_{6}\right)$, and Zhang et al. [8) [9] established $R\left(K_{1, n}, W_{8}\right)$.
In this note we first give a lower bound on $R\left(K_{1, n}, W_{m}\right)$ for even $m \leq 2 n-2$. One can check that when $m=6,8$, the lower bound on $R\left(K_{1, n}, W_{m}\right)$ in Theorem 4 is the exact value, see [2, (9, 8].

Theorem 4. If $6 \leq m \leq 2 n-2$ and $m$ is even, then

$$
R\left(K_{1, n}, W_{m}\right) \geq \begin{cases}2 n+m / 2-1, & \text { if both } n \text { and } m / 2 \text { are even } ; \\ 2 n+m / 2, & \text { otherwise. }\end{cases}
$$

Moreover, we establish the exact values when $n+2 \leq m \leq 2 n-2$. We will show that the lower bound in Theorem 4 is the exact value if $m \geq n+2$.

Theorem 5. If $n+2 \leq m \leq 2 n-2$ and $m$ is even, then

$$
R\left(K_{1, n}, W_{m}\right)= \begin{cases}2 n+m / 2-1, & \text { if both } n \text { and } m / 2 \text { are even } ; \\ 2 n+m / 2, & \text { otherwise. }\end{cases}
$$

## 2 Preliminaries

We denote by $\nu(G)$ the order of $G$, by $\delta(G)$ the minimum degree of $G, c(G)$ the circumference of $G$, and $g(G)$ the girth of $G$, respectively. The graph $G$ is said to be pancyclic if $G$ contains cycles of every length between 3 and $\nu(G)$, and weakly pancyclic if $G$ contains cycles of every length between $g(G)$ and $c(G)$.

We will use the following results.

Theorem 6 (Dirac [3]). Every 2-connected graph $G$ has circumference $c(G) \geq \min \{2 \delta(G), \nu(G)\}$.

Theorem 7 (Brandt et al. [1]). Every non-bipartite graph $G$ with $\delta(G) \geq(\nu(G)+2) / 3$ is weakly pancyclic and has girth 3 or 4.

Theorem 8 (Jackson [6]). Let $G$ be a bipartite graph with partition sets $X$ and $Y, 2 \leq$ $|X| \leq|Y|$. If for every vertex $x \in X, d(x) \geq \max \{|X|,|Y| / 2+1\}$, then $G$ has a cycle containing all vertices in $X$, (i.e., of length $2|X|$ ).

A graph $G$ is said to be $k$-regular if every vertex of $G$ has degree $k$.

Lemma 1. Let $k$ and $n$ be two integers with $n \geq k+1$ and $k$ or $n$ is even. Then there is a $k$-regular graph of order $n$ each component of which is of order at most $2 k+1$.

Proof. We first assume that $k+1 \leq n \leq 2 k+1$. If $k$ is even, then let $G$ be the graph with vertex set $\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ and every vertex $v_{i}$ is adjacent to the $k$ vertices in $\left\{v_{i \pm 1}, v_{i \pm 2}, \ldots, v_{i \pm k / 2}\right\}$, where the subscripts are taken modulo $n$. Then $G$ is a $k$-regular graph of order $n$. If $k$ is odd, then $n$ is even and $n-1-k$ is even. Similarly as above we can get a $(n-1-k)$-regular graph $H$ of order $n$. Then $G=\bar{H}$ is a $k$-regular graph of order $n$. Since $n \leq 2 k+1$, every component of $G$ has order at most $2 k+1$.

Now we assume that $n \geq 2 k+2$.
If $k$ is even, then let

$$
n=q(2 k+1)+r, 0 \leq r \leq 2 k
$$

Note that $q \geq 1$. If $r=0$, then the union of $q$ copies of a $k$-regular graph of order $2 k+1$ is a required graph. If $k+1 \leq r \leq 2 k$, then the union of $q$ copies of a $k$-regular graph of order $2 k+1$ and one copy of a $k$-regular graph of order $r$ is a required graph. Now we assume that $1 \leq r \leq k$. Note that $k+1 \leq k+r \leq 2 k$. Then the union of $q-1$ copies of a $k$-regular graph of order $2 k+1$, one copy of a $k$-regular graph of order $k+1$, and one copy of a $k$-regular graph of order $k+r$, is a required graph.

If $k$ is odd, then $n$ is even. Let

$$
n=2 q k+r, 0 \leq r<2 k .
$$

Clearly $r$ is even. If $r=0$ then the union of $q$ copies of a $k$-regular graph of order $2 k$ is a required graph. If $k+1 \leq r<2 k$, then the union of $q$ copies of a $k$-regular graph of order $2 k$ and one copy of a $k$-regular graph of order $r$ is a required graph. Now we assume that $2 \leq r \leq k-1$. Note that $k+1 \leq k+r-1 \leq 2 k$. Then the union of $q-1$ copies of a $k$-regular graph of order $2 k$, one copy of a $k$-regular graph of order $k+1$, and one copy of a $k$-regular graph of order $k+r-1$, is a required graph.

## 3 Proof of Theorem 4]

For convenience we define a constant $\theta$ such that $\theta=1$ if both $n$ and $m / 2$ are even, and $\theta=0$ otherwise. We will construct a graph $G$ of order $2 n+m / 2-\theta-1$ such that $G$ contains no $K_{1, n}$ and $\bar{G}$ contains no $W_{m}$.

It is easy to check that $m / 2-1$ or $n+m / 2-\theta-1$ is even. By Lemma Let $H$ be an $(m / 2-1)$-regular graph of order $n+m / 2-\theta-1$ such that each component of which has order at most $m-1$. Let $G=\bar{H} \cup K_{n}$. Then $\nu(G)=2 n+m / 2-\theta-1$.

We first show that $G$ contains no $K_{1, n}$. Clearly $K_{n}$ contains no $K_{1, n}$. Note that every vertex in $H$ has degree $m / 2-1$, and then every vertex in $\bar{H}$ has degree $\nu(H)-1-m / 2+1=$ $n-\theta-1$. Thus $\bar{H}$ contains no $K_{1, n}$.

Second we show that $\bar{G}$ contains no $W_{m}$. Suppose to contrary that $\bar{G}$ contains a $W_{m}$. Let $x$ be the hub of the $W_{m}$. If $x$ is contained in $K_{n}$, then all vertices of the wheel other than $x$ are in $V(H)$. This implies that $H$ has a cycle $C_{m}$. But every component of $H$ has order less than $m$, a contradiction. So we assume that $x \in V(H)$. Note that $x$ has $m / 2-1$ neighbors in $H$. At least $m / 2+1$ vertices of the wheel are in the $K_{n}$. This implies that there are two vertices in the $K_{n}$ such that they are adjacent in $\bar{G}$, a contradiction.

This implies that $R\left(K_{1, n}, W_{m}\right) \geq 2 n+m / 2-\theta$.

## 4 Proof of Theorem 5

Note that by our assumption $n \geq 4$ and $m \geq 6$. We already showed $R\left(K_{1, n}, W_{m}\right) \geq$ $2 n+m / 2-\theta$ in Theorem 4. Now we prove that $R\left(K_{1, n}, W_{m}\right) \leq 2 n+m / 2-\theta$. Let $G$ be a graph of order

$$
\nu(G)=2 n+m / 2-\theta .
$$

Suppose that $\bar{G}$ has no $K_{1, n}$, i.e.,

$$
\begin{equation*}
\delta(G) \geq n+m / 2-\theta \text {. } \tag{1}
\end{equation*}
$$

We will prove that $G$ has a $W_{m}$. We assume to the contrary that $G$ contains no $W_{m}$. We choose such a $G$ with minimum size.

Let $u$ be a vertex of $G$ with maximum degree. Set

$$
H=G[N(u)] \text { and } I=V(G) \backslash(\{u\} \cup N(u)) .
$$

Note that $\nu(H)=d(u)$.
Claim 1. $d(u) \geq n+m / 2$; and for every $v \in V(H), d(v)=n+m / 2-\theta$.
Proof. If $\theta=0$, then by (1), $d(u) \geq n+m / 2$. If $\theta=1$, then $n$ and $m / 2$ are both even. Thus $\nu(G)=2 n+m / 2-1$ is odd. If every vertex of $G$ has degree $2 n+m / 2-1$, then $G$ has an even order, a contradiction. This implies $d(u) \geq n+m / 2$.

Let $v$ be a vertex in $H$. Clearly $d(v) \geq \delta(G) \geq n+m / 2-\theta$. If $d(v) \geq n+m / 2-\theta+1$, then $d(u) \geq d(v) \geq n+m / 2-\theta+1$. Thus $G^{\prime}=G-u v$ has size less than $G$ with $\delta\left(G^{\prime}\right) \geq n+m / 2-\theta$. Since $G^{\prime}$ is a subgraph of $G$, it contains no $W_{m}$, a contradiction.

By Claim 1 we assume that

$$
\begin{equation*}
\nu(H)=n+m / 2+\tau, \text { where } \tau \geq 0 \tag{2}
\end{equation*}
$$

Claim 2. $\delta(H) \geq m / 2+\tau$.
Proof. Let $v$ be an arbitrary vertex of $H$. By Claim 团 $d(v)=n+m / 2-\theta$. Note that $\nu(G-H)=(2 n+m / 2-\theta)-(n+m / 2+\tau)=n-\theta-\tau$. Thus

$$
d_{H}(v) \geq d(v)-\nu(G-H)=(n+m / 2-\theta)-(n-\theta-\tau)=m / 2+\tau .
$$

Thus the claim holds.
Claim 3. $H$ is separable.
Proof. By (2), $\nu(H) \geq m \geq 3$. Suppose to contrary that $H$ is 2 -connected. By Claim 2 and Theorem 6, $c(G) \geq m$. Also note that

$$
3 \delta(H) \geq 3 m / 2+3 \tau \geq n+m / 2+3 \tau+2 \geq \nu(H)+2
$$

i.e., $\delta(H) \geq(\nu(H)+2) / 3$.

If $H$ is non-bipartite, then by Theorem 3, $H$ is weakly pancyclic and of girth 3 or 4 . Thus $H$ contains $C_{m}$. Note that $u$ is adjacent to every vertex of the $C_{m}$, hence $G$ contains a $W_{m}$, a contradiction.

If $H$ is bipartite, say with partition sets $X$ and $Y$, then $|X| \geq m / 2+\tau$ and

$$
|Y|=\nu(H)-|X| \leq(n+m / 2+\tau)-(m / 2+\tau)=n,
$$

since $\delta(H) \geq m / 2+\tau$. Let $X^{\prime}$ be a subset of $X$ with $\left|X^{\prime}\right|=m / 2$. Note that for every vertex $x$ of $X^{\prime}$,

$$
d_{Y}(x)=d_{H}(x) \geq m / 2 \geq n / 2+1 \geq|Y| / 2+1 .
$$

By Theorem 图, the subgraph of $H$ induced by $X^{\prime} \cup Y$ contains a $C_{m}$. Thus $G$ contains a $W_{m}$, a contradiction.

If $H$ is disconnected, then $H$ has at least two components; if $H$ is connected, then $H$ has at least two end-blocks. Now let $D$ be a component or an end-block of $H$ such that $\nu(D)$ is as small as possible. We define a constant $\varepsilon$ such that $\varepsilon=1$ if $D$ is an end-block of $H$, and $\varepsilon=0$ otherwise. Thus

$$
\begin{equation*}
\nu(D) \leq(\nu(H)+\varepsilon) / 2 . \tag{3}
\end{equation*}
$$

If $D$ is an end-block of $H$, then let $z$ be the cut-vertex of $H$ contained in $D$.
Claim 4. For every two vertices $v, w \in V(D)$ which are not cut-vertices of $H, \mid N_{I}(v) \cap$ $N_{I}(w) \mid \geq m / 2-1$.

Proof. Note that $d_{I}(v)=d(v)-1-d_{H}(v) \geq d(v)-\nu(D)$, and $d_{I}(w) \geq d(w)-\nu(D)$.

$$
\begin{aligned}
\left|N_{I}(v) \cap N_{I}(w)\right| & \geq d_{I}(v)+d_{I}(w)-|I| \geq d(v)+d(w)-2 \nu(D)-|I| \\
& \geq 2 \delta(G)-(\nu(H)+\varepsilon)-|I|=2 \delta(G)-\nu(G)+1-\varepsilon \\
& =2(n+m / 2-\theta)-(2 n+m / 2-\theta)+1-\varepsilon \\
& =m / 2+1-\theta-\varepsilon \geq m / 2-1 .
\end{aligned}
$$

Thus the claim holds.
Suppose that there is a vertex $v \in V(D)$ which is not a cut-vertex of $H$ such that $v$ has $m / 2$ neighbors in $V(D)$ each of which is not a cut-vertex of $H$. Then let $X$ be the set of such $m / 2$ neighbors of $v$ and $Y=\{u\} \cup N_{I}(v)$. Let $B$ be the bipartite subgraph of $G$ with partition sets $X$ and $Y$, and for any two vertices $x \in X$ and $y \in Y, x y \in E(B)$ if and only if $x y \in E(G)$.

Note that $|X|=m / 2$. By Claim 4 , every vertex of $X$ has at least $m / 2$ neighbors in $Y$. By Claims 1 and 2, $d(v)=n+m / 2-\theta$ and $d_{H}(v) \geq m / 2+\tau$. Thus $|Y|=d(v)-d_{H}(v) \leq$ $n-\theta-\tau$. Since $m \geq n+2, m / 2 \geq|Y| / 2+1$. By Theorem ${ }^{8}, B$ contains a $C_{m}$. Note that $v$ is adjacent to every vertex of the $C_{m}$, hence $G$ has a $W_{m}$, a contradiction.

So we conclude that $D$ is an end-block of $H$ (i.e., $\varepsilon=1$ ), and every vertex $v \in V(D) \backslash\{z\}$ has at most $m / 2-1$ neighbors in $V(D) \backslash\{z\}$. By Claim 2, we can see that $z$ is adjacent to every vertex in $V(D) \backslash\{z\}$ and every vertex in $V(D) \backslash\{z\}$ has degree in $H$ exactly $m / 2$ and $\tau=0$.

Claim 5. Every vertex in $V(D) \backslash\{z\}$ is adjacent to every vertex in $I$.
Proof. Let $v$ be a vertex in $V(D) \backslash\{z\}$. Since $d(v)=n+m / 2-\theta$ and $d_{H}(v)=m / 2$. we have

$$
d_{I}(v)=d(v)-1-d_{H}(v)=n-1-\theta .
$$

Also note that

$$
|I|=\nu(G)-1-\nu(H)=(2 n+m / 2-\theta)-1-(n+m / 2)=n-1-\theta .
$$

This implies that $v$ is adjacent to every vertex in $I$.
Case 1. $N_{I}(z) \neq \emptyset$.
Note that $|I|=n-1-\theta \geq m / 2-1$. Let $v \in V(D) \backslash\{z\}$ and $u_{1}, u_{2}, \ldots, u_{m / 2-1}$ be $m / 2-1$ vertices in $I$ such that $z u_{1} \in E(G)$, and let $v_{1}, v_{2} \ldots, v_{m / 2-1}$ be $m / 2-1$ vertices in $N_{D}(v) \backslash\{z\}$. Then $u z u_{1} v_{1} u_{2} v_{2} \cdots u_{m / 2-1} v_{m / 2-1} u$ is a $C_{m}$. Since $v$ is adjacent to every vertex of the $C_{m}, G$ contains a $C_{m}$, a contradiction.

Case 2. $N_{I}(z)=\emptyset$ and $G[I]$ is not empty.
Let $v \in V(D) \backslash\{z\}$ and $u_{1}, u_{2}, \ldots, u_{m / 2-1}$ be $m / 2-1$ vertices in $I$ such that $u_{1} u_{2} \in$ $E(G)$, and let $v_{1}, v_{2} \ldots, v_{m / 2-1}$ be $m / 2-1$ vertices in $N_{D}(v) \backslash\{z\}$. Then $u z v_{1} u_{1} u_{2} v_{2} u_{3} v_{3} \ldots$ $u_{m / 2-1} v_{m / 2-1} u$ is a $C_{m}$. Since $v$ is adjacent to every vertex of the $C_{m}, G$ contains a $C_{m}$, a contradiction.

Case 3. $N_{I}(z)=\emptyset$ and $G[I]$ is empty.
Let $w$ be an arbitrary vertex in $I$. Note that $w$ is nonadjacent to every vertex in $\{u, z\} \cup I$. Hence

$$
d(w) \leq \nu(G)-2-|I|=(2 n+m / 2-\theta)-2-(n-1-\theta)=n+m / 2-1 .
$$

Since $d(w) \geq \delta(G)=n+m / 2-\theta$, we can see that $\theta=1$ and $w$ is adjacent to every vertex of $V(H) \backslash\{z\}$. Moreover, every vertex in $I$ is adjacent to every vertex in $V(H) \backslash\{z\}$.

Since $\theta=1$, by Claim $d(u)=n+m / 2$ and $d(z)=n+m / 2-1$. Thus there is a vertex $x \in V(H) \backslash\{z\}$ such that $x z \notin E(G)$. By Claim 2, let $v_{1}, v_{2}, \ldots, v_{m / 2}$ be $m / 2$ vertices in $N_{H}(x)$ and $u_{1}, u_{2}, \ldots, u_{m / 2}$ be $m / 2$ vertices in $\{u\} \cup I$. Then $u_{1} v_{1} u_{2} v_{2} \cdots u_{m / 2} v_{m / 2} u_{1}$ is a $C_{m}$. Since $x$ is adjacent to every vertex of the $C_{m}, G$ contains a $W_{m}$, a contradiction.

The proof is complete.

## References

[1] S. Brandt, R.J. Faudree and W. Goddard, Weakly pancyclic graphs, J. Graph Theory 27 (1998) 141-176.
[2] Y. Chen, Y. Zhang and K. Zhang, The Ramsey numbers of stars versus wheels, European J. Combin. 25 (2004) 1067-1075.
[3] G.A. Dirac, Some theorems on abstract graphs, Proc. London. Math. Soc. 2 (1952) 69-81.
[4] Hasmawati, Bilangan Ramsey untuk graf bintang terhadap graf roda, Tesis Magister, Departemen Matematika ITB, Indonesia, 2004.
[5] Hasmawati, E.T. Baskoro, H. Assiyatun, Star-wheel Ramsey numbers, J. Combin. Math. Combin. Comput. 55 (2005), 123-128.
[6] B. Jackson, Cycles in bipartite graphs, J. Comb. Theory, Ser. B 30 (3) (1981) 332-342.
[7] Surahmat, E.T. Baskoro, On the Ramsey number of path or star versus $W_{4}$ or $W_{5}$, in: Proceedings of the 12th Australasian Workshop on Combinatorial Algorithms (Bandung, Indonesia, 2001) 174-179.
[8] Y. Zhang, Y. Chen and K. Zhang, The Ramsey numbers for stars of even order versus a wheel of order nine, European J. Combin. 29 (2008) 1744-1754.
[9] Y. Zhang, T.C.E Cheng and Y. Chen, The Ramsey numbers for stars of odd order versus a wheel of order nine, Discrete Math., Alg. and Appl. 1 (3) (2009) 413-436.


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