# Distance magic labeling in complete 4-partite graphs 

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#### Abstract

Let $G$ be a complete $k$-partite simple undirected graph with parts of sizes $p_{1} \leqslant p_{2} \cdots \leqslant p_{k}$. Let $P_{j}=\sum_{i=1}^{j} p_{i}$ for $j=1, \ldots, k$. It is conjectured that $G$ has distance magic labeling if and only if $\sum_{i=1}^{P_{j}}(n-i+1) \geqslant j\binom{n+1}{2} / k$ for all $j=1, \ldots, k$. The conjecture is proved for $k=4$, extending earlier results for $k=2,3$.


## 1 Introduction

Let $G=(V, E)$ be a finite, simple, undirected graph of order $n$. Denoting $[n]=\{1,2, \ldots, n\}$, a distance magic labeling of $G$ [6] (or sigma labeling [4]) is a bijection $f: V \rightarrow[n]$ such that for all $x \in V$,

$$
\begin{equation*}
\sum_{y \in N(x)} f(y)=\mathbf{c} \tag{1}
\end{equation*}
$$

for a constant $\mathbf{c}$, independent of $x(N(x)$ is the set of vertices adjacent to $x)$.
Miller, Rodger and Simanjuntak [6] showed that if $G$ is the $2 k r$-regular multipartite graph $H \times \overline{K_{2 k}}$ then $G$ has a distance magic labeling ( $\overline{K_{2 k}}$ is the complement of the graph $K_{2 k}$ ). They also showed that for the complete symmetric multipartite graph $H_{n, p}$ with $p$ parts and $n$ vertices in each part, $H_{n, p}$ has a distance magic labeling if and only if either $n$ is even or both $n$ and $p$ are odd. In addition, they gave necessary and sufficient conditions for a complete multipartite graph $K_{p_{1}, p_{2}, \ldots, p_{k}}$ (where the parts are not necessarily of equal sizes) to have a distance magic labeling for $k=2,3$. The result for $k=2$ also appears in [4. For more results and surveys on distance magic labeling see [2, 3, 5].

It has been observed that the problem of characterizing the complete multipartite graphs which have a distance magic labeling is equivalent to a problem on partitions of [n]: let $V=V_{1} \cup V_{2} \cup \cdots \cup V_{k}$ be the parts of $G=K_{p_{1}, p_{2}, \ldots, p_{k}}$ with sizes $p_{1}, p_{2}, \ldots, p_{k}$, respectively, so that $p_{1}+p_{2}+\cdots+p_{k}=n$. Then, $G$ has a distance magic labeling if and only if there exists a bijection $f: V \rightarrow[n]$ such that for all $j=1, \ldots, k$, $\sum_{i=1, i \neq j} \sum_{x \in V_{i}} f(x)=\mathbf{c}$, where $\mathbf{c}$ is a constant. This is equivalent to $\mathbf{c}=\binom{n+1}{2}-\sum_{x \in V_{j}} f(x)$ for all $j=1, \ldots, k$, or $\sum_{x \in V_{j}} f(x)=\binom{n+1}{2} / k$ for all $j=1, \ldots, k$. Denote $s^{n, k}=\binom{n+1}{2} / k$. The problem can be reformulated as follows:
Problem 1. Let $n, k$ and $p_{1}, \ldots, p_{k}$ be positive integers such that $p_{1}+\cdots+p_{k}=n$ and $s^{n, k}$ is an integer. When is it possible to find a partition of the set $[n]$ into $k$ subsets of sizes $p_{1}, \ldots, p_{k}$, respectively, such that the sum of the elements in each subset is $s^{n, k}$ ?

Anholcer, Cichacz and Peterin [1] related this problem to a different problem in vertex labeling: let $G$ be the graph obtained from the cycle $C_{k}$ by replacing every vertex $v_{i}$ by a clique $K\left[v_{i}\right]$ of some order $p_{i}$, and joining all the vertices of each $K\left[v_{i}\right]$ with all the vertices of $K\left[v_{j}\right]$ whenever $v_{j}$ is a neighbor of $v_{i}$ in $C_{k}$. Then, consider the problem of finding a closed distance magic labeling of $G$, that is, a magic labeling where the sum in (1) includes $f(x)$. Clearly, if the partition in Problem 1 is solved for $k$, then $G$ has a closed distance magic labeling. A necessary condition for such a partition to exist was observed in [1]:
Observation 1.1. Assume the setup of Problem 1 with $p_{1}, \ldots, p_{k}$ given in a non-decreasing order. Let $P_{j}=\sum_{i=1}^{j} p_{i}$ for $j=1, \ldots, k$. If the mentioned partition of $[n]$ exists, then

$$
\begin{equation*}
\sum_{i=1}^{P_{j}}(n-i+1) \geqslant j s^{n, k} \quad \text { for all } \quad j=1, \ldots, k \tag{2}
\end{equation*}
$$

It is conjectured here (Conjecture 1.5) that this condition is sufficient. Before formulating the conjecture we shall need some notation and definitions:
Definition 1.2. Let $n$ be a positive integer and let $\mathcal{P}=\left\{p_{1}, p_{2}, \cdots, p_{k}\right\}$ be a set of positive integers satisfying $\sum_{i=1}^{k} p_{i}=n$. We say that a partition $\mathcal{A}=\left\{A_{1}, A_{2}, \ldots, A_{k}\right\}$ of $[n]$ implements the sequence $\left\{p_{i}\right\}_{i=1}^{k}$, if there is a bijection $f: \mathcal{A} \rightarrow \mathcal{P}$ such that $\left|A_{i}\right|=f\left(A_{i}\right)$ for all $i=1, \ldots, k$.
Definition 1.3. For any set of integers $A$, the sum of $A$, denoted $S(A)$, is the sum of the elements in $A$.
Definition 1.4. Let $n$ and $k$ be positive integer such that $s^{n, k}$ is an integer. We say that the partition $\mathcal{A}=\left\{A_{1}, A_{2}, \ldots, A_{k}\right\}$ of $[n]$ is equitable if $S\left(A_{i}\right)=s^{n, k}$ for all $i=1, \ldots, k$.

Conjecture 1.5. Let $k<n$ be positive integers such that $s^{n, k}$ is an integer. Let $1<p_{1} \leqslant p_{2} \leqslant \cdots \leqslant p_{k}$ be positive integers such that $\sum_{i=1}^{k} p_{i}=n$. There exists an equitable partition of $[n]$ implementing $\left\{p_{i}\right\}_{i=1}^{k}$ if and only if (2) holds.

This conjecture was in fact proved in [6] for $k=2,3$. The case $k=2$ was independently proved in [4]. The main result in this paper is:
Theorem 1.6. Conjecture 1.5 holds for $k=4$.
The approach used for proving Theorem 1.6 also provides simple proofs for the cases $k=2,3$ (Remarks 2.8 and 2.12.

Thus, the $k$-partite complete graphs for which a magic distance labeling exits are characterized for $k \leqslant 4$. This also solves the above mentioned closed distance magic labeling problem from 1 for $k=4$.

Note that the case where $p_{1}=1$ is left out in Conjecture 1.5, as this case is different and straightforward. Suppose $p_{1}=1$. In order for a partition which implements $\left\{p_{i}\right\}_{i=1}^{k}$ to be equitable, the set of size 1 must be $\{n\}$, and thus, $n=s^{n, k}$, which is equivalent to $k=(n+1) / 2$. Since there can be only one set of size 1 in an equitable partition, all the other sets must be of size 2 . Thus, we can partition the remaining $[n-1]$ into $k-1$ pairs, in the obvious way, to get an equitable partition.

## 2 Preliminary results and notation

We first introduce some notation and definitions and prove some preliminary results for general $k$. We assume that $k$ and $n$ are such that $s^{n, k}$ is an integer.
Notation 2.1. For a partition $\mathcal{A}=\left\{A_{1}, A_{2}, \ldots, A_{k}\right\}$ of $[n]$, we denote $d(\mathcal{A})=\sum_{i=1}^{k}\left(S\left(A_{i}\right)-s^{n, k}\right)^{2}$.
For any set $A$ and an element $x$ we use the notation $A-x$ for $A \backslash\{x\}$ and $A \cup x$ for $A \cup\{x\}$.
Notation 2.2. Let $\mathcal{A}=\left\{A_{1}, A_{2}, \ldots, A_{k}\right\}$ be a partition of $[n]$ and suppose $a \in A_{i}$ and $b \in A_{j}$, where $a<b$ and $i \neq j$. We shall denote by $\chi_{a, b}$ the operator that acts on $\mathcal{A}$ by exchanging $a$ and $b$ between $A_{i}$ and $A_{j}$. The result is a new partition $\chi_{a, b}(\mathcal{A})$, which we shall denote $\mathcal{A}_{a, b}$, where $A_{i}$ is replaced by $A_{i}-a \cup b$ and $A_{j}$ is replaced by $A_{j}-b \cup a$.
Observation 2.3. Let $\mathcal{A}=\left\{A_{1}, A_{2}, \ldots, A_{k}\right\}$ be partition of [n]. Suppose $a \in A_{i}$ and $b \in A_{j}$ where $i \neq j$. Then, $d\left(\mathcal{A}_{a, b}\right)=d(\mathcal{A})$ if $b-a=S\left(A_{j}\right)-S\left(A_{i}\right), d\left(\mathcal{A}_{a, b}\right)>d(\mathcal{A})$ if $b-a>S\left(A_{j}\right)-S\left(A_{i}\right)$, and $d\left(\mathcal{A}_{a, b}\right)<d(\mathcal{A})$ if $b-a<S\left(A_{j}\right)-S\left(A_{i}\right)$.

Proof. Let $t=b-a$ and $u=S\left(A_{j}\right)-S\left(A_{i}\right)$. We have $d\left(\mathcal{A}_{a, b}\right)=d(\mathcal{A})+2 t(t-u)$.
Definition 2.4. For a given sequence $\mathcal{P}=\left\{p_{i}\right\}_{i=1}^{k}$ satisfying $\sum_{i=1}^{k} p_{i}=n$, let $d_{\min }^{\mathcal{P}}$ be the minimal value of $d(\mathcal{A})$ over all partitions of $[n]$ which implement $\mathcal{P}$. A partition $\mathcal{A}$ implementing $\mathcal{P}$, such that $d(\mathcal{A})=d_{\text {min }}^{\mathcal{P}}$ will be called a minimal partition.
Definition 2.5. Let $\mathcal{A}$ be a partition of $[n]$ and let $A \in \mathcal{A}$. We say that $A$ is low if $S(A)<s^{n, k}, A$ is high if $S(A)>s^{n, k}$, and $A$ is exact if $S(A)=s^{n, k}$.
Observation 2.6. Suppose there is no equitable partition of $[n]$ implementing $\left\{p_{i}\right\}_{i=1}^{k}$ and let $\mathcal{A}$ be a minimal partition. Then there is no $c \in[n]$ such that $c$ is in a low set of $\mathcal{A}$ and $c+1$ is in a high set.
Proof. Assume the contrary, so that $c \in A_{i}$ and $c+1 \in A_{j}$ where $A_{i}$ is a low set and $A_{j}$ is a high set. We have $S\left(A_{j}\right)-S\left(A_{i}\right)>1$. So, $d\left(\mathcal{A}_{c, c+1}\right)<d(\mathcal{A})$, by Observation 2.3 contradicting the minimality of $\mathcal{A}$.

Lemma 2.7. Let $\mathcal{P}=\left\{p_{i}\right\}_{i=1}^{k}$ be such that $\sum_{i=1}^{k} p_{i}=n$ and satisfies (2). Suppose there is no equitable partition implementing $\mathcal{P}$ and let $\mathcal{A}$ be a minimal partition of $[n]$ implementing $\mathcal{P}$. Then, there exists a number $t$ in a low set of $\mathcal{A}$ such that $t+1$ is in an exact set of $\mathcal{A}$ and there exists a number $s$ in an exact set of $\mathcal{A}$ such that $s+1$ in a high set of $\mathcal{A}$.

Proof. Let $\mathcal{A}^{l}$ and $A^{e}$ be the collections of low and exact sets of $\mathcal{A}$, respectively. Let $l=\left|\mathcal{A}^{l}\right|$ and $e=\left|A^{e}\right|$. Let $L=\sum_{A \in \mathcal{A}^{l}}|A|$ and $E=\sum_{A \in \mathcal{A}^{e}}|A|$. If all the elements in the low sets of $\mathcal{A}$ are greater than all the elements in the other sets, then $\bigcup_{A \in \mathcal{A}^{l}} A=\{n-L+1, \ldots, n-1, n\}$, and we have

$$
\sum_{i=1}^{P_{l}}(n-i+1)=\sum_{i=1}^{p_{1}+\cdots+p_{l}}(n-i+1)<l \cdot s^{n, k}
$$

contradicting (2). So there must be an element $t$ in a low set such that $t+1$ is not in a low set. By Observation 2.6, $t+1$ must be in an exact set. Now, suppose, for contradiction, that all the elements in the low and exact sets of $\mathcal{A}$ are greater than all the elements in the high sets of $\mathcal{A}$. We have

$$
\sum_{i=1}^{P_{l+e}}(n-i+1)=\sum_{i=1}^{p_{1}+\cdots+p_{l+e}}(n-i+1)<(l+e) s^{n, k}
$$

contradicting (2). So, there must be $s$ in a low or exact set such that $s+1$ is in a high set. By Observation 2.6, $s$ must be in an exact set.

Remark 2.8. It follows from Lemma 2.7 that if $\mathcal{P}$ satisfies (2) and there is no equitable partition implementing $\mathcal{P}$, then $k \geqslant 3$. This implies Conjecture 1.5 for $k=2$.
Lemma 2.9. Let $\mathcal{P}=\left\{p_{i}\right\}_{i=1}^{k}$ be such that $\sum_{i=1}^{k} p_{i}=n$ and satisfies 2). If there is no equitable partition implementing $\mathcal{P}$, then every minimal partition implementing $\mathcal{P}$ contains at least one exact set, one low set with sum $s^{n, k}-1$, and one high with sum $s^{n, k}+1$.

Proof. Let $\mathcal{A}$ be a minimal partition implementing $\mathcal{P}$. By Lemma 2.7, there must be a low set $X$, an exact set $Z$ and a number $a \in X$ such that $a+1 \in Z$ and an exact set $W$, a high set $Y$, and a number $b \in W$ such that $b+1 \in Y$. Since $\mathcal{A}$ is minimal, we must have $S(X)=s^{n, k}-1$ and $S(Y)=s^{n, k}+1$, by Observation 2.3

In the discussion ahead we shall use diagrams with two horizontal lines. High sets with sum $s^{n, k}+1$ appear above the lines, exact sets appear between the lines, and low sets with sum $s^{n, k}-1$ are drawn below the lines. An arrow points from a number to its successor. For example, Figure 1 illustrates the setup in the proof of Lemma 2.9 .


Figure 1

Definition 2.10. We say that two partitions $\mathcal{A}$ and $\mathcal{A}^{\prime}$ of $[n]$ are equivalent, denoted $\mathcal{A} \equiv \mathcal{A}^{\prime}$, if there exists a bijection $f: \mathcal{A} \rightarrow \mathcal{A}^{\prime}$ such that $S(f(A))=S(A)$ for all $A \in \mathcal{A}$.

Clearly, if $\mathcal{A} \equiv \mathcal{A}^{\prime}$, then $d(\mathcal{A})=d\left(\mathcal{A}^{\prime}\right)$.


Figure 2

Lemma 2.11. If $\mathcal{A}$ is a minimal partition then the configurations illustrated in Figure 2 are not possible, given that $b \neq a+1$ in all figures, $c \neq b+1$ in Figure 2(b), $x \neq b+1$ in Figures 2(c) and 2(e), and $x+2 \neq a$ in Figures 2(d) and $2(e)$.

Proof. Suppose the configuration in Figure $2(\mathrm{a})$ exists. Applying $\chi_{a, a+1}$ we obtain an equivalent partition with $b$ in a low set and $b+1$ in a high set, contradicting Observation 2.6. Suppose the configuration in Figure 2(b) exists. Applying $\chi_{c, c+1}$, the element $b+1$ gets pushed up to the high set and we obtain a configuration as in Figure 2(a). Suppose the configuration in Figure 2(c) or Figure 2(d) exists. Applying $\chi_{x, x+2}$, we obtain an equivalent partition with a configuration as in Figure 2(b). If the Configuration in Figure 2(e) exits, then applying $\chi_{x, x+2}$ yields an equivalent partition with a configuration as in Figure 2(a).

In general, if $A, B \in \mathcal{A}, S(B)=S(A)+1, a \in A$ and $a+1 \in B$, then, clearly $\mathcal{A}_{a, a+1} \equiv \mathcal{A}$. Such actions of the form $\chi_{a, a+1}$ will be common in our discussion and we won't always mention the obvious fact that the partitions are equivalent.
Remark 2.12. Let $\mathcal{P}=\left\{p_{1}, p_{2}, p_{3}\right\}$ be such that $p_{1}+p_{2}+p_{3}=n$ and satisfies (2) for $k=3$. Suppose there is no equitable partition implementing $\mathcal{P}$ and let $\mathcal{A}$ be a minimal partition implementing $\mathcal{P}$. By Observation 2.6. Lemma 2.9 and Lemma 2.11(a) we must have the setup illustrated in Figure 3. Note that we may assume that $n \in A_{3}$ (if $n \in A_{2}$ we apply $\chi_{t+1, t+2}$ and if $n \in A_{1}$ we apply $\chi_{t+1, t+2} \circ \chi_{t, t+1}$ ). Let $s$ be the maximal element in $A_{2}-(t+1)$. Since $s \neq n$ and $n \notin A_{1}$ we must have $s+1 \in A_{3}$, by Observation 2.6. But this yields a configuration as in Lemma 2.11 (a). Thus, Conjecture 1.5 holds for $k=3$.


Figure 3

## 3 Some lemmas for the case $k=4$

Lemma 3.1. Let $\mathcal{P}=\left\{p_{i}\right\}_{i=1}^{4}$ be such that $\sum_{i=1}^{4} p_{i}=n$ and satisfies (2). If there is no equitable partition of $[n]$ implementing $\mathcal{P}$, then every minimal partition implementing $\mathcal{P}$ has one low set with sum $s^{n, 4}-1$, one high set with sum $s^{n, 4}+1$, and two exact sets.

Proof. By Lemma 2.9 there is one low set and one high set with the indicated properties, and one exact set. Since the sum of all the elements is $4 s^{n, 4}$, the fourth set must be exact.

Definition 3.2. Let $\mathcal{A}$ be a partition of $[n]$ implementing the sequence $\left\{p_{i}\right\}_{i=1}^{k}$ and assume $\mathcal{A}$ is not equitable. We define the width of $\mathcal{A}$, denoted $\omega(\mathcal{A})$, as the minimal value of $y-x$ over all $x, y \in[n]$ such that $y>x$ and such that $y$ is in a high set of $\mathcal{A}$ and $x$ is in a low set. If there are no such $x$ and $y$ we define $\omega(\mathcal{A})=\infty$.

Lemma 3.3. Let $\mathcal{P}=\left\{p_{i}\right\}_{i=1}^{4}$ be such that $\sum_{i=1}^{4} p_{i}=n$ and satisfies 2) for $k=4$. Suppose there is no equitable partition of $[n]$ implementing $\mathcal{P}$. Then, there exists a minimal partition of $[n]$ with finite width.

Proof. Let $\mathcal{A}=\left\{A_{1}, \ldots, A_{4}\right\}$ be a minimal partition of $[n]$ implementing $\left\{p_{i}\right\}_{i=1}^{4}$ and assume $\omega(\mathcal{A})=\infty$. We shall show that there is an equivalent partition with finite width. By Lemma 3.1, we may assume that $S\left(A_{1}\right)=s^{n, 4}-1, S\left(A_{2}\right)=S\left(A_{3}\right)=s^{n, 4}$ and $S\left(A_{4}\right)=s^{n, 4}+1$. By Lemma 2.7 there exist $a \in A_{1}$ such that $a+1 \in A_{2} \cup A_{3}$, and $b \in A_{2} \cup A_{3}$ such that $b+1 \in A_{4}$. We may assume that $a+1 \in A_{2}$. If $b \in A_{2}$, then, by Lemma 2.11(a), we must have $b=a+1$. But then $\omega(\mathcal{A}) \leqslant 2$, contradicting our assumption. So, we must have $b \in A_{3}$. If there exist $z<w$ such that $z \in A_{2}$ and $w \in A_{4}$, then $z \neq a+1$, since $\omega(\mathcal{A})=\infty$. Applying $\chi_{a, a+1}$ will yield an equivalent partition with finite width, since it has $z$ in a low set and $w$ in a high set. If there are $z<w$ such that $z \in A_{1}$ and $w \in A_{3}$, then $w \neq b$, since $\omega(\mathcal{A})=\infty$, and applying $\chi_{b, b+1}$ will yield another equivalent partition with finite width. Thus, there must exist $z<w$ such that $z \in A_{2}$ and $w \in A_{3}$ (otherwise, all the elements of $A_{1} \cup A_{2}$ are greater than all the elements in $A_{3} \cup A_{4}$, contradicting (2)). If $w=b$, then $z<b+1 \in A_{4}$ and we have already considered such a setting. So we assume $w \neq b$. In this case we can apply $\chi a, a+1 \circ \chi_{b, b+1}$ and obtain an equivalent partition with $w$ in the high set and $z$ in the low set, and thus, of finite width.

Lemma 3.4. Let $\mathcal{A}=\{X, Y, Z, W\}$ be a minimal partition of $[n]$ implementing $\left\{p_{i}\right\}_{i=1}^{4}$, of minimal width among all such minimal partitions. Assume that $S(X)=s^{n, 4}-1, S(Y)=s^{n, 4}+1$ and $S(Z)=$ $S(W)=s^{n, 4}$. Let $x \in X$ and $y \in Y$ satisfy $y-x=\omega(\mathcal{A})$. Suppose $x+1 \in A$ and $y-1 \in B$, where $A, B \in\{Z, W\}$. Then
(i) $y-x=2$ or 3, and $y-x=2$ if and only if $A=B$.
(ii) $A-(x+1)$ contains no element a such that $a-1 \in X$
(iii) $A-(x+1)$ contains no element a such that $a+1 \in(X-x)$.
(iv) $B-(y-1)$ contains no element $b$ such that $b+1 \in Y$
(v) $B-(y-1)$ contains no element $b$ such that $b-1 \in(Y-y)$.
(vi) If $A=B$, then $Y-y$ contains no element $c$ such that $c+1 \in X-x$.

Figure 4 indicates in dotted lines the illegal configurations of (ii)-(vi) for the two cases implied by (i). Note that in Figure $4(b)$ the cases (ii) and (iv) are already known from Lemma 2.11(a).


Figure 4

Proof. First note that $\mathcal{A}$ exists by Lemma 3.3. (i) The existence of $z \in A$ satisfying $x+1<z<y$ is not possible since $z$ would be in the low set of $\mathcal{A}_{x, x+1}$ while $y$ is in the high set, contradicting the minimality of $\omega(\mathcal{A})$. Similarly, the existence of $w \in B$ such that $x<w<y-1$ would imply that $w$ is in the high set of $\mathcal{A}_{y-1, y}$ and $x$ in the low set. Again, a contradiction. Thus, $y-x \leqslant 3$. If $x+1$ and $y-1$ are in the same set, then they are equal, by Lemma 2.11(a), and in this case $y-x=2$.

Now, let $a \in A-(x+1)$. (ii) If $a-1 \in X$, then $\mathcal{A}_{a, a-1}$ has $x+1$ in a low set and $y$ in a high set, contradicting the minimality of $\omega(\mathcal{A})$. (iii) If $a+1 \in X-x$ we apply $\chi_{x, x+1}$ and obtain a setup as in (ii). The proofs of (iv) and (v) are similar.
(vi) Suppose such $c \in Y-y$ exists. Note that $\mathcal{A}_{x, x+2} \equiv \mathcal{A}$, but now $c$ is in the low set and $c+1$ is in the high set, contradicting Observation 2.6 .

Lemma 3.5. Let $\mathcal{A}=\left\{A_{i}\right\}_{i=1}^{4}$ be a minimal partition of $[n]$ implementing $\left\{p_{i}\right\}_{i=1}^{4}$, of minimal width among all such minimal partitions. Then, the configurations illustrated in Figure 5 are not possible, assuming $x+2 \neq d$ (Figures 5(d) and 5(f)) and $x \neq d+2$ (Figures 5(e) and 5(f)).


Figure 5

Proof. Assume the configuration in Figure 5(a) exists. Since there is symmetry between the roles of $x$ and $d$, we may assume that $x>d$. Assume there exists $c \neq n$ in $A_{2}-(x+1)$ such that $c+1 \notin A_{2}$. We have $c+1 \notin A_{1}-x($ Lemma 3.4 (iii) $), c+1 \notin A_{3}\left(\right.$ Lemma 2.11(b)) and $c+1 \notin A_{4}$ (Lemma 2.11(a)).

Thus, we must have $c+1=x$. Let $k \geqslant 0$ be minimal such that $c-k \in A_{2}$ but $c-k-1 \notin A_{2}$ (assuming there exists such $k$ ) and denote $u=c-k$. We have, $u-1 \notin A_{1}$, by Lemma 3.4 (ii), and $u-1 \notin A_{4}$, by Lemma $3.4(\mathrm{v})$. If $u-1 \in A_{3}$ it contradicts Lemma 2.11(b). We conclude that such $k$ does not exist.

Thus, if $A_{2}$ contains any element smaller than $x$, it must contain $\{1,2, \ldots, x-1\}$. This is impossible, since $d<x$. Hence, such $c$ does not exist and we must have that $A_{2}=\{x+1, n-l, \ldots, n\}$ for some $l \geqslant 0$. Let $u=n-l$. As in the previous paragraph, $u-1 \notin A_{1} \cup\left(A_{4}-(x+2)\right) \cup A_{3}$. So, we must have $u-1=x+2$. That is, $A_{2}=\{x+1, x+3, \ldots, n\}$.

Using a similar argument as for $A_{2}$ above we conclude that $A_{3}=\{1,2, \ldots, d-1, d+1\}$. Thus, $A_{1} \cup A_{4}=\{d, d+2, \ldots, x, x+2\}$. Since $d+2 \in A_{4}$ and $x \in A_{1}$, there must exist $a \in A_{4}$ such that $a+1 \in A_{1}$. This would contradict Lemma 3.4(vi), unless $a=d+2$ and $a+1=x$. In this case we must have $A_{1}=\{x, d\}$ and $A_{4}=\{x+2, d+2\}$. Hence, $S\left(A_{1}\right)=x+d$ and $S\left(A_{4}\right)=x+d+4$. This yields a contradiction since $S\left(A_{1}\right)=S\left(A_{4}\right)-2$. We conclude that a configuration as in Figure 5(a) is not possible.

Now, if a configuration as in Figure 5(b) exists, applying $\chi_{x, x+1}$ yields an equivalent partition with a configuration similar to the one in Figure 5(a). If a configuration as in Figure 5(c) exists, applying $\chi_{x+1, x+2}$ yields a configuration similar to the one in Figure 5(a). If a configuration as in Figure 5(d) exists, applying $\chi_{x, x+2}$ yields an equivalent partition with a configuration similar to the one in Figure 5(c). If a configuration as in Figure 5(e) exists, applying $\chi_{x, x+2}$ yields a configuration similar to the one in Figure 5(b). Finally, If a configuration as in Figure 5(f) exists, applying $\chi_{x, x+2}$ yields a configuration similar to the one in Figure $5(\mathrm{a})$.

## 4 Proof of Theorem 1.6

We assume, for contradiction, that $d_{\text {min }}^{\mathcal{P}}>0$ for the given $\left\{p_{i}\right\}_{i=1}^{4}$. Let $\mathcal{A}=\left\{A_{1}, A_{2}, A_{3}, A_{4}\right\}$ be a minimal partition of $[n]$ implementing $\left\{p_{i}\right\}_{i=1}^{4}$, such that $\omega(\mathcal{A})$ is minimal among all such minimal partitions. By Lemma 3.1. we may assume that $S\left(A_{1}\right)=s^{n, 4}-1, S\left(A_{2}\right)=S\left(A_{3}\right)=s^{n, 3}$ and $S\left(A_{4}\right)=s^{n, 4}+1$. Let
$x \in A_{1}$ and $y \in A_{4}$ be such that $y-x=\omega(\mathcal{A})$. We may assume that $x+1 \in A_{2}$. By Lemma 3.4(i), there are only two cases to consider: $y-x=2$ and $y-x=3$.

First assume $y-x=2$. That is, $x \in A_{1}, x+1 \in A_{2}$ and $x+2 \in A_{4}$. It will be convenient to notice that we have the same setup as in Figure 4(b), with $X=A_{1}, A=B=A_{2}$ and $Y=A_{4}$. Let $B_{1}=A_{1}-x$, $B_{2}=A_{2}-(x+1), B_{3}=A_{3}$ and $B_{4}=A 4-(x+2)$. We may make two assumptions:

Assumption 1: $n \notin A_{1}$
Assumption 2: $\max _{A_{2}}<\max _{A_{4}}$.
(If $n \in A_{1}$ we apply $\chi_{x, x+2}$, which ensures both assumptions. If $n \notin A_{1}$ and Assumption 2 does not hold, we apply $\chi_{x+1, x+2}$.)

Let $m_{i}$ be the maximal elements in $B_{i}$ for $i=1, \ldots, 4$. By our assumptions, $m_{1}, m_{2}<n$. By Lemma 2.11(a), $m_{2}+1 \notin A_{4}$ and by Lemma 3.4(iii), $m_{2}+1 \notin B_{1}$. Thus, either $m_{2}+1 \in A_{3}$ or $m_{2}+1=x$. Suppose $m_{2}+1 \in A_{3}$. By Assumption 2, there exists $k \geqslant 1$ such that $m_{2}+k \in A_{3}$ and $m_{2}+k+1 \notin A_{3}$ (otherwise $m_{2}+1, m_{2}+2, \ldots, n$ are all in $A_{3}$ and $m_{2}$ would be larger than any element of $A_{4}$ ). Suppose $m_{2}+k+1 \in A_{4}$. If $k=1$ it contradicts Lemma 3.5 (b), and if $k>1$ it contradicts Lemma 2.11(b). Now, suppose $m_{2}+k+1 \in A_{1}$ and $m_{2}+k+1 \neq x$. If $k=1$ it contradicts Lemma 3.5(e), and if $k>1$ we contradict Lemma 2.11(c). Thus, $m_{2}+k+1=x$ for some $k \geqslant 0$. In any case we have $x-1 \in A_{2} \cup A_{3}$. In particular, $x-1 \notin A_{1}$.

Now, by Lemma 2.11(a), $m_{1}+1 \notin A_{2}$, and by Observation 2.6, $m_{1}+1 \notin A_{4}$. It follows that $m_{1}+1 \in A_{3}$. Suppose there exists $l \geqslant 1$ such that $m_{1}+l \in A_{3}$ and $m_{1}+l+1 \notin A_{3}$. Note that $m_{1}+l+1 \neq x$, since we already know that $m_{2}+k+1=x$. Since $m_{1}$ is maximal in $A_{1}-x$, we must have $m_{1}+l+1 \in A_{2} \cup A_{4}$. Suppose $m_{1}+l+1 \in A_{4}$. If $l=1$, then $m_{1}+2 \in A_{4}$ and we contradict Lemma 3.5 (a). If $l>1$, then $m_{1}+1 \neq m_{1}+l$ and we contradict Lemma 2.11 (a). Now, suppose $m_{1}+l+1 \in A_{2}$. If $l=1$ it contradicts Lemma 3.5.(c). If $l>1$, it contradicts Lemma 2.11(b). We conclude that $m_{1}+1, \ldots, n \in A_{3}$ and thus, $m_{4}<n$.

We have $m_{4}+1 \notin A_{2}$, by Assumption 2, and $m_{4}+1 \notin A_{1}$, by Lemma 3.4(vi) and the fact that $x-1 \in A_{2} \cup A_{3}$. So, we must have $m_{4}+1 \in A_{3}$ and there exists $t \geqslant 1$ such that $m_{4}+t \in A_{3}$ and $m_{4}+t+1 \notin A_{3}$ (since $m_{1}+1, \ldots, n \in A_{3}$ ). Clearly, $m_{4}+t+1 \neq x+2$, so $m_{4}+t+1$ can only be in $A_{1}$, by Assumption 2. Now, $m_{4}+t+1 \neq x$, since $m_{2}+k+1=x$. So, $m_{4}+t+1 \in B_{1}$. If $t=1$ it contradicts Lemma 3.5 (f), and if $t>1$, we have a contradiction to Lemma 2.11(e). This concludes the case where $y-x=2$.

Now assume $y-x=3$. By Lemma 3.4 we have $x+1 \in A_{2}, x+2 \in A_{3}$ and $y=x+3 \in A_{4}$. It will be convenient to notice that we have a setup similar to the one in Figure 4(a) with $X=A_{1}$, $A=A_{2}, B=A_{3}, Y=A_{4}$ and $y=x+3$. Let $B_{1}=A_{1}-x, B_{2}=A_{2}-(x+1), B_{3}=A_{3}-(x+2)$ and $B_{4}=A_{4}-(x+3)$.

Let $z \in B_{1}$ and assume $z \neq n$. We know that $z+1 \notin A_{4}$ by Observation [2.6, $z+1 \notin A_{2}$ by Lemma 3.4(ii), and $z+1 \notin A_{3}$ by Lemma 2.11(a). Thus, $z+1 \in A_{1}$. Now, let $z \in B_{2}$ and $z \neq n$. We have, $z+1 \notin A_{4}$ by Lemma 2.11 (a), and $z+1 \notin B_{1}$ by Lemma 3.4 (iii). Also, $z+1 \notin A_{3}$, by Lemma 2.11(b). Thus, $z+1 \in B_{2}$ or $z+1=x$. It follows that either $B_{1}$ or $B_{2}$ is equal to $\{t, t+1, \ldots, x-1\}$ for some $t \geqslant 1$ and the other is equal to $\{s, s+1, \ldots, n\}$ for some $s>x+3$. We may assume $A_{1}=\{x, s, s+1, \ldots, n\}$ and $A_{2}=\{t, t+1, \ldots, x-1, x+1\}$ (by applying $\chi_{x, x+1}$ if necessary).

Let $z>1$ be an element of $B_{4}$. We have $z-1 \notin A_{1}$ by Observation 2.6. $z-1 \notin A_{2}$ by Lemma 2.11(a), and $z-1 \notin A_{3}$ by Lemma 3.4(iv). Thus, $z-1 \in A_{4}$. Let $z>1$ be an element of $B_{3}$. We have $z-1 \notin A_{1}$ by Lemma 2.11(a), $z-1 \notin B_{4}$ by Lemma 3.4(v), and $z-1 \notin A_{2}$, by Lemma 2.11(b). Thus, $z-1 \in B_{3}$ or $z-1=x+3$. We conclude that either $B_{3}$ or $B_{4}$ is equal to $\{1,2, \ldots, t-1\}$ and the other is equal to $\{x+4, \ldots, s-1\}$ for the same $t$ and $s$ as above. Since none of the $A_{i}$ 's has size 1 , we must have that $1<t \leqslant x-1$ and $x+3<s \leqslant n$. By applying $\chi_{x+2, x+3}$ if necessary, we may assume that $A_{3}=\{1,2, \ldots, t-1, x+2\}$ and $A_{4}=\{x+3, x+4, \ldots, s-1\}$ (Figure 6(a).

Note that $s^{n, 4}=S\left(A_{4}\right)-1 \geqslant(x+3)+(x+4)-1=2 x+6$. Thus, $\left|A_{2}\right|>2$, that is, $t<x-1$. Let $\mathcal{A}^{\prime}$ be partition consisting of $A_{1}^{\prime}=A_{1}, A_{2}^{\prime}=A_{2} \backslash\{t, x+1\} \cup\{t-1, x+2\}, A_{3}^{\prime}=A_{3} \backslash\{t-1, x+2\} \cup\{t, x+1\}$, and $A_{4}^{\prime}=A_{4}$ (Figure 6(b)). Note that $S\left(A_{i}^{\prime}\right)=S\left(A_{i}\right)$ for $i=1, \ldots, 4$, and thus $\mathcal{A}^{\prime}$ is also minimal with the same minimal width. Since $t<x-1$, we have $t+1 \in A_{2}^{\prime}$ and we have a contradiction to Lemma 2.11(b). This completes the proof.

Remark 4.1. In the case $k=2$ there is yet another simple proof:


Figure 6

Suppose $p_{1}+p_{2}=n, p_{1} \leqslant p_{2}$ and $\sum_{i=1}^{p_{1}}(n-i+1) \geqslant s^{n, 2}$ (Condition (2)). Consider the following two partitions of $[n]$ : $\mathcal{A}=\left\{A_{1}, A_{2}\right\}$ and $\mathcal{B}=\left\{B_{1}, B_{2}\right\}$, such that $\left|A_{i}\right|=\left|B_{i}\right|=p_{i}$, for $i=1,2$, $A_{1}=\left\{1, \ldots, p_{1}\right\}$ and $B_{1}=\left\{n-p_{1}+1, \ldots, n\right\}$. We have $\left|A_{1}\right|<s^{n, 2}$ since $p_{1} \leqslant p_{2}$ and $\left|B_{1}\right| \geqslant s^{n, 2}$ by (21). We show that we can switch from $\mathcal{A}$ to $\mathcal{B}$ by a sequence of operations of the form $\chi_{a, a+1}$. Thus, at some point along the way we must have an equitable partition.

We start with partition $\mathcal{A}$ and apply $\chi_{1,2} \circ \chi_{2,3} \circ \cdots \circ \chi_{p_{1}-1, p_{1}} \circ \chi_{p_{1}, p_{1}+1}$. This results in the partition consisting of $\left\{2,3, \ldots, p_{1}, p_{1}+1\right\}$ and $\left\{1, p_{1}+2, \ldots, n\right\}$. Then, we apply $\chi_{2,3} \circ \cdots \circ \chi_{p_{1}+1, p_{1}+2}$, resulting in $\left\{3,4, \ldots, p_{1}+2\right\}$ and $\left\{1,2, p_{1}+3, \ldots, n\right\}$, and so on. Eventually we arrive at partition $\mathcal{B}$.

It might be possible to generalize this continuity approach to higher $k$ 's by applying a higher dimensional continuity technique, such as Sperner's theorem. The problem is to define the right division into ( $k-1$ )-dimensional simplices.

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