# Difference Covering Arrays and PseudoOrthogonal Latin Squares 

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#### Abstract

Difference arrays are used in applications such as software testing, authentication codes and data compression. Pseudo-orthogonal Latin squares are used in experimental designs. A special class of pseudo-orthogonal Latin squares are the mutually nearly orthogonal Latin squares (MNOLS) first discussed in 2002, with general constructions given in 2007. In this paper we develop row complete MNOLS from difference covering arrays. We will use this connection to settle the spectrum question for sets of 3 mutually pseudo-orthogonal Latin squares of even order, for all but the order 146.


## 1 Introduction

Difference matrices are a fundamental tool used in the construction of combinatorial objects, generating a significant body of research that has identified a number of existence constraints. These difference matrices have been used for diverse applications, for instance, in the construction of authentication codes without secrecy [15], software testing [4], [5] and data compression [10]. This diversity of applications, coupled with existence constraints, has motivated authors to generalise the definition to holey difference matrices, difference covering arrays and difference packing arrays, to mention just a few.

In the current paper we are interested in constructing subclasses of cyclic difference covering arrays and exploiting these structures to emphasize new connections with other combinatorial objects, such as pseudoorthogonal Latin squares. We use this connection to settle the existence spectrum for sets of 3 mutually pseudo-orthogonal Latin squares of even order, in all but one case. We begin with the formal definitions.

A difference matrix (DM) over an abelian group $(G,+)$ of order $n$ is defined to be an $n \times k$ matrix $Q=[q(i, j)]$ with entries from $G$ such that, for all pairs of columns $0 \leq j, j^{\prime} \leq k-1, j \neq j^{\prime}$, the difference set

$$
\Delta_{j, j^{\prime}}=\left\{q(i, j)-q\left(i, j^{\prime}\right) \mid 0 \leq i \leq n-1\right\}
$$

contains every element of $G$ equally often, say $\lambda$ times. (See, for instance, [6], [8] and [9].) Note that we label the rows from 0 to $n-1$ and the columns 0 to $k-1$. Also to be consistent with later sections involving Latin squares and covering arrays our definition uses the transpose of the matrix given in [6] and [8]. Since the addition of a constant vector, over $G$, to all rows and a constant vector to any column does not alter the set $\Delta_{j, j^{\prime}}$, we may assume that one row and one column contain only 0 , the identity element of $G$. More precisely, to simplify later calculations, we will assume that all entries in the last row and last column of $Q$ are 0 . A difference matrix will be denoted $\mathrm{DM}(n, k ; \lambda)$. If $(G,+)$ is the cyclic group we refer to a cyclic difference matrix.

Theorem 1.1. [6, Thm 17.5, p 411] $A \operatorname{DM}(n, k ; \lambda)$ does not exist if $k>\lambda n$.
In the main, we will use difference matrices with $k=4, \lambda=1$ and where possible we will work with cyclic difference matrices. In Section 3 we list a number of existence results that will be relevant to the current paper.

A holey difference matrix (HDM) over an abelian group ( $G,+$ ) of order $n$ with a subgroup $H$ of order $h$ is defined to be an $(n-h) \times k$ matrix $Q=[q(i, j)]$ with entries from $G$ such that, for all pairs of columns $0 \leq j, j^{\prime} \leq k-1, j \neq j^{\prime}$, the difference set

$$
\Delta_{j, j^{\prime}}=\left\{q(i, j)-q\left(i, j^{\prime}\right) \mid 0 \leq i \leq n-h-1\right\}
$$

contains every element of $G \backslash H$ equally often, say $\lambda$ times. A holey difference matrix will be denoted $\operatorname{HDM}(k, n ; h)$, where $|G|=n$ and $|H|=h$. If $G$ is the cyclic group then we refer to a cyclic holey difference matrix.
Remark 1.2. As before a constant vector may be added to any column without affecting $\Delta_{j, j^{\prime}}$ so we may assume that all entries in the last column of $Q$ are equal to 0 . However since $H$ is a subgroup, 0 belongs to the hole. Consequently 0 does not occur in $\Delta_{j, j^{\prime}}$, and thus there will be no row containing two or more 0 's. Further since $\Delta_{j, k-1}=G \backslash H, 0 \leq j \leq k-2$, the entries of $H$ do not occur in the first $k-1$ columns of $Q$.

A difference covering (packing) array over an abelian group $(G,+)$ of order $n$ is defined to be an $\eta \times k$ matrix $Q=[q(i, j)]$ with entries from $G$ such that, for all pairs of distinct columns $0 \leq j, j^{\prime} \leq k-1$, the difference set

$$
\Delta_{j, j^{\prime}}=\left\{q(i, j)-q\left(i, j^{\prime}\right) \mid 0 \leq i \leq \eta-1\right\}
$$

contains every element of $G$ at least (at most) once. (See, for instance, [19] and [20].) A difference covering array will be denoted $\operatorname{DCA}(k, \eta ; n)$ and a difference packing array will be denoted $\mathrm{DPA}(k, \eta ; n)$. If $(G,+)$ is the cyclic group, then the difference covering (packing) array is said to be cyclic.

Difference covering arrays have been studied in their own right and are related to mutually orthogonal partial Latin squares and transversal coverings, with applications in information technology, see [1] and [14].

As before we may assume that the last row and last column of a $\mathrm{DCA}(k, \eta ; n)$ contain only 0 .
In the papers [19] and [20, Yin constructs cyclic $\operatorname{DCA}(4, n+1 ; n)$ for all even integers $n$, with similar results for cyclic difference packing arrays. Yin documents a number of product constructions for difference covering arrays, some of which will be reviewed in Section 3 and then adapted to construct difference covering arrays with specific properties; properties that build connections with pseudo-orthogonal Latin squares.

The additional properties that we seek are that 0 (the entry relating to identity element of $G$ ) occurs at least twice in each column of the $\operatorname{DCA}(k, n+1 ; n)$ and for pairs of columns, not including the last column, the repeated difference is not the element 0 . Formally we are interested in $\mathrm{DCA}(k, n+1 ; n), Q=[q(i, j)]$, ( $0 \leq i \leq n, 0 \leq j \leq k-1$ ) satisfying the properties:

P1. the entry $0 \in G$ occurs at least twice in each column of $Q$, and
P2. for all pairs of distinct columns $j$ and $j^{\prime}, j \neq k-1 \neq j^{\prime}, \Delta_{j, j^{\prime}}=\left\{q(i, j)-q\left(i, j^{\prime}\right) \mid 0 \leq i \leq n-1\right\}=G \backslash\{0\}$,
Note that this last property implies that $\Delta_{j, j^{\prime}}$ contains a repeated difference that is not 0 .
The following example, of cyclic $\mathrm{DCA}(4,7 ; 6)$ that satisfies P 1 and P 2 , is taken from 13 .

$$
B^{T}=\left[\begin{array}{lllllll}
0 & 1 & 2 & 3 & 4 & 5 & 0 \\
1 & 3 & 5 & 0 & 2 & 4 & 0 \\
3 & 0 & 4 & 1 & 5 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

In the next lemma we show that if $G$ is the cyclic group over $\mathbb{Z}_{n}$, then these conditions imply that for all distinct columns $j$ and $j^{\prime}, j \neq k-1 \neq j^{\prime}$,

$$
\Delta_{j, j^{\prime}}=\{0,1,2, \ldots, n / 2, n / 2, \ldots, n-1\}
$$

with repetition retained.
Lemma 1.3. If there exists a cyclic $D C A(k, n+1 ; n), Q=[q(i, j)],(0 \leq i \leq n, 0 \leq j \leq k-1)$ satisfying Properties P1 and P2, then $n$ is even. Further, given $d_{0}$ such that $d_{0}=q(i, j)-q\left(i, j^{\prime}\right)=q\left(i^{\prime}, j\right)-q\left(i^{\prime}, j^{\prime}\right)$, for $i \neq i^{\prime}$ and $k-1 \neq j \neq j^{\prime} \neq k-1$, then $d_{0}=n / 2$.

Proof. Let $Q=[q(i, j)](0 \leq i \leq n, 0 \leq j \leq k-1)$ represent the difference covering array. The definition requires that $\mathbb{Z}_{n} \subseteq \Delta_{j, j^{\prime}}$ and since column $k-1$ of $Q$ contains all zeros, Property P1 implies that the remaining columns are permutations of the multi-set $\{0,0,1,2, \ldots, n-1\}$.

Let $d_{0} \in \mathbb{Z}_{n} \backslash\{0\}$ represent the repeated difference in $\Delta_{j, j^{\prime}}$. Suppose $n$ is odd and, without loss of generality, that column 0 is in standard form. Then, for all $0<j \leq k-2, \sum_{i=0}^{n-1} q(i, j)=\frac{(n-1) n}{2}$ and

$$
\sum_{i=0}^{n-1}(i-q(i, j)) \equiv \frac{(n-1) n}{2}+d_{0} \quad \bmod n
$$

Consequently $2 d_{0}=n(2 u-n+1)$, or equivalently $n \mid 2 d_{0}$. But since $n$ is odd, this leads to the contradiction, $d_{0} \in \mathbb{Z}_{n}$ and $n \mid d_{0}$. Thus $n$ is $2 p$ for some integer $p$, where $p$ divides $d_{0}$, implying $d_{0}=p$.

The remainder of this paper is organised as follows. In Section 2 we will draw the connection between $\operatorname{DCA}(k, n+1 ; n)$ and sets of mutually pseudo-orthogonal Latin squares and for a subclass of squares settle the spectrum question for all but a single order, namely 146. In Section 3 we review some of the general constructions for difference covering arrays and show that these constructions can be used to construct $\mathrm{DCA}(k, n+1 ; n)$ that satisfy Properties P1 and P2. In Section 4 we give three new constructions for $\operatorname{DCA}(4, n+1 ; n)$ 's and consequently new families of mutually pseudo-orthogonal Latin squares.

The notation $[a, b]=\{a, a+1, \ldots, b-1, b\}$ refers to the closed interval of integers from $a$ to $b$.

## 2 Pseudo-orthogonal Latin squares and difference covering arrays

In this section we verify that cyclic difference covering arrays can be used to construct pseudo-orthogonal Latin squares.

A Latin square of order $n$ is an $n \times n$ array in which each of the symbols of $\mathbb{Z}_{n}$ occurs once in every row and once in every column. Two Latin squares $A=[a(i, j)]$ and $B=[b(i, j)]$, of order $n$, are said to be orthogonal if

$$
O=\{(a(i, j), b(i, j)) \mid 0 \leq i, j \leq n-1\}=\mathbb{Z}_{n} \times \mathbb{Z}_{n}
$$

A set of $t$ Latin squares is said to be mutually orthogonal, $t-\operatorname{MOLS}(n)$, if they are pairwise orthogonal. A set of $t$ idempotent $\operatorname{MOLS}(n)$, denoted $t-\operatorname{IMOLS}(n)$, is a set of $t-\operatorname{MOLS}(n)$ each of which is idempotent; that is, the cell $(i, i)$ contains the entry $i$, for all $0 \leq i \leq n-1$.

It is well known that difference matrices can be used to construct sets of mutually orthogonal Latin squares, see for instance [9, Lemma 6.12].

While the applications of orthogonal Latin squares are well documented, there are still many significant existence questions unanswered. For instance, it is known that there is no pair of MOLS(6), however it is not known if there exists a set of three MOLS(10), or four MOLS(22), see [6]. The existence of a set of four MOLS(14) was established by Todorov [16] in 2012, but it is not known if there exists a set of five MOLS(14). Many of the existence results have been obtained using quasi-difference matrices or difference matrices with holes, see 6].

The importance and applicability of MOLSs combined with these difficult open questions has motivated authors, such as Raghavarao, Shrikhande and Shrikhande [13] and Bate and Boxall [2], to slightly vary the orthogonality condition to that of pseudo-orthogonal. A pair of Latin squares, $A=[a(i, j)]$ and $B=[b(i, j)]$, of order $n$, is said to be pseudo-orthogonal if given $O=\{(a(i, j), b(i, j)) \mid 0 \leq i, j \leq n-1\}$, for all $a \in \mathbb{Z}_{n}$

$$
|\{(a, b(i, j)) \mid(a, b(i, j)) \in O\}|=n-1
$$

That is, each symbol in $A$ is paired with every symbol in $B$ precisely once, except for one symbol with which it is paired twice and one symbol with which it is not paired at all. A set of $t$ Latin squares, of order $n$, are said to be mutually pseudo-orthogonal if they are pairwise pseudo-orthogonal.

The value and applicability of pseudo-orthogonal Latin squares has been established through applications to multi-factor crossover designs in animal husbandry [2], and strongly regular graphs [3] (though the definition varies here). Mutually nearly orthogonal Latin squares (MNOLS) are a special class of pseudo-orthogonal Latin squares, in that the set $O$ does not contain the pair $(a, a)$, for any $a \in \mathbb{Z}_{n}$. Mutually nearly orthogonal Latin squares (MNOLS) were first discussed in a paper by Raghavarao, Shrikhande and Shirkhande in 2002 [13].

A natural question to ask is: Can we use difference techniques to construct mutually pseudo-orthogonal Latin squares? Raghavarao, Shrikhande and Shirkhande did precisely this and constructed mutually pseudoorthogonal Latin squares from cyclic $\operatorname{DCA}(k, n+1 ; n)$ termed $(k, n)$-difference sets in [13. The Raghavarao, Shrikhande and Shirkhande result is as follows.

Theorem 2.1. If there exists a cyclic $D C A(t+1,2 p+1 ; 2 p), Q^{\prime}=\left[q^{\prime}(i, j)\right]$, that satisfies P1 and P2, then there exists a set of $t$ pseudo-orthogonal Latin squares of order $2 p$.

Proof. Recall that without loss of generality we may assume that the last row and column of $Q^{\prime}$ contain all zeros. Construct a new matrix $Q=[q(i, j)]$ by removing the last row and last column from $Q^{\prime}$ and define a set of $t$ arrays, $L_{s}=\left[l_{s}(i, j)\right], 0 \leq s \leq t-1$, of order $2 p$, by

$$
\begin{equation*}
l_{s}(i, j)=q(i, s)+j(\bmod 2 p), 0 \leq i, j \leq 2 p-1 \tag{2.1}
\end{equation*}
$$

It is easy to see that each column of $L_{s}$ is a permutation of $\mathbb{Z}_{2 p}$ and so $L_{s}$ is a Latin square. By Lemma 1.3

$$
\Delta_{j, j^{\prime}}=\left\{q^{\prime}(i, j)-q^{\prime}\left(i, j^{\prime}\right) \mid 1 \leq i \leq 2 p\right\}=\left(\mathbb{Z}_{2 p} \backslash\{0\}\right) \cup\{p\}
$$

implying that when any two Latin squares are superimposed we obtain the set of ordered pairs $\left(\left\{\mathbb{Z}_{2 p} \times \mathbb{Z}_{2 p}\right\} \backslash\right.$ $\{(x, x) \mid 0 \leq x \leq 2 p-1\}) \cup\{(x, x+p) \mid 0 \leq x \leq 2 p-1\}$ with repetition retained.

If there exists a pair of pseudo-orthogonal Latin squares generated from cyclic difference covering arrays satisfying P1 and P2, then there exists a pair of nearly orthogonal Latin squares. Conversely, a pair of nearly orthogonal Latin squares are necessarily pseudo-orthogonal Latin squares. Given this and the strong connection with papers 11 and 13 we will state all results in terms of mutually nearly orthogonal Latin squares.

Raghavarao, Shrikhande and Shirkhande established bounds on the maximum number of Latin squares in a set of mutually nearly orthogonal Latin squares. This result provides bounds on $k$ for $\mathrm{DCA}(k, n+1 ; n)$ that satisfy P1 and P2.
Lemma 2.2. Let $p \geq 2$ be a positive integer. If there exists a cyclic $D C A(k+1,2 p+1 ; 2 p)$ that satisfies $P 1$ and P2, then $k \leq p+1$. Further if $p$ is even and there exists a $D C A(k, 2 p+1 ; 2 p)$, then $k<p+1$.

Proof. If $k>p+1$ then there exists a set of more than $(p+1)$ - $\operatorname{MNOLS}(2 p)$, which contradicts Raghavarao, Shrikhande and Shrikhande result. Similarly for the second statement.

### 2.1 Some interesting facts

It is also interesting to note that the MNOLS $(2 p)$ constructed from cyclic difference covering arrays are essentially copies of the cyclic group. Consequently, these Latin squares are all bachelor squares, in that they have no orthogonal mate.

We also note that these sets of Latin squares are row complete. A row complete Latin square, $L=[l(i, j)]$ is one in which the columns can be reordered in such a way that the set $\left\{\left(l(i, j), l(i, j+1) \mid i \in \mathbb{Z}_{n}, 0 \leq j \leq\right.\right.$ $n-1\}=\mathbb{Z}_{n} \times \mathbb{Z}_{n}$. So the set of entries obtained by taking pairs of adjacent cells in the same row, for all rows, gives the set of all ordered pairs on $\mathbb{Z}_{n}$.

Williams [18] verified that the columns of the Latin square corresponding to the Cayley table of the cyclic group can be rearranged to obtain a row complete Latin square.

Each of the $k$ MNOLS $(2 p)$ constructed from cyclic $\mathrm{DCA}(k+1,2 p+1 ; 2 p)$ can be obtained by reordering the rows of the Cayley table of the cyclic group, without touching the columns. Hence simultaneously reordering the columns of these nearly orthogonal Latin squares will also produce row complete pseudo-orthogonal Latin squares.

## 3 The spectrum for sets of 3 mutually nearly orthogonal Latin squares

In 2007, Li and van Rees [11] continued the study of $3-\operatorname{MNOLS}(n)$ and conjectured that they exist for all even $n \geq 6$. In a partial answer to this question, Li and van Rees proved the existence for small orders and orders greater than 356 , (see also [12]).
Theorem 3.1. [11, Thm 4.8] If $2 p \geq 358$, then there exists a $3-M N O L S(2 p)$.

This work was extended in 2014, when Demirkale, Donovan and Khodkar [7] developed further constructions for cyclic $\mathrm{DCA}(4,2 p+1 ; 2 p)$ proving:

Theorem 3.2. 77] There exist $3-M N O L S(2 p)$, where $2 p \equiv 14,22,38,46 \bmod 48$.
The next result lists known values for cyclic $\mathrm{DCA}(4,2 p+1 ; 2 p)$ satisfying P 1 and P 2 , with $2 p \leq 356$.
Lemma 3.3. There exists cyclic $D C A(4,2 p+1 ; 2 p)$ for $2 p=6,8, \ldots, 20,22,38,46,62,70,86,94,110,118$, $134,142,158,166,182,190,206,214,230,238,254,262,278,286,302,310,326,334,350$.

Proof. The existence of orders 6 and 8 was given in [13] and orders 10, 12, 14, 16, 18, 20 in [11]. All the remaining cases were shown to exist in [7].
van Rees recently summarised these results and indicated that 3 -MNOLS $(2 p)$ exist for all orders except possibly those given below.

Lemma 3.4. 17 A set of $3-M N O L S(2 p)$ exists except possibly when $2 p=24,26,28,30,34,36,42,50,52,54$, $58,66,74,82,92,102,106,114,116,122,124,130,138,146,148,170,172,174,178$.

In this section we will settle the existence question for all even orders except $2 p=146$. For completeness we list all values less than $2 p=358$ and, given the connection with row complete Latin squares, where possible we will use cyclic difference covering arrays to construct the Latin squares. Therefore we begin this section by reviewing relevant results from [8], [19] and [20, and adapting these to construct cyclic DCA $(4,2 m+1 ; 2 m)$ that satisfy P1 and P2. Where appropriate we will indicate how these results can be used to settle the spectrum question.

We begin with the following straight forward result that is analogous to [8, Lem 2.3].

Lemma 3.5. Suppose that there exists a $\operatorname{HDM}(k, n ; h)$ over the group $(G,+)$ with a hole over the subgroup $H$. Further suppose there exists a $D C A(k, h+1 ; h)$ over $H$ satisfying $P 1$ and $P 2$. Then there exists a $D C A(k, n+1 ; n)$ over $G$ satisfying P1 and P2. Further suppose that the $\operatorname{HDM}(k, n ; h)$ and $D C A(k, h+1 ; h)$ are cyclic. Then there exists a cyclic $D C A(k, n+1 ; n)$ satisfying P1 and P2.

Proof. In the cyclic case, let $A=[a(i, j)](0 \leq i \leq n-1-h, 0 \leq j \leq k-1)$ represent the cyclic $\operatorname{HDM}(k, n ; h)$ and $B=[b(i, j)](0 \leq i \leq h, 0 \leq j \leq k-1)$ represent the cyclic $\operatorname{DCA}(k, h+1 ; h)$. The definition of cyclic implies that $H=\{0, u, 2 u, \ldots(h-1) u\}$, where $n=u h$ and the proof of Lemma 1.3 implies that $h$ is even and the repeated difference in $\Delta_{j, j^{\prime}}, j \neq k-1 \neq j^{\prime}$, of $B$ is $h u / 2=n / 2$.

Set $Q=[q(i, j)](0 \leq i \leq n, 0 \leq j \leq k)$ to be the concatenation of $A$ with $B$ and we obtain a cyclic $\mathrm{DCA}(k, n+1 ; n)$ that satisfies P1 and P2.

The non-cyclic case follows similarly.

Next we give a general product type construction taken from [8] and adapt it to construct cyclic difference covering arrays that satisfy P1 and P2.

Lemma 3.6. [8, Lem 2.6] If both a cyclic $\operatorname{HDM}(k, n ; h)$ and a cyclic $D M\left(n^{\prime}, k ; 1\right)$ exist, then so does a cyclic $\operatorname{HDM}\left(k, n n^{\prime} ; h n^{\prime}\right)$. In particular, if there exists a cyclic $D M(n, k ; 1)$ and a cyclic $D M\left(n^{\prime}, k ; 1\right)$ then there exists cyclic $D M\left(n n^{\prime}, k ; 1\right)$.

The first statement of Lemma 3.6 coupled with Lemma 3.5 leads to the following straightforward result.
Corollary 3.7. Suppose that there exists a cyclic $\operatorname{HDM}(k, n ; h)$, a cyclic $D M\left(n^{\prime}, k ; 1\right)$ and a cyclic $D C A\left(k, h n^{\prime}+\right.$ 1 ; $h n^{\prime}$ ) that satisfies P1 and P2. Then there exists a cyclic $D C A\left(k, n n^{\prime}+1 ; n n^{\prime}\right)$ that satisfies P1 and P2.

The second statement of Lemma 3.6 can also be adapted.
Lemma 3.8. Suppose a cyclic $D M(n, k ; 1)$, a cyclic $D M\left(n^{\prime}, k ; 1\right)$ and a cyclic $D C A\left(k, n^{\prime}+1 ; n^{\prime}\right)$ satisfying P1 and P2 exist. Then there exists a cyclic $D C A\left(k, n n^{\prime}+1 ; n n^{\prime}\right)$ that satisfies P1 and P2.

Proof. This result can be obtained by taking a hole of size 1 in Corollary 3.7 or as follows. Let $A=[a(i, j)]$ $(0 \leq i \leq n-1,0 \leq j \leq k-1)$ represent the $\operatorname{cyclic} \operatorname{DM}(n, k ; 1), B=[b(i, j)]\left(0 \leq i \leq n^{\prime}-1,0 \leq j \leq k-1\right)$ represent the cyclic $\operatorname{DM}\left(n^{\prime}, k ; 1\right)$ and $C=[c(i, j)]\left(0 \leq i \leq n^{\prime}, 0 \leq j \leq k-1\right)$ represent the cyclic $\mathrm{DCA}\left(k, n^{\prime}+1 ; n^{\prime}\right)$. Recall that $a(n-1, j)=b\left(n^{\prime}-1, j\right)=c\left(n^{\prime}, j\right)=0$ for all $0 \leq j \leq k-1$ and $a(i, k-1)=b\left(i^{\prime}, k-1\right)=c\left(i^{\prime \prime}, k-1\right)=0$ for all $0 \leq i \leq n-1,0 \leq i^{\prime} \leq n^{\prime}-1$, and $0 \leq i^{\prime \prime} \leq n^{\prime}$.

Construct a matrix $Q=[q(i, j)]$ where, for $0 \leq i \leq n-1,0 \leq i^{\prime} \leq n^{\prime}-1,0 \leq i^{\prime \prime} \leq n^{\prime}$ and $0 \leq j \leq k-1$,

$$
\begin{aligned}
q\left(i+i^{\prime} n, j\right) & =a(i, j)+b\left(i^{\prime}, j\right) n, \text { when } i \neq n-1 \\
q\left(n-1+i^{\prime \prime} n, j\right) & =c\left(i^{\prime \prime}, j\right) n .
\end{aligned}
$$

Then

$$
\begin{aligned}
\Delta_{j, j^{\prime}}= & \left\{a(i, j)+b\left(i^{\prime}, j\right) n-a\left(i, j^{\prime}\right)-b\left(i^{\prime}, j^{\prime}\right) n \mid 0 \leq i \leq n-2,0 \leq i^{\prime} \leq n^{\prime}-1\right\} \\
& \cup\left\{c\left(i^{\prime \prime}, j\right) n-c\left(i^{\prime \prime}, j^{\prime}\right) n \mid 0 \leq i^{\prime \prime} \leq n^{\prime}\right\} \\
= & \mathbb{Z}_{n} \backslash\left\{0, n, 2 n, \ldots, n\left(n^{\prime}-1\right)\right\} \cup\left\{0, n, 2 n, \ldots, n n^{\prime} / 2, n n^{\prime} / 2, \ldots\right. \\
& \left.n\left(n^{\prime}-1\right)\right\}
\end{aligned}
$$

Properties P1 and P2 follow as in the proof of Lemma 3.5,

This result can be generalised to construct non-cyclic difference covering arrays as in 19 .
We now combine these results with various results of [6], 8] and [20] to obtain results for cyclic DCA $(4,2 p+$ $1 ; 2 p$ ), satisfying P1 and P2, implying new existence results for row complete 3 -MNOLS $(2 p)$, where $2 p<358$. In the lists below $*$ indicates that the existence of $3-\operatorname{MNOLS}(2 p)$ was previously unknown.

In doing this we will settle the remaining cases in Lemma 3.4 with the exception of $2 p=146$. Here we believe that there exists a $\operatorname{DCA}(4,147 ; 146)$ satisfying P 1 and P 2 but have been unable to verify it.

Theorem 3.9. [6, Thm 17.6, $p$ 411] If $n$ is a prime greater than or equal to $k$, then there exists a cyclic $D M(n, k ; 1)$.

Lemma 3.10. [20, Lem 2.5] Let $n \geq 5$ be prime. Then there exists a cyclic $\operatorname{HDM}(4,2 n ; 2)$.
Lemma 3.11. There exists cyclic $D C A(4,2 p+1 ; 2 p)$ for $2 p=50^{*}, ~ 98,170^{*}, 242,290,338$.
Proof. Corollary 3.7 together with Theorem 3.9 and Lemma 3.10 can be used first to construct a cyclic $\operatorname{HDM}(4,2 p ; h)$ with $2 p(h)=50(10), 98(14), 170(10), 242(22), 290(10), 338(26)$ and then the required $\mathrm{DCA}(4,2 p+$ $1 ; 2 p)$.

Lemmas 3.13, 3.15, 3.17 do not document any new existence results, however they do verify that for the given orders cyclic $\mathrm{DCA}(4,2 p+1 ; 2 p)$ satisfying P 1 and P 2 exist.

Lemma 3.12. [20, Thm 2.1] Let $n$ be an odd positive integer satisfying $\operatorname{gcd}(n, 9) \neq 3$. Then there exists a cyclic $\operatorname{HDM}(4,2 n ; 2)$.

Lemma 3.13. There exists cyclic $D C A(4,2 p+1 ; 2 p)$ for $2 p=90,126,198,234,306,342$.
Proof. Corollary 3.7 together with Theorem 3.9 and Lemma 3.12 can be used first to construct a cyclic $\operatorname{HDM}(4,2 p ; h)$ with $2 p(h)=90(10), 126(14), 198(22), 234(26), 306(34), 342(38)$ and then the required $\mathrm{DCA}(4,2 p+$ $1 ; 2 p)$.

Theorem 3.14. [20, Thm 2.3] Let $n \geq 4$ and $n=2^{\alpha} 3^{\beta} p_{1}^{\alpha_{1}} \ldots p_{t}^{\alpha_{t}}$, where $(\alpha, \beta) \neq(1,0), \alpha_{i} \geq 0$ and the prime factors $p_{i} \geq 5$ for $1 \leq i \leq t$. Then there exists a cyclic $\operatorname{HDM}(4,2 n ; 2)$.

Lemma 3.15. There exists cyclic $D C A(4,2 p+1 ; 2 p)$ for $2 p=60,80,84,100,112,120,132,156,160,168$, $176,180,204,208,224,228,240,252,264,272,276,300,304,312,320,336,352$.

Proof. Corollary 3.7 together with Theorem 3.9 and Theorem 3.14 can be used first to construct a cyclic $\operatorname{HDM}(4,2 p ; h)$ with $2 p(h)=60(10), 80(10), 84(14), 100(10), 112(14), 120(10), 132(22), 156(26), 160(10)$, $168(14)$, $176(22)$, $180(10), 204(34)$, $208(26)$, $224(14)$, $228(38), 240(10), 252(14), 264(22), 272(34), 276(46)$, $300(10), 304(38), 312(26), 320(10), 336(14), 352(22)$ and then the required $\mathrm{DCA}(4,2 p+1 ; 2 p)$.

Theorem 3.16. [20, Thm 2.4] Let $n=p_{1}^{\alpha_{1}} \ldots p_{t}^{\alpha_{t}}$, where $\alpha_{i} \geq 0$ and the prime factors $p_{i} \geq 5$ for $1 \leq i \leq t$. Then there exists a cyclic $\operatorname{HDM}(4,4 n ; 4)$.

Lemma 3.17. There exists cyclic $D C A(4,2 p+1 ; 2 p)$ for $2 p=140,196,220,260,308,340$.
Proof. Corollary 3.7 together with Theorem 3.9 and Theorem 3.16 can be used first to construct a cyclic $\operatorname{HDM}(4,2 p ; h)$ with $2 p(h)=140(28), 196(28), 220(20), 260(20), 308(28), 340(20)$ and then the required DCA $(4,2 p+$ $1 ; 2 p)$.

Theorem 3.18. [8, Thm 3.10] A cyclic $D M\left(3^{i}, 5 ; 1\right)$ exists for all $i \geq 3$.
Lemma 3.19. There exists cyclic $D C A(4,2 p+1 ; 2 p)$ for $2 p=216,270,324$.
Proof. Corollary 3.7 together with Theorem 3.18 and Theorem 3.14 can be used to first construct a cyclic $\operatorname{HDM}(4,2 p ; h)$ with $2 p(h)=216(54), 270(54), 324(54)$ and then the required $\mathrm{DCA}(4,2 p+1 ; 2 p)$.

The next result from Yin's paper [20] is interesting in that it allows us to construct difference covering arrays and so nearly orthogonal Latin squares of order $6 n$, and gives many values that were previously unresolved (the obstruction was the non-existence of MNOLS of order 6).

Theorem 3.20. [20, Thm 2.2] Let $n$ be a positive integer of the form $p_{1}^{\alpha_{1}} p_{2}^{\alpha_{2}} \ldots p_{t}^{\alpha_{t}}$, where $\alpha_{i} \geq 0$ and the prime factors $p_{i} \geq 5$ for $1 \leq i \leq t$. Then there exists a cyclic $\operatorname{HDM}(4,6 n ; 6)$.
Corollary 3.21. Let $n$ be an integer of the form $p_{1}^{\alpha_{1}} p_{2}^{\alpha_{2}} \ldots p_{t}^{\alpha_{t}}$, where $\alpha_{i} \geq 0$ and the prime factors $p_{i} \geq 5$ for $1 \leq i \leq t$. Then there exists a cyclic $D C A(4,6 n+1 ; 6 n)$ satisfying P1 and P2. Consequently there exists cyclic $D C A(4,2 p+1 ; 2 p)$ for $2 p=30^{*}, 42^{*}, 66^{*}, 78,102^{*}, 114^{*}, 138^{*}, 150,174^{*}$,
$186,210,222,246,258,282,294,318,330,354$.
The following result can be verified using direct constructions given in the later Section 4.
Lemma 3.22. There exists cyclic $D C A(4,2 p+1 ; 2 p)$ for $2 p=26^{*}, 266,2 p=40,56,88,104,136,152,184$, $200,232,248,280,296,328,344$ and $2 p=34^{*}, 58^{*}, 82^{*}, 106^{*}, 130^{*}, 154,178^{*}, 202,226,250,274,298,322$, 346.

Proof. Corollary 4.5 given in Section 4. verifies the existence of cyclic DCA $(4,2 p+1 ; 2 p)$ with $2 p=26^{*}, 266$.
Corollary 4.8, given in Section 4, verifies the existence of cyclic $\operatorname{DCA}(4,2 p+1 ; 2 p)$ with $2 p=40,56,88,104$, $136,152,184,200,232,248,280,296,328,344$.

Theorem 4.9, given in Section4] verifies the existence of cyclic DCA $(4,2 p+1 ; 2 p)$ with $2 p=34^{*}, 58^{*}, 82^{*}, 106^{*}, 130^{*}$, $154,178^{*}, 202,226,250,274,298,322,346$.

Lemma 3.23. There exists cyclic $D C A(4,2 p+1 ; 2 p)$ for $2 p=24^{*}, 28^{*}, 32,36^{*}, 44,48,52^{*}, 54^{*}$.
Proof. These results have been verified by computer searches. The first column of the $\operatorname{DCA}(4,2 p+1,2 p)$ is given by $[0,1,2, \ldots, 2 p-1]$, the second column by $[1,3, \ldots, 2 p-1,2,4, \ldots, 2 p-2]$ and the third column by $2 p=24:[2,0,3,1,14,21,20,19,23,15,6,18,16,10,17,8,11,22,5,13,4,9,7,12]$ $2 p=28:[2,0,3,1,11,16,22,25,20,23,4,8,21,5,18,10,19,13,24,27,7,26,15,9,6,14,17,12]$ $2 p=32:[2,0,3,6,1,13,22,30,21,25,28,26,7,5,23,20,12,10,24,17,31,15,29,27,11,14,4,9,8,19$, $18,16]$
$2 p=36:[5,35,13,20,11,9,1,31,10,2,30,33,4,34,32,25,28,16,27,22,3,29,19,24,18,15,6,23,17,7$, $0,8,14,12,21,26]$
$2 p=44:[39,13,26,21,35,3,17,16,40,28,38,25,6,10,34,5,18,30,43,15,19,36,7,24,32,14,4,0,31$, $12,2,9,23,37,11,42,41,29,20,1,33,27,8,22]$
$2 p=48:[5,41,23,40,1,39,34,25,28,8,4,9,21,30,43,18,12,2,42,45,32,37,33,0,26,15,13,22,10$, $35,44,7,36,16,27,19,46,38,3,47,31,29,17,14,11,24,20,6]$
$2 p=52:[18,12,50,37,16,6,45,4,31,34,47,21,29,2,5,22,38,3,39,27,0,15,51,7,28,24,42,40,48$, $32,9,26,20,11,1,41,19,35,43,13,49,33,14,17,46,8,36,23,10,30,25,44]$
$2 p=54:[6,5,31,27,20,38,19,4,30,51,3,52,49,14,48,23,41,12,25,0,32,40,21,50,9,45,16,1,46$, $11,28,42,47,35,39,2,22,13,34,33,24,44,15,53,7,17,37,36,26,18,10,43,29,8]$.

For the remaining values $64,68,72,74,76,92,96,108,116,122,124,128,144,146,148,162,164,172,188$, $192,194,212,218,236,244,256,268,284,288,292,314,316,332,348,356$ we were unable to construct cyclic difference covering arrays however for completeness and to answer questions about the spectrum we give full details verifying existence. It should be noted that it is possible to construct $\mathrm{DCA}(4,2 p+1 ; 2 p)$ satisfying P 1 and P2 for some of these orders however our construction does not give cyclic difference covering arrays and so the details have been omitted here.

For the Li and van Rees conjecture [11, Conjecture 5.1] we require two more results from their paper. The second result uses group divisible designs: A $\mathcal{K}$-group divisible design of type $g_{1}^{a_{1}} g_{2}^{a_{2}} \ldots g_{s}^{a_{s}}$ is a partition $\mathcal{G}$ of a finite set $\mathcal{V}$, of cardinality $v=\sum_{i=1}^{s} a_{i} g_{i}$, into $a_{i}$ groups of size $g_{i}, 1 \leq i \leq s$, together with a family of subsets (blocks) $\mathcal{B}$ of $\mathcal{V}$ such that: 1) if $B \in \mathcal{B}$, then $|B| \in \mathcal{K}, 2$ ) every pair of distinct elements of $\mathcal{V}$ occurs in 1 block of $\mathcal{B}$ or 1 group of $\mathcal{G}$ but not both, and 3) $|\mathcal{G}|>1$.

Theorem 3.24. [11, Thm 4.1] Suppose there exists $k-M N O L S(2 p)$, $k-M O L S(2 p)$, and $k-M O L S(n)$. Then there exists $k-M N O L S(2 p n)$.

Theorem 3.25. [11, Thm 4.5] Suppose there exists a $\mathcal{K}$-GDD of type $g_{1}^{a_{1}} \ldots g_{s}^{a_{s}}$. Further suppose that for any group size $g_{i}$ there exists a s-MNOLS $\left(g_{i}\right)$ and for any block size $k \in \mathcal{K}$ there exists a s-IMOLS $(k)$. Then there are $s-\operatorname{MNOLS}\left(\sum_{i=1}^{s} a_{i} g_{i}\right)$.

Lemma 3.26. There exists $3-N M O L S(2 p)$ for $2 p=76,92^{*}, 96,108,116^{*}, 124^{*}, 128,144,148^{*}, 164,172^{*}$, 188, 192, 212, 236, 244, 256, 268, 284, 288, 292, 316, 332, 348 and 356.

Proof. The 3-MNOLS $(2 p), 2 p=76\left(12^{5} 16^{1}\right), 92\left(20^{4} 12\right), 96\left(16^{5} 16^{1}\right), 108\left(20^{5} 8^{1}\right)$,
$116\left(20^{5}, 16\right), 124\left(20^{5} 24^{1}\right), 128\left(24^{5} 8^{1}\right), 144\left(24^{5} 24^{1}\right), 148\left(24^{5} 28\right), 164\left(28^{5} 24^{1}\right)$,
$172\left(32^{5} 12\right), 188\left(32^{5} 28^{1}\right), 192\left(32^{5} 36^{1}\right), 212\left(36^{5} 32^{1}\right), 236\left(40^{5} 36^{1}\right), 244\left(40^{5} 44^{1}\right)$,
$256\left(44^{5} 36^{1}\right), 268\left(44^{5} 48^{1}\right), 284\left(48^{5} 44^{1}\right), 288\left(48^{5} 48^{1}\right), 292\left(48^{5} 52^{1}\right), 316\left(52^{5} 56^{1}\right)$,
$332\left(56^{5} 52^{1}\right), 348\left(56^{5} 68^{1}\right)$ and $356\left(60^{5} 56^{1}\right)$ can be constructed applying 5-GDD, that exist by 6. Thm 4.17, p 258], in Theorem 3.25. Here the bracketed information gives the type of the GDD.

Lemma 3.27. There exists $3-N M O L S(2 p)$ for $2 p=64,68,72,74^{*}, 122,162,194,218$ and 314.
Proof. 3-NMOLS $(2 p)$ for $2 p=64,68,72,74^{*}, 122,162,194$ and 218 can be constructed by applying $8-G D D\left(8^{8}\right)$, $\{7,8,9\}-\operatorname{GDD}\left(8^{7} 6^{2}\right), 9-\operatorname{GDD}\left(8^{9}\right),\{7,8,9\}-\operatorname{GDD}\left(8^{7} 6^{3}\right),\{7,8\}-\operatorname{GDD}\left(16^{7} 10^{1}\right),\{11,12,13\}-\operatorname{GDD}\left(12^{12} 10^{1} 8^{1}\right)$, $\{11,12,13\}-\operatorname{GDD}\left(16^{11} 10^{1} 8^{1}\right),\{13,14\}-\operatorname{GDD}\left(16^{13} 10^{1}\right)$ and $\{8,9,10,11\}-$
$\operatorname{GDD}\left(32^{8} 14^{2} 10^{1}\right)$, that exist by finite field constructions, in Theorem 3.25 respectively.
All the results of this section combine to the following theorem.
Theorem 3.28. There exists a set of $3-M N O L S(2 p)$ for each positive integers $p \geq 3$, except possibly $p=73$.
Conjecture 3.29. There exists $D C A(4,2 p+1 ; 2 p)$ satisfying P1 and P2 of all positive integers $p \geq 3$.

## 4 Construction of difference covering arrays $\operatorname{DCA}(4,2 m+1 ; 2 m)$

This section is devoted to giving new constructions for families of cyclic $\operatorname{DCA}(4,2 m+1 ; 2 m)$ when $m=2 k+1$, $m=8 k+4$ and $m=3 k+2$, respectively. In each of these cases the difference covering arrays satisfy P1 and P2 and so they can be used to construct MNOLS ( $2 m$ ).

When using a cyclic $\operatorname{DCA}(4,2 m+1 ; 2 m)$ to construct nearly orthogonal Latin squares we strip off the last row and last column of zeros. Thus to reduce the complexity of the notation and to avoid confusion, we will assume that we are constructing a $2 m \times 3$ array $Q=[q(i, j)]$ that satisfies:

- each column is a permutation of $Z_{2 m}$ and
- $\Delta_{j, j^{\prime}}=\left\{q(i, j)-q\left(i, j^{\prime}\right)| | 0 \leq i \leq 2 m-1\right\}=\{1,2, \ldots, m, m, \ldots, 2 m-1\}$, with repetition retained.

Also we will use the following notation: $q(a, 0)=a($ or $q(\alpha, 0)=a(\alpha)), q(a, 1)=b(a)($ or $q(\alpha, 1)=b(\alpha))$, and $q(a, 2)=c(a)($ or $q(\alpha, 2)=c(\alpha))$.

The following lemmas document some well known results, stated without proof, which will be used extensively in the proof of subsequent results.

Lemma 4.1. For all integers $x, y, z, \operatorname{gcd}(x+y z, z)=\operatorname{gcd}(x, z)$.
Lemma 4.2. Let $g$ and $p$ be positive integers and $h$ a non-negative integer. Working modulo $2 p$, if $g c d(g, 2 p)=1$ then

$$
\{g x+h \mid 0 \leq x \leq 2 p-1\}=\mathbb{Z}_{2 p}
$$

or if $\operatorname{gcd}(g, 2 p)=r$ and $h \equiv s \bmod r$ then

$$
\{g x+h \mid 0 \leq x \leq 2 p / r-1\}=\{r x+s \mid 0 \leq x \leq 2 p / r-1\}
$$

### 4.1 Construction for general families $\operatorname{DCA}(4,2 m+1 ; 2 m)$ for some odd $m$

In this subsection we give a general construction for a difference covering array $\mathrm{DCA}(4,2 m+1 ; 2 m)$, for $m$ odd. The proof that such a difference covering array exists uses the results presented in the following lemma. Note that in this section unless otherwise stated all arithmetic is modulo $2 m$. In particular, for $i \not \equiv 2 \bmod 3 \mathrm{a}$ non-negative integer, and $k=2 i^{2}+7 i+6$, we present an infinite family of $\mathrm{DCA}(4,2 m+1 ; 2 m)$ for $m=2 k+1$.

| Intervals | $I_{1}$ | $I_{2}$ | $I_{3}$ | $I_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $q(a, 0)=a$ | $[0, m+f]$ | $[m+f+1, m-1]$ | $[m, m-f-1]$ | $[m-f, 2 m-1]$ |
| $q(a, 1)=b(a)$ | $a f+m$ | $a f+m$ | $(a+1) f$ | $(a+1) f$ |
|  |  |  | $+m-1$ | $+m-1$ |
| $q(a, 2)=c(a)$ | $-(a-1)(f+1)$ | $-(a-1)(f+1)$ | $-a(f+1)$ | $-a(f+1)$ |
|  | -2 | $+m-2$ | $+m$ |  |
| $b(a)-a$ | $a(f-1)$ | $a(f-1)$ | $(a+1)(f-1)$ | $(a+1)(f-1)$ |
|  | $+m$ | $+m$ | $+m$ | $+m$ |
| $c(a)-a$ | $-(a-1)(f+2)$ | $-(a-1)(f+2)$ | $-a(f+2)$ | $-a(f+2)$ |
|  | -3 | $+m-3$ | $+m$ |  |
| $c(a)-b(a)$ | $-a(2 f+1)$ | $-a(2 f+1)$ | $-a(2 f+1)$ | $-a(2 f+1)$ |
|  | $+f-1+m$ | $+f-1$ | $-(f-1)$ | $-(f-1)+m$ |

Figure 1: Entries are elements of $\mathbb{Z}_{2 m}$, where $q(a, 0)=a, q(a, 1)=b(a)$ and $q(a, 2)=c(a)$ in the array $Q=[q(i, j)]$. Rows 5 to 7 give the differences.

Lemma 4.3. Let $f$ and $m$ be integers such that $\operatorname{gcd}(f, 2 m)=2, \operatorname{gcd}(f+2,2 m)=2$, and $f^{2}+f+1 \equiv m$ $\bmod 2 m$. Then

$$
\begin{align*}
\operatorname{gcd}(f, m) & =1  \tag{4.1}\\
\operatorname{gcd}(f+1, m) & =1  \tag{4.2}\\
\operatorname{gcd}(f-1, m) & =1  \tag{4.3}\\
\operatorname{gcd}(2 f+1, m) & =1  \tag{4.4}\\
m f & \equiv 0 \quad \bmod 2 m . \tag{4.5}
\end{align*}
$$

Proof. Eq4.1. Note that since $f$ is even, $f^{2}+f+1$ is odd implying $m$ is odd and hence $\operatorname{gcd}(f, m)=1$.
Eq4.2. Since $f+1 \equiv-f^{2} \bmod m$ and $\operatorname{gcd}(f, m)=1$ we have $1=\operatorname{gcd}(f, m)=\operatorname{gcd}\left(f^{2}, m\right)=\operatorname{gcd}(f+1, m)$.
Eq4.3. Since $f-1=\left(-f^{2}-f-1\right)+f^{2}+2 f \equiv f(f+2) \bmod m, \operatorname{gcd}(f, m)=1$ and $\operatorname{gcd}(f+2, m)=1$, we have $1=\operatorname{gcd}(f(f+2), m)=\operatorname{gcd}(f-1, m)$.

Eq4.4 Since $2 f+1 \equiv-f(f-1) \bmod m, \operatorname{gcd}(f, m)=1$ and $\operatorname{gcd}(f-1, m)=1,1=\operatorname{gcd}(f(f-1), m)=$ $\operatorname{gcd}(2 f+1, m)$.

Eq4.5. This follows from the fact that $f$ is even.
For a suitable choice of $f$, we divide the domain of $a$ into the subintervals $[0, m+f],[m+f+1, m-1]$, [ $m, m-f-1$ ] and $[m-f, 2 m-1]$ where all endpoints are included.
Example 1. To aid understanding we begin with an example where $m=13$ and $f=16$ and give the transpose of the difference covering array $\operatorname{DCA}(4,27 ; 26)$. The key to understanding the proof is to recognise that within the subintervals $I_{1}=[0, \ldots, 3], I_{2}=[4, \ldots, 12], I_{3}=[13, \ldots, 22]$ and $I_{4}=[23, \ldots, 25]$, the value of $a, b(a)$ and $c(a)$ increases by a constant "jump", respectively $1, f=16$ and $-(f+1)=9$. This implies that the differences will also increase by a constant. By carefully choosing the start value on each subinterval it is possible to obtain the required values in $\mathbb{Z}_{2 m}$. The value $m=13$ is boldfaced in the differences.

|  | $I_{1}$ | $I_{2}$ | $I_{3}$ | $I_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $a$ | 0 1 2 3 | $\begin{array}{llllllllll}4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12\end{array}$ | 13141516171819202122 | 232425 |
| $b(a)$ | $13 \begin{array}{llll}13 & 3 & 19 & 9\end{array}$ |  | $21_{2}$ | $6 \quad 2212$ |
| $c(a)$ | $15 \quad 24 \quad 7 \quad 16$ | $12 \begin{array}{llllllllll}12 & 4 & 13 & 22 & 5 & 14 & 23 & 6\end{array}$ | $\begin{array}{lllllllllll}0 & 9 & 18 & 1 & 10 & 19 & 2 & 11 & 20 & 3\end{array}$ | $\begin{array}{llll}25 & 8 & 17\end{array}$ |
| $b(a)-a$ | $13 \quad 217$ | $\begin{array}{lllllllllll}21 & 10 & 25 & 14 & 3 & 18 & 7 & 22 & 11\end{array}$ | $\begin{array}{lllllllllll}15 & 4 & 19 & 8 & 23 & 12 & 1 & 16 & 5 & 20\end{array}$ | 92413 |
| $c(a)-a$ | $15 \begin{array}{llll}15 & 23 & 5 & 13\end{array}$ | 8161624661412241220 |  | 21018 |
| $c(a)-b(a)$ | 2 21147 | $\left\lvert\, \begin{array}{llllll}13 & 6 & 25 & 18 & 11 & 4 \\ 23 & 16 & 9\end{array}\right.$ | $\|$24 17 10 3 22 15 8 1 20 <br> 13         | 1912 |

Theorem 4.4. Let $f$ and $m$ be natural numbers such that $\operatorname{gcd}(f, 2 m)=2, \operatorname{gcd}(f+2,2 m)=2, f^{2}+f+1 \equiv m$ $\bmod 2 m$, and $m+3 \leq f \leq 2 m-4$. Then a cyclic $D C A(4,2 m+1 ; 2 m)$ satisfying P1 and P2 exists.

Proof. The proof is by construction with the values of $\operatorname{DCA}(4,2 m+1 ; 2 m)$ as given in Figure 1, with the 3 columns of $Q=[q(i, j)]$ given by $q(a, 0)=a, q(a, 1)=b(a), q(a, 2)=c(a)$. We will show that $Q$ has the required properties.

If $m+3 \leq f \leq 2 m-4$ and $f$ is even then, working modulo $2 m, 3 \leq m+f \leq m-4$ and so the intervals $[0, m+f]$ and $[m+f+1, m-1]$ are non-empty. Further, $m+4 \leq m-f \leq 2 m-3$ and so the intervals [ $m, m-f-1$ ] and $[m-f, 2 m-1]$ are non-empty.

Given that $f$ is even and $m$ is odd, by Lemma 4.2

$$
\begin{gathered}
a f+m \equiv 1 \quad \bmod 2 \Longrightarrow\left\{b(a) \mid a \in I_{1} \cup I_{2}\right\}=\{2 g+1 \mid 0 \leq g \leq m-1\} \\
(a+1) f+m-1 \equiv 0 \quad \bmod 2 \Longrightarrow\left\{b(a) \mid a \in I_{3} \cup I_{4}\right\}=\{2 g \mid 0 \leq g \leq m-1\},
\end{gathered}
$$

and so $\{b(a) \mid 0 \leq a \leq 2 m-1\}=[2 m]$.
On each of the subintervals $c(a)$ takes the form $a g+h$, where $g=-(f+1)$, so the "jump" size is $-(f+1)$ and

$$
\begin{aligned}
c(m)= & 0 \\
c(m+f+1)-c(m-f-1)= & -m-f(f+1)+m-2+m \\
& -f(f+1)-(f+1)-m \\
= & -2 f^{2}-2 f-2-(f+1)=-(f+1), \\
c(0)-c(m-1)= & -(f+1), \\
c(m-f)-c(m+f)= & -(f+1), \\
c(m)-c(2 m-1)= & -(f+1) .
\end{aligned}
$$

Thus by reordering the subintervals as $I_{3}, I_{2}, I_{1}, I_{4}$, and noting for instance, $c(m-f-1)-(f+1)=c(m+f+1)$, we get $\{c(a) \mid 0 \leq a \leq 2 m-1\}=[2 m]$.

For $b(a)-a, c(a)-a$ and $c(a)-b(a)$ we are required to show that for $a \in[2 m]$ the differences cover the multiset $\{1,2, \ldots, m-1, m, m, m+1, \ldots, 2 m-1\}=([2 m] \backslash\{0\}) \cup\{m\}$.

For $b(a)-a$, the subintervals are taken in natural order $I_{1}, I_{2}, I_{3}, I_{4}$. Starting at $a=0$ and finishing at $a=2 m-1$, we have $b(0)-0=m=(2 m-1+1)(f-1)+m=b(2 m-1)-(2 m-1)$, so the difference $m$ occurs twice. Further, the $\operatorname{gcd}(f-1,2 m)=1$ implies that $a(f-1)+m, 0 \leq a \leq m-1$, are all distinct, as are $(a+1)(f-1)+m, m \leq a \leq 2 m-1$ and

$$
\begin{aligned}
& b(m)-m=(m+1)(f-1)+m=f-1, \\
& b(m-1)-(m-1)=(m-1)(f-1)+m=-(f-1) .
\end{aligned}
$$

Thus there is a jump of $-2(f-1)$ between $a=m-1$ and $a=m$ and the difference 0 is omitted, implying $\{b(a)-a \mid 0 \leq a \leq 2 m-1\}=([2 m] \backslash\{0\}) \cup\{m\}$.

For $c(a)-a$, since $f+2$ is even and $\operatorname{gcd}(f+2, m)=1$, these values are all distinct on each of the subintervals, $\left|I_{3} \cup I_{1}\right|=m+1,\left|I_{4} \cup I_{2}\right|=m-1$, and

$$
\begin{aligned}
c(m)-m & =-m(f+2)+m=m, \\
c(m+f)-(m+f) & =-(m+f-1)(f+2)-3=m, \\
(c(0)-0)-(c(m-f-1)-(m-f-1)) & =m-f(f+2)-3=-(f+2), \\
c(2 m-1)-(2 m-1) & =-(2 m-1)(f+2)=f+2, \\
c(m+f+1)-(m+f+1) & =-f^{2}-2 f+m-3=-(f+2) .
\end{aligned}
$$

Thus $-(a-1)(f+2)-3,-a(f+2)+m \equiv 1 \bmod 2$, hence,

$$
\left\{c(a)-a \mid a \in I_{3} \cup I_{1}\right\}=\{2 g+1 \mid 0 \leq g \leq m-1\} \cup\{m\}
$$

In addition, $-a(f+2)+m-1,-a(f+2) \equiv 0 \bmod 2$ implies that

$$
\left\{c(a)-a \mid a \in I_{4} \cup I_{2}\right\}=\{2 g \mid 1 \leq g \leq m-1\}
$$

giving $\{c(a)-a \mid 0 \leq a \leq 2 m-1\}=([2 m] \backslash\{0\}) \cup\{m\}$.
For $c(a)-b(a)$, since $\operatorname{gcd}(2 f+1,2 m)=1$, these values are all distinct on the subintervals, $c(m+f+1)-$ $b(m+f+1)=-2 f^{2}-2 f-2+m=m=m+2 f^{2}+2 f+2=c(m-f-1)-b(m-f-1)$, and

$$
\begin{aligned}
c(0)-b(0)-(c(m-1)-b(m-1)) & =-(2 f+1) \\
c(m+f)-b(m+f) & =2 f+1 \\
c(m-f)-b(m-f) & =-(2 f+1)
\end{aligned}
$$

Thus when the subintervals are reordered to $I_{2}, I_{1}, I_{4}$, $I_{3}$ we may verify that $\{c(a)-b(a) \mid 0 \leq a \leq 2 m-1\}=$ $([2 m] \backslash\{0\}) \cup\{m\}$. Note that the values of $c(a)-b(a)$ start and finish on $m$ and the value 0 is omitted between $a=m+f$ and $a=m-f$.

Corollary 4.5. Let $i \not \equiv 2 \bmod 3$ be a non-negative integer, $k=2 i^{2}+7 i+6$ and $m=2 k+1$. Then there exists an infinite family of cyclic $D C A(4,2 m+1 ; 2 m)$ 's satisfying P1 and P2.

Proof. Taking $f=m+3+2 i$, then $f$ is even. In addition $m+3 \leq f \leq 2 m-4$, since $i \geq 0$ and $2 m-4=$ $(m+3)+(m-7)=(m+3)+\left(4 i^{2}+14 i+6\right) \geq m+3+2 i=f$. Now

$$
\begin{aligned}
f^{2}+f+1 & \equiv 4 i^{2}+14 i+13 \bmod 2 \mathrm{~m} \\
& =2\left(2 i^{2}+7 i+6\right)+1=m \bmod 2 \mathrm{~m}
\end{aligned}
$$

Further, applying Lemma 4.1 repeatedly,

$$
\begin{aligned}
\operatorname{gcd}(f, 2 m) & =2\left(\operatorname{gcd}\left(2 i^{2}+8 i+8,\left(2 i^{2}+8 i+8\right)+2 i^{2}+6 i+5\right)\right) \\
& =2\left(\operatorname{gcd}\left(2 i+3,2 i^{2}+4 i+2+(2 i+3)\right)\right) \\
& =2\left(\operatorname{gcd}\left(2 i+3,2(i+1)^{2}\right)\right)=2(\operatorname{gcd}(2 i+3, i+1))=2
\end{aligned}
$$

Also,

$$
\begin{aligned}
\operatorname{gcd}(f+2,2 m) & =2\left(\operatorname{gcd}\left(2 i^{2}+8 i+9,\left(2 i^{2}+8 i+9\right)+2 i^{2}+6 i+4\right)\right) \\
& =2(\operatorname{gcd}(2 i+5,2(i+2)(i+1))) \\
& =2(\operatorname{gcd}(2 i+5,(i+2)(i+1)))
\end{aligned}
$$

Now $\operatorname{gcd}(2 i+5, i+2)=\operatorname{gcd}(2(i+2)+1, i+2)=1$. Whereas

$$
\begin{aligned}
\operatorname{gcd}(2 i+5, i+1) & =\operatorname{gcd}(2(i+1)+3, i+1))=\operatorname{gcd}(3, i+1) \\
& \neq 1 \text { when } i+1 \equiv 0 \bmod 3 \text { or equivalently } \mathrm{i} \equiv 2 \bmod 3
\end{aligned}
$$

Thus taking $i \not \equiv 2 \bmod 3$ we can construct a $\operatorname{DCA}(4,2 m+1 ; 2 m)$ as per the Theorem 4.4.

### 4.2 Construction of difference covering arrays DCA $(4,4 m+1 ; 4 m)$

In this subsection we give a general construction for a difference covering array $\mathrm{DCA}(4,4 m+1 ; 4 m)$. It will be shown that for all non-negative integers $k$, such that $3 \nmid(2 k+1)$, this construction gives an infinite family of $\mathrm{DCA}(4,16 k+9 ; 16 k+8)$. The proof that such a difference covering array exists uses the results presented in the following lemma. Note that in this section unless otherwise stated all arithmetic is modulo 4 m .

Lemma 4.6. Let $f$ and $m$ be natural numbers such that $m \equiv 2 \bmod 4, \operatorname{gcd}(f$,
$4 m)=2, \operatorname{gcd}(f-1,4 m)=1$, and $f^{2}+f-2 \equiv 2 m \bmod 4 m$. Then

$$
\begin{align*}
\operatorname{gcd}(2 m+2-f, 4 m) & =4  \tag{4.6}\\
\operatorname{gcd}(2 m-f+1,4 m) & =1  \tag{4.7}\\
\operatorname{gcd}(2 m-2 f+2,4 m) & =2  \tag{4.8}\\
m f & \equiv 2 m \quad \bmod 4 m . \tag{4.9}
\end{align*}
$$

Proof. Eq4.6. Rewriting $2 m+2-f=f^{2}+f-2+2-f=f^{2} \bmod 4 m$ and assuming $\operatorname{gcd}(f, 4 m)=2$ gives $\operatorname{gcd}\left(f^{2}, 4 m\right)=4$.

Eq4.7. Since $2 m-f+1$ is odd, the $\operatorname{gcd}(2 m-f+1,4 m)$ is odd. Assume there exists an odd $x$ such that $x \mid 4 m$ and $x \mid(2 m-f+1)$, then $x \mid m$ and so $x \mid(f-1)$. But the $\operatorname{gcd}(f-1,4 m)=1$, so $x=1$.

Eq4.8 Assume that there exists $x$ such that $x \mid(m-f+1)$ and $x \mid 2 m$. Since $m-f+1$ is odd, $x$ is odd and so $x \mid m$. Consequently $x \mid(f-1)$ and $x \mid 4 m$, implying $x=1$.

Eq4.9. It follows that $m f \equiv m\left(2 m-f^{2}+2\right)=2 m^{2}-m f^{2}+2 m \equiv 2 m \bmod 4 m$.
Theorem 4.7. Let $f$ be a natural number and $m=4 k+2$, where $k$ is a non-negative integer, such that $\operatorname{gcd}(f, 4 m)=2, \operatorname{gcd}(f-1,4 m)=1$, and $f^{2}+f-2 \equiv 2 m \bmod 4 m$. Then a cyclic $D C A(4,4 m+1 ; 4 m)$ satisfying P1 and P2 exists.
$\left.\begin{array}{|c|c|c|c|c|}\hline \text { Intervals } & I_{1} & I_{2} & I_{3} & I_{4} \\ \hline q(a, 0)=a & {[0, m-1]} & {[m, 2 m-1]} & {[2 m, 3 m-1]} & {[3 m, 4 m-1]} \\ \hline q(a, 1)=b(a) & (a+1) f-1 & (a+1) f-1 & a f & a f \\ \hline q(a, 2)=c(a) & (a+1)(2 m-f+2) & a(2 m-f+2) \\ & -1 & -m & (a+1)(2 m-f+2) & a(2 m-f+2) \\ +m-1\end{array}\right]$

Figure 2: Entries are elements of $\mathbb{Z}_{4 m}$, where $q(a, 0)=a, q(a, 1)=b(a)$ and $q(a, 2)=c(a)$ in the array $Q=[q(i, j)]$.

Proof. The proof is by construction with the values of $\operatorname{DCA}(4,4 m+1 ; 4 m)$ as given in Figure 2, with the 3 columns of $Q=[q(i, j)]$ given by $q(a, 0)=a, q(a, 1)=b(a), q(a, 2)=c(a)$. We will show that $Q$ has the required properties.

For $b(a)$, since $f$ is even, Lemma 4.2 implies that

$$
\begin{aligned}
& (a+1) f-1 \equiv 1 \quad \bmod 2 \Longrightarrow\left\{b(a) \mid a \in I_{1} \cup I_{2}\right\}=\{2 g+1 \mid 0 \leq g \leq 2 m-1\}, \\
& a f \equiv 0 \bmod 2 \Longrightarrow\left\{b(a) \mid a \in I_{3} \cup I_{4}\right\}=\{2 g \mid 0 \leq g \leq 2 m-1\},
\end{aligned}
$$

and $\{b(a) \mid 0 \leq a \leq 4 m-1\}=\mathbb{Z}_{4 m}$.
For $c(a)$, since $\operatorname{gcd}(2 m-f+2,4 m)=4$, Lemma 4.2 implies that

$$
\begin{aligned}
&(a+1)(2 m-f+2)+m-1 \equiv 1 \quad \bmod 4 \Longrightarrow\left\{c(a) \mid a \in I_{3}\right\}=\{4 g+1 \mid 0 \leq \\
& \\
& a(2 m-f+2) \equiv 0 \bmod 4 \Longrightarrow\left\{c(a) \mid a \in I_{4}\right\}=\{4 g \mid 0 \leq g \leq \\
&m-1\}, \\
&(a+1)(2 m-f+2)-1 \equiv 3 \bmod 4 \Longrightarrow\left\{c(a) \mid a \in I_{1}\right\}=\{4 g+3 \mid 0 \leq \\
&g \leq m-1\}, \\
& a(2 m-f+2)-m \equiv 2 \bmod 4 \Longrightarrow
\end{aligned}
$$

Thus the set of values $\left\{c(a) \mid a \in \mathbb{Z}_{4 m}\right\}=\mathbb{Z}_{4 m}$.
For $b(a)-a, c(a)-a$ and $c(a)-b(a)$ we are required to show that for $a \in \mathbb{Z}_{4 m}$ the differences cover the multiset $\{1,2, \ldots, 2 m-1,2 m, 2 m, 2 m+1, \ldots, 4 m-1\}=\left(\mathbb{Z}_{4 m} \backslash\{0\}\right) \cup\{2 m\}$.

For $b(a)-a$, the $\operatorname{gcd}(f-1,4 m)=1$, and

$$
\begin{aligned}
b(2 m)-2 m & =2 m(f-1)=2 m \\
b(2 m-1)-(2 m-1) & =(2 m-1+1) f-1-(2 m-1)=(2 m)(f-1)=2 m, \\
b(4 m-1)-(4 m-1) & =-(f-1) \\
b(0)-0 & =f-1
\end{aligned}
$$

So using a "jump" of $f-1$ and ordering the subintervals as $I_{3}, I_{4}, I_{1}, I_{2}$ we obtain the difference $2 m$ twice and the difference 0 is omitted between $a=4 m-1$ and $a=0$ implying that $\{b(a)-a \mid 0 \leq a \leq 4 m-1\}=$ $\left(\mathbb{Z}_{4 m} \backslash\{0\}\right) \cup\{2 m\}$.

For $c(a)-a$, the $\operatorname{gcd}(2 m-f-1,4 m)=1$, and

$$
\begin{aligned}
c(m)-m & =m(2 m-f+1)-m=2 m, \\
c(3 m-1)-(3 m-1) & =(3 m-1+1)(2 m-f+1)+m \\
& =2 m, \\
c(3 m)-3 m-(c(2 m-1)-(2 m-1)) & =(m+1)(2 m-f+1)+m \\
& =2 m-f+1, \\
c(0)-0 & =2 m-f+1, \\
c(4 m-1)-(4 m-1) & =(4 m-1)(2 m+f-1) \\
& =-(2 m-f+1), \\
c(2 m)-2 m-(c(m-1)-(m-1)) & =(m+1)(2 m-f+1)+m \\
& =2 m-f+1 .
\end{aligned}
$$

So using a "jump" of $2 m-f+1$ and ordering the subintervals as $I_{2}, I_{4}, I_{1}, I_{3}$ we obtain the difference $2 m$ twice and the difference 0 is omitted between $a=4 m-1$ and $a=0$, implying $\{c(a)-a \mid 0 \leq a \leq 4 m-1\}=$ $\left(\mathbb{Z}_{4 m} \backslash\{0\}\right) \cup\{2 m\}$.

For $c(a)-b(a)$, and

$$
\begin{aligned}
c(3 m)-b(3 m) & =m(2 m-2 f+2)=2 m, \\
c(m-1)-b(m-1) & =(m-1+1)(2 m-2+2)=2 m, \\
c(4 m-1)-b(4 m-1) & =-(2 m-2 f+2) \\
c(0)-b(0) & =2 m-2 f+2, \\
c(2 m)-b(2 m)-(c(2 m-1)-b(2 m-1)) & =2 m-2 f+2
\end{aligned}
$$

Then since

$$
\begin{aligned}
2 m-2 f-2 \equiv 0 \quad \bmod 2 \Longrightarrow\left\{c(a)-b(a) \mid a \in I_{4} \cup I_{1}\right\}= & \{2 g \mid 1 \leq g \leq \\
& 2 m-1\} \cup\{2 m\}, \\
-f+1 \equiv 1 \bmod 2 \Longrightarrow\left\{c(a)-b(a) \mid a \in I_{2} \cup I_{3}\right\}= & \{2 g+1 \mid 0 \leq g \leq \\
& 2 m-1\}
\end{aligned}
$$

implying $\{c(a)-b(a) \mid 0 \leq a \leq 4 m-1\}=\left(\mathbb{Z}_{4 m} \backslash\{0\}\right) \cup\{2 m\}$.
Corollary 4.8. For $k \geq 0$ such that $k \not \equiv 1 \bmod 3$ a cyclic $D C A(4,4 m+1 ; 4 m)$ satisfying P1 and P2 can be constructed as described in Theorem 4.7.

Proof. Given $m=4 k+2$, take $f=2 m-2$. Then $f=8 k+2$, and

$$
\operatorname{gcd}(f, 4 m)=2(\operatorname{gcd}(4 k+1,8 k+4))=2(\operatorname{gcd}(4 k+1,2(4 k+1)+2))=2
$$

In addition

$$
\begin{aligned}
\operatorname{gcd}(f-1,4 m) & =\operatorname{gcd}(2 m-3,4 m)=\operatorname{gcd}(8 k+1,16 k+8) \\
& =\operatorname{gcd}(8 k+1,2 k+1) \text { since } 2 \nmid(8 \mathrm{k}+1) \\
& =\operatorname{gcd}(6 k, 2 k+1)=1, \text { if } 3 \nmid(2 \mathrm{k}+1) .
\end{aligned}
$$

Also

$$
f^{2}+f-2=64 k^{2}+32 k+4+8 k+2-2=4 k(16 k+8)+8 k+4 \equiv 2 m \bmod 4 \mathrm{~m} .
$$

Hence, $f=2 m-2$ satisfies the assumptions of Theorem4.7 and we can construct a $\mathrm{DCA}(4,16 k+9 ; 16 k+8)$ for $k$ such that $3 \nmid(2 k+1)$ as described in this theorem.

### 4.3 Construction of difference covering arrays $\mathbf{D C A}(4,2 m+1 ; 2 m)$, where $m=3 \mu+2$

In this subsection we give a general construction for a difference covering array $\mathrm{DCA}(4,2 m+1 ; 2 m)$, where $m=3 \mu+2$. The proof that such a difference covering array exists uses the result presented in the following lemma. Note that in this section unless otherwise stated all arithmetic is modulo $12 k+10, k \geq 0$.

Theorem 4.9. Let $\mu$ be an odd positive integer. Then there exists a cyclic $D C A(4,6 \mu+5 ; 6 \mu+4)$ satisfying P1 and P2.

Proof. Since $\mu \geq 1, n=6 \mu+4 \geq 10$, since 3 is prime, $\operatorname{gcd}(3,6 \mu+4)=1$. Hence $\operatorname{gcd}(3 \mu+4,6 \mu+4)=$ $\operatorname{gcd}(3 \mu, 6 \mu+4)=1$.

Let $\mu=2 k+1, k \geq 0$, then

$$
\begin{aligned}
\mu(3 \mu+2) & =(2 k+1)(6 k+5)=3 \mu+2, \text { and } \\
(3 \mu+2)^{2} & =3 \mu+2
\end{aligned}
$$

That is, $(n / 2)^{2} \equiv n / 2 \bmod n$.

| Intervals for <br> $\alpha$ | $I_{1}$ <br> $[0, \mu-1]$ | $I_{2}$ <br> $[\mu, 2 \mu]$ | $I_{3}$ <br> $[2 \mu+1,3 \mu+1]$ |
| :---: | :---: | :---: | :---: |
| $q(\alpha, 0)=a(\alpha)$ | $3 \alpha+3 \mu+4$ | $3 \alpha+2$ | $3 \alpha+3 \mu+4$ |
| $q(\alpha, 1)=b(\alpha)$ | $3 \alpha(\mu+1)+2 \mu+2$ | $3 \alpha(\mu+1)+2 \mu+2$ | $3 \alpha(\mu+1)+2 \mu+2$ |
| $q(\alpha, 2)=c(\alpha)$ | $\alpha(3 \mu+4)+5 \mu+4$ | $\alpha(3 \mu+4)+5 \mu+4$ | $\alpha(3 \mu+4)+5 \mu+4$ |
| $b(\alpha)-a(\alpha)$ | $3 \alpha \mu-\mu-2$ | $3 \alpha \mu+2 \mu$ | $3 \alpha \mu-\mu-2$ |
| $c(\alpha)-a(\alpha)$ | $\alpha(3 \mu+1)+2 \mu$ | $\alpha(3 \mu+1)+5 \mu+2$ | $\alpha(3 \mu+1)+2 \mu$ |
| $c(\alpha)-b(\alpha)$ | $\alpha+3 \mu+2$ | $\alpha+3 \mu+2$ | $\alpha+3 \mu+2$ |
| Intervals for | $I_{4}$ | $I_{5}$ | $I_{6}$ |
| $\alpha$ | $[3 \mu+2,4 \mu+2]$ | $[4 \mu+3,5 \mu+2]$ | $[5 \mu+3,6 \mu+3]$ |
| $q(\alpha, 0)=a(\alpha)$ | $3 \alpha+3 \mu+3$ | $3 \alpha+1$ | $3 \alpha+3 \mu+3$ |
| $q(\alpha, 1)=b(\alpha)$ | $3 \alpha(\mu+1)+2 \mu+1$ | $3 \alpha(\mu+1)+2 \mu+1$ | $3 \alpha(\mu+1)+2 \mu+1$ |
| $q(\alpha, 2)=c(\alpha)$ | $\alpha(3 \mu+4)+5 \mu+4$ | $\alpha(3 \mu+4)+5 \mu+4$ | $\alpha(3 \mu+4)+5 \mu+4$ |
| $b(\alpha)-a(\alpha)$ | $3 \alpha \mu-\mu-2$ | $3 \alpha \mu+2 \mu$ | $3 \alpha \mu-\mu-2$ |
| $c(\alpha)-a(\alpha)$ | $\alpha(3 \mu+1)+2 \mu+1$ | $\alpha(3 \mu+1)+5 \mu+3$ | $\alpha(3 \mu+1)+2 \mu+1$ |
| $c(\alpha)-b(\alpha)$ | $\alpha+3 \mu+3$ | $\alpha+3 \mu+3$ | $\alpha+3 \mu+3$ |

Figure 3: Entries are elements of $\mathbb{Z}_{6 \mu+4}$, where $q(\alpha, 0)=a(\alpha), q(\alpha, 1)=b(\alpha)$ and $q(\alpha, 2)=c(\alpha)$ in the array $Q=[q(i, j)]$.

The proof is by construction with the values for $\operatorname{DCA}(4,6 \mu+5 ; 6 \mu+4)$ as given in Figure 3 with the three columns of $Q=[q(i, j)]$ given by $q(\alpha, 0)=a(\alpha), q(\alpha, 1)=b(\alpha)$ and $q(\alpha, 2)=c(\alpha)$.

Since $\operatorname{gcd}(3, n)=1,\{3 \alpha \mid 0 \leq \alpha \leq n-1\}=\mathbb{Z}_{n}$, by Lemma 4.2, Further $a(3 \mu+2)=3(3 \mu+2)+3 \mu+3=1$ and $a(2 \mu)=6 \mu+2$ and there is a "jump" of 3 between $a(4 \mu+2)$ and $a(0) ; a(\mu-1)$ and $a(5 \mu+3) ; a(6 \mu+3)$ and $a(2 \mu+1) ; a(3 \mu+1)$ and $a(4 \mu+3) ; a(5 \mu+2)$ and $a(\mu)$, respectively. Thus reordering the subintervals as $I_{4}, I_{1}, I_{6}, I_{3}, I_{5}, I_{2}$ gives $\{a(\alpha) \mid 0 \leq \alpha \leq n-1\}=\mathbb{Z}_{n}$.

For $b(\alpha)$,

$$
\begin{aligned}
3 \alpha(\mu+1)+2 \mu+2 \equiv 0 \quad \bmod 2 \Longrightarrow & \left\{b(\alpha) \mid \alpha \in I_{1} \cup I_{2} \cup I_{3}\right\} \\
& =\{2 g \mid 0 \leq g \leq 3 \mu+2\} \\
3 \alpha(\mu+1)+2 \mu+1 \equiv 1 \quad \bmod 2 \Longrightarrow & \left\{b(\alpha) \mid \alpha \in I_{4} \cup I_{5} \cup I_{6}\right\} \\
& =\{2 g+1 \mid 0 \leq g \leq 3 \mu+1\}
\end{aligned}
$$

For $c(\alpha)$, since $\operatorname{gcd}(3 \mu+4,6 \mu+4)=1$ Lemma 4.2 implies that $\{c(\alpha) \mid 0 \leq \alpha \leq 6 \mu+3\}=\mathbb{Z}_{6 \mu+4}$.
For $b(\alpha)-a(\alpha)$, since $\operatorname{gcd}(3 \mu, 6 \mu+4)=1$,

$$
\begin{aligned}
b(\mu)-a(\mu) & =3 \mu+2, \\
b(4 \mu+2)-a(4 \mu+2) & =3 \mu+2, \\
b(5 \mu+3)-a(5 \mu+3)-(b(2 \mu)-a(2 \mu)) & =3 \mu \\
b(4 \mu+3)-a(4 \mu+3)-(b(\mu-1)-a(\mu-1)) & =6 \mu \\
b(2 \mu+1)-a(2 \mu+1)-(b(5 \mu+2)-a(5 \mu+2)) & =3 \mu \\
b(0)-a(0)-(b(6 \mu+3)-a(6 \mu+3) & =3 \mu .
\end{aligned}
$$

So using a "jump" of $3 \mu$ and ordering the subintervals as $I_{2}, I_{6}, I_{1}, I_{5}, I_{3}, I_{4}$, we obtain the difference $3 \mu+2$ twice and since there is $6 \mu$ between $b(\alpha)-a(\alpha)$ for $\alpha=\mu-1$ and $\alpha=4 \mu+3$ the difference 0 is omitted implying that $\{b(\alpha)-a(\alpha) \mid 0 \leq \alpha \leq 6 \mu+3\}=\left(\mathbb{Z}_{n} \backslash\{0\}\right) \cup\{n / 2\}$.

For $c(\alpha)-b(\alpha)$, since $\operatorname{gcd}(3 \mu+1,6 \mu+4)=2$, we have
and

$$
\begin{aligned}
c(5 \mu+3)-a(5 \mu+3) & =3 \mu+2, \\
c(2 \mu)-a(2 \mu) & =3 \mu+2, \\
c(3 \mu+2)-a(3 \mu+2)-(c(6 \mu+3)-a(6 \mu+3)) & =-(3 \mu+3) \\
c(\mu)-a(\mu)-(c(4 \mu+2)-a(4 \mu+2)) & =-(3 \mu+3) \\
c(2 \mu+1)-a(2 \mu+1) & =3 \mu+1 \\
c(5 \mu+2)-c(5 \mu+2) & =3 \mu+3 \\
c(0)-a(0)-(c(3 \mu+1)-a(3 \mu+1)) & =-(3 \mu+3) \\
c(\mu-1)-a(\mu-1)-(c(4 \mu+3)-a(4 \mu+3)) & =-(3 \mu+3) .
\end{aligned}
$$

Reordering the intervals as $I_{6}, I_{4}, I_{2}$ and $I_{3}, I_{1}, I_{5}$ and using a regular "jump" of $-(3 \mu+3)$, with the jump of $6 \mu+2$ between $\alpha=2 \mu+1$ and $\alpha=5 \mu+2$, being the exception, we have the difference $3 \mu+2$ twice and the difference 0 omitted, thus $\{c(\alpha)-a(\alpha) \mid 0 \leq \alpha \leq 6 \mu+3\}=\left(\mathbb{Z}_{6 \mu+4} \backslash\{0\}\right) \cup\{3 \mu+2\}$ with repetition retained.

For $c(\alpha)-b(\alpha)$, we note that

$$
\begin{aligned}
c(0)-b(0) & =3 \mu+2, \\
c(3 \mu+1)-b(3 \mu+1) & =-1
\end{aligned}
$$

and so the values of $c(\alpha)-a(\alpha)$ on the subinterval $I_{1} \cup I_{2} \cup I_{3}$ cover the set $\{3 \mu+2, \ldots,-1\}$. Also

$$
\begin{aligned}
& c(3 \mu+2)-b(3 \mu+2)=1 \\
& c(6 \mu+3)-b(6 \mu+3)=3 \mu+2
\end{aligned}
$$

and so the values of $b(\alpha)-c(\alpha)$ on the subinterval $I_{4} \cup I_{5} \cup I_{6}$ cover the set $\{3 \mu+2, \ldots, 6 \mu-1\}$. Consequently $\{b(\alpha)-c(\alpha) \mid 0 \leq \alpha \leq 6 \mu+3\}=([6 \mu+4] \backslash\{0\}) \cup\{3 \mu+2\}$ with repetition retained.

### 4.4 Infinite families

The construction of Theorem 4.9 constructs sets of three MNOLS of orders $10,22,34,46 \bmod 48$. The construction of Corollary 4.8 constructs sets of three MNOLS of orders $8,40 \bmod 48$. Combined with the constructions of [7], there is a construction of three MNOLS for $8,10,14,22,34,38,40,46 \bmod 48$. There are infinite families constructed from Corollary 4.5 and from results of Li and van Rees [11], but these cannot be described mod 48 . It is an open question as to why 48 features in many of the constructions.

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