The rainbow connection number of the power graph of a finite group

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Abstract

This paper studies the rainbow connection number of the power graph Γ_G of a finite group G. We determine the rainbow connection number of Γ_G if G has maximal involutions or is nilpotent, and show that the rainbow connection number of Γ_G is at most three if G has no maximal involutions. The rainbow connection numbers of power graphs of some nonnilpotent groups are also given.

Key words: Rainbow path; rainbow connection number; finite group; power graph.

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1 Introduction

Given a connected graph Γ , denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and edge set, respectively. Define a coloring $\zeta : E(\Gamma) \to \{1, 2, \ldots, k\}, k \in \mathbb{N}$, where adjacent edges may be colored the same. A path P is *rainbow* if any two edges in P are colored distinct. If Γ has a rainbow path from u to v for each pair of vertices u and v, then Γ is *rainbow-connected* under the coloring ζ , and ζ is called a *rainbow* k-coloring of Γ . The *rainbow connection number* of Γ , denoted by $rc(\Gamma)$, is the minimum k for which there exists a rainbow k-coloring of Γ .

The rainbow connection number of a graph Γ was introduced by Chartrand et al. [7]. It was showed in [5, 20] that computing $\operatorname{rc}(\Gamma)$ is NP-hard. Moreover, it has been proved in [20], that for any fixed $t \geq 2$, deciding if $\operatorname{rc}(\Gamma) = t$ is NP-complete. Some topics on restrict graphs are as follows: oriented graphs [8], graph products [14], hypergraphs [4], corona graphs [9], line graphs [24], Cayley graphs [21], dense graphs [22] and sparse random graphs [12]. Most of the results and papers that dealt with it can be found in [23].

In this paper we study the rainbow connection number of the power graph of a finite group. We always use G to denote a finite group with the identity e. The power graph Γ_G has the vertex set G and two distinct elements are adjacent if one is

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a power of the other. The concept of a power graph was introduced in [16]. Recently, many interesting results on power graphs have been obtained, see [2, 3, 6, 10, 11, 17–19, 25]. A detailed list of results and open questions on power graphs can be found in [1]. (Since our paper deals only with undirected graphs, for convenience throughout we use the term "power graph" to refer to an undirected power graph defined as above, see also [16], Section 3).

A finite group is called a *p*-group if its order is a power of *p*, where *p* is a prime. In *G*, an element of order 2 is called an involution. An involution *x* is *maximal* if the only cyclic subgroup containing *x* is the subgroup $\langle x \rangle$ generated by *x*. Denote by M_G the set of all maximal involutions of *G*. We use M_G to discuss the rainbow connection number of Γ_G .

This paper is organized as follows. In Section 2 we express $\operatorname{rc}(\Gamma_G)$ in terms of $|M_G|$ if $M_G \neq \emptyset$. In Section 3 we show that $\operatorname{rc}(\Gamma_G) \leq 3$ if $M_G = \emptyset$. In particular, we determine $\operatorname{rc}(\Gamma_G)$ if G is nilpotent. The rainbow connection numbers of power graphs of some nonnilpotent groups are also given.

2 $M_G \neq \emptyset$

In this section we prove the following theorem.

Theorem 2.1 Let G be a finite group of order at least 3. Then

$$\operatorname{rc}(\Gamma_G) = \begin{cases} 3, & \text{if } 1 \le |M_G| \le 2; \\ |M_G|, & \text{if } |M_G| \ge 3. \end{cases}$$

We begin with the following lemma.

Lemma 2.2 $\operatorname{rc}(\Gamma_G) \geq |M_G|$.

Proof. Let $M_G = \{z_1, \ldots, z_m\}$. Observe that e is the unique vertex adjacent to z_i in Γ_G , where $i = 1, \ldots, m$. Hence, for each pair of maximal involutions z_i and z_j , the path from z_i to z_j is unique, which is (z_i, e, z_j) . Suppose ζ is a rainbow k-coloring of Γ_G . Then $|\{\zeta(\{z_i, e\}) : i = 1, \ldots, m\}| = m$, and so $k \ge m$, as desired. \Box

For $x \in G$, let $[x] = \{y \in G : \langle y \rangle = \langle x \rangle\}$. Then $\{[x] : x \in G\}$ is a partition of G.

Lemma 2.3 $rc(\Gamma_G) \le max\{|M_G|, 3\}.$

Proof. Suppose that $\{[x_1], \ldots, [x_s]\}$ and $\{[x_{s+1}], \ldots, [x_{s+t}]\}$ are partitions of $\{x \in G : |x| \text{ is even, } |x| \ge 4\}$ and $\{x \in G : |x| \text{ is odd, } |x| \ge 3\}$, respectively. For $1 \le i \le s$, let u_i be the involution in $\langle x_i \rangle$. Write $M_G = \{z_1, \ldots, z_m\}$ and

$$E_1 = \{\{e, x\} : x \in \bigcup_{i=1}^{s+t} ([x_i] \setminus \{x_i\})\} \cup \{\{u_i, x_i\} : i = 1, \dots, s\},\$$

$$E_2 = \{\{e, x_i\} : i = 1, \dots, s+t\} \cup (\bigcup_{i=1}^s \{\{u_i, x\} : x \in [x_i] \setminus \{x_i\}\}),\$$

$$E_3 = E(\Gamma_G) \setminus (E_1 \cup E_2 \cup \{\{e, z_j\} : j = 1, \dots, m\}).$$

The sets of edges E_1, E_2 and $\{\{e, z_j\} : j = 1, \dots, m\}$ are showed in Figure 1.



Figure 1: The set of edges $E_1 \cup E_2 \cup \{\{e, z_j\} : j = 1, \dots, m\}$

Let $k = \max\{|M_G|, 3\}$. Define a coloring

$$\zeta: E(\Gamma_G) \longrightarrow \{1, \dots, k\}, \quad f \longmapsto \begin{cases} i, & \text{if } f \in E_i, \\ j, & \text{if } f = \{e, z_j\}, \\ \end{cases} \text{ where } i = 1, 2, 3;$$

In order to get the desired inequality, we only need to show that ζ is a rainbow kcoloring of Γ_G . Pick a pair of nonadjacent vertices v and w of Γ_G . It suffices to find a rainbow path from v to w under the coloring ζ . If $\zeta(\{e, v\}) \neq \zeta(\{e, w\})$, then (v, e, w)is a desired rainbow path. Now suppose $\zeta(\{e, v\}) = \zeta(\{e, w\})$. Then $\{v, w\} \not\subseteq$ $(M_G \cup \{e\})$. Without loss of generality, we may assume that $v \in V(\Gamma_G) \setminus (M_G \cup \{e\})$. As shown in Figure 1, there exists a vertex $v' \in V(\Gamma_G) \setminus (M_G \cup \{e\})$ such that

$$\{\zeta(\{e,v\}), \zeta(\{e,v'\}), \zeta(\{v,v'\})\} = \{1,2,3\},\$$

which implies that (v, v', e, w) is a rainbow path, as desired.

Combining Lemmas 2.2 and 2.3, we get the following.

Proposition 2.4 If $|M_G| \ge 3$, then $\operatorname{rc}(\Gamma_G) = |M_G|$.

For a prime p, let $s_p(G)$ denote the number of subgroups of order p in G.

Lemma 2.5 ([13, Section 4, I]) Let p be a prime dividing the order of G. Then

$$s_p(G) \equiv 1 \pmod{p}.$$

Lemma 2.6 Let p be a prime dividing |G|. If $rc(\Gamma_G) = 2$, then $s_p(G) = 1$.

Proof. Suppose for the contrary that $s_p(G) \neq 1$. It follows from Lemma 2.5 that $s_p(G) \geq 3$. Let $\langle y_1 \rangle$, $\langle y_2 \rangle$ and $\langle y_3 \rangle$ be pairwise distinct subgroups of order p in G. Note that, for $i \neq j$, there is no cyclic subgroup containing $\langle y_i \rangle$ and $\langle y_j \rangle$. Hence, the path from y_i to y_j with length 2 is unique, which is (y_i, e, y_j) . For any rainbow k-coloring ζ of Γ_G , we deduce that $\zeta(\{e, y_1\}), \zeta(\{e, y_2\})$ and $\zeta(\{e, y_3\})$ are pairwise distinct, which implies that $k \geq 3$, contrary to $\operatorname{rc}(\Gamma_G) = 2$.

By Lemmas 2.2, 2.3 and 2.6, we get the following result.

Proposition 2.7 If $|M_G| = 2$, then $\operatorname{rc}(\Gamma_G) = 3$.

Proposition 2.8 If $|G| \ge 3$ and $|M_G| = 1$, then $\operatorname{rc}(\Gamma_G) = 3$.

Proof. It follows from Lemma 2.3 that $rc(\Gamma_G) \leq 3$. Suppose for the contrary that $rc(\Gamma_G) \leq 2$. If $rc(\Gamma_G) = 1$, then Γ_G is a complete graph, and so G is a cyclic group of prime power order by [6, Theorem 2.12], contrary to $|G| \geq 3$ and $|M_G| = 1$. In the following assume that $rc(\Gamma_G) = 2$.

Suppose that G is a 2-group. By Lemma 2.6, the involution is unique, which implies that G is cyclic or generalised quaternion by [15, Theorem 5.4.10 (ii)], a contradiction.

Suppose that |G| has a prime divisor p at least 3. Let x be an element of G with |x| = p. Write $M_G = \{z\}$. It follows from Lemma 2.6 that $\langle x \rangle$ and $\langle z \rangle$ are normal subgroups in G. Note that $\langle x \rangle \cap \langle z \rangle = \langle e \rangle$. So $\langle x \rangle \langle z \rangle$ is a cyclic group, contrary to the fact that z is maximal.

Proof of Theorem 2.1 follows from Propositions 2.4, 2.7 and 2.8.

For $n \geq 3$, let D_{2n} denote the dihedral group of order 2n, where

$$D_{2n} = \langle a, b : a^n = b^2 = 1, bab = a^{-1} \rangle.$$

 \mathbb{Z}_2^n denotes the elementary abelian 2-group. Note that $M_{D_{2n}} = \{b, ab, a^2b, \ldots, a^{n-1}b\}$ and $M_{\mathbb{Z}_2^n}$ consists of all nonidentity elements. By Theorem 2.1, we get the following.

Example 1 For $n \geq 3$, we have $\operatorname{rc}(\Gamma_{D_{2n}}) = n$ and $\operatorname{rc}(\Gamma_{\mathbb{Z}_2^n}) = 2^n - 1$.

3 $M_G = \emptyset$

In this section we study the rainbow connection number of Γ_G when G has no maximal involutions.

For a positive integer n, let D(n) be the set of all divisors of n. Denote by ϕ the Euler's totient function. In view of [26, Part VIII, Problem 45], one has $\phi(n) \ge |D(n)| - 2$. For $x \in G$, recall that $[x] = \{y \in G : \langle y \rangle = \langle x \rangle\}$. Write

$$E_1(\langle x \rangle) = \bigcup_{i=1}^{|D(|x|)|-2} \{ \{x_i, y\} : y \in \langle x \rangle, |y| = d_i \},$$

$$(1)$$

where $[x] = \{x_1, \dots, x_{\phi(|x|)}\}$ and $D(|x|) = \{1, d_1, \dots, d_{|D(|x|)|-2}, |x|\}$ (see Figure 2).



Figure 2: The partition of $V(\Gamma_{\langle x \rangle})$ and the set of edges $E_1(\langle x \rangle)$

Theorem 3.1 Let G be a finite group with no maximal involutions.

(i) If G is cyclic, then

$$\operatorname{rc}(\Gamma_G) = \begin{cases} 1, & \text{if } |G| \text{ is a prime power;} \\ 2, & \text{otherwise.} \end{cases}$$

(ii) If G is noncyclic, then $rc(\Gamma_G) = 2$ or 3.

Proof. (i) Write $G = \langle x \rangle$. If |x| is a prime power, then Γ_G is a complete graph by [6, Theorem 2.12], and so $\operatorname{rc}(\Gamma_G) = 1$. Now suppose that |x| is not a power of any prime. Then $\operatorname{rc}(\Gamma_G) \geq 2$. With reference to (1), write $E_1 = E_1(\langle x \rangle)$. It is clear that $E_1 \subseteq E(\Gamma_G)$. Let $E_2 = E(\Gamma_G) \setminus E_1$. Define a coloring

$$\zeta: E(\Gamma_G) \longrightarrow \{1, 2\}, \quad f \longmapsto i \text{ if } f \in E_i.$$

In order to get the desired result, we only need to show that ζ is a rainbow 2-coloring. For any pair of nonadjacent vertices v and w, there exist distinct indices i and j in $\{1, \ldots, |D(|x|)| - 2\}$ such that $|v| = d_i$ and $|w| = d_j$. It follows from Figure 2 that (v, x_i, w) is a rainbow path under the coloring ζ , as desired.

(ii) It is immediate from Lemma 2.3.

We first give two examples for computing $rc(\Gamma_G)$ when G is noncyclic with no maximal involutions. The generalized quaternion group is defined by

$$Q_{4n} = \langle x, y : x^n = y^2, x^{2n} = 1, y^{-1}xy = x^{-1} \rangle, \qquad n \ge 2.$$
(2)

Example 2 If *n* is odd, then $rc(\Gamma_{Q_8 \times \mathbb{Z}_n}) = 2$.

Proof. There are exactly three maximal cyclic subgroup in $Q_8 \times \mathbb{Z}_n$, which we denote by $\langle x_1 \rangle$, $\langle x_2 \rangle$ and $\langle x_3 \rangle$. It is easy to see that $|x_1| = |x_2| = |x_3| = 4n$. Let C be a subgroup of order 2n in $\langle x_1 \rangle$. Then $C = \langle x_i \rangle \cap \langle x_j \rangle$ for $1 \le i < j \le 3$. Write $D(n) = \{d_1, \ldots, d_t\}$. Let B_i , C_i and D_i be the set of generators of the subgroup of order $4d_i$ in $\langle x_1 \rangle$, $\langle x_2 \rangle$ and $\langle x_3 \rangle$, respectively. Consequently, we have

$$V(\Gamma_{Q_8 \times \mathbb{Z}_n}) = C \cup \bigcup_{i=1}^t (B_i \cup C_i \cup D_i),$$

$$E(\Gamma_{Q_8 \times \mathbb{Z}_n}) = E(\Gamma_{\langle x_1 \rangle}) \cup E(\Gamma_{\langle x_2 \rangle}) \cup E(\Gamma_{\langle x_3 \rangle}).$$

The partition of $V(\Gamma_{Q_8 \times \mathbb{Z}_n})$ is showed in Figure 3, where u is the unique involution. With reference to (1), there exists a unique vertex $x'_3 \in [x_3]$ such that $\{u, x'_3\} \in E_1(\langle x_3 \rangle)$. Write

$$E'_{1} = \bigcup_{i=1}^{t} \{\{e, x\} : x \in B_{i}\} \cup \{\{u, x\} : x \in C_{i}\}\},\$$

$$E_{1} = E'_{1} \cup E_{1}(\langle x_{1} \rangle) \cup E_{1}(\langle x_{2} \rangle) \cup (E_{1}(\langle x_{3} \rangle) \setminus \{\{u, x'_{3}\}\}).$$



Figure 3: The partition of $V(\Gamma_{Q_8 \times \mathbb{Z}_n})$ and the set of edges E'_1

It is clear that $E_1 \subseteq E(\Gamma_{Q_8 \times \mathbb{Z}_n})$. Write $E_2 = E(\Gamma_{Q_8 \times \mathbb{Z}_n}) \setminus E_1$. Define a coloring

$$\zeta: E(\Gamma_{Q_8 \times \mathbb{Z}_n}) \longrightarrow \{1, 2\}, \quad f \longmapsto k \text{ if } f \in E_k$$

For i = 1, 2, 3, let Δ_i be the subgraph of $\Gamma_{\langle x_i \rangle}$ induced by $V(\Gamma_{\langle x_i \rangle}) \setminus \{e, u\}$. Similar to the proof of Theorem 3.1 (i), we deduce that $\zeta|_{E(\Delta_i)}$ is a rainbow 2-coloring of Δ_i . If vertices v and w satisfy $u \notin \{v, w\}$ and $\{v, w\} \not\subseteq V(\Delta_i)$ for any $i \in \{1, 2, 3\}$, then (v, e, w) or (v, u, w) is a rainbow path under ζ from Figure 3. If v is a vertex that is not adjacent to u, there exists a vertex $x''_3 \in [x_3] \setminus \{x'_3\}$ such that $\{x''_3, v\} \in E_1(\langle x_3 \rangle)$, and so (u, x''_3, v) is a rainbow path under ζ . It follows that ζ is a rainbow 2-coloring of $\Gamma_{Q_8 \times \mathbb{Z}_n}$. This completes the proof. \Box

Example 3 If $n \geq 3$, then $rc(\Gamma_{Q_{4n}}) = 3$.

Proof. With reference to (2), we have $y^{-1} = x^n y$ and $(x^i y)^{-1} = x^{2n-i} y$ for $i \in \{1, \ldots, n-1\}$, which implies that

$$V(\Gamma_{Q_{4n}}) = \{e, x, \dots, x^{2n-1}\} \cup (\bigcup_{i=0}^{n-1} \{x^i y, (x^i y)^{-1}\}),$$
$$E(\Gamma_{Q_{4n}}) = E(\Gamma_{\langle x \rangle}) \cup \bigcup_{i=0}^{n-1} E(\Gamma_{\langle x^i y \rangle}),$$

as shown in Figure 4. It follows from Theorem 3.1 that $rc(\Gamma_{Q_{4n}}) = 2$ or 3. Suppose



Figure 4: $\Gamma_{Q_{4n}}$

for the contrary that there exists a rainbow 2-coloring ζ of $\Gamma_{Q_{4n}}$.

Assume that n = 3. Without loss of generality, let $\zeta(\{e, x^2\}) = 1$. Then $\zeta(\{e, x^iy\}) = 2$ for $i \in \{0, 1, 2\}$. Hence, for $0 \le i < j \le 2$, the rainbow path from x^iy to x^jy is (x^iy, x^3, x^jy) , which implies that $\zeta(\{y, x^3\}), \zeta(\{xy, x^3\})$ and $\zeta(\{x^2y, x^3\})$ are pairwise distinct, a contradiction. Therefore $\operatorname{rc}(\Gamma_{Q_{12}}) = 3$.

In the following, assume that $n \geq 4$. Let Δ be the induced subgraph of $\Gamma_{Q_{4n}}$ on the vertices $\{e, x, y, xy, x^2y, x^3y, x^n\}$. Then $\zeta|_{E(\Delta)}$ is a rainbow 2-coloring of Δ .

We claim that there exists a rainbow path from e to x^n with length 2 under $\zeta|_{E(\Delta)}$ in Δ . In fact, if $\zeta|_{E(\Delta)}(\{e, x^iy\}) = \zeta|_{E(\Delta)}(\{x^iy, x^n\})$ for each $i \in \{0, 1, 2, 3\}$, then there exist two distinct indices j and k in $\{0, 1, 2, 3\}$ such that

$$\zeta|_{E(\Delta)}(\{e, x^j y\}) = \zeta|_{E(\Delta)}(\{x^j y, x^n\}) = \zeta|_{E(\Delta)}(\{e, x^k y\}) = \zeta|_{E(\Delta)}(\{x^k y, x^n\}),$$

which implies that there is no rainbow path from $x^j y$ to $x^k y$ under $\zeta|_{E(\Delta)}$ in Δ , a contradiction. Hence, the claim is valid.

Let Δ_0 be the graph obtained from Δ by deleting the edge $\{e, x_n\}$. Then Δ_0 is isomorphic to the complete bipartite graph $K_{2,5}$. By the claim above, we have $\operatorname{rc}(K_{2,5}) = 2$, contrary to [7, Theorem 2.6].

For a noncyclic group G with no maximal involutions, it is difficult for us to determine which groups G satisfy $rc(\Gamma_G) = 2$. However, we give a sufficient condition.

Proposition 3.2 If G is a group of order $p^n q$ for positive integer n, where p, q are distinct primes and p < q, such that the following conditions hold, then $rc(\Gamma_G) = 2$.

- (i) Each Sylow p-subgroup is cyclic and the Sylow q-subgroup is unique.
- (ii) The intersection of all Sylow p-subgroups is of order p^{n-1} .
- (iii) $p^{n-1} \ge q$.

Proof. Note that the number of Sylow *p*-subgroups is *q*. Suppose that $\{P_1, \ldots, P_q\}$ is the set of all Sylow *p*-subgroups, and *Q* is the unique Sylow *q*-subgroup. Then $\bigcap_{i=1}^{q} P_i$ and *Q* are cyclic and normal in *G*. Hence, there exists an element *x* of order $p^{n-1}q$ such that $(\bigcap_{i=1}^{q} P_i)Q = \langle x \rangle$, and so the set of all cyclic subgroups of *G* is

$$\{P_1,\ldots,P_q\} \cup \{\langle y \rangle : y \in \langle x \rangle\}.$$

For $1 \leq i \leq q$, let A_i be the set of all generators of P_i . By (iii) we choose pairwise distinct elements u_1, \ldots, u_{q-1} in $(\bigcap_{i=1}^q P_i) \setminus \{e\}$. With reference to (1), write

$$E'_{1} = \{\{e, y\} : y \in \bigcup_{i=1}^{q} A_{i}\} \cup \bigcup_{i=1}^{q-1} \{\{u_{i}, y\} : y \in A_{i}\},\$$
$$E_{1} = E'_{1} \cup E_{1}(\langle x \rangle).$$

The set E'_1 is showed in Figure 5. It is clear that $E_1 \subseteq E(\Gamma_G)$. Let $E_2 = E(\Gamma_G) \setminus E_1$. Define a coloring

$$\zeta: E(\Gamma_G) \longrightarrow \{1, 2\}, \quad f \longmapsto k \text{ if } f \in E_k.$$

In order to get the desired result, we only need to show that ζ is a rainbow 2-coloring of Γ_G . It follows from Theorem 3.1 that $\zeta|_{E(\Gamma_{\langle x \rangle})}$ is a rainbow 2-coloring



Figure 5: $V(\Gamma_G)$ and the set of edges E'_1

of $\Gamma_{\langle x \rangle}$. Pick any pair of nonadjacent vertices z and w such that $\{z, w\} \not\subseteq V(\Gamma_{\langle x \rangle})$. It suffices to find a rainbow path from z to w under ζ . Without loss of generality, assume that $z \in \bigcup_{i=1}^{q} A_i$. If $w \in \bigcup_{i=1}^{q} A_i$, then there exist indices i and j in $\{1, \ldots, q\}$ with i < j such that $z \in A_i$ and $w \in A_j$, and so (z, u_i, w) is a desired rainbow path. If $w \in V(\Gamma_{\langle x \rangle})$, then (z, e, w) is a desired rainbow path. \Box

By Proposition 3.2, we have the following example.

Example 4 Let $G = \langle a, b : a^{27} = b^7 = e, a^{-1}ba = b^2 \rangle \cong \mathbb{Z}_{27} \ltimes \mathbb{Z}_7$. Then $\operatorname{rc}(\Gamma_G) = 2$.

The following sufficient condition for $rc(\Gamma_G) = 3$ is immediate from Theorem 3.1 and Lemma 2.6.

Proposition 3.3 Suppose that G is a noncyclic group with no maximal involutions. If there exists a prime p dividing |G| such that the subgroup of order p in G is not unique, then $\operatorname{rc}(\Gamma_G) = 3$.

Finally, we determine the rainbow connection number of the power graph of a nilpotent group.

Corollary 3.4 Let G be a noncyclic nilpotent group with no maximal involutions. Then

$$\operatorname{rc}(\Gamma_G) = \begin{cases} 2, & \text{if } G \text{ is isomorphic to } Q_8 \times \mathbb{Z}_n \text{ for some odd number } n; \\ 3, & \text{otherwise.} \end{cases}$$

Proof. It follows from Theorem 3.1 that $\operatorname{rc}(\Gamma_G) = 2$ or 3. Suppose $\operatorname{rc}(\Gamma_G) = 2$. Then for any prime p dividing |G|, the subgroup of order p in G is unique by Proposition 3.3. By [15, Theorem 5.4.10 (ii)], the Sylow p-subgroups are cyclic for any odd prime p, which implies that 2 is a divisor of |G| and the Sylow 2-subgroup is isomorphic to Q_{2^m} for $m \geq 3$. Hence we get $G \cong Q_{2^m} \times \mathbb{Z}_n$ for some odd number n. Let H be a subgroup of G that is isomorphic to Q_{2^m} .

We claim that for any pair of nonadjacent vertices x and y of Γ_H , there does not exist a vertex in $G \setminus H$ adjacent to both x and y in Γ_G . Suppose for the contrary that $\{\{x, z\}, \{y, z\}\} \subseteq E(\Gamma_G)$ for some $z \in G \setminus H$. Then $x = z^s$ and $y = z^t$ for some integers s and t, which implies that $x, y \in \langle z \rangle$. Note that |x| and |y| are powers of 2. It follows that |x| is divisible by |y|, or |y| is divisible by |x|. Since $x, y \in \langle z \rangle$, one has $\langle x \rangle \subseteq \langle y \rangle$ or $\langle y \rangle \subseteq \langle x \rangle$. Thus x and y are adjacent, a contradiction. Hence, the claim is valid. By the claim, one gets $\operatorname{rc}(\Gamma_H) = 2$. It follows from Example 3 that m = 3, and so $G \cong Q_8 \times \mathbb{Z}_n$. By Example 2, we get the desired result. \Box

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