# The rainbow connection number of the power graph of a finite group 

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#### Abstract

This paper studies the rainbow connection number of the power graph $\Gamma_{G}$ of a finite group $G$. We determine the rainbow connection number of $\Gamma_{G}$ if $G$ has maximal involutions or is nilpotent, and show that the rainbow connection number of $\Gamma_{G}$ is at most three if $G$ has no maximal involutions. The rainbow connection numbers of power graphs of some nonnilpotent groups are also given.


Key words: Rainbow path; rainbow connection number; finite group; power graph.

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## 1 Introduction

Given a connected graph $\Gamma$, denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and edge set, respectively. Define a coloring $\zeta: E(\Gamma) \rightarrow\{1,2, \ldots, k\}, k \in \mathbb{N}$, where adjacent edges may be colored the same. A path $P$ is rainbow if any two edges in $P$ are colored distinct. If $\Gamma$ has a rainbow path from $u$ to $v$ for each pair of vertices $u$ and $v$, then $\Gamma$ is rainbow-connected under the coloring $\zeta$, and $\zeta$ is called a rainbow $k$-coloring of $\Gamma$. The rainbow connection number of $\Gamma$, denoted by $\operatorname{rc}(\Gamma)$, is the minimum $k$ for which there exists a rainbow $k$-coloring of $\Gamma$.

The rainbow connection number of a graph $\Gamma$ was introduced by Chartrand et al. [7]. It was showed in [5, 20] that computing $\operatorname{rc}(\Gamma)$ is NP-hard. Moreover, it has been proved in [20], that for any fixed $t \geq 2$, deciding if $\operatorname{rc}(\Gamma)=t$ is NP-complete. Some topics on restrict graphs are as follows: oriented graphs [8], graph products [14], hypergraphs [4], corona graphs [9], line graphs [24], Cayley graphs [21], dense graphs [22] and sparse random graphs [12]. Most of the results and papers that dealt with it can be found in [23].

In this paper we study the rainbow connection number of the power graph of a finite group. We always use $G$ to denote a finite group with the identity $e$. The power graph $\Gamma_{G}$ has the vertex set $G$ and two distinct elements are adjacent if one is

[^0]a power of the other. The concept of a power graph was introduced in [16]. Recently, many interesting results on power graphs have been obtained, see $[2,3,6,10,11,17$ $19,25]$. A detailed list of results and open questions on power graphs can be found in [1]. (Since our paper deals only with undirected graphs, for convenience throughout we use the term "power graph" to refer to an undirected power graph defined as above, see also [16], Section 3).

A finite group is called a $p$-group if its order is a power of $p$, where $p$ is a prime. In $G$, an element of order 2 is called an involution. An involution $x$ is maximal if the only cyclic subgroup containing $x$ is the subgroup $\langle x\rangle$ generated by $x$. Denote by $M_{G}$ the set of all maximal involutions of $G$. We use $M_{G}$ to discuss the rainbow connection number of $\Gamma_{G}$.

This paper is organized as follows. In Section 2 we express $\operatorname{rc}\left(\Gamma_{G}\right)$ in terms of $\left|M_{G}\right|$ if $M_{G} \neq \emptyset$. In Section 3 we show that $\mathrm{rc}\left(\Gamma_{G}\right) \leq 3$ if $M_{G}=\emptyset$. In particular, we determine $\operatorname{rc}\left(\Gamma_{G}\right)$ if $G$ is nilpotent. The rainbow connection numbers of power graphs of some nonnilpotent groups are also given.

## $2 \quad M_{G} \neq \emptyset$

In this section we prove the following theorem.
Theorem 2.1 Let $G$ be a finite group of order at least 3. Then

$$
\operatorname{rc}\left(\Gamma_{G}\right)= \begin{cases}3, & \text { if } 1 \leq\left|M_{G}\right| \leq 2 \\ \left|M_{G}\right|, & \text { if }\left|M_{G}\right| \geq 3\end{cases}
$$

We begin with the following lemma.
Lemma $2.2 \operatorname{rc}\left(\Gamma_{G}\right) \geq\left|M_{G}\right|$.
Proof. Let $M_{G}=\left\{z_{1}, \ldots, z_{m}\right\}$. Observe that $e$ is the unique vertex adjacent to $z_{i}$ in $\Gamma_{G}$, where $i=1, \ldots, m$. Hence, for each pair of maximal involutions $z_{i}$ and $z_{j}$, the path from $z_{i}$ to $z_{j}$ is unique, which is ( $z_{i}, e, z_{j}$ ). Suppose $\zeta$ is a rainbow $k$-coloring of $\Gamma_{G}$. Then $\left|\left\{\zeta\left(\left\{z_{i}, e\right\}\right): i=1, \ldots, m\right\}\right|=m$, and so $k \geq m$, as desired.

For $x \in G$, let $[x]=\{y \in G:\langle y\rangle=\langle x\rangle\}$. Then $\{[x]: x \in G\}$ is a partition of $G$.
Lemma $2.3 \mathrm{rc}\left(\Gamma_{G}\right) \leq \max \left\{\left|M_{G}\right|, 3\right\}$.
Proof. Suppose that $\left\{\left[x_{1}\right], \ldots,\left[x_{s}\right]\right\}$ and $\left\{\left[x_{s+1}\right], \ldots,\left[x_{s+t}\right]\right\}$ are partitions of $\{x \in$ $G:|x|$ is even, $|x| \geq 4\}$ and $\{x \in G:|x|$ is odd, $|x| \geq 3\}$, respectively. For $1 \leq i \leq$ $s$, let $u_{i}$ be the involution in $\left\langle x_{i}\right\rangle$. Write $M_{G}=\left\{z_{1}, \ldots, z_{m}\right\}$ and

$$
\begin{aligned}
& E_{1}=\left\{\{e, x\}: x \in \cup_{i=1}^{s+t}\left(\left[x_{i}\right] \backslash\left\{x_{i}\right\}\right)\right\} \cup\left\{\left\{u_{i}, x_{i}\right\}: i=1, \ldots, s\right\}, \\
& E_{2}=\left\{\left\{e, x_{i}\right\}: i=1, \ldots, s+t\right\} \cup\left(\cup_{i=1}^{s}\left\{\left\{u_{i}, x\right\}: x \in\left[x_{i}\right] \backslash\left\{x_{i}\right\}\right\}\right), \\
& E_{3}=E\left(\Gamma_{G}\right) \backslash\left(E_{1} \cup E_{2} \cup\left\{\left\{e, z_{j}\right\}: j=1, \ldots, m\right\}\right) .
\end{aligned}
$$

The sets of edges $E_{1}, E_{2}$ and $\left\{\left\{e, z_{j}\right\}: j=1, \ldots, m\right\}$ are showed in Figure 1.


Figure 1: The set of edges $E_{1} \cup E_{2} \cup\left\{\left\{e, z_{j}\right\}: j=1, \ldots, m\right\}$

Let $k=\max \left\{\left|M_{G}\right|, 3\right\}$. Define a coloring

$$
\zeta: E\left(\Gamma_{G}\right) \longrightarrow\{1, \ldots, k\}, \quad f \longmapsto\left\{\begin{array}{ll}
i, & \text { if } f \in E_{i},
\end{array} \quad \text { where } i=1,2,3 ; ~ 子, ~ i e, ~ w h e r e ~ j=1, \ldots, m .\right.
$$

In order to get the desired inequality, we only need to show that $\zeta$ is a rainbow $k$ coloring of $\Gamma_{G}$. Pick a pair of nonadjacent vertices $v$ and $w$ of $\Gamma_{G}$. It suffices to find a rainbow path from $v$ to $w$ under the coloring $\zeta$. If $\zeta(\{e, v\}) \neq \zeta(\{e, w\})$, then $(v, e, w)$ is a desired rainbow path. Now suppose $\zeta(\{e, v\})=\zeta(\{e, w\})$. Then $\{v, w\} \nsubseteq$ $\left(M_{G} \cup\{e\}\right)$. Without loss of generality, we may assume that $v \in V\left(\Gamma_{G}\right) \backslash\left(M_{G} \cup\{e\}\right)$. As shown in Figure 1, there exists a vertex $v^{\prime} \in V\left(\Gamma_{G}\right) \backslash\left(M_{G} \cup\{e\}\right)$ such that

$$
\left\{\zeta(\{e, v\}), \zeta\left(\left\{e, v^{\prime}\right\}\right), \zeta\left(\left\{v, v^{\prime}\right\}\right)\right\}=\{1,2,3\}
$$

which implies that $\left(v, v^{\prime}, e, w\right)$ is a rainbow path, as desired.
Combining Lemmas 2.2 and 2.3, we get the following.
Proposition 2.4 If $\left|M_{G}\right| \geq 3$, then $\operatorname{rc}\left(\Gamma_{G}\right)=\left|M_{G}\right|$.
For a prime $p$, let $s_{p}(G)$ denote the number of subgroups of order $p$ in $G$.
Lemma 2.5 ([13, Section 4, I]) Let p be a prime dividing the order of $G$. Then

$$
s_{p}(G) \equiv 1 \quad(\bmod p) .
$$

Lemma 2.6 Let $p$ be a prime dividing $|G|$. If $\operatorname{rc}\left(\Gamma_{G}\right)=2$, then $s_{p}(G)=1$.
Proof. Suppose for the contrary that $s_{p}(G) \neq 1$. It follows from Lemma 2.5 that $s_{p}(G) \geq 3$. Let $\left\langle y_{1}\right\rangle,\left\langle y_{2}\right\rangle$ and $\left\langle y_{3}\right\rangle$ be pairwise distinct subgroups of order $p$ in $G$. Note that, for $i \neq j$, there is no cyclic subgroup containing $\left\langle y_{i}\right\rangle$ and $\left\langle y_{j}\right\rangle$. Hence, the path from $y_{i}$ to $y_{j}$ with length 2 is unique, which is $\left(y_{i}, e, y_{j}\right)$. For any rainbow $k$-coloring $\zeta$ of $\Gamma_{G}$, we deduce that $\zeta\left(\left\{e, y_{1}\right\}\right), \zeta\left(\left\{e, y_{2}\right\}\right)$ and $\zeta\left(\left\{e, y_{3}\right\}\right)$ are pairwise distinct, which implies that $k \geq 3$, contrary to $\operatorname{rc}\left(\Gamma_{G}\right)=2$.

By Lemmas 2.2, 2.3 and 2.6, we get the following result.

Proposition 2.7 If $\left|M_{G}\right|=2$, then $\operatorname{rc}\left(\Gamma_{G}\right)=3$.
Proposition 2.8 If $|G| \geq 3$ and $\left|M_{G}\right|=1$, then $\operatorname{rc}\left(\Gamma_{G}\right)=3$.
Proof. It follows from Lemma 2.3 that $\operatorname{rc}\left(\Gamma_{G}\right) \leq 3$. Suppose for the contrary that $\operatorname{rc}\left(\Gamma_{G}\right) \leq 2$. If $\operatorname{rc}\left(\Gamma_{G}\right)=1$, then $\Gamma_{G}$ is a complete graph, and so $G$ is a cyclic group of prime power order by [6, Theorem 2.12], contrary to $|G| \geq 3$ and $\left|M_{G}\right|=1$. In the following assume that $\mathrm{rc}\left(\Gamma_{G}\right)=2$.

Suppose that $G$ is a 2 -group. By Lemma 2.6, the involution is unique, which implies that $G$ is cyclic or generalised quaternion by [15, Theorem 5.4.10 (ii)], a contradiction.

Suppose that $|G|$ has a prime divisor $p$ at least 3 . Let $x$ be an element of $G$ with $|x|=p$. Write $M_{G}=\{z\}$. It follows from Lemma 2.6 that $\langle x\rangle$ and $\langle z\rangle$ are normal subgroups in $G$. Note that $\langle x\rangle \cap\langle z\rangle=\langle e\rangle$. So $\langle x\rangle\langle z\rangle$ is a cyclic group, contrary to the fact that $z$ is maximal.

Proof of Theorem 2.1 follows from Propositions 2.4, 2.7 and 2.8.
For $n \geq 3$, let $D_{2 n}$ denote the dihedral group of order $2 n$, where

$$
D_{2 n}=\left\langle a, b: a^{n}=b^{2}=1, b a b=a^{-1}\right\rangle .
$$

$\mathbb{Z}_{2}^{n}$ denotes the elementary abelian 2-group. Note that $M_{D_{2 n}}=\left\{b, a b, a^{2} b, \ldots, a^{n-1} b\right\}$ and $M_{\mathbb{Z}_{2}^{n}}$ consists of all nonidentity elements. By Theorem 2.1, we get the following.

Example 1 For $n \geq 3$, we have $\operatorname{rc}\left(\Gamma_{D_{2 n}}\right)=n$ and $\operatorname{rc}\left(\Gamma_{\mathbb{Z}_{2}^{n}}\right)=2^{n}-1$.
$3 \quad M_{G}=\emptyset$
In this section we study the rainbow connection number of $\Gamma_{G}$ when $G$ has no maximal involutions.

For a positive integer $n$, let $D(n)$ be the set of all divisors of $n$. Denote by $\phi$ the Euler's totient function. In view of [26, Part VIII, Problem 45], one has $\phi(n) \geq|D(n)|-2$. For $x \in G$, recall that $[x]=\{y \in G:\langle y\rangle=\langle x\rangle\}$. Write

$$
\begin{equation*}
E_{1}(\langle x\rangle)=\bigcup_{i=1}^{|D(|x|)|-2}\left\{\left\{x_{i}, y\right\}: y \in\langle x\rangle,|y|=d_{i}\right\}, \tag{1}
\end{equation*}
$$

where $[x]=\left\{x_{1}, \ldots, x_{\phi(|x|)}\right\}$ and $D(|x|)=\left\{1, d_{1}, \ldots, d_{|D(|x|)|-2},|x|\right\}$ (see Figure 2).


Figure 2: The partition of $V\left(\Gamma_{\langle x\rangle}\right)$ and the set of edges $E_{1}(\langle x\rangle)$

Theorem 3.1 Let $G$ be a finite group with no maximal involutions.
(i) If $G$ is cyclic, then

$$
\operatorname{rc}\left(\Gamma_{G}\right)= \begin{cases}1, & \text { if }|G| \text { is a prime power } ; \\ 2, & \text { otherwise } .\end{cases}
$$

(ii) If $G$ is noncyclic, then $\operatorname{rc}\left(\Gamma_{G}\right)=2$ or 3 .

Proof. (i) Write $G=\langle x\rangle$. If $|x|$ is a prime power, then $\Gamma_{G}$ is a complete graph by [6, Theorem 2.12], and so $\operatorname{rc}\left(\Gamma_{G}\right)=1$. Now suppose that $|x|$ is not a power of any prime. Then $\operatorname{rc}\left(\Gamma_{G}\right) \geq 2$. With reference to (1), write $E_{1}=E_{1}(\langle x\rangle)$. It is clear that $E_{1} \subseteq E\left(\Gamma_{G}\right)$. Let $E_{2}=E\left(\Gamma_{G}\right) \backslash E_{1}$. Define a coloring

$$
\zeta: E\left(\Gamma_{G}\right) \longrightarrow\{1,2\}, \quad f \longmapsto i \text { if } f \in E_{i} .
$$

In order to get the desired result, we only need to show that $\zeta$ is a rainbow 2-coloring. For any pair of nonadjacent vertices $v$ and $w$, there exist distinct indices $i$ and $j$ in $\{1, \ldots,|D(|x|)|-2\}$ such that $|v|=d_{i}$ and $|w|=d_{j}$. It follows from Figure 2 that $\left(v, x_{i}, w\right)$ is a rainbow path under the coloring $\zeta$, as desired.
(ii) It is immediate from Lemma 2.3.

We first give two examples for computing $\operatorname{rc}\left(\Gamma_{G}\right)$ when $G$ is noncyclic with no maximal involutions. The generalized quaternion group is defined by

$$
\begin{equation*}
Q_{4 n}=\left\langle x, y: x^{n}=y^{2}, x^{2 n}=1, y^{-1} x y=x^{-1}\right\rangle, \quad n \geq 2 . \tag{2}
\end{equation*}
$$

Example 2 If $n$ is odd, then $\operatorname{rc}\left(\Gamma_{Q_{8} \times \mathbb{Z}_{n}}\right)=2$.
Proof. There are exactly three maximal cyclic subgroup in $Q_{8} \times \mathbb{Z}_{n}$, which we denote by $\left\langle x_{1}\right\rangle,\left\langle x_{2}\right\rangle$ and $\left\langle x_{3}\right\rangle$. It is easy to see that $\left|x_{1}\right|=\left|x_{2}\right|=\left|x_{3}\right|=4 n$. Let $C$ be a subgroup of order $2 n$ in $\left\langle x_{1}\right\rangle$. Then $C=\left\langle x_{i}\right\rangle \cap\left\langle x_{j}\right\rangle$ for $1 \leq i<j \leq 3$. Write $D(n)=\left\{d_{1}, \ldots, d_{t}\right\}$. Let $B_{i}, C_{i}$ and $D_{i}$ be the set of generators of the subgroup of order $4 d_{i}$ in $\left\langle x_{1}\right\rangle,\left\langle x_{2}\right\rangle$ and $\left\langle x_{3}\right\rangle$, respectively. Consequently, we have

$$
\begin{aligned}
& V\left(\Gamma_{Q_{8} \times \mathbb{Z}_{n}}\right)=C \cup \bigcup_{i=1}^{t}\left(B_{i} \cup C_{i} \cup D_{i}\right), \\
& E\left(\Gamma_{Q_{8} \times \mathbb{Z}_{n}}\right)=E\left(\Gamma_{\left\langle x_{1}\right\rangle}\right) \cup E\left(\Gamma_{\left\langle x_{2}\right\rangle}\right) \cup E\left(\Gamma_{\left\langle x_{3}\right\rangle}\right) .
\end{aligned}
$$

The partition of $V\left(\Gamma_{Q_{8} \times \mathbb{Z}_{n}}\right)$ is showed in Figure 3, where $u$ is the unique involution. With reference to (1), there exists a unique vertex $x_{3}^{\prime} \in\left[x_{3}\right]$ such that $\left\{u, x_{3}^{\prime}\right\} \in$ $E_{1}\left(\left\langle x_{3}\right\rangle\right)$. Write

$$
\begin{aligned}
& E_{1}^{\prime}=\bigcup_{i=1}^{t}\left(\left\{\{e, x\}: x \in B_{i}\right\} \cup\left\{\{u, x\}: x \in C_{i}\right\}\right), \\
& E_{1}=E_{1}^{\prime} \cup E_{1}\left(\left\langle x_{1}\right\rangle\right) \cup E_{1}\left(\left\langle x_{2}\right\rangle\right) \cup\left(E_{1}\left(\left\langle x_{3}\right\rangle\right) \backslash\left\{\left\{u, x_{3}^{\prime}\right\}\right\}\right) .
\end{aligned}
$$



Figure 3: The partition of $V\left(\Gamma_{Q_{8} \times \mathbb{Z}_{n}}\right)$ and the set of edges $E_{1}^{\prime}$

It is clear that $E_{1} \subseteq E\left(\Gamma_{Q_{8} \times \mathbb{Z}_{n}}\right)$. Write $E_{2}=E\left(\Gamma_{Q_{8} \times \mathbb{Z}_{n}}\right) \backslash E_{1}$. Define a coloring

$$
\zeta: E\left(\Gamma_{Q_{8} \times \mathbb{Z}_{n}}\right) \longrightarrow\{1,2\}, \quad f \longmapsto k \text { if } f \in E_{k} .
$$

For $i=1,2,3$, let $\Delta_{i}$ be the subgraph of $\Gamma_{\left\langle x_{i}\right\rangle}$ induced by $V\left(\Gamma_{\left\langle x_{i}\right\rangle}\right) \backslash\{e, u\}$. Similar to the proof of Theorem 3.1 (i), we deduce that $\left.\zeta\right|_{E\left(\Delta_{i}\right)}$ is a rainbow 2-coloring of $\Delta_{i}$. If vertices $v$ and $w$ satisfy $u \notin\{v, w\}$ and $\{v, w\} \nsubseteq V\left(\Delta_{i}\right)$ for any $i \in\{1,2,3\}$, then $(v, e, w)$ or $(v, u, w)$ is a rainbow path under $\zeta$ from Figure 3. If $v$ is a vertex that is not adjacent to $u$, there exists a vertex $x_{3}^{\prime \prime} \in\left[x_{3}\right] \backslash\left\{x_{3}^{\prime}\right\}$ such that $\left\{x_{3}^{\prime \prime}, v\right\} \in E_{1}\left(\left\langle x_{3}\right\rangle\right)$, and so $\left(u, x_{3}^{\prime \prime}, v\right)$ is a rainbow path under $\zeta$. It follows that $\zeta$ is a rainbow 2 -coloring of $\Gamma_{Q_{8} \times \mathbb{Z}_{n}}$. This completes the proof.

Example 3 If $n \geq 3$, then $\operatorname{rc}\left(\Gamma_{Q_{4 n}}\right)=3$.
Proof. With reference to (2), we have $y^{-1}=x^{n} y$ and $\left(x^{i} y\right)^{-1}=x^{2 n-i} y$ for $i \in$ $\{1, \ldots, n-1\}$, which implies that

$$
\begin{aligned}
& V\left(\Gamma_{Q_{4 n}}\right)=\left\{e, x, \ldots, x^{2 n-1}\right\} \cup\left(\bigcup_{i=0}^{n-1}\left\{x^{i} y,\left(x^{i} y\right)^{-1}\right\}\right), \\
& E\left(\Gamma_{Q_{4 n}}\right)=E\left(\Gamma_{\langle x\rangle}\right) \cup \bigcup_{i=0}^{n-1} E\left(\Gamma_{\left\langle x^{i} y\right\rangle}\right),
\end{aligned}
$$

as shown in Figure 4. It follows from Theorem 3.1 that $\mathrm{rc}\left(\Gamma_{Q_{4 n}}\right)=2$ or 3 . Suppose


Figure 4: $\Gamma_{Q_{4 n}}$
for the contrary that there exists a rainbow 2-coloring $\zeta$ of $\Gamma_{Q_{4 n}}$.

Assume that $n=3$. Without loss of generality, let $\zeta\left(\left\{e, x^{2}\right\}\right)=1$. Then $\zeta\left(\left\{e, x^{i} y\right\}\right)=2$ for $i \in\{0,1,2\}$. Hence, for $0 \leq i<j \leq 2$, the rainbow path from $x^{i} y$ to $x^{j} y$ is $\left(x^{i} y, x^{3}, x^{j} y\right)$, which implies that $\zeta\left(\left\{y, x^{3}\right\}\right), \zeta\left(\left\{x y, x^{3}\right\}\right)$ and $\zeta\left(\left\{x^{2} y, x^{3}\right\}\right)$ are pairwise distinct, a contradiction. Therefore $\operatorname{rc}\left(\Gamma_{Q_{12}}\right)=3$.

In the following, assume that $n \geq 4$. Let $\Delta$ be the induced subgraph of $\Gamma_{Q_{4 n}}$ on the vertices $\left\{e, x, y, x y, x^{2} y, x^{3} y, x^{n}\right\}$. Then $\left.\zeta\right|_{E(\Delta)}$ is a rainbow 2-coloring of $\Delta$.

We claim that there exists a rainbow path from $e$ to $x^{n}$ with length 2 under $\left.\zeta\right|_{E(\Delta)}$ in $\Delta$. In fact, if $\left.\zeta\right|_{E(\Delta)}\left(\left\{e, x^{i} y\right\}\right)=\left.\zeta\right|_{E(\Delta)}\left(\left\{x^{i} y, x^{n}\right\}\right)$ for each $i \in\{0,1,2,3\}$, then there exist two distinct indices $j$ and $k$ in $\{0,1,2,3\}$ such that

$$
\left.\zeta\right|_{E(\Delta)}\left(\left\{e, x^{j} y\right\}\right)=\left.\zeta\right|_{E(\Delta)}\left(\left\{x^{j} y, x^{n}\right\}\right)=\left.\zeta\right|_{E(\Delta)}\left(\left\{e, x^{k} y\right\}\right)=\left.\zeta\right|_{E(\Delta)}\left(\left\{x^{k} y, x^{n}\right\}\right),
$$

which implies that there is no rainbow path from $x^{j} y$ to $x^{k} y$ under $\left.\zeta\right|_{E(\Delta)}$ in $\Delta$, a contradiction. Hence, the claim is valid.

Let $\Delta_{0}$ be the graph obtained from $\Delta$ by deleting the edge $\left\{e, x_{n}\right\}$. Then $\Delta_{0}$ is isomorphic to the complete bipartite graph $K_{2,5}$. By the claim above, we have $\operatorname{rc}\left(K_{2,5}\right)=2$, contrary to [7, Theorem 2.6].

For a noncyclic group $G$ with no maximal involutions, it is difficult for us to determine which groups $G$ satisfy $\operatorname{rc}\left(\Gamma_{G}\right)=2$. However, we give a sufficient condition.

Proposition 3.2 If $G$ is a group of order $p^{n} q$ for positive integer $n$, where $p, q$ are distinct primes and $p<q$, such that the following conditions hold, then $\operatorname{rc}\left(\Gamma_{G}\right)=2$.
(i) Each Sylow p-subgroup is cyclic and the Sylow $q$-subgroup is unique.
(ii) The intersection of all Sylow p-subgroups is of order $p^{n-1}$.
(iii) $p^{n-1} \geq q$.

Proof. Note that the number of Sylow $p$-subgroups is $q$. Suppose that $\left\{P_{1}, \ldots, P_{q}\right\}$ is the set of all Sylow $p$-subgroups, and $Q$ is the unique Sylow $q$-subgroup. Then $\cap_{i=1}^{q} P_{i}$ and $Q$ are cyclic and normal in $G$. Hence, there exists an element $x$ of order $p^{n-1} q$ such that $\left(\cap_{i=1}^{q} P_{i}\right) Q=\langle x\rangle$, and so the set of all cyclic subgroups of $G$ is

$$
\left\{P_{1}, \ldots, P_{q}\right\} \cup\{\langle y\rangle: y \in\langle x\rangle\} .
$$

For $1 \leq i \leq q$, let $A_{i}$ be the set of all generators of $P_{i}$. By (iii) we choose pairwise distinct elements $u_{1}, \ldots, u_{q-1}$ in $\left(\cap_{i=1}^{q} P_{i}\right) \backslash\{e\}$. With reference to (1), write

$$
\begin{aligned}
& E_{1}^{\prime}=\left\{\{e, y\}: y \in \bigcup_{i=1}^{q} A_{i}\right\} \cup \bigcup_{i=1}^{q-1}\left\{\left\{u_{i}, y\right\}: y \in A_{i}\right\} \\
& E_{1}=E_{1}^{\prime} \cup E_{1}(\langle x\rangle)
\end{aligned}
$$

The set $E_{1}^{\prime}$ is showed in Figure 5. It is clear that $E_{1} \subseteq E\left(\Gamma_{G}\right)$. Let $E_{2}=E\left(\Gamma_{G}\right) \backslash E_{1}$. Define a coloring

$$
\zeta: E\left(\Gamma_{G}\right) \longrightarrow\{1,2\}, \quad f \longmapsto k \text { if } f \in E_{k} .
$$

In order to get the desired result, we only need to show that $\zeta$ is a rainbow 2-coloring of $\Gamma_{G}$. It follows from Theorem 3.1 that $\left.\zeta\right|_{E\left(\Gamma_{\langle x\rangle}\right)}$ is a rainbow 2-coloring


Figure 5: $\quad V\left(\Gamma_{G}\right)$ and the set of edges $E_{1}^{\prime}$
of $\Gamma_{\langle x\rangle}$. Pick any pair of nonadjacent vertices $z$ and $w$ such that $\{z, w\} \nsubseteq V\left(\Gamma_{\langle x\rangle}\right)$. It suffices to find a rainbow path from $z$ to $w$ under $\zeta$. Without loss of generality, assume that $z \in \cup_{i=1}^{q} A_{i}$. If $w \in \cup_{i=1}^{q} A_{i}$, then there exist indices $i$ and $j$ in $\{1, \ldots, q\}$ with $i<j$ such that $z \in A_{i}$ and $w \in A_{j}$, and so $\left(z, u_{i}, w\right)$ is a desired rainbow path. If $w \in V\left(\Gamma_{\langle x\rangle}\right)$, then $(z, e, w)$ is a desired rainbow path.

By Proposition 3.2, we have the following example.
Example 4 Let $G=\left\langle a, b: a^{27}=b^{7}=e, a^{-1} b a=b^{2}\right\rangle \cong \mathbb{Z}_{27} \ltimes \mathbb{Z}_{7}$. Then $\operatorname{rc}\left(\Gamma_{G}\right)=2$.
The following sufficient condition for $\operatorname{rc}\left(\Gamma_{G}\right)=3$ is immediate from Theorem 3.1 and Lemma 2.6.

Proposition 3.3 Suppose that $G$ is a noncyclic group with no maximal involutions. If there exists a prime $p$ dividing $|G|$ such that the subgroup of order $p$ in $G$ is not unique, then $\operatorname{rc}\left(\Gamma_{G}\right)=3$.

Finally, we determine the rainbow connection number of the power graph of a nilpotent group.

Corollary 3.4 Let $G$ be a noncyclic nilpotent group with no maximal involutions. Then

$$
\operatorname{rc}\left(\Gamma_{G}\right)= \begin{cases}2, & \text { if } G \text { is isomorphic to } Q_{8} \times \mathbb{Z}_{n} \text { for some odd number } n ; \\ 3, & \text { otherwise } .\end{cases}
$$

Proof. It follows from Theorem 3.1 that $\operatorname{rc}\left(\Gamma_{G}\right)=2$ or 3 . Suppose $\operatorname{rc}\left(\Gamma_{G}\right)=2$. Then for any prime $p$ dividing $|G|$, the subgroup of order $p$ in $G$ is unique by Proposition 3.3. By [15, Theorem 5.4.10 (ii)], the Sylow $p$-subgroups are cyclic for any odd prime $p$, which implies that 2 is a divisor of $|G|$ and the Sylow 2-subgroup is isomorphic to $Q_{2^{m}}$ for $m \geq 3$. Hence we get $G \cong Q_{2^{m}} \times \mathbb{Z}_{n}$ for some odd number $n$. Let $H$ be a subgroup of $G$ that is isomorphic to $Q_{2^{m}}$.

We claim that for any pair of nonadjacent vertices $x$ and $y$ of $\Gamma_{H}$, there does not exist a vertex in $G \backslash H$ adjacent to both $x$ and $y$ in $\Gamma_{G}$. Suppose for the contrary that $\{\{x, z\},\{y, z\}\} \subseteq E\left(\Gamma_{G}\right)$ for some $z \in G \backslash H$. Then $x=z^{s}$ and $y=z^{t}$ for some integers $s$ and $t$, which implies that $x, y \in\langle z\rangle$. Note that $|x|$ and $|y|$ are powers of
2. It follows that $|x|$ is divisible by $|y|$, or $|y|$ is divisible by $|x|$. Since $x, y \in\langle z\rangle$, one has $\langle x\rangle \subseteq\langle y\rangle$ or $\langle y\rangle \subseteq\langle x\rangle$. Thus $x$ and $y$ are adjacent, a contradiction. Hence, the claim is valid. By the claim, one gets $\mathrm{rc}\left(\Gamma_{H}\right)=2$. It follows from Example 3 that $m=3$, and so $G \cong Q_{8} \times \mathbb{Z}_{n}$. By Example 2, we get the desired result.

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