

Rainbow connection number and independence number of a graph*

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Abstract

Let G be an edge-colored connected graph. A path of G is called rainbow if its every edge is colored by a distinct color. G is called rainbow connected if there exists a rainbow path between every two vertices of G . The minimum number of colors that are needed to make G rainbow connected is called the rainbow connection number of G , denoted by $rc(G)$. In this paper, we investigate the relation between the rainbow connection number and the independence number of a graph. We show that if G is a connected graph, then $rc(G) \leq 2\alpha(G) - 1$. Two examples G are given to show that the upper bound $2\alpha(G) - 1$ is equal to the diameter of G , and therefore the best possible since the diameter is a lower bound of $rc(G)$.

Keywords: rainbow coloring, rainbow connection number, independence number, connected dominating set

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. The following notation and terminology are needed in the sequel. Let $u \in V$ and $v \in V$ be two distinct vertices of a graph $G = (V, E)$. The *distance between u and v* in G , denoted by $d(u, v)$, is the length of a shortest path connecting them in G . A (u, v) -*path* is a path with initial vertex u and terminal vertex v , denoted by $P[u, v]$. Let $P_H[u, v]$ denote the path in H connecting u and v , where H is the subgraph of G . For two subsets U and W of V , a (U, W) -*path* is a path which starts at a vertex of U and ends at a vertex of W , and

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whose internal vertices belong to neither U nor W . We use $E[U, W]$ to denote the set of edges of G with one end in U and the other end in W , and $e(U, W) = |E[U, W]|$. Let $G[U]$ denote the subgraph of G whose vertex set is U and whose edge set consists of all such edges of G that have both ends in U . The following notions are from [13]. A set $D \subseteq V(G)$ is called a k -step connected dominating set of G , if every vertex in $G \setminus D$ is at a distance at most k from D , where $G[D]$ is connected. The k -step open neighborhood of a set D is $N^k(D) := \{x \in V(G) | d(x, D) = k\}$, where $k \in \mathbb{N}$. We often use $e(G)$ to denote the number of edges in a graph G and $|G|$ to denote the order of G . For undefined terminology and notation, we refer to [1].

Let $G = (V, E)$ be a connected graph with vertex set V and edge set E . A k -edge coloring of G is a mapping $c : E \rightarrow C$, where C is a set of k distinct colors. In [8] Chartrand, Johns, McKeon and Zhang introduced a new concept about the connectivity and coloring of a graph, which is given follows. A path of G is called *rainbow* if every edge of it is colored by a distinct color. For every two vertices u and v of G , if there exists a rainbow path between them, we say that G is *rainbow connected*. The *rainbow connection number* $rc(G)$ is defined as the smallest number of colors that are needed to make G rainbow connected. An edge coloring is called a *rainbow coloring* if it makes G rainbow connected. From the definition of rainbow connection, we can see that the diameter $diam(G) \leq rc(G) \leq e(G)$. For more knowledge on the rainbow connection, we refer to [15, 14].

In [5] Chakraborty, Fischer, Matsliah and Yuster showed that given a graph G , deciding if $rc(G) = 2$ is NP-complete, in particular, computing $rc(G)$ is NP-hard, which were conjectured by Caro, Lev, Roditty, Tuza and Yuster [4]. So, to obtain upper bounds for the rainbow connection number $rc(G)$ of a graph G becomes interesting. Therefore, many good upper bounds have been obtained in terms of other graph parameters. Caro, Lev, Roditty, Tuza and Yuster [4] conjectured that if G is a connected graph with n vertices and $\delta(G) \geq 3$, then $rc(G) < \frac{3}{4}n$. Schhiermeyer [21] confirmed the conjecture and showed that if G is a connected graph with n vertices and $\delta(G) \geq 3$, then $rc(G) \leq \frac{3n-1}{4}$, and there are examples to show that $\frac{3}{4}$ cannot be replaced with a smaller constant. In [6] Chandran, Das, Rajendraprasad and Varma obtained a good relation between the rainbow connection number and the minimum degree of a graph. They showed that if G is a connected graph of order n and minimum degree $\delta(G)$, then $rc(G) \leq 3n/(\delta(G) + 1) + 3$, and the bound is tight up to additive factors. Later, we [10] studied the relation between the rainbow connection number and the minimum degree sum, a generalized result of the minimum degree version. We showed that if G is a graph with k independent vertices, then $rc(G) \leq \frac{3kn}{\sigma_k(G)+k} + 6k - 3$. In [3], Basavaraju, Chandran, Rajendraprasad and Ramaswamy investigated the relation between the rainbow connection number and the radius of a bridgeless graph. They showed that for every bridgeless graph G with radius $rad(G)$, $rc(G) \leq rad(G)(rad(G) + 2)$, and gave an example to show that the bound is tight. In [7] Li, Liu, Chandran, Mathew and Rajendraprasad showed that if G is a 2-connected graph of order n ($n \geq 3$), then $rc(G) \leq \lceil \frac{n}{2} \rceil$, and the upper bound is tight for $n \geq 4$. Later, Ekstein et al. [11] got the same result. Li et al. [7] also got some relations

between the rainbow connection number and the connectivity of a graph. Schiermeyer [20] obtained a relation between the rainbow connection number of a graph G and the chromatic number of the complement of G , i.e., $rc(G) \leq 2\chi(\bar{G}) - 1$.

This paper intends to give a relation between the rainbow connection number and the independence number of a graph. Recall that an *independent set* of a graph G is a set of vertices such that any two of these vertices are non-adjacent in G . The *independence number* $\alpha(G)$ of G is the cardinality of a maximum independent set of G . The independence number of a graph is an important parameter, and the investigation on the relations between the independence number and other graph parameters is interesting, see [2, 9, 12, 16, 17, 18, 19]. We obtain the following result.

Theorem 1 *If G is a connected graph, then $rc(G) \leq 2\alpha(G) - 1$, and the bound is the best possible.*

Two examples are given to show that our bound $2\alpha(G) - 1$ is exactly equal to the diameter of G , and therefore our bound is the best possible since the diameter is a lower bound of $rc(G)$. Moreover, for these examples some good bounds in terms of other parameters can be arbitrarily bad. As we know, even for the chromatic number, every upper bound is attained yet arbitrarily bad for many graphs. We also give an example, Example 3, to show that our result $rc(G) \leq 2\alpha(G) - 1$ could be arbitrarily bad. This example also shows that the bounds in terms of other parameters are arbitrarily bad.

Example 1: Let $P_{2t} = v_1v_2v_3 \cdots v_{2t-1}v_{2t}$ be a path of length $2t - 1$, and let G_1, G_2, \dots, G_t be t ($t \geq 2$) complete graphs with $|G_1| = 2$ and $|G_i| = s$ for $2 \leq i \leq t$. For every i with $1 \leq i \leq t$, we join each vertex of G_i to every vertex of v_{2i-1} and v_{2i} . The obtained graph is denoted by G . One can see that G is connected with $\delta(G) = 3$, and $I(G) = \{v_2, v_4, v_6, \dots, v_{2t}\}$ is a maximum independent set, that is, $\alpha(G) = t$. We also know that the distance $d(v_1, v_{2t}) = 2t - 1$. So, $rc(G) \geq 2t - 1$. Now we use $2t - 1$ distinct colors to give G an edge coloring. Let $1, 2, \dots, 2t - 1$ be $2t - 1$ distinct colors. We use the $2t - 1$ colors to color all the edges of P_{2t} , and each edge with a distinct color. Thus, P_{2t} is rainbow connected. Then we use color $2i - 1$ to color every edge of $E[V(G_i), \{v_{2i-1}, v_{2i}\}]$. Finally, we use color 1 to color every edge of $G[V(G_i)]$. One can show that G is rainbow connected. For each pair $(u, v) \in (G_i, P_{2t})$, either the edge uv_{2i} together with the path in P_{2t} connecting v_{2i} and v forms a rainbow path, or the edge uv_{2i-1} together with the path in P_{2t} connecting v_{2i-1} and v forms a rainbow path. For each pair $(u, v) \in (G_i, G_j)$ with $1 \leq i < j \leq t$, the edges uv_{2i} and vv_{2j-1} together with the path in P_{2t} connecting v_{2i} and v_{2j-1} form a rainbow path. So, G is rainbow connected, and hence $rc(G) \leq 2t - 1$. Thus, $rc(G) = 2t - 1 = 2\alpha(G) - 1$. Note that, $diam(G) = 2t - 1$, $rad(G) = t$, $\delta(G) = 3$ and for any $v \in V(G) \setminus (\{v_1, v_2\} \cup V(G_1))$, the degree of v is at least $s + 1$.

The following facts can be easily seen. When s is very large, the order n of G is very large. The upper bounds in [6, 21] can be arbitrarily bad, because $3n/(\delta(G) + 1) + 3 = 3n/4 + 3$ and $\frac{3n-1}{4}$, both bounds are large by selecting s to be large; When $k \geq 2$,

$\sigma_k(G) \geq 3 + s + 1$, the bound in [10] is $\frac{3kn}{\sigma_k(G)+k} + 6k - 3 \leq \frac{3kn}{s+4+k} + 6k - 3$, better than the bounds of [6, 21], when s is very large, but it is also far from the diameter of G ; The bound in [3] is $rad(G)(rad(G) + 2) = t(t + 2)$, a square of t . However, our result $rc(G) \leq 2\alpha(G) - 1$ is the best, which is equal to $diam(G)$.

Example 2: For $1 \leq i \leq 2t$ ($t \geq 2$), let the sets V_1, V_2, \dots, V_{2t} be pairwise disjoint, $|V_1| = |V_2| = 2$ and for $i \geq 3$, $|V_i| = s$ ($s \geq 2$). Join each vertex of V_i to every vertex of V_{i+1} for $1 \leq i \leq 2t - 1$. Suppose that every two vertices of V_i are adjacent for each i . We denote the resulting graph by G . Note that, G is 2-connected, $\alpha(G) = t$, $diam(G) = 2t - 1$ and $rad(G) = t$. For any vertex $v \in V_1$, the degree of v is 3, for any vertex $v \in V_2$, the degree of v is $s + 3$, for any vertex $v \in V \setminus \{V_1, V_2, V_{2t}\}$, the degree of v is $3s - 1$, and for any vertex $v \in V_{2t}$, the degree of v is $2s - 1$. Color each edge in $E[V_i, V_{i+1}]$ with color i for $1 \leq i \leq 2t - 1$ and each edge in $G[V_i]$ with color 1 for $1 \leq i \leq 2t$. It is not difficult to check that $rc(G) = 2\alpha(G) - 1 = 2t - 1 = diam(G)$.

One can see the following facts. When s is very large, the order n of G is very large. The upper bounds in [6, 7, 21] can be arbitrarily bad, because $3n/(\delta(G)+1)+3 = 3n/4+3$, $\lceil \frac{n}{2} \rceil$ and $\frac{3n-1}{4}$ are all large by selecting s to be large; When s is very large, the bound in [10] is $\frac{3kn}{\sigma_k(G)+k} + 6k - 3 \leq \frac{3kn}{s+6+k} + 6k - 3$, better than the bounds of [6, ?], but it is also far from $diam(G)$. The bound in [3] is $rad(G)(rad(G) + 2) = t(t + 2)$, a square of t , far from $diam(G)$. However, our result $rc(G) \leq 2\alpha(G) - 1$ is the best, which is equal to $diam(G)$.

Example 3: Consider the graph $G = K_{1,1,s}$ with partition sets V_1, V_2 and V_3 , and $|V_1| = |V_2| = 1$, and $|V_3| = s$ ($s > 3$). Then G is 2-connected, $\delta(G) = 2$, $\alpha(G) = s$. and $rc(G) = 3$. We can see that the upper bounds of [6, 7, 21] and our result can be arbitrarily bad, because $3n/(\delta(G)+1)+3 = 3n/4+3$, $\frac{3n-1}{4}$, $\lceil \frac{n}{2} \rceil$ and $2s - 1$ are very large by selecting s to be large.

2 Proof of Theorem 1

Before proving our main result, we first prove a lemma. Although this lemma can be found in [6], here we use a new technique to give it another proof.

Lemma 1 *Let G be a connected graph with $\delta(G) \geq 2$, and let D be a connected dominating set of G . Then $rc(G) \leq rc(G[D]) + 3$.*

Proof. At first, we use $rc(G[D])$ different colors to give $G[D]$ a rainbow coloring. Then, let 1, 2 and 3 be three distinct fresh colors. Since D is a connected dominating set of G , $V(G) = D \cup N(D)$. We will perform the following procedure:

Procedure 1:

$F = N(D)$, $i = 1$,
while there exists a vertex $w_i \in F$ with $d_{G[F]}(w_i) \geq 1$ do

$$H_i = N[w_i] \cap F, F = F \setminus H_i, \\ i = i + 1.$$

While the above procedure ends, we get an empty graph $G[F]$, and a sequence of vertices w_1, w_2, \dots, w_t and a sequence of sets H_1, H_2, \dots, H_t . So, we have partitioned $N(D)$ into some disjoint subsets H_1, H_2, \dots, H_t, F . Now we will give colors to the remaining uncolored edges of G . We use color 1 to color every edge in $E[w_i, D]$, and use color 2 to color every edge in $G[N(D)]$. Finally, we use color 3 to color every edge in $E[H_i \setminus \{w_i\}, D]$. It is not difficult to check that G is rainbow connected. ■

Before giving the proof of Theorem 1, we need the following observation.

Observation. Let G be a graph. If G has a cut edge uv , then we replace uv by a clique of order at least 3, i.e., we add to G some new vertices w_1, w_2, \dots, w_q with $q \geq 1$ such that u, v, w_1, \dots, w_q form a complete subgraph. The new graph is denoted by G' , and is called a *blow-up graph* of G at the cut edge uv . It is not difficult to check that $rc(G') = rc(G)$ and $\alpha(G') = \alpha(G)$. In this way, we need only to consider graphs without any cut edge, and therefore without any pendant edge, i.e., its minimum degree is at least 2.

Proof of Theorem 1. If G is a complete graph, then $\alpha(G) = 1$ and $rc(G) = 1$, Theorem 1 follows. Now assume that G is a non-complete graph with $\delta(G) \geq 2$.

We will perform the following procedure to obtain a tree T whose vertex set D is a connected dominating set of G . Let $y_0 \in V(G)$ with $d(y_0) = \delta(G)$. Since G is a non-complete graph, $N^2(y_0) \neq \emptyset$. We look at the following procedure:

Procedure 2:

$$D = \{y_0\}, T = y_0, X = \emptyset, Y = \{y_0\}.$$

While $N^2(D) \neq \emptyset$

take any vertex $v \in N^2(D)$, let $P = vhu$ be a path of length 2,

where $h \in N^1(D)$ and $u \in D$. Let $D = D \cup V(P)$,

$$T = T \cup P, X = X \cup \{h\}, Y = Y \cup \{v\}.$$

If $u \in X$, we call u an X -knot vertex. Note that $N^2(D)$ does not contain any neighbor of Y . When the above procedure ends, the algorithm runs $|X|$ rounds. Thus we get $V(G) = D \cup N^1(D)$, where D is a connected dominating set. Note that Y is an independent set and $|Y| = |X| + 1$. So $|Y| \leq \alpha(G)$ and $|D| = |Y| + |X| = 2|Y| - 1$. Note that T is a spanning tree of $G[D]$ and the pendant vertices of T are all in Y , and also note that if x is an X -knot vertex, then x is adjacent to at least three vertices of T .

In the following, D , T , Y and X are always the same as those obtained in the above algorithm. In order to continue our proof, we need the following lemmas.

Lemma 2 *If there exists a vertex $w \in N^1(D)$ such that $e(w, Y) = 0$, then $rc(G) \leq 2\alpha(G) - 1$.*

Proof. Let $I = Y \cup \{w\}$. Then I is an independent set and $|I| = |Y| + 1$. So $|Y| = |I| - 1 \leq \alpha(G) - 1$. By Lemma 1, we can get that $rc(G) \leq rc(G[D]) + 3 \leq |D| + 2 = 2|Y| + 1$.

Hence, $rc(G) \leq 2(\alpha(G) - 1) + 1 = 2\alpha(G) - 1$. \blacksquare

From the proof of Lemmas 2, we can see that if we can find an independent set I satisfying $|I| = |Y| + 1$, then we will get $rc(G) \leq 2\alpha(G) - 1$.

Lemma 3 *If $G[D] = T$ and there exist two vertices $w, w' \in N^1(D)$ such that $ww' \notin E(G)$, $e(w, Y) = 1$, $e(w', Y) = 1$, $e(w, X) = 0$ and $e(w', X) = 0$, then $rc(G) \leq 2\alpha(G) - 1$.*

Proof. Let $wy \in E(G)$ and $w'y' \in E(G)$ where $y \in D$ and $y' \in D$. Since $G[D] = T$, let $P_T[y, y']$ denote the only path connecting y and y' in T . If there do not exist two successive vertices of X in $P_T[y, y']$, then let $I = \{w, w'\} \cup (V(P_T[y, y']) \cap X) \cup (Y \setminus (V(P_T[y, y']) \cap Y))$. One can see that I is an independent set and $|I| = |Y| + 1$, and so $rc(G) \leq 2\alpha(G) - 1$. Otherwise, suppose that there exist two successive vertices of X in $P_T[y, y']$. By the structure of T , we can conclude that there must be an X -knot vertex between the two successive vertices. Then there must be a segment in $P_T[y, y']$, without loss of generality, say $P_T[y, x] \subset P_T[y, y']$, where x is an X -knot vertex, and in $P_T[y, x]$ there is an vertex x' of X adjacent to x , and x', x are the only two successive vertices in $P_T[y, x]$. Then $I = \{w\} \cup (V(P_T[y, x']) \cap X) \cup (Y \setminus (V(P_T[y, x']) \cap Y))$ is an independent set and $|I| = |Y| + 1$. So $rc(G) \leq 2\alpha(G) - 1$. \blacksquare

Lemma 4 *If $G[D] = T$ and there exist vertices $w_1, w_2 \in N^1(D)$ and $y_1, y_2 \in D$ such that $w_1w_2 \notin E(G)$, and $N(w_1) \cap D = N(w_2) \cap D = \{y, y'\}$, then $rc(G) \leq 2\alpha - 1$.*

Proof. Let $P_T[y, y']$ denote the only path connecting y and y' in T . If there do not exist two successive vertices of X in $P_T[y, y']$, then let $I = \{w_1, w_2\} \cup (V(P_T[y, y']) \cap X) \cup (Y \setminus (V(P_T[y, y']) \cap Y))$. One can see that I is an independent set and $|I| = |Y| + 1$, and so $rc(G) \leq 2\alpha(G) - 1$. Otherwise, there must exist two successive vertices of X in $P_T[y, y']$. By the structure of T , we can conclude that there must be an X -knot vertex between the two successive vertices. Then there must be a segment in $P_T[y, y']$, without loss of generality, say $P_T[y, x] \subset P_T[y, y']$, where x is an X -knot vertex, and in $P_T[y, x]$ there is an vertex x' of X adjacent to x , and x', x are the only two successive vertices in $P_T[y, x]$. Then $I = \{w, w'\} \cup (V(P_T[y, x']) \cap X) \cup ((Y \setminus ((V(P_T[y, x']) \cap Y) \cup \{y'\}))$ is an independent set and $|I| = |Y| + 1$. So $rc(G) \leq 2\alpha(G) - 1$. \blacksquare

Let $N^1(D) = A \cup B$ where $w \in A$ if and only if $e(w, D) \geq 2$, and $w \in B$ if and only if $e(w, D) = 1$. By Lemma 2, we can get that every vertex $w \in B$ satisfies $e(w, X) = 0$, and so $e(w, Y) = 1$. By Lemma 3, we can get that $G[B]$ is a complete subgraph. In the following we will divide two cases to finish our proof.

Case 1. $e(G[D]) \geq e(T) + 1$.

Let $a_1a_2 \in E(G[D])$ and $a_1a_2 \notin E(T)$. Note that T is a spanning tree of $G[D]$. So $T \cup a_1a_2$ contains a cycle, say C , and $a_1a_2 \in E(C)$. Let $G' = T \cup a_1a_2$. Then

$rc(G[D]) \leq rc(G')$. Since $rc(G') \leq e(T) - (|C| - 1) + rc(C)$ and $rc(C) \leq \lceil \frac{|C|}{2} \rceil$ when $|C| \geq 4$, we can get

$$rc(G') \leq \begin{cases} e(T) - \frac{|C|}{2} + 1, & |C| \text{ is even} \\ e(T) - \frac{|C|-3}{2}, & |C| \text{ is odd and } |C| \neq 3 \\ e(T) - 1, & |C| = 3 \end{cases}$$

Hence, $rc(G[D]) \leq rc(G') \leq e(T) - 1$.

Now we color every edge of G in the following way. First, we use $rc(G[D])$ distinct colors to rainbow color $G[D]$. Then let c', c'' be two fresh colors. For any vertex $w \in A$, let $w', w'' \in D$ with $ww', ww'' \in E(G)$, set $c(ww') = c'$ and $c(ww'') = c''$. For any vertex $b \in B$, let $b' \in D$ with $bb' \in E(G)$, set $c(bb') = c'$. For the remaining uncolored edges of $E(G)$, we use a used color to color them. Thus we have colored all the edges of G .

We will show that G is rainbow connected. For each pair $(u, v) \in (N(D) \times D)$, the edge uu' together with the path in G' connecting u' and v forms a rainbow path, where $c(uu') = c'$ and $u' \in D$. For each pair $(u, v) \in (A \times A)$, the edges uu' and vv'' together with the path in G' connecting u' and v'' form a rainbow path, where $c(uu') = c'$ and $c(vv'') = c''$. For each pair $(u, v) \in (A \times B)$, the edges uu'' and vv' together with the path in G' connecting u'' and v' form a rainbow path, where $c(uu'') = c''$ and $c(vv') = c'$. Thus we have showed that G is rainbow connected.

In the above coloring, we used at most $rc(G[D]) + 2 \leq e(T) + 1$ colors. Hence, $rc(G) \leq e(T) + 1$, that is $rc(G) \leq |D|$. Since $|D| = 2|Y| - 1 \leq 2\alpha(G) - 1$, we can get $rc(G) \leq 2\alpha(G) - 1$.

Case 2. $e(G[D]) = e(T)$.

Let $1, 2, c_1, c_2$ and a be 5 distinct colors, and in the following proof, we use a to color each edge of $E[B, D]$, and use c_1 to color each edge of $E(G[B])$.

Choose a longest path P in $G[D]$ such that two ends of P are pendant vertices. We know that the two pendant vertices belong to Y , and $|P| \geq 3$. Let $P = y_1x_1x_2 \cdots x_ky_2$. We look at two subcases:

Subcase 2.1. $V(P) \subset D$.

Since P is a longest path and $|Y| = |X| + 1$, we can get $|P| \geq 4$. Choose a pendant edge in T , say y_3x , which is not in P . Let P' be a path passing through y_3x in T with $|V(P) \cap V(P')| = 1$, and let $V(P) \cap V(P') = \{x'\}$. Without loss of generality, let $|P[y_1, x']| \geq 3$, and let $c(y_1x_1) = 1$, $c(x_1x_2) = c_1$, $c(x_ky_2) = 2$ and $c(xy_3) = c_2$.

Let A_1, A_2, A_3 and A_4 be the subsets of A . $w_1 \in A_1$ if and only if $w_1y_1 \in E(G)$ and w_1 is adjacent to only one vertex of $D \setminus \{y_1, y_2\}$. Let $c(w_1y_1) = c_2$ and $c(w_1w'_1) = 2$ where $w'_1 \in D$; $w_2 \in A_2$ if and only if $w_2y_2 \in E(G)$ and w_2 is adjacent to only one vertex of $D \setminus \{y_1, y_2\}$. Let $c(w_2y_2) = c_2$ and $c(w_2w'_2) = 1$ where $w'_2 \in D$; $w_3 \in A_3$ if and only if

$w_3y_1 \in E(G)$, $w_3y_2 \in E(G)$ and $e(w_3, D) = 2$. Let $c(w_3y_1) = 2$ and $c(w_3y_2) = 1$. $w_4 \in A_4$ if and only if w_4 is adjacent to at least two vertices w'_4 and w''_4 of $D \setminus \{y_1, y_2\}$. Assume that the distance between w'_4 and y_1 in T is not more than the distance between w''_4 and y_1 in T . Let $c(w_4w'_4) = a$ and $c(w_4w''_4) = 1$. Let B_1 , B_2 and B_3 be the subsets of B . $b_1 \in B_1$ if and only if b_1 is only adjacent to y_1 ; $b_2 \in B_2$ if and only if b_2 is only adjacent to y_2 ; $b_3 \in B_3$ if and only if b_3 is only adjacent to some vertex of $Y \setminus \{y_1, y_2\}$. Thus we get $A = A_1 \cup A_2 \cup A_3 \cup A_4$, $B = B_1 \cup B_2 \cup B_3$. From Lemma 4 we know that $G[A_3]$ is a complete subgraph, and from Lemma 3 we get that $G[B]$ is a complete subgraph. It is easy to check that for any vertex of $N(D)$ is rainbow connection to any vertex of D .

Subsubcase 2.1.1. $B_1 \neq \emptyset$, $B_2 \neq \emptyset$ and $B_3 \neq \emptyset$.

First, we will show that $G[A_1]$, $G[A_2]$ and $G[A_4]$ is rainbow connected, respectively. For each pair $(u, v) \in (A_1 \times A_1)$, let $u', v' \in D$ with $uu', vv' \in E(G)$, if $u' \neq v'$, then without loss of generality, we assume that the path in T from v' to y_1 does not contain the edge y_3x . Thus the edges uy_1 and vv' together with the path in T connecting y_1 and v' form a rainbow path between u and v ; if $u' = v'$ and $v'y_2 \in E(G)$, then the edges uy_1 and vy_2 together with the path P form a rainbow path between u and v ; if $u' = v'$ and $u'y_2 \in E(G)$, similarly, there is a rainbow path between them; if $u' = v'$ and assume that $v'y_2 \notin E(G)$ and $u'y_2 \notin E(G)$, then from Lemma 4, we can get $uv \in E(G)$. So for each pair $(u, v) \in (A_1 \times A_1)$, there is a rainbow path connecting them. For each pair $(u, v) \in (A_2 \times A_2)$, similar to $(u, v) \in (A_1 \times A_1)$, we can get a rainbow path between u and v ; For each pair $(u, v) \in (A_4 \times A_4)$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = a$ and $c(vv') = 1$.

Second, we will show that for any vertex $u \in A_1$, there is a rainbow path connecting it to any vertex of $A_2 \cup A_3 \cup A_4 \cup B$. For each pair $(u, v) \in (A_1 \times (A_2 \cup A_4))$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 2$ and $c(vv') = 1$. For each pair $(u, v) \in (A_1 \times A_3)$, uy_1v is a rainbow path. For each pair $(u, v) \in (A_1 \times B_1)$, uy_1v is a rainbow path. For each pair $(u, v) \in (A_1 \times B_2)$, the edges uy_1 and vy_2 together with the path P form a rainbow path. For each pair $(u, v) \in (A_1 \times B_3)$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 2$ and $c(vv') = a$.

Third, we will show that for any vertex $u \in A_2$, there is a rainbow path connecting it to any vertex of $A_3 \cup A_4 \cup B$. For each pair $(u, v) \in (A_2 \times A_4)$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 1$ and $c(vv') = a$. For each pair $(u, v) \in (A_2 \times A_3)$, uy_2v is a rainbow path. For each pair $(u, v) \in (A_2 \times B_1)$, the edges uy_2 and vy_1 together with the path P form a rainbow path. For each pair $(u, v) \in (A_2 \times B_2)$, uy_2v is a rainbow path. For each pair $(u, v) \in (A_2 \times B_3)$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 1$ and $c(vv') = a$.

Fourth, we will show that for any vertex $u \in A_3$, there is a rainbow path connecting it to any vertex of $A_4 \cup B$. For each pair $(u, v) \in (A_3 \times A_4)$, the edges uy_1 and vv'

together with the path in T connecting y_1 and v' form a rainbow path, where $c(uy_1) = 2$ and $c(vv') = a$. For each pair $(u, v) \in (A_3 \times B_1)$, uy_1v is a rainbow path. For each pair $(u, v) \in (A_3 \times B_2)$, uy_2v is a rainbow path. For each pair $(u, v) \in (A_3 \times B_3)$, the edges uy_1 and vv' together with the path in T connecting y_1 and v' form a rainbow path.

Finally, we will show that for any vertex $u \in A_4$, there is a rainbow path connecting it to any vertex of B . For each pair $(u, v) \in (A_4 \times B_1)$, the edges uu' , vb_2 and b_2y_2 together with the path in T connecting u' and y_2 form a rainbow path, where $c(uu') = 1$ and $b_2 \in B_2$. For each pair $(u, v) \in (A_4 \times B_2)$, the edges uu' and vy_2 together with the path in T connecting u' and y_2 form a rainbow path, where $c(uu') = 1$. For each pair $(u, v) \in (A_4 \times B_3)$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 1$ and $c(vv') = a$.

Thus, we have proved that G is rainbow connected.

From the proof above, we can see the following facts: for any vertex of A there is a rainbow path connecting it to any vertex of G , and the internal vertex of the rainbow path does not contain any vertex of B ; for any vertex of B_2 , there is a rainbow path connecting it to any vertex of G , and the rainbow path does not contain any vertex of $B_1 \cup B_3$; for any vertex of B_3 , there is a rainbow path connecting it to any vertex of G , and the rainbow path does not contain any vertex of $B_1 \cup B_2$. Hence, in the following proof, we can assume that $B_3 = \phi$ and $B_2 = \phi$.

Subsubcase 2.1.2. $B_1 \neq \phi$.

We still use the above mentioned subsets A_1, A_2, A_3, A_4 and B_1 , and we still color the edges of G in the above way except for setting $c(w_4w'_4) = a$ and $c(w_4w''_4) = 2$. Thus we only need to show that for any vertex of A_4 , there is a rainbow path connecting it to any vertex of G . We will give the proof as follows. For each pair $(u, v) \in (A_4 \times A_4)$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = a$ and $c(vv') = 2$. For each pair $(u, v) \in (A_4 \times A_3)$, the edges uu' and vy_1 together with the path in T connecting u' and y_1 form a rainbow path, where $c(uu') = a$. For each pair $(u, v) \in (A_4 \times A_2)$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = a$ and $c(vv') = 1$. For each pair $(u, v) \in (A_4 \times A_1)$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = a$ and $c(vv') = 2$. For each pair $(u, v) \in (A_4 \times B_1)$, the edges uu' and vy_1 together with the path in T connecting u' and y_1 form a rainbow path, where $c(uu') = 2$. Hence, we have showed that G is rainbow connected.

Subcase 2.2. $V(P) = D$.

Since $V(P) = D$ and $|Y| = |X| + 1$, P is a (Y, X) -alternate path. Let $A_1, A_2, A_3, A_4, B_1, B_2$ and B_3 be the above mentioned subsets.

Subsubcase 2.2.1. $|P| = 3$.

Let $P = y_1x_1y_2$. We use color 1 to color edge y_1x_1 and use color 2 to color y_2x_1 . Let A_1, A_2, A_3, B_1 and B_2 be the above mentioned subsets. Note that: $A_4 = \phi, B_3 = \phi, G[A_1 \cup B_1]$ is a complete subgraph, and $G[A_2 \cup B_2]$ is a complete subgraph. For any $w_1 \in A_1$ and $w_2 \in A_2$, let $c(w_1y_1) = a, c(w_1x_1) = 1, c(w_2y_2) = a$ and $c(w_2x_1) = 2$. It is obvious that for each vertex of $A \cup B$, there is a rainbow path connecting it to any vertex of P . For each pair $(u, v) \in (A_1 \times A_2)$, ux_1v is a rainbow path. For each pair $(u, v) \in (A_1 \times A_3)$, uy_1v is a rainbow path. For each pair $(u, v) \in (A_1 \times B_2)$, ux_1y_2v is a rainbow path. For each pair $(u, v) \in (A_2 \times A_3)$, uy_2v is a rainbow path. For each pair $(u, v) \in (A_2 \times B_1)$, ux_1y_1v is a rainbow path. For each pair $(u, v) \in (A_3 \times B_1)$, uy_1v is a rainbow path. For each pair $(u, v) \in (A_3 \times B_2)$, uy_2v is a rainbow path. Thus, we have showed that G is rainbow connected.

Subsubcase 2.2.2. $|P| \geq 5$.

Let $c(y_1x_1) = 1, c(x_1y'_1) = c_1, c(y_2x_2) = 2$ and $c(x_2y'_2) = c_2$ where $y'_1, y'_2 \in V(P)$. We color the edges of G in the following way: We use a to color each edge of $E[B, D]$, and use c_1 to color each edge of $G[B]$. For any $w_1 \in A_1$, let $c(w_1y_1) = 2$ and $c(w_1w'_1) = a$ where $w'_1 \in D$; For any $w_2 \in A_2$, let $c(w_2y_2) = 1$ and $c(w_2w'_2) = a$ where $w'_2 \in D$; For any $w_3 \in A_3$, let $c(w_3y_1) = 2$ and $c(w_3y_2) = 1$; For any $w_4 \in A_4$, assume that the distance between w'_4 and y_1 in P is not more than the distance between w''_4 and y_1 in P , let $c(w_4w'_4) = a$ and $c(w_4w''_4) = 1$ where $w'_4, w''_4 \in D$. We will divide three cases to show that G is rainbow connected.

Subsubsubcase 2.2.2.1. $B_1 \neq \phi, B_2 \neq \phi$ and $B_3 \neq \phi$.

First, we can easily check that for each vertex of $A \cup B$, there is a rainbow path connecting it to any vertex of P .

Second, we will show that $G[A_1], G[A_2]$ and $G[A_4]$ are rainbow connected, respectively. For each pair $(u, v) \in (A_1 \times A_1)$, the edges uy_1 and vv' together with the path in T connecting y_1 and v' form a rainbow path, where $c(uy_1) = 2$ and $c(vv') = a$; For each pair $(u, v) \in (A_2 \times A_2)$, the edges uy_2 and vv' together with the path in T connecting y_2 and v' form a rainbow path, where $c(uy_2) = 1$ and $c(vv') = a$; For each pair $(u, v) \in (A_4 \times A_4)$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 1$ and $c(vv') = a$.

Third, we will show that for each vertex of A_1 , there is a rainbow path connecting it to any vertex of $A_2 \cup A_3 \cup A_4 \cup B$. For each pair $(u, v) \in (A_1 \times A_2)$, the edges uu' and vy_2 together with the path in T connecting u' and y_2 form a rainbow path, where $c(uu') = a$. For each pair $(u, v) \in (A_1 \times A_3)$, the edges uu' and vy_1 together with the path in T connecting u' and y_1 form a rainbow path, where $c(uu') = a$. For each pair $(u, v) \in (A_1 \times A_4)$, the edges uy_1 and vv' together with the path in T connecting y_1 and v' form a rainbow path, where $c(vv') = a$. For each pair $(u, v) \in (A_1 \times B_1)$, uy_1v is a rainbow path. For each pair $(u, v) \in (A_1 \times B_2)$, uy_1b_1v is a rainbow path, where $b_1 \in B_1$. For each pair $(u, v) \in (A_1 \times B_3)$, the edges uy_1 and vv' together with the path in T connecting y_1 and v' form a rainbow path.

Fourth, we will show that for each vertex of A_2 , there is a rainbow path connecting it to any vertex of $A_3 \cup A_4 \cup B$. For each pair $(u, v) \in (A_2 \times A_3)$, the edges uu' and vy_1 together with the path in T connecting u' and y_1 form a rainbow path, where $c(uu') = a$. For each pair $(u, v) \in (A_2 \times A_4)$, the edges uy_2 and vv' together with the path in T connecting y_2 and v' form a rainbow path, where $c(vv') = a$. For each pair $(u, v) \in (A_2 \times B_1)$, uy_2b_2v is a rainbow path, where $b_2 \in B_2$. For each pair $(u, v) \in (A_2 \times B_2)$, uy_2v is a rainbow path. For each pair $(u, v) \in (A_2 \times B_3)$, the edges uy_1 and vv' together with the path in T connecting y_1 and v' form a rainbow path.

Fifth, we will show that for each vertex of A_3 , there is a rainbow path connecting it to any vertex of $A_4 \cup B$. For each pair $(u, v) \in (A_3 \times A_4)$, the edges uy_1 and vv' together with the path in T connecting y_1 and v' form a rainbow path, where $c(vv') = a$. For each pair $(u, v) \in (A_3 \times B_1)$, uy_1v is a rainbow path. For each pair $(u, v) \in (A_3 \times B_2)$, uy_2v is a rainbow path. For each pair $(u, v) \in (A_3 \times B_3)$, the edges uy_1 and vv' together with the path in T connecting y_1 and v' form a rainbow path.

Finally, we will show that for each vertex of A_4 , there is a rainbow path connecting it to any vertex of B . For each pair $(u, v) \in (A_4 \times B_1)$, the edges uu' , vb_2 and b_2y_2 together with the path in T connecting y_2 and u' form a rainbow path, where $c(uu') = 1$. For each pair $(u, v) \in (A_4 \times B_2)$, the edges uu' and vy_2 together with the path in T connecting u' and y_2 form a rainbow path, where $c(uu') = 1$. For each pair $(u, v) \in (A_4 \times B_3)$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 1$.

Hence, we have showed that G is rainbow connected.

From the proof above, we can see the following facts: for any vertex of A there is a rainbow path connecting it to any vertex of G , and the internal vertex of the rainbow path does not contain any vertex of B ; for any vertex of B_3 , there is a rainbow path connecting it to any vertex of G , and the rainbow path does not contain any vertex of $B_1 \cup B_2$. Hence, in the following proof we can assume that $B_3 = \phi$.

Subsubsubcase 2.2.2.2. $B_1 = \phi$ and $B_2 \neq \phi$.

We still make use of the above coloring way except for the edges of $E[A_1, D]$. We now color the edges of $E[A_1, D]$ as follows: For any vertex $w_1 \in A_1$, if $w_1x_1 \in E(G)$, then let $c(w_1y_1) = a$ and $c(w_1x_1) = 1$; if $w_1x_1 \notin E(G)$, then let $w'_1 \in D \setminus \{y_1, x_1, y_2\}$ with $w_1w'_1 \in E(G)$, and let $P[y_1, w'_1]$ be a subpath of P , $z \in V(P[y_1, w'_1])$ with $zw'_1 \in E(G)$, then let $c(w_1y_1) = c(zw'_1)$, $c(w_1w'_1) = 1$. From the coloring, one can easily check that $G[A_1]$ is rainbow connected.

Now, we show that for each vertex of A_1 , there is a rainbow path connecting it to any vertex of $A_2 \cup A_3 \cup A_4 \cup B$. For each pair $(u, v) \in (A_1 \times (A_2 \cup A_4))$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 1$ and $c(vv') = a$. For each pair $(u, v) \in (A_1 \times A_3)$, uy_1v is a rainbow path. For each pair $(u, v) \in (A_1 \times B_2)$, the edges uu' and vy_2 together with the path in T connecting u' and

y_2 form a rainbow path, where $c(uu') = 1$.

Thus we have proved that G is rainbow connected.

Subsubcase 2.2.2.3 $B_1 \neq \emptyset$ and $B_2 = \emptyset$.

We still make use of the above coloring way except for the edges of $E[A_2, D]$ and the edges of $E[A_4, D]$. For any vertex $w_4 \in A_4$, we let $c(w_4w_4') = a$ and $c(w_4w_4'') = 2$. For any vertex $w_2 \in A_2$, we will color the edges of $E[A_2, D]$ in the following way: If $w_2x_2 \in E(G)$, then let $c(w_2y_2) = a$ and $c(w_2x_2) = 2$; If $w_2x_2 \notin E(G)$, then let $w_2' \in D \setminus \{y_1, x_2, y_2\}$ with $w_2w_2' \in E(G)$, and let $P[y_2, w_2']$ be a subpath of P , $z' \in V(P[y_2, w_2'])$ with $z'w_2' \in E(G)$, then let $c(w_2y_2) = c(z'w_2')$. One can easily check that $G[A_2]$ and $G[A_4]$ are rainbow connected, respectively.

Then, we will show that for each vertex of A_2 , there is a rainbow path connecting it to any vertex of $A_1 \cup A_3 \cup A_4 \cup B$. For each pair $(u, v) \in (A_2 \times (A_1 \cup A_4))$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 2$ and $c(vv') = a$. For each pair $(u, v) \in (A_2 \times A_3)$, uy_2v is a rainbow path. For each pair $(u, v) \in (A_2 \times B_1)$, the edges uu' and vy_1 together with the path in T connecting u' and y_1 form a rainbow path, where $c(uu') = 2$.

Finally, we will show that for each vertex of A_4 , there is a rainbow path connecting it to any vertex of $A_1 \cup A_3 \cup B$. For each pair $(u, v) \in (A_4 \times A_1)$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 2$ and $c(vv') = a$. For each pair $(u, v) \in (A_4 \times A_3)$, the edges uu' and vy_1 together with the path in T connecting u' and y_1 form a rainbow path, where $c(uu') = a$. For each pair $(u, v) \in (A_4 \times B_1)$, the edges uu' and vy_1 together with the path in T connecting u' and y_1 form a rainbow path, where $c(uu') = 2$.

Thus we have proved that G is rainbow connected.

In the above coloring, we used $e(T) + 1$ colors. Hence, $rc(G) \leq e(T) + 1$, and so we can get $rc(G) \leq 2\alpha(G) - 1$.

Combining the above Cases 1 and 2, we have completed the proof of Theorem 1. ■

Since the independence number $\alpha(G)$ is at most the number of cliques that partition the vertex set of a graph G , and the minimum of such partitions is the chromatic number of the complement \bar{G} of G , we can get the following corollary, which is Theorem 10 of [20].

Corollary 1 (Theorem 10, [20]) *Let G be a connected graph with chromatic number $\chi(G)$. Then $rc(G) \leq 2\chi(\bar{G}) - 1$.*

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