On graph theory Mertens' theorems

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Abstract

In this paper, we study graph-theoretic analogies of the Mertens' theorems by using basic properties of the Ihara zeta-function. One of our results is a refinement of a special case of the dynamical system Mertens' second theorem due to Sharp and Pollicott.

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1 Introduction

Throughout this paper, we use the notation in the textbook [11] of Terras for graph theory and the (Ihara) zeta-function, and we often refer to basic facts included in this textbook.

In 1874, Mertens proved the so-called Mertens' first/second/third theorems (the equalities (5)(13)(14) in [7], respectively). In 1991, Sharp studied the dynamical-systemic analogues of Mertens' second/third theorems (Theorem 1 in [10]), and in 1992, Pollicott improved the error terms in the theorems of Sharp as follows (Theorem and Remark in [9]):

- Dynamical system Mertens' second theorem: For a hyperbolic (and so geodesic) flow (which is not necessarily topologically weak-mixing) restricted to a basic set with closed orbits τ of least period $\lambda(\tau)$ and topological entropy $h > 0$, as $x \to \infty$,

$$\sum_{N(\tau)\le x} \frac{1}{N(\tau)} = \log(\log x) + \gamma + \log\left(\operatorname*{Res}_{s=1} \zeta(s)\right) - \sum_{\tau}\sum_{n\ge 2} \frac{1}{n}\cdot\frac{1}{N(\tau)^n} + O\left(\frac{1}{\log x}\right),$$

where $N(\tau) = e^{h\lambda(\tau)}$, γ is the Euler-Mascheroni constant,

$$\zeta(s) = \prod_{\tau}\left(1 - \frac{1}{N(\tau)^s}\right)^{-1},$$

and $\operatorname*{Res}_{s=1}\zeta(s)$ denotes the residue of ζ at $s = 1$.

- Dynamical system Mertens' third theorem: For the same flow, as $x \to \infty$,

$$\prod_{N(\tau)\le x}\left(1 - \frac{1}{N(\tau)}\right) = \frac{e^{-\gamma}}{\operatorname*{Res}_{s=1}\zeta(s)}\cdot\frac{1}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

(For the notation of dynamical systems, see the textbook [8] of Parry-Pollicott.) Note that Sharp and Pollicott did not explicitly write the dynamical system Mertens' first theorem.

From the second theorem of Pollicott, the constant term (so-called Mertens constant) can be explicitly known, but the coefficients of $1/\log^k x$ can not be computed. Our purpose in this paper is to present graph-theoretic analogies of the Mertens' theorems whose coefficients can be explicitly known. So, our second theorem is a refinement of a special case of the theorem due to Pollicott in the sense that the coefficients of $1/\log^k x$ can be computed.

In the rest of this section, we introduce the notation of graph theory, next recall the notation and properties of the (Ihara) zeta-function, and last state the main theorem.

First, we recall the notation of graphs. Let X be an undirected graph with vertex set V of $\nu := |V|$ and edge set E of $\epsilon := |E|$. Simply, such a graph X is denoted by $X := (V, E)$.

A directed edge (or an arc) a from a vertex u to a vertex v is denoted by $a = (u, v)$, and the inverse of a is denoted by $a^{-1} = (v, u)$. The origin (resp. terminus) of a is denoted by $o(a) := u$ (resp. $t(a) := v$).

We can direct the edges of X, and label the edges as follows:

$$\vec{E} := \{e_1, \quad e_2, \quad \ldots, \quad e_\epsilon, \quad e_{\epsilon+1} = e_1^{-1}, \quad e_{\epsilon+2} = e_2^{-1} \quad \ldots, \quad e_{2\epsilon} = e_\epsilon^{-1}\}.$$

A path $C = a_1 \cdots a_s$, where the a_i are directed edges, is said to have a backtrack (resp. tail) if $a_{j+1} = a_j^{-1}$ for some j (resp. $a_s = a_1^{-1}$), and a path C is called a cycle (or closed path) if $o(a_1) = t(a_s)$. The length $\ell(C)$ of a path $C = a_1 \cdots a_s$ is defined by $\ell(C) := s$.

A cycle C is called prime (or primitive) if it satisfies the following:

- C does not have backtracks and a tail;
- no cycle D exists such that $C = D^f$ for some $f > 1$.

The equivalence class $[C]$ of a cycle $C = a_1 \cdots a_s$ is defined as the set of cycles

$$[C] := \{a_1 a_2 \cdots a_{s-1} a_s, \qquad a_2 \cdots a_{s-1} a_s a_1, \qquad \ldots, \qquad a_s a_1 a_2 \cdots a_{s-1}\},$$

and an equivalence class $[P]$ of a prime cycle P is called prime.

Let Δ_X and $\pi_X(n)$ denote

$$\Delta = \Delta_X := \gcd\{\ell(P) : [P] \text{ is a prime equivalence class in } X\},$$
$$\pi(n) = \pi_X(n) := |\{[P] : [P] \text{ is a prime equivalence class in } X \text{ with } \ell(P) = n\}|.$$

Throughout this paper, we always assume that X is a finite, connected, noncycle and undirected graph without degree-one vertices, and we denote by a symbol $[P]$ a prime equivalence class.

Next, we recall the zeta-function of $X = (V, E)$. The (Ihara) zeta-function of X is defined as follows (the equality (9) in [4], and also see Definition 2.2 in [11]):

$$Z_X(u) := \prod_{[P]} (1 - u^{\ell(P)})^{-1}$$

with $|u|$ sufficiently small, where $[P]$ runs through all prime equivalence classes in X. The radius of convergence of $Z_X(u)$ is denoted by R_X. Note that $0 < R_X < 1$ since X is a noncycle graph (see, for example, page 197 in [11]).

Let $W = W_X := (w_{ij})$ denote the edge adjacency matrix of a graph X, that is, a $2\epsilon \times 2\epsilon$ matrix defined by

$$w_{ij} := \begin{cases} 1 & \text{if } t(e_i) = o(e_j) \text{ and } e_j \neq e_i^{-1} \text{ for } e_i, e_j \in \vec{E}, \\ 0 & \text{otherwise} \end{cases}$$

(see page 28 in [11]).

In this paper, our main theorem is:

Main Theorem. *Suppose that $X=(V,E)$ is a finite, connected, noncycle and undirected graph without degree-one vertices. Set*

$$a=a(N):=\left\{\frac{N}{\Delta_X}\right\}\Delta_X \quad \left(=N-\left[\frac{N}{\Delta_X}\right]\Delta_X\right),$$

where $[x]$ (resp. $\{x\}$) denotes the integer (resp. fractional) part of the real number x, and thus $0\le a(N)<\Delta_X$. Then, the following items (1)(2)(3) hold:

(1) (Graph theory Mertens' first Theorem) As $N\to\infty$,

$$\sum_{n\le N} n\cdot\pi_X(n)R_X^n = N-a(N)+A_X+K_X+O\left((\rho_X R_X)^N\right)$$
$$\left(=\left[\frac{N}{\Delta_X}\right]\Delta_X+A_X+K_X+O\left((\rho_X R_X)^N\right)\right),$$

where the constants A_X, K_X and ρ_X are defined by

$$A_X:=\sum_{\substack{\lambda\in Spec(W),\\ |\lambda|<1/R_X}}\frac{\lambda R_X}{1-\lambda R_X},\qquad K_X:=\sum_{n=1}^{\infty}\left(n\cdot\pi_X(n)-\sum_{\lambda\in \mathrm{Spec}(W)}\lambda^n\right)R_X^n,$$

and

$$\rho_X:=\max\{|\lambda|:\lambda\in Spec(W),\ |\lambda|<1/R_X\},$$

respectively. (The convergence of K_X is shown in Section 2.)

(2) (Graph theory Mertens' second Theorem) As $N\to\infty$,

$$\sum_{n\le N}\pi_X(n)R_X^n=\log N+\gamma+\log C_X-H_X$$
$$-\sum_{s=1}^{k}\left(\frac{a^s}{s}+\sum_{m=0}^{s-1}\binom{s-1}{m}\frac{a^m B_{s-m}\Delta_X^{s-m}}{s-m}\right)\frac{1}{N^s}+O\left(\frac{1}{N^{k+1}}\right)$$

for each $k\ge 1$, where γ is the Euler-Mascheroni constant, B_s are the s-th Bernoulli numbers defined by

$$\frac{t}{e^t-1}=\sum_{s=0}^{\infty}B_s\frac{t^s}{s!},$$

and the constants C_X and H_X are defined by

$$C_X:=-\frac{1}{R_X}\operatorname*{Res}_{u=R_X}Z_X(u),\qquad H_X:=-\sum_{n\ge 1}\frac{1}{n}\left(n\cdot\pi_X(n)-\sum_{\lambda\in Spec(W)}\lambda^n\right)R_X^n,$$

respectively. (The convergence of H_X is shown in Section 2.)

In particular ($k=0$), as $N\to\infty$,

$$\sum_{n\le N}\pi_X(n)R_X^n=\log N+\gamma+\log C_X-H_X+O\left(\frac{1}{N}\right).$$

(3) (Graph theory Mertens' third Theorem, [2]) As $N \to \infty$,

$$\prod_{n \le N} (1 - R_X^n)^{\pi_X(n)} = \frac{e^{-\gamma}}{C_X} \cdot \frac{1}{N}\left(1 + O\left(\frac{1}{N}\right)\right).$$

Note that our first theorem (1) is a refinement of a result in our previous paper [2] in the sense that the constant term $A_X + K_X$ is explicitly written, and note that our second theorem (2) is a refinement of a special case of the result due to Pollicott (Theorem (i) and Remark in [9]) in the sense that all the coefficients of $1/N^k$ can be explicitly computed. Our proofs in this paper are elementary (without the theory of the Ihara prime zeta-function which is studied in [2]), and moreover they are completely different from previous proofs.

Our theorems (1)(2) can be simplified under the assumption which all the degrees of vertices are greater than 2: If X is bipartite, then $\Delta_X = 2$, and so $a(2N) = 0$ or $a(2N+1) = 1$. Otherwise, $\Delta_X = 1$, and therefore $a(N) = 0$. (See, for details, Proposition 3.2 in [5].)

The contents of this paper are as follows. In the next section, we first prove a keylemma, which plays an important role in the proof of the main theorem, and next introduce the constants in the main theorem. In Section 3, we give the proof of the main theorem.

2 KeyLemma

In this section, in order to show the theorem, we introduce a keylemma and two constants.

The following facts are often used in this paper.

Fact. *Suppose that $X = (V, E)$ satisfies the same conditions as the main theorem.*

(1) (Theorem 1.4 in [5], and also see Theorem 8.1 (3) in [11]) The poles of $Z_X(u)$ on the circle $|u| = R_X$ have the form $R_X e^{2\pi i a/\Delta_X}$, where $a = 1, 2, \ldots, \Delta_X$.

(2) (Orthogonality relation, for example, see Exercise 10.1 in [11])

$$\sum_{a=1}^{\Delta_X} e^{2\pi i a n/\Delta_X} = \begin{cases} \Delta_X & \text{if } \Delta_X \mid n; \\ 0 & \text{otherwise.} \end{cases}$$

(3) (Two-term determinant formula, [3] and [1], and also see (4.4) in [11]) The zeta-function $Z_X(u)$ can be written as

$$Z_X(u) = 1/\det(I_{2\epsilon} - Wu) = \prod_{\lambda \in Spec(W)} (1 - \lambda u)^{-1}.$$

The following keylemma plays an important role in the proof of the main theorem.

KeyLemma. *Suppose that $X = (V, E)$ satisfies the same conditions as the main theorem.*

(a) As $N \to \infty$,

$$\sum_{n=1}^{N} \sum_{\lambda \in Spec(W)} (\lambda R_X)^n = \left[\frac{N}{\Delta_X}\right] \Delta_X + A_X + O\left((\rho_X R_X)^N\right),$$

where $[x]$ denotes the integer part of the real number x.

(b) As $N \to \infty$,

$$\sum_{n=1}^{N} \frac{1}{n} \sum_{\lambda \in Spec(W)} (\lambda R_X)^n = \sum_{n=1}^{[N/\Delta_X]} \frac{1}{n} + \log C_X + \log \Delta_X + O\left((\rho_X R_X)^N\right).$$

(c) ([2]) Let $0 < \alpha < 1/2$ be a real number, and fix it. Then, there exists a natural number $N_0 = N_0(\alpha)$ such that for any $n \geq N_0$,

$$\left| n \cdot \pi_X(n) - \sum_{\lambda \in Spec(W)} \lambda^n \right| < 2\epsilon \left(\frac{1}{R_X} \right)^{(1-\alpha)n}.$$

(d) (cf. Section 2 in [6]) Set

$$a = a(N) := N - \left[\frac{N}{\Delta_X} \right] \Delta_X,$$

and thus $0 \leq a(N) < 1$. Then,

$$\sum_{n=1}^{[N/\Delta_X]} \frac{1}{n} = \log N - \log \Delta_X + \gamma$$
$$- \sum_{s=1}^{k} \left(\frac{a^s}{s} + \sum_{m=0}^{s-1} \binom{s-1}{m} \frac{a^m B_{s-m} \Delta_X^{s-m}}{s-m} \right) \frac{1}{N^s} + O\left(\frac{1}{N^{k+1}} \right)$$

for each $k \geq 1$.

Proof. In this proof, we abbreviate the suffix X, that is, $R = R_X$, $\Delta = \Delta_X$, etc.

Let $\{a_n\}$ be a sequence of real numbers with $0 < a_n \leq 1$ for any n. Then, it follows from Facts (1)(2)(3) that we obtain the equality

$$\sum_{n=1}^{N} a_n \sum_{\substack{\lambda \in \mathrm{Spec}(W), \\ |\lambda| = 1/R}} (\lambda R)^n = \sum_{n=1}^{N} a_n \sum_{a=1}^{\Delta} e^{-2\pi i a n/\Delta} = \Delta \sum_{n=1}^{[N/\Delta]} a_{n\Delta}. \tag{1}$$

Moreover, it follows by the triangle inequality that we obtain the inequality

$$\left| \sum_{n>N} a_n \sum_{\substack{\lambda \in \mathrm{Spec}(W), \\ |\lambda| < 1/R}} (\lambda R)^n \right| \leq \sum_{n>N} a_n \sum_{\substack{\lambda \in \mathrm{Spec}(W), \\ |\lambda| < 1/R}} (|\lambda| R)^n$$
$$< 2\epsilon \sum_{n>N} (\rho R)^n = \frac{2\epsilon \rho R}{1 - \rho R} (\rho R)^N. \tag{2}$$

In the proofs of the items (a)(b), we use the equality (1) and the inequality (2).

(a) Note that

$$\left| \sum_{n=1}^{N} \sum_{|\lambda| = 1/R} (\lambda R)^n - \left[\frac{N}{\Delta} \right] \Delta \right| = \left| \sum_{n=1}^{[N/\Delta]} \Delta - \left[\frac{N}{\Delta} \right] \Delta \right| = 0$$

by the equality (1). On the other hand, note that

$$\left|\sum_{n=1}^{N}\sum_{|\lambda|<1/R}(\lambda R)^n - A\right| = \left|\sum_{n>N}\sum_{|\lambda|<1/R}(\lambda R)^n\right| < \frac{2\epsilon\rho R}{1-\rho R}(\rho R)^N$$

from the inequality (2). By combining these, the item (a) follows from the triangle inequality.

(b) Set the sums

$$S_1(N) := \sum_{n=1}^{N}\frac{1}{n}\sum_{|\lambda|=1/R}(\lambda R)^n \qquad \text{and} \qquad S_2(N) := \sum_{n=1}^{N}\frac{1}{n}\sum_{|\lambda|<1/R}(\lambda R)^n.$$

First, we consider the sum $S_1(N)$. It follows from the equality (1) that

$$S_1(N) = \sum_{n=1}^{N}\frac{1}{n}\sum_{|\lambda|=1/R}(\lambda R)^n = \sum_{n=1}^{[N/\Delta]}\frac{1}{n}.$$

Next, we compute the sum $S_2(N)$. We now consider the constant defined by

$$F := \log\prod_{|\lambda|<1/R}\frac{1}{1-\lambda R} \quad \left(= -\sum_{|\lambda|<1/R}\log(1-\lambda R) = \sum_{|\lambda|<1/R}\sum_{n\geq 1}\frac{1}{n}(\lambda R)^n\right),$$

and then we obtain $F = \log C_X + \log\Delta$. This is proved as follows: Note that

$$\begin{aligned} C_X = \lim_{u\uparrow R}\frac{(R-u)Z_X(u)}{R} &= \lim_{u\uparrow R}\left(1-\frac{u}{R}\right)\prod_{\lambda\in\mathrm{Spec}(W)}\frac{1}{1-\lambda u} \\ &= \lim_{u\uparrow R}\prod_{\substack{\lambda\in\mathrm{Spec}(W),\\ \lambda\neq 1/R}}\frac{1}{1-\lambda u} = \prod_{\substack{\lambda\in\mathrm{Spec}(W),\\ \lambda\neq 1/R}}\frac{1}{1-\lambda R} \end{aligned}$$

by the definition of C_X and Facts (1)(3). It is well known that

$$\sum_{n=0}^{\Delta-1}X^n = \frac{X^\Delta - 1}{X-1} = \prod_{a=1}^{\Delta-1}\left(X - e^{-2\pi i a/\Delta}\right), \quad \text{and so} \quad \Delta = \prod_{a=1}^{\Delta-1}\left(1-e^{-2\pi i a/\Delta}\right).$$

Combining these equalities, we obtain

$$\begin{aligned} \prod_{|\lambda|<1/R}\frac{1}{1-\lambda R} &= \prod_{\substack{\lambda\in\mathrm{Spec}(W),\\ \lambda\neq 1/R}}\frac{1}{1-\lambda R}\cdot\prod_{\substack{|\lambda|=1/R,\\ \lambda\neq 1/R}}(1-\lambda R) \\ &= \prod_{\substack{\lambda\in\mathrm{Spec}(W),\\ \lambda\neq 1/R}}\frac{1}{1-\lambda R}\cdot\prod_{a=1}^{\Delta-1}\left(1-e^{-2\pi i a/\Delta}\right) = C_X\cdot\Delta, \end{aligned}$$

and thus $F = \log C_X + \log\Delta$.

It follows from the inequality (2) that we obtain the inequality

$$|S_2(N) - F| = \left|\sum_{|\lambda|<1/R}\sum_{n>N}\frac{1}{n}(\lambda R)^n\right| < \frac{2\epsilon(\rho R)^{N+1}}{1-\rho R},$$

that is,

$$S_2(N) = F + O\left((\rho R)^N\right) = \log C_X + \log\Delta + O\left((\rho R)^N\right).$$

Hence, by combining the above results, we obtain

$$\begin{aligned}\sum_{n=1}^{N}\frac{1}{n}\sum_{\lambda\in\mathrm{Spec}(W)}(\lambda R)^n &= S_1(N)+S_2(N)\\ &= \sum_{n=1}^{[N/\Delta]}\frac{1}{n} + \log C_X + \log\Delta + O\left((\rho R)^N\right).\end{aligned}$$

(c) Let $\mu(n)$ denote the Möbius function. Note that $\sum_{d|n}|\mu(d)|\le n$. It is known that

$$\pi(n) = \frac{1}{n}\sum_{d|n}\mu(d)N_{n/d}, \qquad \text{and} \qquad N_n = \sum_{\lambda\in\mathrm{Spec}(W)}\lambda^n$$

(see (10.3) and (10.4) in [11], respectively). Combining these equalities, we obtain

$$n\cdot\pi(n) = \sum_{\lambda\in\mathrm{Spec}(W)}\sum_{d|n}\mu(d)\lambda^{n/d},$$

and therefore

$$\begin{aligned}\left|n\cdot\pi(n) - \sum_{\lambda\in\mathrm{Spec}(W)}\lambda^n\right| &= \left|\sum_{\lambda\in\mathrm{Spec}(W)}\sum_{\substack{d|n,\\ d\ge2}}\mu(d)\lambda^{n/d}\right|\\ &\le \sum_{\lambda\in\mathrm{Spec}(W)}\sum_{\substack{d|n,\\ d\ge2}}|\mu(d)|\cdot|\lambda|^{n/d} \le \sum_{\lambda\in\mathrm{Spec}(W)}\sum_{\substack{d|n,\\ d\ge2}}|\mu(d)|\cdot|\lambda|^{n/2}\\ &< n\sum_{\lambda\in\mathrm{Spec}(W)}\left(\frac{1}{R}\right)^{n/2} \le 2\epsilon n\left(\frac{1}{R}\right)^{n/2}.\end{aligned}$$

On the other hand, since $0<R<1$ and $0<\alpha<1/2$ by our assumptions, there exists a natural number $N_0 = N_0(\alpha)$ such that for any $n\ge N_0$,

$$n \le \left(\frac{1}{R}\right)^{(1/2-\alpha)n}, \qquad \text{and so} \qquad n\left(\frac{1}{R}\right)^{n/2} \le \left(\frac{1}{R}\right)^{(1-\alpha)n}.$$

Hence, for any $n\ge N_0$,

$$\left|n\cdot\pi(n) - \sum_{\lambda\in\mathrm{Spec}(W)}\lambda^n\right| \le 2\epsilon\left(\frac{1}{R}\right)^{(1-\alpha)n},$$

and the assertion of the item (c) follows.

(d) It is known from the equality (9) in [6] that

$$\left|\int_{[N/\Delta]}^{\infty}\frac{P_{2k+1}(x)}{x^{2k+2}}dx\right| = O\left(\frac{1}{N^{2k+1}}\right) \quad \left(= O\left(\frac{1}{N^{k+1}}\right)\right), \tag{3}$$

where $P_{2k+1}(x)$ is a periodic Bernoulli polynomial. Note that $[N/\Delta] = (N-a)/\Delta$. Recall that the $(2s-1)$-th Bernoulli numbers B_{2s-1} $(s\ge1)$ are given by

$$B_1 = -1/2 \qquad \text{and} \qquad B_{2s-1} = 0 \quad (s\ge2).$$

Then, it follows from the equality (7) in [6] and the above equality (3) that

$$\begin{aligned}\sum_{n=1}^{[N/\Delta]}\frac{1}{n}-\gamma &= \log\left[\frac{N}{\Delta}\right]+\frac{1}{2[N/\Delta]}-\sum_{s=1}^{k}\frac{B_{2s}}{2s[N/\Delta]^{2s}}+O\left(\frac{1}{N^{k+1}}\right)\\ &= \log\left[\frac{N}{\Delta}\right]-\sum_{s=1}^{2k}\frac{B_s}{s[N/\Delta]^s}+O\left(\frac{1}{N^{k+1}}\right),\end{aligned}$$

and therefore

$$\begin{aligned}\sum_{n=1}^{[N/\Delta]}\frac{1}{n}-\gamma &= \log\left[\frac{N}{\Delta}\right]-\sum_{s=1}^{2k}\frac{B_s}{s[N/\Delta]^s}+O\left(\frac{1}{N^{k+1}}\right)\\ &= \log\left[\frac{N}{\Delta}\right]-\sum_{s=1}^{k}\frac{B_s}{s[N/\Delta]^s}+O\left(\frac{1}{N^{k+1}}\right)\\ &= \log N+\log\left(1-\frac{a}{N}\right)-\log\Delta\\ &\qquad -\sum_{s=1}^{k}\frac{B_s\Delta^s}{sN^s}\frac{1}{(1-a/N)^s}+O\left(\frac{1}{N^{k+1}}\right)\\ &= \log N-\log\Delta-\sum_{s=1}^{k}\frac{a^s}{sN^s}\\ &\qquad -\sum_{s=1}^{k}\frac{B_s\Delta^s}{sN^s}\sum_{m\geq 0}\binom{s-1+m}{m}\left(\frac{a}{N}\right)^m+O\left(\frac{1}{N^{k+1}}\right). \end{aligned}\tag{4}$$

On the other hand, since the inequality

$$\binom{s-1+m}{m}=\frac{s-1+m}{m}\cdot\frac{s-2+m}{m-1}\cdots\frac{s+1}{2}\cdot\frac{s}{1}\leq s^m$$

holds, we obtain the inequalities

$$\begin{aligned}\sum_{s=1}^{k}\frac{B_s\Delta^s}{sN^s}\sum_{m>k-s}\binom{s-1+m}{m}\left(\frac{a}{N}\right)^m &\leq \sum_{s=1}^{k}\frac{B_s\Delta^s}{sN^s}\sum_{m>k-s}\left(\frac{sa}{N}\right)^m\\ &= \sum_{s=1}^{k}\frac{B_s\Delta^s}{sN^s}\left(\frac{sa}{N}\right)^{k-s+1}\frac{1}{1-sa/N}\\ &\leq \frac{1}{N^{k+1}}\cdot\frac{N}{N-ka}\sum_{s=1}^{k}\frac{B_s\Delta^s}{s}(sa)^{k-s+1},\end{aligned}$$

that is,

$$\sum_{s=1}^{k}\frac{B_s\Delta^s}{sN^s}\sum_{m>k-s}\binom{s-1+m}{m}\left(\frac{a}{N}\right)^m=O\left(\frac{1}{N^{k+1}}\right).\tag{5}$$

Hence, combining the equalities (4)(5), we obtain

$$\begin{aligned}\sum_{n=1}^{[N/\Delta]}\frac{1}{n} &= \log N-\log\Delta+\gamma\\ &\qquad -\sum_{s=1}^{k}\left(a^s+B_s\Delta^s\sum_{m=0}^{k-s}\binom{s+m-1}{m}\left(\frac{a}{N}\right)^m\right)\frac{1}{sN^s}+O\left(\frac{1}{N^{k+1}}\right),\end{aligned}$$

and the assertion of the item (d) follows after an elementary computation. ∎

By using KeyLemma (c), we can prove the convergences of constants.

Lemma 1. *Suppose that $X = (V, E)$ satisfies the same conditions as the main theorem.*

(1) The series

$$K_X = \sum_{n\geq 1} \left(n \cdot \pi_X(n) - \sum_{\lambda \in Spec(W)} \lambda^n \right) R_X^n$$

is convergent.

(2) The series

$$H_X = -\sum_{n\geq 1} \frac{1}{n} \left(n \cdot \pi_X(n) - \sum_{\lambda \in Spec(W)} \lambda^n \right) R_X^n$$

is convergent. Moreover,

$$H_X = \sum_{[P]} \sum_{m\geq 2} \frac{1}{m} R_X^{m\ell(P)}$$

holds.

Proof. Let $\{a_n\}$ be a sequence of real numbers with $0 < a_n \leq 1$ for any n. Note that

$$\begin{aligned}
\left| \sum_{n\geq N_0} a_n \left(n \cdot \pi(n) - \sum_{\lambda \in \mathrm{Spec}(W)} \lambda^n \right) R^n \right| &\leq \sum_{n\geq N_0} \left| n \cdot \pi(n) - \sum_{\lambda \in \mathrm{Spec}(W)} \lambda^n \right| R^n \\
&< 2\epsilon \sum_{n\geq N_0} \left(\frac{1}{R} \right)^{(1-\alpha)n} R^n \\
&\leq 2\epsilon \sum_{n\geq 1} R^{\alpha n} = \frac{2\epsilon R^\alpha}{1 - R^\alpha}
\end{aligned}$$

by KeyLemma (c). Hence, the convergences of the items (1)(2) hold from this inequality.

Next, we show the equality of the item (2). Assume that $|u| < R$. It follows from Fact (3) and the definition of $Z_X(u)$ that

$$\begin{aligned}
\sum_{n\geq 1} \sum_{\lambda \in \mathrm{Spec}(W)} \frac{\lambda^n}{n} u^n &= \sum_{\lambda \in \mathrm{Spec}(W)} \log(1 - \lambda u)^{-1} \\
&= \log Z_X(u) = \sum_{[P]} \log(1 - u^{\ell(P)})^{-1} \\
&= \sum_{[P]} \sum_{m\geq 1} \frac{1}{m} u^{m\ell(P)} = \sum_{[P]} u^{\ell(P)} + \sum_{[P]} \sum_{m\geq 2} \frac{1}{m} u^{m\ell(P)} \\
&= \sum_{n\geq 1} \pi(n) u^n + \sum_{[P]} \sum_{m\geq 2} \frac{1}{m} u^{m\ell(P)},
\end{aligned}$$

and therefore

$$-\sum_{n\geq 1} \frac{1}{n} \left(n \cdot \pi_X(n) - \sum_{\lambda \in \mathrm{Spec}(W)} \lambda^n \right) u^n = \sum_{[P]} \sum_{m\geq 2} \frac{1}{m} u^{m\ell(P)}. \tag{6}$$

On the other hand, by the graph theory prime-number theorem (see Theorem 10.1 in [11]), the radius of convergence of the function

$$P(u) = \sum_{[P]} u^{\ell(P)} = \sum_{n \geq 1} \pi(n) u^n$$

is equal to R. Note that $R^2 < R$ since $0 < R < 1$. Then,

$$\begin{aligned}
\sum_{[P]} \sum_{m \geq 2} \frac{1}{m} R^{m\ell(P)} &\leq \sum_{[P]} \sum_{m \geq 2} R^{m\ell(P)} = \sum_{[P]} \frac{R^{2\ell(P)}}{1 - R^{\ell(P)}} \\
&\leq \frac{1}{1-R} \sum_{[P]} R^{2\ell(P)} \leq \frac{1}{1-R} P(R^2) < +\infty.
\end{aligned}$$

Hence, since both sides of the equality (6) are also convergent for $u = R$, the assertion follows by the uniqueness of analytic continuation (namely, the principle of uniqueness). ∎

3 Proof of the main theorem

In this section, we show the main theorem.

Proof. (The main theorem) (1) Assume that N is sufficiently large. Then, it follows from KeyLemma (c) that we obtain

$$\begin{aligned}
\left| \sum_{n=1}^{N} n \cdot \pi(n) R^n - \sum_{n=1}^{N} \sum_{\lambda \in \mathrm{Spec}(W)} (\lambda R)^n - K \right| &= \left| \sum_{n > N} \left(n \cdot \pi(n) - \sum_{\lambda \in \mathrm{Spec}(W)} \lambda^n \right) R^n \right| \\
&\leq \sum_{n > N} \left| n \cdot \pi(n) - \sum_{\lambda \in \mathrm{Spec}(W)} \lambda^n \right| R^n \\
&< 2\epsilon \sum_{n > N} \left(\frac{1}{R} \right)^{(1-\alpha)n} R^n \\
&= 2\epsilon \sum_{n > N} R^{\alpha n} = \frac{2\epsilon R^{\alpha(N+1)}}{1 - R^{\alpha}},
\end{aligned}$$

and therefore by KeyLemma (a), we obtain

$$\begin{aligned}
\sum_{n=1}^{N} n \cdot \pi(n) R^n &= \sum_{n=1}^{N} \sum_{\lambda \in \mathrm{Spec}(W)} (\lambda R)^n + K + O\left((\rho R)^N\right) \\
&= \left[\frac{N}{\Delta} \right] \Delta + A + K + O\left((\rho R)^N\right)
\end{aligned}$$

as $N \to \infty$. Hence, the assertion of the item (1) follows.

(2) Suppose that N is sufficiently large. Then, it follows from KeyLemma (c) that

$$\begin{aligned}\left|\sum_{n=1}^{N}\pi(n)R^n-\sum_{n=1}^{N}\frac{1}{n}\sum_{\lambda\in\mathrm{Spec}(W)}(\lambda R)^n+H\right| &= \left|\sum_{n>N}\left(\pi(n)-\frac{1}{n}\sum_{\lambda\in\mathrm{Spec}(W)}\lambda^n\right)R^n\right|\\ &= \left|\sum_{n>N}\frac{1}{n}\left(n\cdot\pi(n)-\sum_{\lambda\in\mathrm{Spec}(W)}\lambda^n\right)R^n\right|\\ &\leq \sum_{n>N}\left|n\cdot\pi(n)-\sum_{\lambda\in\mathrm{Spec}(W)}\lambda^n\right|R^n\\ &< 2\epsilon\sum_{n>N}\left(\frac{1}{R}\right)^{(1-\alpha)n}R^n\\ &= 2\epsilon\sum_{n>N}R^{\alpha n}=\frac{2\epsilon R^{\alpha(N+1)}}{1-R^{\alpha}},\end{aligned}$$

and therefore we obtain

$$\sum_{n=1}^{N}\pi(n)R^n=\sum_{n=1}^{N}\frac{1}{n}\sum_{\lambda\in\mathrm{Spec}(W)}(\lambda R)^n-H+O\left(R^{\alpha N}\right)$$

as $N\to\infty$. Hence, it follows from KeyLemmas (b)(d) that the assertion holds.

(3) Assume that N is sufficiently large, and define the following functions:

$$H^{\leq N}=\sum_{n\leq N}\pi(n)\sum_{m\geq 2}\frac{1}{m}R^{mn},\qquad\text{and}\qquad H^{>N}=\sum_{n>N}\pi(n)\sum_{m\geq 2}\frac{1}{m}R^{mn}.$$

Note that $H=H^{\leq N}+H^{>N}$ by Lemma 1 (2). It follows from the graph theory prime-number theorem (see Theorem 10.1 in [11]) that there exists a constant $c_1>0$ such that for any $n>N$,

$$\pi(n)<\frac{c_1}{R^n}.$$

Recall that $0<R<1$ since X is a noncycle graph. Then, we obtain

$$\begin{aligned}H^{>N} &= \sum_{n>N}\pi(n)\sum_{m\geq 2}\frac{1}{m}R_X^{mn}<c_1\sum_{n>N}\sum_{m\geq 2}R^{(m-1)n}\\ &= c_1\sum_{n>N}\frac{R^n}{1-R^n}<\frac{c_1}{1-R}\sum_{n>N}R^n=\frac{c_1R}{(1-R)^2}R^N,\end{aligned}$$

and therefore $H^{>N}=O(R^N)$.

From the item (2) and the above result, we obtain

$$\begin{aligned}\sum_{n\leq N}\pi(n)R^n+H^{\leq N} &= \log N+\gamma+\log C_X-H^{>N}+O\left(\frac{1}{N}\right)\\ &= \log N+\gamma+\log C_X+O\left(\frac{1}{N}\right).\end{aligned}$$

Since the left-hand side of the above equality is equal to

$$\begin{aligned}\sum_{n\le N}\pi(n)R^n + H^{\le N} &= \sum_{n\le N}\pi(n)\sum_{m=1}^{\infty}\frac{1}{m}R^{mn}\\ &= -\sum_{n\le N}\pi(n)\log\left(1-R^n\right) = -\log\left(\prod_{n\le N}\left(1-R^n\right)^{\pi(n)}\right),\end{aligned}$$

we obtain

$$\prod_{n\le N}\left(1-R^n\right)^{\pi(n)} = \frac{e^{-\gamma}}{C_X}\cdot\frac{1}{N}\exp\left(O\left(\frac{1}{N}\right)\right) = \frac{e^{-\gamma}}{C_X}\cdot\frac{1}{N}\left(1+O\left(\frac{1}{N}\right)\right),$$

and the assertion of the item (3) follows. ∎

Remark. *(1) When $k=0$, our second theorem just corresponds to a special case of the second theorem due to Pollicott (see Section 1 in this paper). This is proved as follows:*

In our case, the topological entropy is equal to the constant $h=-\log R>0$. We now define $u=R^s$, $N(P)=e^{h\ell(P)}=R^{-\ell(P)}$ and $x=e^{hN}$. Then, the left-hand side is equal to

$$\sum_{n\le N}\pi(n)R^n = \sum_{\ell(P)\le N}R^{\ell(P)} = \sum_{N(P)\le x}\frac{1}{N(P)}.$$

On the other hand, the right-hand side can be transformed as follows. Note that

$$\begin{aligned}C_X &= -\frac{1}{R}\cdot\lim_{u\uparrow R}(u-R)Z_X(u)\\ &= -\frac{1}{R}\cdot\lim_{s\downarrow 1}\frac{R^s-R}{s-1}\cdot\lim_{s\downarrow 1}(s-1)Z_X(R^s) = h\cdot\operatorname*{Res}_{s=1}Z_X(R^s).\end{aligned}$$

It follows from Lemma 1 (2) that

$$H_X = \sum_{[P]}\sum_{n\ge 2}\frac{1}{n}R^{n\ell(P)} = \sum_{[P]}\sum_{n\ge 2}\frac{1}{n}\cdot\frac{1}{N(P)^n}.$$

By combining the above results, we obtain

$$\begin{aligned}&\log N+\gamma+\log C_X - H_X + O\left(\frac{1}{N}\right)\\ &\quad = \log\left(\log x\right)+\gamma+\log\left(\operatorname*{Res}_{s=1}Z_X(R^s)\right)-\sum_{[P]}\sum_{n\ge 2}\frac{1}{n}\cdot\frac{1}{N(P)^n}+O\left(\frac{1}{\log x}\right).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\sum_{N(P)\le x}\frac{1}{N(P)} &= \log\left(\log x\right)+\gamma+\log\left(\operatorname*{Res}_{s=1}Z_X(R^s)\right)\\ &\qquad -\sum_{[P]}\sum_{n\ge 2}\frac{1}{n}\cdot\frac{1}{N(P)^n}+O\left(\frac{1}{\log x}\right).\end{aligned}$$

(2) The error term $O(1/N)$ in our second theorem can not be replaced by $o(1/N)$ since in general, the coefficient $\Delta/2-a(N)$ of $1/N$ is not equal to zero.

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