On the choosability of claw-free perfect graphs

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Abstract

It has been conjectured that for every claw-free graph G the choice number of G is equal to its chromatic number. We focus on the special case of this conjecture where G is perfect. Claw-free perfect graphs can be decomposed via clique-cutset into two special classes called elementary graphs and peculiar graphs. Based on this decomposition we prove that the conjecture holds true for every claw-free perfect graph with maximum clique size at most 4.

1 Introduction

We consider finite, undirected graphs, without loops. Given a graph G and an integer k, a *k-coloring* of the vertices of G is a mapping $c : V(G) \to \{1, 2, \dots, k\}$ for which every pair of adjacent vertices x, y satisfies $c(x) \neq c(y)$. A *coloring* is a k-coloring for any k. The graph G is called *k-colorable* if it admits a k-coloring. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k such that G is k-colorable.

The *list-coloring* variant of the coloring problem, introduced by Erdős, Rubin and Taylor [4] and by Vizing [9], is as follows. Assume that each vertex v has a list $L(v)$ of prescribed colors; then we want to find a coloring c such that $c(v) \in L(v)$ for all $v \in V(G)$. When such a coloring exists we say that the graph G is *L-colorable* and that c is an *L-coloring* of G. Given an integer k, a graph G is *k-choosable* if it is L-colorable for every assignment L that satisfies $|L(v)| = k$ for all $v \in V(G)$ (equivalently, if it is L-colorable for every assignment L that satisfies $|L(v)| = k$ for all $v \in V(G)$). The *choice number* or *list-chromatic number* $ch(G)$ of G is the smallest k such that G is k-choosable. It is easy to see that every k-choosable graph G is k-colorable (consider the assignment $L(v) = \{1, 2, \dots, k\}$ for all $v \in V(G)$), and so $\chi(G) \leq ch(G)$ holds for every graph. There are graphs for which the difference between $ch(G)$ and $\chi(G)$ is arbitrarily large. (For example, it is easy to see that the choice number of the complete bipartite graph K_{p,p^p} is $p + 1$.)

The above notions can be extended to the problem of coloring the edges of a graph. The least number of colors necessary to color all edges of a graph in such a way that no two adjacent edges receive the same color is its *chromatic index*

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$\chi'(G)$. The least k such that G is L'-edge-colorable for any assignment L' of colors to the edges of G with $|L'(e)| = k$ for all $e \in E$ is called the *choice index* or *list-chromatic index* of G. Vizing (see [9]), proposed the following conjecture:

Conjecture 1.1. *Every graph G satisfies $ch'(G) = \chi'(G)$.*

The special case of this conjecture dealing with list-coloring the edges of a complete bipartite graph was known as the Dinitz conjecture, as it was equivalent to a problem on Latin squares posed by Jeffrey Dinitz. Galvin [5] established the following more general result.

Theorem 1.2 (Galvin [5])**.** *Every bipartite graph G satisfies $ch'(G) = \chi'(G)$.*

The problem of edge-coloring can be reduced to a special instance of the problem of vertex-coloring via the line-graph. Given a graph H, the *line-graph* $\mathcal{L}(H)$ of H is the graph whose vertices are the edges of H and whose edges are the pairs of adjacent edges of H. Conversely, H is called the *root* graph of $\mathcal{L}(H)$. It is clear that $\chi(\mathcal{L}(H)) = \chi'(H)$ and $ch(\mathcal{L}(H)) = ch'(H)$.

In a graph G, we say that a vertex v is *complete* to a set $S \subseteq V(G)$ when v is adjacent to every vertex in S, and *anticomplete* to S when v has no neighbor in S. Given two sets $S, T \subseteq V(G)$ we say that S is *complete* to T is every vertex in S is adjacent to every vertex in T, and *anticomplete* to T when no vertex in S is adjacent to any vertex in T. The neighborhood of a vertex v is denoted by $N_G(v)$ (and the subscript G may be dropped when there is no ambiguity). The complement of graph G is denoted by $\overline{G}$.

A graph is *cobipartite* if its complement is bipartite, in other words if its vertex-set can be partitioned into at most two cliques. We let P_n, C_n and K_n respectively denote the path, cycle and complete graph on n vertices.

Given any graph F, a graph G is *F-free* if no induced subgraph of G is isomorphic to F. The *claw* is the graph with four vertices a, b, c, d and edges ab, ac, ad; vertex a is called the *center* of the claw.

A graph G is *perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$. A *Berge* graph is any graph that does not contain as an induced subgraph an odd cycle of length at least five or the complement of an odd cycle of length at least five. Chudnovsky, Robertson, Seymour, Thomas solved the long-standing and famous problem known as the Strong Perfect Graph Conjecture by proving the following theorem.

Theorem 1.3 ([3])**.** *A graph G is perfect if and only if it is Berge.*

The special case of the Strong Perfect Graph Conjecture concerning claw-free graphs had been resolved much earlier by Parthasarathy and Ravindra.

Theorem 1.4 (Parthasarathy and Ravindra [12])**.** *Every claw-free Berge graph G is perfect.*

Here we are interested in a restricted version of a question posed by two of us [6, 7], asking whether it is true that every claw-free graph G satisfies $ch(G) = \chi(G)$.

Conjecture 1.5. *Every claw-free perfect graph G satisfies $ch(G) = \chi(G)$.*

This conjecture was proved in [8] for every claw-free perfect graph G with $\omega(G) \leq 3$. Here we will prove it for the case $\omega(G) \leq 4$. Our main result is the following.

Theorem 1.6. *Let G be a claw-free perfect graph with $\omega(G) \leq 4$. Then $ch(G) = \chi(G)$.*

Our proof is based on a decomposition theorem for claw-free perfect graphs due to Chvátal and Sbihi [2]. They proved that every claw-free perfect graph either admits a clique cutset or belongs to two specific classes of graphs, which we defined precisely below.

Definition (Clique cutset)**.** *A* clique cutset *in a graph G is a clique C of G such that $G \setminus C$ is disconnected. A* minimal clique cutset *is a clique cutset that does not contain another clique cutset.*

If C is a minimal clique cutset in a graph G and $A_1, \ldots, A_k$ are the vertex-sets of the components of $G \setminus C$, we consider that G is decomposed into the collection of induced subgraphs $G[A_i \cup C]$ for $i = 1, \ldots, k$. These subgraphs themselves may admit clique cutsets, so the decomposition (via minimal clique cutsets) can be applied further. This decomposition can be represented by a tree, where each non-leaf node corresponds to an induced subgraph G' of G and a minimal clique cutset C' of G', and the children of the node are the induced subgraphs into which G' is decomposed along C'. The leaves of T are indecomposable subgraphs of G (subgraphs that have no clique cutset), which we call *atoms*. (This tree may not be unique, depending on the choice of a clique cutset at each node.) Whitesides [15] and Tarjan [14] proved that for every graph G on n vertices every clique-cutset decomposition tree has at most n leaves and that such a decomposition can be obtained in polynomial time $O(n^3)$. A nice feature is that every graph G admits an *extremal* clique cutset, that is, a minimal clique cutset C such that there is a component H of $G \setminus C$ such that $G[V(H) \cup C]$ is an atom.

Definition (Elementary graph [2])**.** *A graph is* elementary *if its edges can be colored with two colors (one color on each edge) in such a way that every induced two-edge path has its two edges colored differently.*

Definition (Peculiar graph [2])**.** *A graph G is* peculiar *if $V(G)$ can be partitioned into nine sets A_i, B_i, Q_i $(i = 1, 2, 3)$ that satisfy the following properties for each i, where subsbcripts are understood modulo 3:*

- *Each of the nine sets is non-empty and induces a clique.*
- *A_i is complete to $B_i \cup A_{i+1} \cup A_{i+2} \cup B_{i+2}$ and not complete to B_{i+1}.*
- *B_i is complete to $A_i \cup B_{i+1} \cup B_{i+2} \cup A_{i+1}$ and not complete to A_{i+2}.*
- *Q_i is complete to $A_{i+1} \cup B_{i+1} \cup A_{i+2} \cup B_{i+2}$ and anticomplete to $A_i \cup B_i \cup Q_{i+1} \cup Q_{i+2}$.*

We say that $(A_1, B_1, A_2, B_2, A_3, B_3, Q_1, Q_2, Q_3)$ is a peculiar partition of G.

Theorem 1.7 (Chvátal and Sbihi [2])**.** *Every claw-free perfect graph either has a clique cutset or is a peculiar graph or an elementary graph.*

The structure of peculiar graphs is clear from their definition. Concerning elementary graphs, their structure was elucidated by Maffray and Reed [11] as follows. Let us say that an edge is *flat* if it is not contained in a triangle.

Definition (Flat edge augmentation)**.** *Let xy be a flat edge in a graph G, and let A be a cobipartite graph such that $V(A)$ is disjoint from $V(G)$ and $V(A)$ can be partitioned into two cliques X, Y. We obtain a new graph G' by removing x and y from G and adding all edges between X and $N_G(x) \setminus \{y\}$ and all edges between Y and $N_G(y) \setminus \{x\}$. This operation is called* augmenting *the flat edge xy with the cobipartite graph A. In G' the pair (X, Y) is called the* augment.

When $x_1y_1, \dots, x_ky_k$ are pairwise non-adjacent flat edges in a graph G, and $A_1, \dots, A_k$ are pairwise vertex-disjoint cobipartite graphs, also vertex-disjoint from G, one can augment each edge x_iy_i with the graph A_i. Clearly the result is the same whatever the order in which the k operations are performed. We say that the resulting graph is an *augmentation* of G.

Theorem 1.8 (Maffray and Reed [11])**.** *A graph G is elementary if and only if it is an augmentation of the line-graph H of a bipartite multigraph B. Moreover we may assume that each augment A_i satisfies the following:*

- *There is at least one pair of non-adjacent vertices in A_i,*
- *The bipartite graph whose vertex-set is $X_i \cup Y_i$ and whose edges are the edges of A_i with one end in X_i and one in Y_i is connected (and consequently both $|X_i|, |Y_i| \geq 2$).*

In a directed graph D, for every vertex v we let $d^+(v)$ denote the number of vertices w such that vw is an arc of D.

Theorem 1.9 (Galvin [5])**.** *Let G be the line-graph of a bipartite graph B, where $V(B)$ is partitioned into two stable set X, Y. Let f be an $\omega(G)$-coloring of the vertices of G, with colors $1, 2, \dots, \omega(G)$. Let D be the directed graph obtained from G by directing every edge uv as follows, assuming that $f(u) < f(v)$: when the common end of edges u, v in B is in X, then give the orientation $u \rightarrow v$, and when it is in Y give the orientation $u \leftarrow v$. Assume that L is a list assignment on $V(G)$ such that every vertex v of G satisfies $|L(v)| \geq d^+_D(v) + 1$. Then G is L-colorable.*

Let G be a graph and let L be a list assignment on $V(G)$. For every set $S \subseteq V(G)$ we set $L(S) = \bigcup_{x \in S} L(x)$. If f is a coloring of G, we set $f(S) = \{f(x) \mid x \in S\}$. If H is an induced subgraph of G, we may also write $L(H)$ and $f(H)$ instead of $L(V(H))$ and $f(V(H))$ respectively.

For the sake of completeness we recall the classical theorems of Kőnig and Hall. Let $X_1, \dots, X_k$ be a family of sets. A *system of distinct representatives* for the family is a subset $\{x_1, \dots, x_k\}$ of k distinct elements of $X_1 \cup \dots \cup X_k$ such that $x_i \in X_i$ for all $i = 1, \dots, k$. Note that if G is a graph and L is a list assignment on $V(G)$, and the family $\{L(v) \mid v \in V(G)\}$ admits a system of distinct representatives, then this is an L-coloring of G.

Theorem 1.10 (Hall's theorem [10, 13])**.** *A family $\mathcal{F}$ of k sets has a system of distinct representatives if and only if, for all $\ell \in \{1, \dots, k\}$, the union of any ℓ members of $\mathcal{F}$ has size at least ℓ.*

A *matching* in a graph G is a set of pairwise non-incident edges.

Theorem 1.11 (Kőnig's theorem [13]). *In a bipartite graph on n vertices, let μ be the size of a maximum matching and α be the size of a maximum stable set. Then $\mu + \alpha = n$.*

2 Peculiar graphs

Lemma 2.1. *Let G be a connected claw-free graph that contains a peculiar subgraph, and assume that G is also C_5-free. Then G is peculiar.*

Proof. Let H be a peculiar subgraph of G that is maximal. If $H = G$ we are done. So let us assume that $H \neq G$. Since G is connected there is a vertex x of $V(G) \setminus V(H)$ that has a neighbor in H. Let $A_1, B_1, A_2, B_2, A_3, B_3, Q_1, Q_2, Q_3$ be nine cliques that form a partition of $V(H)$ as in the definition of a peculiar graph. For $i = 1, 2, 3$ we pick a pair of non-adjacent vertices $a_i \in A_i$ and $b_{i+1} \in B_{i+1}$, and we pick any $q_i \in Q_i$. (All subscripts are modulo 3.)

If x has no neighbor in $Q_1 \cup Q_2 \cup Q_3$, then it has a neighbor a in $A_i \cup B_i$ for some i; but then $\{a, x, q_{i+1}, q_{i+2}\}$ induces a claw. Therefore x has a neighbor in $Q_1 \cup Q_2 \cup Q_3$.

Suppose that x has a neighbor k in Q_1 and none in $Q_2 \cup Q_3$. Then x has no neighbor z in $A_1 \cup B_1$, for otherwise $\{z, x, q_2, q_3\}$ induces a claw. Also x is adjacent to one of a_2, b_3, for otherwise $\{x, k, a_2, b_3\}$ induces a claw; up to symmetry we assume that x is adjacent to a_2. Then x is adjacent to every vertex $a \in A_3$, for otherwise $\{a_2, q_3, a, x\}$ induces a claw; and to every vertex $y \in A_2 \cup B_2 \cup Q_1$, for otherwise $\{a_3, y, x, q_2\}$ induces a claw; and to every vertex $b \in B_3$, for otherwise $\{b_2, b, q_3, x\}$ induces a claw. Hence x is complete to $A_2 \cup B_2 \cup A_3 \cup B_3 \cup Q_1$ and anticomplete to $A_1 \cup B_1 \cup Q_2 \cup Q_3$. So $V(H) \cup \{x\}$ induces a peculiar subgraph of G, because x can be added to Q_1, a contradiction to the choice of H.

Therefore we may assume up to symmetry that x has a neighbor $k \in Q_1$ and a neighbor $k' \in Q_2$. Note that x has no neighbor $k'' \in Q_3$, for otherwise $\{x, k, k', k''\}$ induces a claw.

Suppose that x has a non-neighbor $a \in A_1$. Then x is adjacent to every vertex $u \in A_2$, for otherwise $\{x, k, u, a, k'\}$ induces a C_5; and then to every vertex $v \in B_2$, for otherwise either $\{a_2, a, x, v\}$ induces a claw (if $av \notin E(G)$) or $\{x, k, v, a, k'\}$ induces a C_5 (if $av \in E(G)$); and then to every vertex $w \in A_3 \cup B_3 \cup Q_1$, for otherwise $\{b_2, x, w, q_3\}$ induces a claw. Then a is adjacent to every vertex $b \in B_2$, for otherwise $\{x, k', a, q_3, b\}$ induces a C_5; and by the same argument the set $A_1 \setminus N(x)$ is complete to B_2. It follows that $a_1 \in N(x)$ since a_1 is not complete to B_2. Then x is adjacent to every vertex $q \in Q_2$, for otherwise $\{a_1, x, q_3, q\}$ induces a claw. But now we observe that $V(H) \cup \{x\}$ induces a larger peculiar subgraph of G, because x can be added to A_3 and the vertices of $A_1 \setminus N(x)$ can be moved to B_1.

Therefore we may assume that x is complete to A_1, and, similarly, to B_2. Then x is adjacent to every vertex u in $Q_2 \cup B_3$, for otherwise $\{a_1, x, u, q_3\}$ induces a claw, and similarly x is complete to $Q_1 \cup A_3$. It cannot be that x has both a non-neighbor $a' \in A_2$ and a non-neighbor $b' \in B_1$, for otherwise $\{x, k, a', b', k'\}$ induces a C_5. So, up to symmetry, x is complete to A_2. But now

$V(H) \cup \{x\}$ induces a larger peculiar subgraph of G, because x can be added to A_3. This completes the proof of the lemma. $\square$

We observe that (up to isomorphism) there is a unique peculiar graph G with $\omega(G) = 4$. Indeed if G is such a graph, with the same notation as in the definition of a peculiar graph, then for each i the set $Q_i \cup A_{i+1} \cup B_{i+1} \cup A_{i+2}$ is a clique, so, since G has no clique of size 5, the four sets $Q_i, A_{i+1}, B_{i+1}, A_{i+2}$ have size 1; and so the nine sets A_i, B_i, Q_i $(i = 1, 2, 3)$ all have size 1. Hence G is the unique peculiar graph on nine vertices.

Lemma 2.2. *Let G be a peculiar graph with $\omega(G) = 4$. Then G is 4-choosable.*

Proof. Let $(A_1, B_1, A_2, B_2, A_3, B_3, Q_1, Q_2, Q_3)$ be a peculiar partition of G. As observed above, we have $|A_i| = |B_i| = |Q_i| = 1$ for all $i = 1, 2, 3$. Hence let $A_i = \{a_i\}$, $B_i = \{b_i\}$ and $Q_i = \{q_i\}$, for all $i = 1, 2, 3$. Recall that a_i is not adjacent to b_{i+1}, for each i. Let $Q = \{q_1, q_2, q_3\}$.

Let L be a list assignment that satisfies $|L(v)| = 4$ for all $v \in V(G)$. Let us prove that G is L-colorable.

First suppose that for some $i \in \{1, 2, 3\}$ we have $L(a_i) \cap L(b_{i+1}) \neq \emptyset$, say for $i = 1$. Pick any $c \in L(a_1) \cap L(b_2)$. Let $G' = G \setminus \{a_1, b_2\}$ and let $L'(x) = L(x) \setminus \{c\}$ for all $x \in V(G')$. Clearly, G' is a claw-free perfect graph and $\omega(G') = 3$. Moreover, G' is elementary. To see this, define an egde coloring of G' by coloring blue the edges in $\{q_3b_1, q_3a_2, b_1a_2, b_3a_3, q_2a_3, b_3q_1\}$ and red the edges in $\{q_2b_1, q_2b_3, b_3b_1, q_1a_2, q_1a_3, a_2a_3\}$; it is a routine matter to check that this edge coloring is an elementary coloring. By [8], G' is 3-choosable, so it admits an L'-coloring. We can extend this coloring to a_1 and b_2 by assigning color c to them. Therefore we may assume that:

$$L(a_i) \cap L(b_{i+1}) = \emptyset \text{ for all } i = 1, 2, 3. \tag{1}$$

Now suppose that there are vertices $u, v \in Q$ such that $L(u) \cap L(v) \neq \emptyset$. Let w be the unique vertex in $Q \setminus \{u, v\}$. Pick any $c \in L(u) \cap L(v)$. Let $G' = G \setminus \{u, v\}$. Let $L'(x) = L(x) \setminus \{c\}$ for all $x \in V(G') \setminus \{w\}$, and let $L'(w) = L(w)$. We claim that the family $\{L'(x) \mid x \in V(G')\}$ admits a system of distinct representatives. Suppose the contrary. By Hall's theorem, there is a set $S \subseteq V(G')$ such that $|L'(S)| < |S|$. Since $|L'(x)| \geq 3$ for all $x \in V(G')$, we have $|L'(S)| \geq 3$, so $|S| \geq 4$; this implies that either (a) $S \supseteq \{a_i, b_{i+1}\}$ for some $i \in \{1, 2, 3\}$ or (b) S contains w. In case (a), (1) implies that c belongs to at most one of $L(a_i)$ and $L(b_{i+1})$, and so $|L'(S)| \geq |L'(a_i) \cup L'(b_{i+1})| \geq 7$, so $|S| \geq 8$, which is impossible because $|V(G')| = 7$. In case (b), since $|L'(w)| = 4$, we have $|L'(S)| \geq 4$, so $|S| \geq 5$, which implies that S satisfies (a) again, a contradiction. Thus the family $\{L'(x) \mid x \in V(G')\}$ admits a system of distinct representatives, which is an L'-coloring of G'. We can extend this coloring to u and v by assigning color c to them. Therefore we may assume that

$$L(u) \cap L(v) = \emptyset \text{ for all } u, v \in Q. \tag{2}$$

We claim that the family $\{L(x) \mid x \in V(G)\}$ admits a system of distinct representatives. Suppose the contrary. By Hall's theorem, there is a set $T \subseteq V(G)$ such that $|L(T)| < |T|$. Since $|L(x)| = 4$ for all $x \in V(G)$, we have $|L(T)| \geq 4$, so $|T| \geq 5$; this implies that either (a) $T \supseteq \{a_i, b_{i+1}\}$ for some $i \in \{1, 2, 3\}$ or (b) T contains two vertices from Q. In either case, (1) or (2) implies that $|L(T)| \geq 8$, so $|T| \geq 9$, that is, $T = V(G)$. But then $T \supset Q$, so (2) implies that $|L(T)| \geq 12$ and $|T| \geq 13$, which is impossible. Thus the family $\{L(x) \mid x \in V(G)\}$ admits a system of distinct representatives, which is an L-coloring of G. $\square$

3 Cobipartite graphs

In this section we analyze the list-colorability of certain cobipartite graphs with certain list assignments.

Lemma 3.1. *Let H be a cobipartite graph, where $V(H)$ is partitioned into two cliques X and Y. Assume that $|X| \leq |Y|$ and that there are $|X|$ non-edges between X and Y and they form a matching in $\overline{H}$. Let L be a list assignment on $V(H)$ such that $|L(x)| \geq |X|$ for all $x \in X$ and $|L(y)| \geq |Y|$ for all $y \in Y$. Then H is L-colorable.*

Proof. Let $X = \{x_1, \ldots, x_p\}$, and let $y_1, \ldots, y_p$ be vertices of Y such that $\{x_1, y_1\}$, $\ldots$, $\{x_p, y_p\}$ are the non-edges of H. The hypothesis implies that $y_1, \ldots, y_p$ are pairwise distinct. Since a clique in H can contain at most one of x_i, y_i for each $i = 1, \ldots, p$, we have $\omega(H) = |Y|$.

We proceed by induction on $|X|$. If $|X| = 0$, then H is a clique with $|L(v)| = |V(H)|$ for all $v \in V(H)$; so H is L-colorable by Hall's theorem. Now suppose that $|X| > 0$. If the family $\{L(v) \mid v \in V(H)\}$ admits a system of distinct representatives, then this is an L-coloring. So suppose the contrary. By Hall's theorem there is a set $T \subseteq V(H)$ such that $|L(T)| < |T|$. Then $|T| > |X|$, so T contains a vertex y from Y, and so $|T| > |L(y)| \geq |Y|$. Since $\omega(H) = |Y|$, it follows that T is not a clique. So T contains non-adjacent vertices x, y with $x \in X$ and $y \in Y$. We have $|L(x) \cup L(y)| \leq |L(T)| < |T| \leq |X| + |Y|$, which implies $L(x) \cap L(y) \neq \emptyset$. Pick a color $c \in L(x) \cap L(y)$. Set $L'(w) = L(w) \setminus \{c\}$ for all $w \in V(H) \setminus \{x, y\}$. Let $X' = X \setminus \{x\}$, $Y' = Y \setminus \{y\}$, and $H' = H \setminus \{x, y\}$. Clearly every vertex $x' \in X'$ satisfies $|L'(x')| \geq |X'|$ and every vertex $y' \in Y'$ satisfies $|L'(y')| \geq |Y'|$, and $|X'| \leq |Y'|$, and there are $|X'|$ non-edges between X' and Y', and they form a matching in $\overline{H'}$. By the induction hypothesis, H' admits an L'-coloring. We can extend it to an L-coloring of H by assigning the color c to x and y. $\square$

Lemma 3.2. *Let H be a cobipartite graph, where $V(H)$ is partitioned into two cliques $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$, and $E(\overline{H}) = \{x_2y_2\}$. Let L be a list assignment on $V(H)$ such that $|L(u)| \geq 2$ for all $u \in V(H)$. Then H is L-colorable if and only if every clique Q of H satisfies $|L(Q)| \geq |Q|$.*

Proof. This is a corollary of Claim 1 in [6]. For completeness, we restate the claim here: *The graph H is not L-colorable if and only if for some $v \in \{x_2, y_2\}$ we have $L(x_1) = L(y_1) = L(v)$ and these three lists are of size two.*

Clearly, if H is L-colorable, then every clique Q of H satisfies $|L(Q)| \geq |Q|$. Conversely, if every clique Q of H satisfies $|L(Q)| \geq |Q|$, then by the above claim, applied to the cliques $\{x_1, y_1, x_2\}$ and $\{x_1, y_1, y_2\}$, we obtain that H is L-colorable. $\square$

Lemma 3.3. *Let H be a cobipartite graph, where $V(H)$ is partitioned into two cliques $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$, and $E(\overline{H}) = \{x_3y_2\}$. Let L be a list assignment on $V(H)$ such that $|L(x)| \geq 3$ for all $x \in X$ and $|L(y)| \geq 2$ for all $y \in Y$. Then H is L-colorable if and only if every clique Q of H satisfies $|L(Q)| \geq |Q|$.*

Proof. If H is L-colorable then clearly every clique Q of H satisfies $|L(Q)| \geq |Q|$. Now let us prove the converse.

First suppose that $L(y_2) \subseteq L(x_3)$. Since $H \setminus \{x_3\}$ is a clique, every subset T of $V(H) \setminus \{x_3\}$ satisfies $|L(T)| \geq |T|$, and so, by Hall's theorem there is an L-coloring of $H \setminus \{x_3\}$. Then we can extend any such coloring by assigning to x_3 the color assigned to y_2.

Now assume that $L(y_2) \not\subseteq L(x_3)$. This implies $|L(x_3) \cup L(y_2)| \geq 4$. Suppose that the family $\{L(x) \mid x \in V(H)\}$ does not have a system of distinct representatives. By Hall's theorem there is a set $T \subseteq V(H)$ such that $|L(T)| < |T|$. By the assumption, T is not a clique, so it contains x_3 and y_2. It follows that $|L(T)| \geq 4$. Hence $|T| = 5$, so $T = V(H)$, and $|L(T)| = 4$, and we may assume that $L(x_3) = \{1, 2, 3\}$ and $L(y_2) = \{3, 4\}$ and $L(T) = \{1, 2, 3, 4\}$. Assign color 3 to x_3 and y_2. Now assign a color c from $L(y_1) \setminus \{3\}$ to y_1 (there may be two choices for c). We may assume that this coloring fails to be extended to $\{x_1, x_2\}$; so it must be that $L(x_1) \setminus \{3, c\}$ and $L(x_2) \setminus \{3, c\}$ are equal and of size 1; so $L(x_1) = L(x_2) = \{b, c, 3\}$ for some $b \neq c$, with $b \in \{1, 2, 4\}$. Suppose that $3 \notin L(y_1)$. Then there is a second choice for c, and we may assume that this attempt fails similarly. Hence $L(y_1) = \{b, c\}$, with $b, c \in \{1, 2, 4\}$. If $\{b, c\} = \{1, 2\}$, then the clique $Q_1 = \{x_1, x_2, x_3, y_1\}$ violates the assumption because $L(Q_1) = \{1, 2, 3\}$. If $\{b, c\} = \{1, 4\}$ or $\{2, 4\}$, then the clique $Q_2 = \{x_1, x_2, y_1, y_2\}$ violates the assumption because $L(Q_2) = \{b, c, 3\}$. So we may assume that $3 \in L(y_1)$, i.e., $L(y_1) = \{c, 3\}$. If $c = 4$, then Q_2 violates the assumption because $L(Q_2) = \{b, 3, 4\}$. So, up to symmetry, $c = 1$. If $b = 2$, then Q_1 violates the assumption because $L(Q_1) = \{1, 2, 3\}$. If $b = 4$, then Q_2 violates the assumption because $L(Q_2) = \{1, 3, 4\}$. Hence the family $\{L(x) \mid x \in V(H)\}$ admits a system of distinct representatives, which is an L-coloring of G. $\square$

Lemma 3.4. *Let H be a cobipartite graph, where $V(H)$ is partitioned into two cliques $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$, and $E(\overline{H}) = \{x_2y_2, x_3y_3\}$. Let L be a list assignment on $V(H)$ such that $|L(x)| \geq 3$ for all $x \in V(H)$. Then H is L-colorable if and only if every clique Q of H satisfies $|L(Q)| \geq |Q|$. In particular, if $|L(x_1) \cup L(y_1)| \geq 4$, then H is L-colorable.*

Proof. If H is L-colorable then clearly every clique Q of H satisfies $|L(Q)| \geq |Q|$. Now let us prove the converse. We first claim that:

$$\text{We may assume that } |L(x_i) \cap L(y_i)| \leq 1 \text{ for each } i \in \{2, 3\}. \tag{1}$$

Suppose on the contrary, and up to symmetry, that $|L(x_2) \cap L(y_2)| \geq 2$. Let $H' = H \setminus \{x_2\}$, and set $L'(y_2) = L(x_2) \cap L(y_2)$ and $L'(u) = L(u)$ for all $u \in \{x_1, x_3, y_1, y_3\}$. Thus H' and L' satisfy the hypothesis of Lemma 3.3. If every clique Q in H' satifies $|L'(Q)| \geq |Q|$, then Lemma 3.3 implies that H' admits an L'-coloring, and we can extend it to an L-coloring of H by giving to x_2 the color assigned to y_2. Hence assume that some clique Q in H' satisfies $|L'(Q)| < |Q|$. We have $|L'(Q)| \geq 2$, so $|Q| \geq 3$, so $3 \leq |L'(Q)| < |Q| \leq 4$, and so $|L'(Q)| = 3$ and $|Q| = 4$. Since x_3 and y_3 play symmetric roles here, we may assume up to symmetry that $Q = \{x_1, y_1, y_2, y_3\}$, and $L'(Q) = \{a, b, c\}$, where a, b, c are three distinct colors. Hence $L(x_1) = L(y_1) = L(y_3) = \{a, b, c\}$. Since $|L(Q)| \geq 4$, there is a color $d \in L(y_2) \setminus \{a, b, c\}$. Since $|L(\{x_1, y_1, x_2, y_3\})| \geq 4$, there is a color $e \in L(y_2) \setminus \{a, b, c\}$. If $a \in L(x_3)$, then we can assign color a to x_3 and y_3, colors b and c to x_1 and y_1, color e to x_2 and color d to y_2. So assume that $a \notin L(x_3)$, and similarly that $b, c \notin L(x_3)$. Then we can assign colors a, b, c to x_1, y_1, y_3, color e to x_2, color d to y_2, and a color from $L(x_3) \setminus \{d, e\}$ to x_3. Thus (1) holds.

It follows from (1) that $|L(x_i) \cup L(y_i)| \geq 5$ for $i = 2, 3$. If the family $\{L(x) \mid x \in V(H)\}$ admits a system of distinct representatives, then this is an L-coloring. So suppose the contrary. By Hall's theorem there is a set $T \subseteq V(H)$ such that $|L(T)| < |T|$. By the assumption, T is not a clique, so it contains x_i and y_i for some $i \in \{2, 3\}$. By (1) we have $|L(T)| \geq 5$, so $|T| \geq 6$, hence $T = V(H)$, and $|L(T)| = 5$, and consequently $|L(x_i)| = |L(y_i)| = 3$ and $|L(x_i) \cap L(y_i)| = 1$ for each $i = 2, 3$. Let $L(x_i) \cap L(y_i) = \{c_i\}$ for $i = 2, 3$.

Suppose that $c_2 \neq c_3$. We assign color c_i to x_i and y_i for each $i = 2, 3$. If this coloring can be extended to $\{x_1, y_1\}$ we are done. So suppose the contrary. Then it must be that $L(x_1) = L(y_1) = \{b, c_2, c_3\}$ for some color $b \in L(H) \setminus \{c_2, c_3\}$. Then we can color H as follows. Assign colors c_2 and c_3 to x_1 and y_1. There are four ways to color x_2 and y_2 with one color from $L(x_2) \setminus \{c_2\}$ for x_2 and one color from $L(y_2) \setminus \{c_2\}$ for y_2; at most two of them use a pair of colors equal to $L(x_3) \setminus \{c_3\}$ or $L(y_3) \setminus \{c_3\}$, so we can choose another way, and there will remain a color for x_3 and a color for y_3.

Now suppose that $c_2 = c_3$; call this color c. Let $L'(v) = L(v) \setminus \{c\}$ for all $v \in V(H) \setminus \{x_3, y_3\}$. We may assume that the graph $H \setminus \{x_3, y_3\}$ does not admit an L'-coloring, for otherwise such a coloring can be extended to H by assigning color c to x_3 and y_3. Hence, by Lemma 3.2 there is a clique Q of size 3 in $H \setminus \{x_3, y_3\}$ such that $|L'(Q)| = 2$, say $L'(Q) = \{a, b\}$. So $L(u) = \{a, b, c\}$ for all $u \in Q$. Moreover Q consists of x_1, y_1 and one of x_2, y_2. We assign color a to x_1, color b to y_1, and color c to x_2 and y_2. Since $|L(Q \cup \{x_3\})| \geq 4$, there is a color $d \in L(x_3) \setminus \{a, b, c\}$, and similarly there is a color $e \in L(y_3) \setminus \{a, b, c\}$. We assign d to x_3 and e to y_3, and we obtain an L-coloring of H.

Finally we prove the last sentence of the lemma. Since x_1 and y_1 are in all cliques of size 4, the assumption that $|L(x_1) \cup L(y_1)| \geq 4$ implies that every clique Q of H satisfies $|L(Q)| \geq |Q|$. So H is L-colorable. $\square$

Lemma 3.5. *Let H be a cobipartite graph with $\omega(H) \leq 4$. Let x, y be two adjacent vertices in H such that $N(x) \setminus \{y\}$ and $N(y) \setminus \{x\}$ are cliques and $V(H) = N(x) \cup N(y)$. Let L be a list assignment such that $|L(x)| \geq 2$, $|L(y)| \geq 2$, and $|L(v)| \geq 4$ for all $v \in V(H) \setminus \{x, y\}$. Then H is L-colorable.*

Proof. Let $X = N(x) \setminus \{y\}$ and $Y = N(y) \setminus \{x\}$. Let $I = X \cap Y$. Since $\{x, y\} \cup I$ is a clique, we have $|I| \leq 2$.

First suppose that $|I| = 2$. Let $I = \{w, w'\}$. Since $\{x\} \cup X$ is a clique that contains I, we have $|X \setminus I| \leq 1$. Likewise $|Y \setminus I| \leq 1$. We may assume that we are in the situation where $X \setminus I$ and $Y \setminus I$ are non-empty and complete to each other, because any other situation can be reduced to that one by adding vertices or edges (which makes the coloring problem only harder). Let $X \setminus I = \{u\}$ and $Y \setminus I = \{v\}$. Suppose that $L(x) \cap L(v) \neq \emptyset$. Pick a color $a \in L(x) \cap L(v)$, assign it to x and v, and remove it from the lists of all other vertices. Pick a color b from $L(y) \setminus \{a\}$, assign it to y and remove it from the list of the vertices in I. Let L' be the reduced list assignment. Then $|L'(w)| \geq 2$, $|L'(w')| \geq 2$, and $|L'(u)| \geq 3$, so we can L'-color greedily w, w', u in this order. Hence assume that $L(x) \cap L(v) = \emptyset$, and similarly that $L(y) \cap L(u) = \emptyset$. Then $|L(x) \cup L(v)| \geq 6$ and $|L(y) \cup L(u)| \geq 6$. It follows that the family $\{L(z) \mid z \in V(H)\}$ satisfies Hall's condition, so H is L-colorable.

Now suppose that $|I| = 1$. Let $I = \{w\}$. Then $|X \setminus \{w\}| \leq 2$ and $|Y \setminus \{w\}| \leq 2$. We may assume that we are in the situation where $X \setminus I$ and $Y \setminus I$ have size 2 and there are three edges between them, because any other situation can be reduced to that one by adding vertices or edges. Let $X \setminus I = \{u, v\}$ and $Y \setminus I = \{s, t\}$, and let $us, ut, vs \in E(H)$ and $vt \notin E(H)$. Suppose that $L(x) \cap L(s) \neq \emptyset$. We pick a color $a \in L(x) \cap L(s)$, assign it to x and s, and remove it from the lists of all other vertices. Then it is easy to see that we can color y, t, w, u, v in this order, using colors from the reduced lists. Hence assume that $L(x) \cap L(s) = \emptyset$, and similarly that $L(y) \cap L(u) = \emptyset$. So $|L(x) \cup L(s)| \geq 6$ and $|L(y) \cup L(u)| \geq 6$.
Suppose that $L(x) \cap L(t) \neq \emptyset$. We pick a color $a \in L(x) \cap L(t)$, assign it to x and t, and remove it from the lists of all other vertices. Since $L(x) \cap L(s) = \emptyset$, the list $L(s)$ loses no color ($a \notin L(s)$). If $L(y) \setminus \{a\}$ and $L(v) \setminus \{a\}$ have a common element b, we assign it to y and v, and it is easy to see that w, u, s can be colored in this order with the reduced lists. On the other hand if $L(y) \setminus \{a\}$ and $L(v) \setminus \{a\}$ are disjoint, then it is easy to see that the family $\{L(z) \setminus \{a\} \mid z \in V(H) \setminus \{x, t\}\}$ satisfies Hall's condition, so H is L-colorable. Hence assume that $L(x) \cap L(t) = \emptyset$, and similarly that $L(y) \cap L(v) = \emptyset$. So $|L(x) \cup L(t)| \geq 6$ and $|L(y) \cup L(v)| \geq 6$.
Suppose that $L(t) \cap L(v) \neq \emptyset$. Pick a color $a \in L(t) \cap L(v)$ and assign it to t and v. Since $L(y) \cap L(v) = \emptyset$ and $L(x) \cap L(t) = \emptyset$ we have $L(y) = L(y) \setminus \{a\}$ and similarly $L(x) = L(x) \setminus \{a\}$. It follows that the family $\{L(z) \setminus \{a\} \mid z \in V(H) \setminus \{t, v\}\}$ satisfies Hall's condition. Finally assume that $L(t) \cap L(v) = \emptyset$. So $|L(t) \cup L(v)| \geq 8$. Then the family $\{L(z) \mid z \in V(H)\}$ satisfies Hall's condition, so H is L-colorable.

Finally suppose that $I = \emptyset$. We may assume that X and Y have size 3 and that the non-edges between them form a matching of size 2, because any other situation can be reduced to that one by adding vertices or edges. Let $X = \{u_1, u_2, u_3\}$, $Y = \{v_1, v_2, v_3\}$, and $E(\overline{H}) = \{u_2v_2, u_3v_3\}$. We can choose a color a from $L(x)$ and a color b from $L(y)$ such that $L(u_1) \setminus \{a\} \neq L(v_1) \setminus \{b\}$. Let $L'(u) = L(u) \setminus \{a\}$ for all $u \in X$ and $L'(v) = L(v) \setminus \{b\}$ for all $v \in Y$. By the last sentence of Lemma 3.4, $H \setminus \{x, y\}$ admits an L'-coloring, and we can extend it to an L-coloring of H by assigning color a to x and color b to y. $\square$

Lemma 3.6. *Let H be a cobipartite graph, where $V(H)$ is partitioned into two cliques $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$, and $E(\overline{H}) = \{x_1y_1, x_2y_2, x_3y_3, x_3y_1, x_1y_2\}$. Let L be a list assignment on $V(H)$ such that $|L(x_3)| = 2$, $|L(y_2)| = 2$, and $|L(w)| = 3$ for every $w \in V(H) \setminus \{x_3, y_2\}$. Then H is L-colorable.*

Proof. Suppose that $L(x_2) \cap L(y_2) \neq \emptyset$. Assign a color a from $L(x_2) \cap L(y_2)$

to x_2 and y_2. Let $L'(u) = L(u) \setminus \{a\}$ for all $u \in \{x_1, x_3, y_1, y_3\}$. Then we can L'-color x_3, x_1, y_3, y_1 greedily in this order, because x_3-x_1-y_3-y_1 is an induced path and the reduced lists' size pattern is $(\geq 1, \geq 2, \geq 2, \geq 2)$. The proof is similar when $L(x_3) \cap L(y_3) \neq \emptyset$. So we may assume that:

$$L(x_2) \cap L(y_2) = \emptyset \text{ and } L(x_3) \cup L(y_3) = \emptyset. \tag{1}$$

Suppose that $L(x_1) \cap L(y_2) \neq \emptyset$. Assign a color a from $L(x_1) \cap L(y_2)$ to x_1 and y_2. Let $L'(u) = L(u) \setminus \{a\}$ for all $u \in \{x_2, x_3, y_1, y_3\}$. By (1), we have $a \notin L(x_2)$, so $L'(x_2) = L(x_2)$, and a is in at most one of $L(x_3)$ and $L(y_3)$. If $a \in L(x_3)$, then we can L'-color greedily x_3, x_2, y_1, y_3 in this order. If $a \in L(y_3)$, then we can L'-color greedily y_3, y_1, x_2, x_3 in this order. The proof is similar when $L(x_3) \cap L(y_1) \neq \emptyset$. So we may assume that:

$$L(x_1) \cap L(y_2) = \emptyset \text{ and } L(x_3) \cap L(y_1) = \emptyset. \tag{2}$$

Suppose that $L(x_1) \cap L(y_1) \neq \emptyset$. Assign a color a from $L(x_1) \cap L(y_1)$ to x_1 and y_1. Let $L'(u) = L(u) \setminus \{a\}$ for all $u \in \{x_2, x_3, y_2, y_3\}$. By (2), we have $a \notin L(x_3)$ and $a \notin L(y_2)$. The graph $H \setminus \{x_1, y_1\}$ is an even cycle, and $|L'(u)| \geq 2$ for every vertex u in that graph, so it is L'-colorable. So we may assume that:

$$L(x_1) \cap L(y_1) = \emptyset. \tag{3}$$

By (1), (2) and (3), we have $|L(u) \cup L(v)| = 5$ whenever $\{u, v\}$ is any of $\{x_2, y_2\}$, $\{x_3, y_3\}$, $\{x_1, y_2\}$, $\{x_3, y_1\}$, and $|L(x_1) \cap L(y_1)| = 6$. It follows that the family $\{L(w) \mid w \in V(H)\}$ admits a system of distinct representatives, which is an L-coloring for H. $\square$

Lemma 3.7. *Let H be a cobipartite graph with $\omega(G) \leq 4$. Let $V(H)$ be partitioned into two cliques X, Y with $X = \{x_1, x_2, x_3\}$, such that x_1 is complete to Y. Let L be a list assignment such that $|L(x_1)| \geq 3$, $|L(x_2)| \geq 2$, $|L(x_3)| \geq 2$, and $|L(y)| \geq 4$ for all $y \in Y$. Then H is L-colorable.*

Proof. Since $Y \cup \{x_1\}$ is a clique, we have $|Y| \leq 3$. If $|Y| \leq 2$, then Lemma 3.3 implies that H is L-colorable. So we may assume that $|Y| = 3$, say $Y = \{y_1, y_2, y_3\}$, and we may assume that $E(\overline{H}) = \{x_2y_2, x_3y_3\}$. If the family $\{L(w) \mid w \in V(H)\}$ admits a system of distinct representatives, then this is an L-coloring of H, so assume the contrary. So there is a set $T \subseteq V(H)$ such that $|L(T)| < |T|$. We have $|L(T)| \geq 2$, so $|T| \geq 3$, so $|L(T)| \geq 3$, so $|T| \geq 4$, so $T \cap Y \neq \emptyset$, so $|L(T)| \geq 4$, and so $|T| \geq 5$. It follows that T is not a clique. Hence assume that $x_2, y_2 \in T$. If $L(x_2) \cap L(y_2) = \emptyset$, then $|L(T)| \geq |L(x_2) \cup L(y_2)| = 6$, so $|T| \geq 7$, which is impossible. Hence $L(x_2) \cap L(y_2) \neq \emptyset$. Assign a color c_2 from $L(x_2) \cap L(y_2)$ to x_2 and y_2. Define $L'(u) = L(u) \setminus \{c_2\}$ for all $u \in V(H) \setminus \{x_2, y_2\}$. If $L'(x_3) \cap L'(y_3) \neq \emptyset$ assign a color c_3 from $L'(x_3) \cap L'(y_3)$ to x_3 and y_3. Then we have $|(L'(x_1) \cup L'(y_1)) \setminus \{c_2\}| \geq 2$, so we can extend the coloring to $\{x_1, y_1\}$. On the other hand, if $L'(x_3) \cap L'(y_3) = \emptyset$, the family $\{L'(w) \mid w \in V(H) \setminus \{x_2, y_2\}\}$ admits a system of distinct representatives. So H admist an L-coloring. $\square$

Lemma 3.8. *Let H be a cobipartite graph, where $V(H)$ is partitioned into two cliques $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$, and $E(\overline{H}) = \{x_1y_1, x_1y_3, x_1y_4, x_2y_2, x_2y_3, x_2y_4, x_3y_3, x_4y_4\}$. Let L be a list assignment on $V(H)$ such that $|L(x_1)| = 2$, $|L(x_2)| = 2$ and $|L(w)| = 4$ for all $w \in V(H) \setminus \{x_1, x_2\}$. Then H is L-colorable.*

Proof. We choose colors c_1, c_2 with $c_1 \in L(x_1)$, $c_2 \in L(x_2)$ and $c_1 \neq c_2$, such that if $|L(y_1) \cap L(y_2)| = 3$, then either $\{c_1\} \neq L(y_2) \setminus L(y_1)$ or $\{c_2\} \neq L(y_1) \setminus L(y_2)$. This is possible as follows: if $|L(y_1) \cap L(y_2)| = 3$, let α be the color in $L(y_1) \setminus L(y_2)$, then choose $c_2 \in L(x_2) \setminus \{\alpha\}$ and $c_1 \in L(x_1) \setminus \{c_2\}$. We assign color c_1 to x_1 and c_2 to x_2. Let $L'(y_1) = L(y_1) \setminus \{c_2\}$, $L'(y_2) = L(y_2) \setminus \{c_1\}$, $L'(x_3) = L(x_3) \setminus \{c_1, c_2\}$, $L'(x_4) = L(x_4) \setminus \{c_1, c_2\}$, $L'(y_3) = L(y_3)$ and $L'(y_4) = L(y_4)$. So $|L'(u)| \geq 2$ for $u \in \{x_3, x_4\}$, $|L'(v)| \geq 3$ for $v \in \{y_1, y_2\}$, and $|L'(w)| = 4$ for $w \in \{y_3, y_4\}$. Note that the choice of c_1 and c_2 implies that $|L'(y_1) \cup L'(y_2)| \geq 4$. Now we show that $H \setminus \{x_1, x_2\}$ is L'-colorable.

Suppose that $L'(x_3) \cap L'(y_3) \neq \emptyset$. Assign a color c_3 from $L'(x_3) \cap L'(y_3)$ to x_3 and y_3. Define $L''(u) = L'(u) \setminus \{c_3\}$ for all $u \in \{x_4, y_1, y_2, y_4\}$. Note that $|L''(x_4)| \geq 1$, $|L''(u)| \geq 2$ for $u \in \{y_1, y_2\}$, and $|L''(y_4)| \geq 3$. Assign a color c_4 from $L''(x_4)$ to x_4. Since $|L'(y_1) \cup L'(y_2)| \geq 4$, it follows that $|(L''(y_1) \cup L''(y_2)) \setminus \{c_4\}| \geq 2$. So we can L''-color greedily $\{y_1, y_2\}$ and then y_4. The proof is similar if $L'(x_4) \cap L'(y_4) \neq \emptyset$. Therefore we may assume that $L'(x_3) \cap L'(y_3) = \emptyset$ and $L'(x_4) \cap L'(y_4) = \emptyset$, and so $|L'(x_3) \cup L'(y_3)| = 6$ and $|L'(x_4) \cup L'(y_4)| = 6$. This and the choice of c_1, c_2 implies that the family $\{L'(w) \mid w \in V(H) \setminus \{x_1, x_2\}\}$ admits a system of distinct representatives. □

Lemma 3.9. *Let H be a cobipartite graph with $\omega(G) \leq 4$. Let C be a clique of size 3 in H such that for every $w \in C$, the set $N(w) \setminus C$ is a clique. Let L be a list assignment such that $|L(w)| = 3$ for all $w \in C$ and $|L(v)| = 4$ for all $v \in V(H) \setminus C$. Then H is L-colorable.*

Proof. If H is not connected, it has two components H_1, H_2 and both are cliques of size at most 4. The hypothesis implies easily that for each $i \in \{1, 2\}$ the family $\{L(u) \mid u \in V(H_i)\}$ satisfies Hall's theorem, and consequently H is L-colorable. Hence we assume that H is connected. Let $n = |V(H)|$ and $V(H) = \{v_1, \ldots, v_n\}$. The hypothesis implies that $n \leq 8$. Let $\mu = n - 4$. Since $\omega(H) = 4$, Kőnig's theorem implies that $\overline{H}$ has a matching of size μ. We may assume that the pairs $\{v_i, v_{i+\mu}\}$ $(i = 1, \ldots, \mu)$ form such a matching. We may also assume that $E(H)$ is maximal under the hypothesis of the lemma, since adding edges can only make the problem harder.

First suppose that $n = 4$. The hypothesis implies that the family $\{L(u) \mid u \in V(H)\}$ satisfies Hall's theorem, and consequently H is L-colorable.

Now suppose that $n = 5$. So $\mu = 1$ and $v_1 v_2 \in E(\overline{H})$. Up to symmetry, we have either $C = \{v_3, v_4, v_5\}$ or $C = \{v_1, v_3, v_4\}$. If $C = \{v_3, v_4, v_5\}$, then we can L-color greedily the vertices v_3, v_4, v_5, v_1, v_2 in this order. If $C = \{v_1, v_3, v_4\}$, then we can L-color greedily the vertices v_1, v_3, v_4, v_5, v_2 in this order.

Now suppose that $n = 6$. So $\mu = 2$ and $\{v_1 v_3, v_2 v_4\} \subseteq E(\overline{H})$. Up to symmetry, we have either $C = \{v_1, v_5, v_6\}$ or $C = \{v_1, v_2, v_5\}$. Suppose that $C = \{v_1, v_5, v_6\}$. Since $\{v_1, v_2, v_4\}$ is not a stable set of size 3 and $N(v_1) \setminus C$ is a clique, v_1 is adjacent to exactly one of v_2, v_4, say to v_4 and not to v_2. Then we can L-color greedily the vertices $v_1, v_5, v_6, v_4, v_3, v_2$ in this order. Suppose that $C = \{v_1, v_2, v_5\}$. By the maximality of $E(H)$ we may assume that $E(\overline{H}) = \{v_1 v_2,\ v_3 v_4\}$. Then Lemma 3.4 (with $X = C$, $Y = V(H) \setminus C$, $x_1 = v_5$ and $y_1 = v_6$) implies that H is L-colorable.

Now suppose that $n = 7$. So $\mu = 3$, and $\{v_1v_4, v_2v_5, v_3v_6\} \subseteq E(\overline{H})$. Up to symmetry, we have either $C = \{v_1, v_2, v_3\}$ or $C = \{v_1, v_2, v_7\}$. If $C = \{v_1, v_2, v_3\}$, then, by the maximality of $E(H)$ we may assume that $E(\overline{H}) = \{v_1v_4,\ v_2v_5,\ v_3v_6\}$, and by Lemma 3.1 (with $X = C$ and $Y = V(H) \setminus C$), H is L-colorable. So suppose that $C = \{v_1, v_2, v_7\}$. For each $i \in \{1, 2\}$, v_i has exactly one neighbor in $\{v_3, v_6\}$, for otherwise either $\{v_i, v_3, v_6\}$ is a stable set of size 3 or $N(v_i) \setminus C$ is not a clique. This leads to the following two cases (a) and (b):

(a) v_1 and v_2 have the same neighbor in $\{v_3, v_6\}$. We may assume that $v_1v_3,\ v_2v_3 \in E(H)$ and $v_1v_6, v_2v_6 \notin E(H)$. Since H is cobipartite, $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$ are cliques, and by the maximality of $E(H)$ we may assume that $\{v_1v_5,\ v_2v_4,\ v_3v_4,\ v_3v_5\} \subseteq E(H)$ and that v_7 is complete to $\{v_1, \ldots, v_6\}$. Pick a color c from $L(v_7)$, assign it to v_7, and set $L'(u) = L(u) \setminus \{c\}$ for all $u \in V(H) \setminus \{v_7\}$. By Lemma 3.1 (with $X = \{v_1, v_2\}$ and $Y = \{v_3, v_4, v_5\}$), $H \setminus \{v_6, v_7\}$ admits an L'-coloring. This can be extended to v_6 since v_6 has only two neighbors in $H \setminus \{v_7\}$. So H is L-colorable.

(b) v_1 and v_2 do not have the same neighbor in $\{v_3, v_6\}$. We may assume that $v_1v_3, v_2v_6 \in E(H)$ and $v_1v_6, v_2v_3 \notin E(H)$. Since H is cobipartite, $\{v_1, v_3, v_5\}$ and $\{v_2, v_4, v_6\}$ are cliques, and by the maximality of $E(H)$ we may assume that $v_4v_5, v_5v_6 \in E(H)$ and that v_7 is complete to $\{v_1, \ldots, v_6\}$. Pick a color c from $L(v_7)$, assign it to v_7, and set $L'(u) = L(u) \setminus \{c\}$ for all $u \in V(H) \setminus \{v_7\}$. By Lemma 3.6, $H \setminus \{v_7\}$ is L'-colorable. So H is L-colorable.

Now suppose that $n = 8$. So $\mu = 4$ and $\{v_1v_5, v_2v_6, v_3v_7, v_4v_8\} \subseteq E(\overline{H})$. Up to symmetry we have $C = \{v_1, v_2, v_3\}$. For each $i \in \{1, 2, 3\}$, v_i has exactly one neighbor in $\{v_4, v_8\}$, for otherwise either $\{v_i, v_4, v_8\}$ is a stable set of size 3 or $N(v_i) \setminus C$ is not a clique. This leads to two cases: (a) v_1, v_2, v_3 have the same neighbor in $\{v_4, v_8\}$; (b) only two of v_1, v_2, v_3 have a common neighbor in $\{v_4, v_8\}$.

Suppose that (a) holds. We may assume that v_1, v_2, v_3 are all adjacent to v_4 and not adjacent to v_8. Since H is cobipartite, $\{v_1, \ldots, v_4\}$ and $\{v_5, \ldots, v_8\}$ are cliques, and by the maximality of $E(H)$ we may assume that $E(\overline{H}) = \{v_1v_5, v_2v_6, v_3v_7, v_4v_8, v_1v_8, v_2v_8, v_3v_8\}$. By Lemma 3.1 (with $X = \{v_1, v_2, v_3\}$ and $Y = \{v_4, v_5, v_6, v_7\}$), $H \setminus \{v_8\}$ admits an L'-coloring. This can be extended to v_8 since v_8 has only three neighbors in H. So H is L-colorable.

Therefore we may assume that (b) holds. We may assume that $v_1v_4,\ v_2v_4,\ v_3v_8 \in E(H)$ and $v_1v_8, v_2v_8, v_3v_4 \notin E(H)$. Since H is cobipartite, $\{v_1, v_2, v_4, v_7\}$ and $\{v_3, v_5, v_6, v_8\}$ are cliques, and by the maximality of $E(H)$ we may assume that $E(\overline{H}) = \{v_1v_5, v_2v_6, v_3v_7, v_4v_8, v_1v_8, v_2v_8, v_3v_4\}$.

Suppose that $L(v_3) \cap L(v_7) \neq \emptyset$. Assign a color c from $L(v_3) \cap L(v_7)$ to v_3 and v_7. Define $L'(w) = L(w) \setminus \{c\}$ for every $w \in V(H) \setminus \{v_3, v_7\}$. By Lemma 3.3, $H \setminus \{v_3, v_7, v_8\}$ admits an L'-coloring. This can be extended to v_8 since v_8 has only two neighbors in $H \setminus \{v_3, v_7\}$. So we may assume that:

$$L(v_3) \cap L(v_7) = \emptyset. \tag{1}$$

Suppose that $L(v_1) \cap L(v_5) \neq \emptyset$. Assign a color c from $L(v_1) \cap L(v_5)$ to v_1 and v_5. Define $L'(w) = L(w) \setminus \{c\}$ for every $w \in V(H) \setminus \{v_1, v_5\}$. By Lemma 3.6 the graph $H \setminus \{v_1, v_5\}$ is L'-colorable. The proof is similar if $L(v_2) \cap L(v_6) \neq \emptyset$. So we may assume that:

$$L(v_1) \cap L(v_5) = \emptyset \text{ and } L(v_2) \cup L(v_6) = \emptyset. \tag{2}$$

Suppose that $L(v_3) \cap L(v_4) \neq \emptyset$. Assign a color c from $L(v_3) \cap L(v_4)$ to v_3 and v_4. Define $L'(w) = L(w) \setminus \{c\}$ for every $w \in V(H) \setminus \{v_3, v_4\}$. By (1), we have $c \notin L(v_7)$, so $L'(v_7) = L(v_7)$. Hence and by (1) and (2), the family $\{L'(w) \mid w \in V(H) \setminus \{v_3, v_4\}\}$ admits a system of distinct representatives. So we may assume that:

$$L(v_3) \cup L(v_4) = \emptyset. \tag{3}$$

Suppose that $L(v_4) \cap L(v_8) \neq \emptyset$. Assign a color c from $L(v_4) \cap L(v_8)$ to v_4 and v_8. Define $L'(w) = L(w) \setminus \{c\}$ for every $w \in V(H) \setminus \{v_4, v_8\}$. By (3), we have $c \notin L(v_3)$, so $L'(v_3) = L(v_3)$. By (1), (2) and (3), the family $\{L'(w) \mid w \in V(H) \setminus \{v_4, v_8\}\}$ admits a system of distinct representatives. So we may assume that:

$$L(v_4) \cup L(v_8) = \emptyset. \tag{4}$$

By (1), (2), (3) and (4), we have $|L(v_i) \cup L(v_j)| = 7$ if the pair $\{i, j\}$ is any of $\{1, 5\}$, $\{2, 6\}$, $\{3, 7\}$ and $\{3, 4\}$, and $|L(v_4) \cup L(v_8)| = 8$. It follows easily that the family $\{L(w) \mid w \in V(H)\}$ admits a system of distinct representatives. $\square$

4 Elementary graphs

Now we can consider the case of any elementary graph G with $\omega(G) \leq 4$.

Theorem 4.1. *Let G be an elementary graph with $\omega(G) \leq 4$. Then $ch(G) = \chi(G)$.*

Proof. This theorem holds for every graph G with $\omega(G) \leq 3$ as proved in [8]. Hence we will assume that $\omega(G) = 4$. By Theorem 1.8, G is the augmentation of the line-graph $\mathcal{L}(H)$ of a bipartite multigraph H. Let $e_1, \dots, e_h$ be the flat edges of $\mathcal{L}(H)$ that are augmented to obtain G. We prove the theorem by induction on h. If $h = 0$, then $G = \mathcal{L}(H)$; in that case the equality $ch(G) = \chi(G)$ follows from Galvin's theorem [5]. Now assume that $h > 0$ and that the theorem holds for elementary graphs obtained by at most $h - 1$ augmentations. Let (X, Y) be the augment in G that corresponds to the edge e_h of $\mathcal{L}(H)$. In $\mathcal{L}(H)$, let $e_h = xy$. So x, y are incident edges of H. In H, let $x = q_x q_{xy}$ and $y = q_y q_{xy}$; so their common vertex q_{xy} has degree 2 in H. Let G_{h-1} be the graph obtained from $\mathcal{L}(H)$ by augmenting only the $h - 1$ other edges $e_1, \dots, e_{h-1}$. So G_{h-1} is an elementary graph.

Let L be a list assignment on $V(G)$ such that $|L(v)| = \omega(G)$ for all $v \in V(G)$. We will prove that G admits an L-coloring.

$$\text{We may assume that } |X \cup Y| > \omega(G). \tag{1}$$

Suppose that $|X \cup Y| \leq \omega(G)$. Let H' be the graph obtained from H by duplicating $|X| - 1$ times the edge x (so that there are exactly $|X|$ parallel edges between the two ends of x in H) and duplicating $|Y| - 1$ times the edge y. Let G'_{h-1} be the graph obtained from $\mathcal{L}(H')$ by augmenting the $h - 1$ edges $e_1, \dots, e_{h-1}$ as in G. Then G'_{h-1} can also be obtained from G by adding all edges between non-adjacent vertices of $X \cup Y$. By the assumption, we have $\omega(G'_{h-1}) = \omega(G)$. By the induction hypothesis, G'_{h-1} admits an L-coloring. Then this is an L-coloring of G. Hence (1) holds.

Let $X = \{x_1, \dots, x_{|X|}\}$ and $Y = \{y_1, \dots, y_{|Y|}\}$. Let $N_X = \{v \in V(G) \setminus (X \cup Y) \mid v$ has a neighbor in $X\}$ and $N_Y = \{v \in V(G) \setminus (X \cup Y) \mid v$ has a neighbor in $Y\}$. By the definition of a line-graph and of an augment, the set N_X is a clique and is complete to X; hence $|N_X| \le \omega(G) - |X|$. Likewise N_Y is a clique and is complete to Y, and $|N_Y| \le \omega(G) - |Y|$. Let μ be the size of a maximum matching in the bipartite graph $\overline{G}[X \cup Y]$. By Kőnig's theorem we have $\mu + \omega(G) = |X| + |Y|$, so $\mu = |X| + |Y| - 4$. Moreover, we may assume that the edges of $\overline{G}[X \cup Y]$ form a matching of size μ (for otherwise we can add some edges to G, in $X \cup Y$, which makes the coloring problem only harder).

The graph $G_{h-1} \setminus \{x, y\}$ is elementary, and it has $h - 1$ augments, so, by the induction hypothesis, it admits an L-coloring f. We will try to extend f to G; if this fails, we will analyse why and then show that we can find another L-coloring of $G_{h-1} \setminus \{x, y\}$ that does extend to G. Let L' be the list assignment defined on $X \cup Y$ as follows: for all $u \in X$, let $L'(u) = L(u) \setminus f(N_X)$, and for all $v \in Y$, let $L'(v) = L(v) \setminus f(N_Y)$. Clearly, f extends to an L-coloring of G if and only if $G[X \cup Y]$ admits an L'-coloring. By (1) and up to symmetry, we may assume that either $|Y| = 4$ (and $|X| \le 4$) or $(|X|, |Y|)$ is equal to $(3, 3)$ or $(2, 3)$. We deal with each case separately.

Case 1: $|Y| = 4$ and $|X| \le 4$. We have $|N_X| \le 4 - |X|$ and $|N_Y| = 0$, so $|L'(u)| \ge |X|$ for all $u \in X$ and $|L'(v)| = 4$ for all $v \in Y$. Since $\omega(G) = 4$, there are $|X|$ non-edges between X and Y that form a matching in $\overline{G}$. By Lemma 3.1, $G[X \cup Y]$ admits an L'-coloring.

Case 2: $|X| = |Y| = 3$. Here we have $\mu = 2$, and we may assume that the non-edges between X and Y are x_2y_2 and x_3y_3. We have $|N_X| \le 1$ and $|N_Y| \le 1$, so $|L'(u)| \ge 3$ for all $u \in X \cup Y$. If $G[X \cup Y]$ is L'-colorable we are done, so assume the contrary. By Lemma 3.4, there is a clique $Q \subset X \cup Y$ such that $|L'(Q)| < |Q|$. Thus $3 \le |L'(Q)| < |Q| \le 4$. This implies that $|Q| = 4$, and in particular Q contains x_1 and y_1. Moreover $|L'(Q)| = 3$, so $L'(x_1)$ and $L'(y_1)$ are equal and have size 3, so $|N_X| = 1$ and $|N_Y| = 1$. Let $N_X = \{u\}$ and $N_Y = \{v\}$. Thus there are colors a, b, c, d, d' such that $L(x_1) = \{a, b, c, d\}$, $L(y_1) = \{a, b, c, d'\}$, $f(u) = d$ and $f(v) = d'$ (possibly $d = d'$). In other words, f satisfies the following "bad" property:

> Either $L(x_1) = L(y_1)$ and $f(u) = f(v)$, or $|L(x_1) \cap L(y_1)| = 3$ and $\{f(u)\} = L(x_1) \setminus L(y_1)$ and $\{f(v)\} = L(y_1) \setminus L(x_1)$. (2)

Let G^* be the graph obtained from G by removing all edges between X and Y and adding two new vertices u^* and v^* with edges u^*v^*, u^*x_i $(i = 1, 2, 3)$ and v^*y_i $(i = 1, 2, 3)$. Let H^* be the graph obtained from H by removing the vertex q_{xy} and adding three vertices q_1, q_2, q_3, with edges q_1q_2 and q_2q_3, plus three parallel edges between q_x and q_1 and three parallel edges between q_3 and q_y. So H^* is bipartite, and it is easy to see that G^* is obtained from $\mathcal{L}(H^*)$ by augmenting $e_1, \dots, e_{h-1}$ as in G. So G^* is elementary.

We define a list assignment L^* on G^* as follows. For all $v \in V(G \setminus (X \cup Y))$, let $L^*(v) = L(v)$. For all $v \in X \cup \{u^*, v^*\}$ let $L^*(v) = \{a, b, c, d\}$, and for all $v \in Y$ let $L^*(v) = \{a, b, c, d'\}$. By the induction hypothesis on h, the graph G^* admits an L^*-coloring f^*. In particular f^* is an L-coloring of $G \setminus (X \cup Y)$. We claim that if $d = d'$ then $f^*(u) \ne f^*(v)$, and if $d \ne d'$ then either $f^*(u) \ne d$ or $f^*(v) \ne d'$. Indeed we have $f^*(X) = \{a, b, c, d\} \setminus \{f^*(u)\}$ and $f^*(Y) = \{a, b, c, d'\} \setminus \{f^*(v)\}$, so if the claim fails then $f^*(X) = f^*(Y)$ and consequently $f^*(u^*) = f^*(v^*)$, a contradiction. So the claim holds. By the claim, we can use f^* instead of f above (as an L-coloring of $G \setminus (X \cup Y)$), because f^* does not satisfy (2); so we can extend it to an L-coloring of G.

Case 3: $|X| = 3$ and $|Y| = 2$. Here we have $\mu = 1$, and we may assume that the only non-edge between X and Y is x_3y_2. We have $|N_X| \leq 1$ and $|N_Y| \leq 2$, so $|L'(u)| \geq 3$ for all $u \in X$ and $|L'(v)| \geq 2$ for all $v \in Y$. If $G[X \cup Y]$ is L'-colorable we are done, so assume the contrary. By Lemma 3.3, there is a clique $Q \subset X \cup Y$ such that $|L'(Q)| < |Q|$. This inequality implies that $Q \not\subseteq Y$, so $Q \cap X \neq \emptyset$. Thus $3 \leq |L'(Q)| < |Q| \leq 4$. This implies that $|Q| = 4$, and in particular Q contains x_1, x_2 and y_1. Moreover $|L'(Q)| = 3$, so $L'(x_1)$ and $L'(x_2)$ are equal and have size 3, so $|N_X| = 1$, and $L'(y_1)$ has size at most 3, so $|N_Y| \geq 1$, and $L'(y_1) \subseteq L'(x_1)$. Let $N_X = \{u\}$. Thus $L(x_1) = L(x_2)$, and f satisfies the following "bad" property:

$$f(u) \in L(x_1) \text{ and } L(y_1) \setminus f(N_Y) \subseteq L(x_1) \setminus \{f(u)\}. \tag{3}$$

Let $G^* = G \setminus \{x_3\}$. Clearly G^* is elementary. Let H^* be the graph obtained from H by duplicating the edge q_xq_{xy} (so that there are two parallel edges between q_x and q_{xy}) and similarly duplicating q_yq_{xy}. It is easy to see that G^* is obtained from $\mathcal{L}(H^*)$ by augmenting $e_1, \ldots, e_{h-1}$ as in G. We define a list assignment L^* on G^* as follows. For all $v \in V(G^*) \setminus \{y_2\}$, let $L^*(v) = L(v)$, and let $L^*(y_2) = L(y_1)$. By the induction hypothesis on h the graph G^* admits an L^*-coloring f^*. We claim that f^* does not satisfy the bad property (3). Indeed if it does, then $f^*(u) \in L^*(x_1)$ and $L^*(y_1) \setminus f^*(N_Y) \subseteq L^*(x_1) \setminus \{f^*(u)\}$. Since $L^*(y_2) = L^*(y_1)$, we also have $L^*(y_2) \setminus f^*(N_Y) \subseteq L^*(x_1) \setminus \{f^*(u)\}$, and this means that the four vertices x_1, x_2, y_1, y_2 (which induce a clique) are colored by f^* using colors from $L^*(x_1) \setminus \{f^*(u)\}$, which has size 3; but this is impossible. So the claim holds. By the claim, we can use f^* instead of f above (as an L-coloring of $G \setminus (X \cup Y)$) and we can extend it to an L-coloring of G. This completes the proof of the theorem. □

5 Claw-free perfect graphs

Now we can prove Theorem 1.6, which we restate here.

Theorem 5.1. *Let G be a claw-free perfect graph with $\omega(G) \leq 4$. Then $ch(G) = \chi(G)$.*

Proof. We may assume that G is connected. Let L be a list assignment on G such that $|L(v)| \geq 4$ for all $v \in V(G)$. Let us prove that G is L-colorable by induction on the number of vertices of G. If G is peculiar, then by Lemma 2.2 we know that the theorem holds. So assume that G is not peculiar. By Theorem 1.7 and Lemma 2.1, we know that G can be decomposed by clique cutsets into elementary graphs. We may assume that:

$$G \text{ has no simplicial vertex.} \tag{1}$$

Suppose that x is a simplicial vertex in G. By the induction hypothesis, $G \setminus \{x\}$ admits an L-coloring f. Since x is simplicial, it has at most three neighbors.

So f can be extended to x by choosing in $L(x)$ a color not assigned by f to its neighbors. Thus (1) holds.

By the discussion after the definition of a clique cutset (Section 1), G admits an extremal cutset C, i.e., a minimal clique cutset such that for some component A of $G \setminus C$ the induced subgraph $G[A \cup C]$ is an atom (i.e., has no clique cutset). Since C is minimal, every vertex x of C has a neighbor in every component of $G \setminus C$ (for otherwise $C \setminus \{x\}$ would be a clique cutset), and it follows that $G \setminus C$ has only two components A_1, A_2 (for otherwise x would be the center of a claw). For $i = 1, 2$ let $G_i = G[C \cup A_i]$. Hence we may assume that G_2 is elementary.

By the induction hypothesis, the graph $G[C \cup A_1]$ is 4-choosable, so it admits an L-coloring f. We will show that we can extend this coloring to G.

By Theorem 1.8, G_2 is obtained by augmenting the line-graph $\mathcal{L}(H)$ of a bipartite graph H. For each augment (X, Y) of G_2, select a pair of adjacent vertices such that one is in X and the other is in Y. Also select all vertices of G_2 that are not in any augment. It is easy to see that $\mathcal{L}(H)$ is isomorphic to the subgraph of G_2 induced by the selected vertices. Without loss it will be convenient to view $\mathcal{L}(H)$ as equal to that induced subgraph. We claim that:

If there is an augment (X, Y) in G_2 such that both $C \cap X$ and $C \cap Y$ are non-empty, then $V(G_2) = X \cup Y$. (2)

Suppose on the contrary, under the hypothesis of (2), that $V(G_2) \neq X \cup Y$. Let $Z = V(G_2) \setminus (X \cup Y)$. Let $Z_X = \{z \in Z \mid z \text{ has a neighbor in } X\}$ and $Z_Y = \{z \in Z \mid z \text{ has a neighbor in } Y\}$. By the definition of an augment, Z_X is complete to X and anticomplete to Y, and Z_Y is complete to Y and anticomplete to X, and $Z_X \cap Z_Y = \emptyset$. Since G_2 is connected, we may assume up to symmetry that $Z_X \neq \emptyset$. Pick any $z \in Z_X$. Since G_2 is an atom, X is not a cutset of G_2 (separating z from Y), so $Z_Y \neq \emptyset$, which restores the symmetry between X and Y. Since C is a clique and has a vertex in Y, C contains no vertex from Z_X; similarly, C contains no vertex from Z_Y; hence $C \subset X \cup Y$. Pick any $x \in C \cap X$. Since C is a minimal cutset, x has a neighor a_1 in A_1. Then a_1 must be adjacent to every neighbor y of x in Y, for otherwise $\{x, a_1, z, y\}$ induces a claw; and it follows that $y \in C$. We can repeat this argument for every vertex in C; by the last item in Theorem 1.8 it follows that every vertex in $X \cup Y$ is adjacent to a_1 and, consequently, is in C. But this is a contradiction because C is a clique and $X \cup Y$ is not a clique. Thus (2) holds.

Now we distinguish two cases.

(I) First suppose that G_2 is not a cobipartite graph.

For every edge uv in the bipartite multigraph H, let C_{uv} be the subset of $V(G_2)$ defined as follows. If v has degree 2 in H, say $N_H(v) = \{u, u'\}$, and $\{vu, vu'\}$ is a flat edge in $\mathcal{L}(H)$ on which an augment (X, X') of G_2 is based (where X corresponds to vu and X' corresponds to vu'), then let $C_{uv} = X$. If uv is not such an edge, then let C_{uv} be the set of parallel edges in H whose ends are u and v. Now for every vertex u in H, let $C_u = \bigcup_{uv \in E(H)} C_{uv}$. Note that C_u is a clique in G_2. We claim that:

There is a vertex u in H such that $C = C_u$. (3)

For every augment (X, Y) in G_2 we have $V(G_2) \neq X \cup Y$, because G_2 is not cobipartite, and so, by (2), either $C \cap X$ or $C \cap Y$ is empty. It follows that there is a vertex u in H such that $C \subseteq C_u$. Suppose that $C \neq C_u$. Then we can pick vertices $x \in C$ and $x' \in C_u \setminus C$ such that H has vertices v, v' with $x \in C_{uv}$ and $x' \in C_{uv'}$. Since C is a minimal cutset, x has a neighbor a_1 in A_1. Since G_2 is an atom, the set $C_u \setminus C_{uv}$ is not a cutset, so x has a neighbor z in $V(G_2) \setminus C_u$. Then $\{x, a_1, x', z\}$ induces a claw, a contradiction. So $C = C_u$ and (3) holds.

By (3), let u be a vertex in H such that $C = C_u$. Let $D = \{d \in A_1 \mid d$ has a neighbor in $C\}$. We claim that:

$$D \cup C \text{ is a clique.} \tag{4}$$

Pick any d in D. First suppose that d is not complete to C. Then we can find vertices $x \in C \cap N(d)$ and $x' \in C \setminus N(d)$ such that H has vertices v, v' with $x \in C_{uv}$ and $x' \in C_{uv'}$. Since G_2 is an atom, the set $C_u \setminus C_{uv}$ is not a cutset, so x has a neighbor z in $V(G_2) \setminus C_u$. Then $\{x, d, x', z\}$ induces a claw, a contradiction. It follows that D is complete to C. Now suppose that D contains non-adjacent vertices d, d'. Pick any $x \in C$. Then x has a neighbor z in $V(G_2) \setminus C_u$. Then $\{x, d, d', z\}$ induces a claw, a contradiction. So D is a clique. Thus (4) holds.

$$G[D \cup C \cup A_2] \text{ is an elementary graph.} \tag{5}$$

Let H^* be the bipartite graph obtained from H by adding $|D|$ vertices of degree 1 adjacent to vertex u. Then it is easy to see (by (3) and (4)) that $G[D \cup C \cup A_2]$ can be obtained from $\mathcal{L}(H^*)$ by augmenting the same flat edges as for G_2 and with the same augments. Thus (5) holds.

Let $D = \{d_1, \dots, d_p\}$. (Actually we have $|C| \geq 2$ by (3) and consequently $|D| \leq 2$ by (4), but we will not use this fact.) Recall that f is an L-coloring of G_1; so for $i = 1, \dots, p$ let $c_i = f(d_i)$.

The maximum degree in H^* is $\Delta(H^*) = \omega(\mathcal{L}(H^*)) \leq \omega(G_2) \leq \omega(G) \leq 4$. So we can color the edges of H^* with 4 colors in such a way that vertices $d_1, \dots, d_p$ receive colors $c_1, \dots, c_p$ respectively. Let L^* be a list assignment on $\mathcal{L}(H^*)$ defined as follows. If $v \in V(\mathcal{L}(H))$, let $L^*(v) = L(v)$. For $i = 1, \dots, p$, let $L^*(d_i) = \{c_1, \dots, c_i\}$. By Theorem 1.9, $\mathcal{L}(H^*)$ admits an L^*-coloring f^*. Now we can use the same technique as in the proof of Theorem 4.1 to extend f^* to an L-coloring of G_2. Moreover, we have $f^*(d_1) = c_1$ and consequently $f^*(d_i) = c_i = f(d_i)$ for all $i = 1, \dots, p$. Let f' be defined as follows. For all $v \in V(G_1) \setminus C$, let $f'(v) = f(v)$, and for all $v \in V(G_2)$, let $f'(v) = f^*(v)$. Then f' is an L-coloring of G. This completes the proof in case (I).

(II) We may now assume that G_2 is a cobipartite graph. Let W be the set of vertices of A_1 that have a neighbor in C. For all $x \in C$, let $N_1(x) = N(x) \cap A_1$, $N_2(x) = N(x) \cap A_2$ and $M_2(x) = A_2 \setminus N(x)$. We observe that:

$$N_1(x) \text{ and } N_2(x) \text{ are non-empty cliques, and } M_2(x) \text{ is a clique.} \tag{6}$$

We know that $N_1(x)$ and $N_2(x)$ are non-empty because C is a minimal cutset. For $i = 1, 2$ pick any $n_i \in N_i(x)$; then $N_i(x)$ is a clique, for otherwise x is the center of a claw with n_{3-i} and two non-adjacent vertices from $N_i(x)$. Also $M_2(x)$ is a clique, for otherwise G_2 contains a stable set of size 3. Thus (6) holds.

Suppose that $|C| = 1$. Let $C = \{x\}$. Then $M_2(x)$ is empty, for otherwise $N_2(x)$ is a clique cutset in G_2 (separating x from $M_2(x)$). So G_2 is a clique. Then every vertex in A_2 is simplicial, a contradiction to (1). So $|C| \geq 2$.

Suppose that two vertices x and y of C have inclusionwise incomparable neighborhoods in A_1. So there is a vertex a in A_1 adjacent to x and not to y, and there is a vertex b in A_1 adjacent to y and not to x. If a vertex u in A_2 is adjacent to x, then it is adjacent to y, for otherwise $\{x, a, y, u\}$ induces a claw, and vice-versa. So $N_2(x) = N_2(y)$, and $|N_2(x)| \leq 2$ (because $N_2(x) \cup \{x, y\}$ is a clique), and $M_2(x) = M_2(y)$. Suppose that $M_2(x) \neq \emptyset$. Let $C' = \{u \in C \setminus \{x, y\} \mid u$ is complete to $N_2(x)\}$. Since $C' \cup N_2(x)$ is a clique, it cannot be a cutset of G_2, so some vertex z in $C \setminus (C' \cup \{x, y\})$ has a neighbor v in $M_2(x)$. Since $z \notin C'$, z has a non-neighbor u in $N_2(x)$. Then za is an edge, for otherwise $\{x, a, z, u\}$ induces a claw. But then $\{z, a, y, v\}$ induces a claw, a contradiction. So $M_2(x) = \emptyset$. Thus $A_2 = N_2(x) = N_2(y)$. If the vertices in A_2 have pairwise comparable neighborhoods in C, then it follows easily that the vertex in A_2 with the smallest degree is simplicial in G, a contradiction to (1). So there are two vertices u, v in A_2 and two vertices z, t in C such that tu, zv are edges and tv, zu are not edges. Clearly $z, t \notin \{x, y\}$, so $|C| = 4$. Then za is an edge, for otherwise $\{x, a, z, u\}$ induces a claw; and similarly, zb, ta, tb are edges. Then ab is an edge, for otherwise $\{z, a, b, v\}$ induces a claw. Recall that since G is perfect and claw-free, the neighborhood of every vertex can be partitioned into two cliques, and consequently (since $\omega(G) \leq 4$) every vertex has degree at most 6. Hence $N(x) = \{y, z, t, a, u, v\}$ (because we already know that x is adjacent to these six vertices), and similarly $N(y) = \{x, z, t, b, u, v\}$, $N(z) = \{x, y, t, a, b, v\}$, and $N(t) = \{x, y, z, a, b, u\}$. It follows that $A_2 = \{u, v\}$ and $W = \{a, b\}$. Here we view f as an L-coloring of $G_1 \setminus (C \cup \{a, b\})$ rather than of G_1, and we try to extend it to $\{a, b\} \cup C \cup A_2$. Let $S = \{s \in V(G_1) \setminus (C \cup \{a, b\}) \mid s$ has a neighbor in $\{a, b\}\}$. If a vertex $s \in S$ is adjacent to a and not to b, then $\{a, s, b, x\}$ induces a claw, a contradiction. By symmetry this implies that S is complete to $\{a, b\}$. Then S is a clique, for otherwise $\{a, s, s', x\}$ induces a claw from some non-adjacent $s, s' \in S$. So $S \cup \{a, b\}$ is a clique, and so $|S| \leq 2$. We remove the colors of $f(S)$ from the lists of a and b. By Lemma 3.8 we can color the vertices of $W \cup C \cup \{u, v\}$ with colors from the lists thus reduced. So G is L-colorable.

Therefore we may assume that any two vertices of C have inclusionwise comparable neighborhoods in A_1. This implies that some vertex a_1 in A_1 is complete to C, and that some vertex x in C is complete to W. Since $\{a_1\} \cup C$ is a clique, we have $|C| \leq 3$. We have $W = N_1(x)$ and, by (6), W is a clique, so $|W| \leq 3$. Here we view f as an L-coloring of $G_1 \setminus C$ rather than of G_1, and we try to extend it to $C \cup A_2$. If $|W| = 1$ (i.e., $W = \{a_1\}$), we remove the color $f(a_1)$ from the list of the vertices in C. Then G_2 is a cobipartite graph which, with the reduced lists, satisfies the hypothesis of Lemma 3.5 or 3.9, so f can be extended to G_2. Hence assume that $|W| \geq 2$.

Suppose that W is complete to C. Then $W \cup C$ is a clique, so $|W| = 2$ and $|C| = 2$. Let $C = \{x, y\}$. Let $X = N_2(x)$, $Y = N_2(y)$, and $Z = A_2 \setminus (X \cup Y)$. Suppose that $Z \neq \emptyset$. By (6) $Z \cup (X \setminus Y)$ is a clique, since it is a subset of $M_2(y)$. Likewise, $Z \cup (Y \setminus X)$ is a clique. Moreover $X \setminus Y$ is complete to $Y \setminus X$, for otherwise $\{x, y, v, z, u\}$ induces a C_5 for some non-adjacent $u \in X \setminus Y$ and

$v \in Y \setminus X$ and for any $z \in Z$. It follows that $X \cup Y$ is a clique cutset in G_2 (separating $\{x, y\}$ from Z), a contradiction. So $Z = \emptyset$, and $A_2 = X \cup Y$. Here we view f as an L-coloring of $G_1 \setminus C$ rather than of G_1, and we try to extend it to $C \cup A_2$. We remove the colors of $f(W)$ from the list of x and y. Since $|W| = 2$, each of these lists loses at most two colors. By Lemma 3.5 we can color the vertices of $C \cup A_2$ with colors from the lists thus reduced. So G is L-colorable.

Now assume that W is not complete to C. So some vertex a_2 in W has a non-neighbor y in C. Then $N_2(x) \cup \{y\}$ is a clique, for otherwise $\{x, a_2, u, v\}$ induces a clique for any two non-adjacent vertices $u, v \in X \cup \{y\}$. Suppose that $M_2(x)$ is empty. So $A_2 = N_2(x)$. Then the vertices in A_2 have comparable neighborhoods in C (because they are complete to $\{x, y\}$ and $|C| \leq 3$), so the vertex in A_2 with the smallest degree is simplicial, a contradiction to (1). Therefore $M_2(x)$ is not empty. Since the clique $\{y\} \cup N_2(x)$ is not a cutset in G_2, some vertex z in $C \setminus \{x, y\}$ has a neighbor v in $M_2(x)$. Hence $|C| = 3$. Then z has a non-neighbor u in $N_2(x)$, for otherwise $\{y, z\} \cup N_2(x)$ is a clique cutset in G_2 (separating x from v). Then za_2 is an edge, for otherwise $\{x, a_2, z, u\}$ induces a claw; and yv is an edge, for otherwise $\{z, a_2, y, v\}$ induces a claw; and uv is an edge since $N_2(y)$ is a clique. Moreover, if $N_2(x)$ contains a vertex u' adjacent to z, then vu' is an edge since $N_2(z)$ is a clique. Since this holds for every vertex in $M_2(x) \cap N(z)$, we deduce that $(M_2(x) \cap N(z)) \cup \{y\} \cup N_2(x)$ is a clique Q. If v' is any non-neighbor of z in $M_2(x)$, then Q is a clique cutset in G_2 (separating $\{x, z\}$ from v'), a contradiction. So $M_2(x) \subset N(z)$. Suppose that $|W| = 3$. Pick $a_3 \in W \setminus \{a_1, a_2\}$. Then $a_3 z$ is not an edge, for otherwise $W \cup \{x, z\}$ is a clique of size 5. So, by the same argument as for a_2, we deduce that $a_3 y$ is an edge. But this means that y and z have inclusionwise incomparable neighborhoods in A_1 (because of a_2, a_3), a contradiction. So $|W| = 2$. We remove the color $f(a_1)$ from the lists of x, y, z and remove the color $f(a_2)$ from the list of x and z. By Lemma 3.7 we can color the vertices of $C \cup A_2$ with colors from the lists thus reduced. So G is L-colorable. This completes the proof of the theorem. $\square$

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