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A refinement of theorems on vertex-disjoint chorded cycles

Theodore Molla^{*}, Michael Santana[†], Elyse Yeager[‡]

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Abstract

In 1963, Corrádi and Hajnal settled a conjecture of Erdős by proving that, for all $k \ge 1$, any graph G with $|G| \ge 3k$ and minimum degree at least 2k contains k vertex-disjoint cycles. In 2008, Finkel proved that for all $k \ge 1$, any graph G with $|G| \ge 4k$ and minimum degree at least 3k contains k vertex-disjoint cycles. Finkel's result was strengthened by Chiba, Fujita, Gao, and Li in 2010, who showed, among other results, that for all $k \ge 1$, any graph G with $|G| \ge 4k$ and minimum Ore-degree at least 6k - 1 contains k vertex-disjoint cycles. We refine this result, characterizing the graphs G with $|G| \ge 4k$ and minimum Ore-degree at least 6k - 2 that do not have k disjoint chorded cycles.

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1 Introduction

All graphs in this paper are simple, unless otherwise noted. Additionally, when referring to cycles in a graph, "disjoint" is always taken to mean "vertex-disjoint." For a graph G, we use V(G) and E(G) to denote the vertices and edges, respectively, and for a vertex v, we will use $v \in G$ to denote $v \in V(G)$. For a vertex $v \in G$, and for a subgraph H of G (where possibly H = G), the neighborhood of v in H is denoted by $N_H(v)$. The number of neighbors of v in H (i.e., $|N_H(v)|$) will be written by $d_H(v)$. Furthermore, we write |G| for the order of a graph G, \overline{G} for its complement, $\delta(G)$ for its minimum degree, and $\alpha(G)$ for its independence number.

The minimum Ore-degree of a non-complete graph G is written $\sigma_2(G)$, and defined as

$$\sigma_2(G) := \min\{d_G(x) + d_G(y) : xy \in E(\overline{G})\};\$$

that is, $\sigma_2(G)$ is the minimum degree-sum of nonadjacent vertices. K_n is the complete graph on n vertices, and K_{s_1,\ldots,s_t} is the complete *t*-partite graph with parts of size s_1,\ldots,s_t . For graphs G and H, G+H is the disjoint union of G and H, and $G \vee H$ is the join of G and H.

In 1963, Corrádi and Hajnal verified a conjecture of Erdős, proving the following.

Theorem 1 (Corrádi-Hajnal, [3]). Every graph G on $|G| \ge 3k$ vertices with $\delta(G) \ge 2k$ contains k disjoint cycles.

This result of Corrádi and Hajnal has been generalized in various ways. One such generalization is a strengthening by Enomoto and Wang, who independently proved the following.

Theorem 2 (Enomoto [5], Wang [14]). Every graph G on $|G| \ge 3k$ vertices with $\sigma_2(G) \ge 4k - 1$ contains k disjoint cycles.

^{*}Department of Mathematics, University of Illinois, Urbana, IL 61801, USA. This author's research is supported in part by the NSF grant DMS-1500121. E-mail address: molla@illinois.edu

[†]Department of Mathematics, University of Illinois, Urbana, IL 61801, USA. This author's research is supported in part by the NSF grant DMS-1266016 "AGEP-GRS". E-mail address: santana@illinois.edu

[‡]Mathematics Department, University of British Columbia, Canada. E-mail address: elyse@math.ubc.ca

Both Theorems 1 and 2 are sharp, leading to the following natural question of Dirac.

Diracq Question 3 (Dirac, [4]). Which (2k-1)-connected multigraphs do not contain k disjoint cycles?

Question 3 was answered in the case of simple graphs in [10], and then in multigraphs in [11]. Indeed, [10] together with [12] answer a more general question for simple graphs, describing graphs with minimum Ore-degree at least 4k - 3 with no k disjoint cycles. To avoid going into too many technical details, we only provide part of this description below.

KKYT Theorem 4 ([10], [12]). Given an integer $k \ge 4$, let G be a graph on $|G| \ge 3k$ vertices with $\sigma_2(G) \ge 4k-3$. Then G contains k disjoint cycles if and only if none of the following hold:

- 1. $\alpha(G) \ge |G| 2k + 1$.
- 2. $G = (K_c + K_{2k-c}) \vee \overline{K_k}$ for some odd c
- 3. $G = (K_1 + K_{2k}) \vee \overline{K_{k-1}}$

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4. |G| = 3k and \overline{G} is not k-colorable

In 2008 Finkel proved the following chorded-cycle analogue to Theorem 1.

FinkelT Theorem 5 (Finkel, [7]). Every graph G on $|G| \ge 4k$ vertices with $\delta(G) \ge 3k$ contains k disjoint chorded cycles.

A stronger vertion of Theorem 5 was conjectured by Bialostocki, Finkel, and Gyárfás in [1], and proved by Chiba, Fujita, Gao, and Li in [2].

CFGLT Theorem 6 (Chiba-Fujita-Gao-Li, [2]). Let r and k be integers with $r + k \ge 1$. Every graph G on $|G| \ge 3r + 4k$ vertices with $\sigma_2(G) \ge 4r + 6k - 1$ contains a collection of r + k disjoint cycles such that k of these cycles are chorded.

In particular, the following corollary holds.

Corollary 7 (Chibia-Fujita-Gao-Li, [2]). Every graph G on $|G| \ge 4k$ vertices with $\sigma_2(G) \ge 6k-1$ contains a collection of k disjoint chorded cycles.

All hypotheses in Theorem 5 and Corollary 7 are sharp. First, since any chorded cycle contains at least four vertices, if |G| < 4k then G does not contain k disjoint chorded cycles. Second, the conditions $\delta(G) \ge 3k$ and $\sigma_2(G) \ge 6k - 1$ are best possible, as demonstrated by the two graphs below.

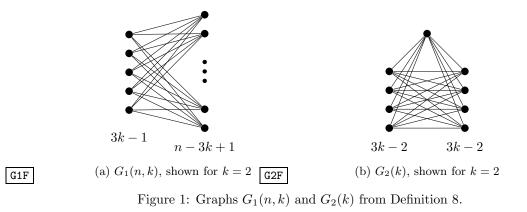
G1 Definition 8. For $n \ge 6k - 2$, define $G_1(n,k) := K_{3k-1,n-3k+1}$ (Figure 1a). For $k \ge 2$, define $G_2(k) := K_{3k-2,3k-2,1}$ (Figure 1b).

For $n \ge 6k - 2$, $|G_1(n,k)| = n \ge 4k$ and $\sigma_2(G_1(n,k)) = 6k - 2$. Each chorded cycle in $G_1(n,k)$ uses at least three vertices from each part, so $G_1(n,k)$ does not contain k disjoint chorded cycles. For $k \ge 2$, $|G_2(k)| = 6k - 3 \ge 4k$ and $\sigma_2(G_2(k)) = 6k - 2$. Each chorded cycle in $G_2(k)$ uses three vertices from each of the big parts, or the dominating vertex and at least two vertices from a big part, so $G_2(k)$ does not contain k chorded cycles.

We can now ask a question similar to Question 3: which graphs G with $\sigma_2(G) \ge 6k - 2$ do not contain k disjoint chorded cycles? Our main result is the following.

main Theorem 9. For $k \ge 2$, let G be a graph with $n := |G| \ge 4k$ and $\sigma_2(G) \ge 6k - 2$. G does not contain k disjoint chorded cycles if and only if $G \in \{G_1(n,k), G_2(k)\}$.

The condition $k \ge 2$ in Theorem 9 is necessary, as subividing every edge of a graph results in a new graph with no chorded cycles. Thus, for k = 1, we obtain the following characterization, which is analogous to the characterization of acyclic graphs as the graphs for which there exists at most one path between every pair of vertices.



Proposition 10. A graph G has no chorded cycle if and only if for all $uv \in E(G)$, G - uv has at most one path between u and v.

Every graph G with $\delta(G) \ge 3k - 1$ also satisfies $\sigma_2(G) \ge 6k - 2$. Therefore, Theorem 9 is a refinement of both Theorem 5 and Corollary 7. Two other immediate corollaries of Theorem 9 are listed here.

Corollary 11. For $k \ge 2$, let G be a graph with $|G| \ge 4k$, $\sigma_2(G) \ge 6k - 2$, and $\alpha(G) \le n - 3k$. Then G contains k disjoint chorded cycles.

Every graph G with $\sigma_2(G) \ge 6k - 2$ also satisfies $\alpha(G) \le n - 3k + 1$. So, requiring $\alpha(G) \le n - 3k$ in Corollary 11 is equivalent to requiring the seemingly weaker condition $\alpha(G) \ne n - 3k + 1$.

Corollary 12. For $k \ge 2$, let G be a graph with $4k \le |G| \le 6k - 4$ and $\sigma_2(G) \ge 6k - 2$. Then G contains k disjoint chorded cycles.

1.1 Outline

We present our result as follows. In Section 2, we detail the setup of our proof and present several important lemmas that will be used throughout our paper. In particular, we find and choose an 'optimal' collection of k-1 disjoint cycles, and use R to denote the subgraph induced by the vertices outside our collection. Then, in Section 3, we consider the case when R does not have a spanning path, and, in Section 4, we consider the case when R has a spanning path. We conclude our paper in Section 5 with some remarks on further extensions.

2 Setup and Preliminaries

2.1 Notation

Let G be a graph, and let $A, B \subseteq V(G)$, not necessarily disjoint. We define $||A, B|| := \sum_{a \in A} |N_G(a) \cap B|$.

When $A = \{a\}$ or A is the vertex set of some subgraph \mathcal{A} , we will often replace A in the above notation with a or \mathcal{A} , respectively. Additionally, if \mathcal{L} is a collection of graphs, then $||A, \mathcal{L}|| = ||A, \bigcup_{L \in \mathcal{L}} V(L)||$. If A is

the vertex set of some subgraph \mathcal{A} , we will write $G[\mathcal{A}]$ for G[A], the subgraph of G induced by the vertices of \mathcal{A} . Furthermore, if \mathcal{B} is a subgraph of G with vertex set B, we will use $\mathcal{A} \setminus \mathcal{B}$ to denote $G[A \setminus B]$, and if $B = \{b_1, \ldots, b_k\}$ and k is small, we will also use $\mathcal{A} - b_1 - \cdots - b_k$. For a vertex v, we additionally write $\mathcal{A} + v$ for $G[A \cup \{v\}]$.

If $P = v_1 \dots v_m$ is a path, then for $1 \leq i \leq j \leq m$, $v_i P v_j$ is the path $v_i \dots v_j$. An *n*-cycle is a cycle with *n* vertices. A singly chorded cycle is a cycle with precisely one chord, and a doubly chorded cycle is a cycle with at least two chords.

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2.2 Setup

We let $k \geq 2$ and consider a graph G' on n vertices such that $n \geq 4k$ and $\sigma_2(G') = 6k - 2$, where G'does not contain k disjoint chorded cycles. We then let G be a graph with vertex set V(G') such that $E(G') \subseteq E(G)$ and G is "edge-maximal" in the sense that, for any $e \in E(\overline{G})$, G + e does contain k disjoint chorded cycles. We then prove that G is $G_1(n,k)$ or $G_2(k)$, which implies that G = G', because any proper spanning subgraph of $G_1(n,k)$ or $G_2(k)$ has minimum Ore-degree less than 6k - 2. Since we have already observed that $G_1(n,k)$ and $G_2(k)$ do not contain k disjoint chorded cycles, this will prove Theorem 9.

Note that $G \not\cong K_n$, else G contains k disjoint chorded cycles. So there exists $e \in E(G)$, and by our edgemaximality condition, G contains k-1 disjoint chorded cycles. Over all possible collections of k-1 disjoint chorded cycles in G, let C be such a collection which satisfies the following conditions when $R := G \setminus C$:

(O1) the number of vertices in \mathcal{C} is minimum,

(O2) subject to (O1), the total number of chords in the cycles of \mathcal{C} is maximum, and

(O3) subject to (O1) and (O2), the length of the longest path in R is maximum.

We use the convention that P is a longest path in R. Since G[P] may have several paths spanning V(P)and the endpoints of such paths will behave in a similar manner, we let

 $\mathcal{P} := \{ v \in V(P) : v \text{ is an endpoint of a path spanning } V(P) \}.$

2.3 Preliminary Results

We begin with a number of observations about G that follow directly from our setup. In the interest of readability, the observations in this paragraph will be used in the text without citation. Since G does not contain k disjoint chorded cycles, R does not contain any chorded cycle, and for any $C \in C$, $G[R \cup C]$ does not contain two disjoint chorded cycles. If p is an endpoint of P and has a neighbor in $R \setminus P$, we can extend P. Thus, ||p, R|| = ||p, P||. If $||p, P|| \ge 3$, then G[P] contains a chorded cycle, so $||p, R|| \le 2$. Similarly, to avoid a chorded cycle in R, $||q, P|| \le 3$ and for any $v \in P$, $||v, P|| \le 4$. If p has two neighbors in P, then G[P] contains two distinct spanning paths.

An immediate corollary of (O1) is that, for any chorded cycle $C \in C$, no vertex of C is incident to two chords; otherwise, we could replace C with a chorded cycle on fewer vertices. We will assume this fact in the proof of the following lemma.

RCedges 4edgesL 3edgesL 3edgesL4 3edgesL5

3edgesL6

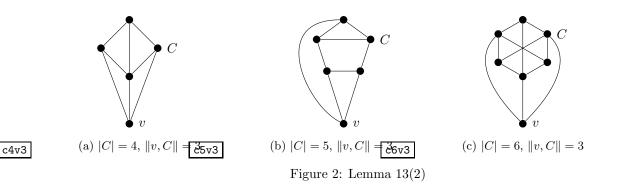
Lemma 13. Let $v \in R$ and $C \in C$.

- (1) If $||v, C|| \ge 4$, then ||v, C|| = 4 = |C|, and $G[C] \cong K_4$.
- (2) If ||v, C|| = 3, then $|C| \in \{4, 5, 6\}$. Moreover:
 - (a) if |C| = 4, then C has a chord incident to the non-neighbor of v (see Figure 2a);
 - (b) if |C| = 5, then C is singly chorded, and the endpoints of the chord are disjoint from the neighbors of v (see Figure 2b); and
 - (c) if |C| = 6, then C has three chords, with $G[C] \cong K_{3,3}$ and $G[C+v] \cong K_{3,4}$ (see Figure 2c).

Proof. If there exist vertices $c_1, c_2 \in C$ that are adjacent along the cycle of C such that $||v, C - c_1 - c_2|| \ge 3$, then $(C - c_1 - c_2) + v$ contains a chorded cycle with strictly fewer vertices than C, contradicting (O1). This proves that if ||v, C|| = 3, then $|C| \le 6$. Similarly, if $||v, C|| \ge 4$, then |C| = 4 and ||v, C|| = 4. If ||v, C|| = 4 and |C| = 4, then v together with a triangle in C give a doubly chorded 4-cycle, so by (O2), $G[C] \cong K_4$.

Suppose ||v, C|| = 3. If |C| = 4, then let $c \in C$ be the non-neighbor of v in C. If c is not incident to a chord, then (C - c) + v gives a doubly chorded 4-cycle, preferable to C by (O2). This proves (a).

So $|C| \in \{5, 6\}$. Since the vertices in $V(C) \setminus N_G(v)$ cannot be adjacent along the cycle C, C - c + v contains a chorded cycle C' of the same length as C, for any $c \in V(C) \setminus N_G(v)$, If c is not incident to a



chord, then C' has strictly more chords than C, violating (O2). So every vertex in $V(C) \setminus N_G(v)$ is incident to a chord.

If |C| = 6, then v is adjacent to every other vertex along the cycle, and every $c \in V(C) \setminus N_G(v)$ is incident to a chord. Since no vertex in C is incident to two chords, (O1) implies (c). If |C| = 5, then (O1) implies that the only possible chord has the two non-neighbors of v as its endpoints, which proves (b).

Lemma 14. Let Q be a path in R such that $|Q| \ge 4$ and let $C \in C$. If $F \subseteq V(Q)$ such that |F| = 4, then $||F, C|| \le 12$. Furthermore, if $G[C] \cong K_4$ and there exists an endpoint v of Q such that $||v, C|| \ge 3$, then $||Q, C|| \le 12$ with ||Q, C|| = 12 only if ||v, C|| = 4.

Proof. Assume $||F, C|| \ge 13$ for some $F \subseteq V(Q)$, |F| = 4, and let u_1, u_2, u_3, u_4 be the vertices of F in the order they appear on the path Q. By Lemma 13, $G[C] \cong K_4$, so there exists $c \in C$ such that $||c, F|| \ge 4$. Since $||\{u_1, u_4\}, C|| \ge 5$, there exists $i \in \{1, 4\}$ such that $||u_i, C|| \ge 3$. So $Q - u_i + c$ and $C - c + u_i$ both contain chorded cycles, a contradiction.

To prove the second statement, suppose $G[C] \cong K_4$ and let v be an endpoint of Q such that $||v, C|| \ge 3$. Note that for every $c \in C$, C - c + v and Q - v + c both contain chorded cycles if $||c, Q - v|| \ge 3$. Thus, $||Q, C|| \le 12$, and furthermore, if ||Q, C|| = 12, then ||c, Q|| = 3 and ||c, v|| = 1 for every $c \in C$.

noC5 Lemma 15. If $C \in C$ and $||v_1, C||, ||v_2, C|| \ge 3$ for distinct $v_1, v_2 \in R$, then $|C| \in \{4, 6\}$.

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Proof. If $C \notin \{4, 6\}$, then |C| = 5 and $N_C(v_1) = N_C(v_2)$, by Lemma 13. Furthemore, Lemma 13, implies that there are two adjacent vertices $c, c' \in N_C(v_1) = N_C(v_2)$, but then $v_1 c v_2 c' v_1$ is a chorded cycle contradicting (O1).

In the following sections, we will often show that every C in C is a 6-cycle. Furthermore, it will often be the case that there exists some $u \in R$ such that ||u, C|| = 3 for every $C \in C$. The following lemma will be useful in considering the neighbors of u in R and their adjacencies in C.

Lemma 16. Let $C \in \mathcal{C}$ with |C| = 6, and let $u, v \in R$ such that $uv \in E(G)$. If ||u, C|| = 3 and $||v, C|| \ge 1$, then $N_C(u) \cap N_C(v) = \emptyset$.

Proof. By Lemma 13, we may assume that $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ are the partite sets of $G[C] \cong K_{3,3}$ with $N_C(u) = A$. Suppose on the contrary that $va_1 \in E(G)$. Then $ua_2b_1a_1vu$ is a 5-cycle with chord ua_1 . This contradicts (O1).

P2P3 Lemma 17. Suppose *H* is a graph with no chorded cycle. Let *U* and *W* be two disjoint paths in *H* and let u_1 and u_2 be the endpoints of *U*. Then $||\{u_1, u_2\}, W|| \leq 3$. If equality holds, then $u_1 \neq u_2$ and for some $i \in [2], ||u_i, W|| = 2$ and $||u_{3-i}, W|| = 1$, with the neighbor of u_{3-i} strictly between the neighbors of u_i on *W*; in addition, ||U, W|| = 3.

Proof. Let $W = w_1 w_2 \dots w_t$ for some $t \ge 1$. $||u_1, W|| \le 2$ and $||u_2, W|| \le 2$, as H does not contain a chorded cycle. Thus, if $||\{u_1, u_2\}, W|| \ge 3$, we may assume that $u_1 \ne u_2$, and, without loss of generality,

that $||u_1, W|| = 2$ and $||u_2, W|| \ge 1$. Suppose $u_1w_i, u_1w_j \in E(H)$ such that i < j, and let $u_2w_\ell \in E(H)$ for some ℓ .

If $\ell \leq i$, then $w_{\ell}Ww_{j}u_{1}Uu_{2}w_{\ell}$ is a cycle with chord $u_{1}w_{i}$. If $\ell \geq j$, then $w_{i}Ww_{\ell}u_{2}Uu_{1}w_{i}$ is a cycle with chord $u_{1}w_{j}$. Thus, the neighbors of u_{2} in W are internal vertices of the path $w_{i}Ww_{j}$. If $||u_{2}, W|| = 2$, then suppose ℓ is the largest index such that $u_{2}w_{\ell} \in E(H)$. However, $w_{i}Ww_{\ell}u_{2}Uu_{1}w_{i}$ is a cycle containing a chord incident to u_{2} . So $||u_{2}, W|| = 1$.

Now if v is an internal vertex on U such that $vw_m \in E(H)$, then by replacing u_2 with v, we deduce that $i \leq m \leq j$. If $m \leq \ell$, then $w_i W w_\ell u_2 U u_1 w_i$ is a cycle with chord vw_m , and if $m > \ell$, then $w_\ell W w_j u_1 U u_2 w_j$ is a cycle with chord vw_m . This proves the lemma.

3 Suppose $V(R) \neq V(P)$.

In this section, we make the assumption that $V(R) \neq V(P)$. That is, there exists some vertex $v \in R \setminus P$. In addition, we will use the convention that p and p' are the endpoints of P, and q (resp. q') is the neighbor of p (resp. p') on P. By the maximality of P, $vp \notin E(G)$ so that $d_G(v) + d_G(p) \geq 6k - 2$. Similarly for v and p'.

Our aim is to show that $G = G_1(n, k)$, which is a complete bipartite graph. To aid us, we define a set of vertices $T := \{v \in R : d_R(v) = 2\}$. We will show that T is contained in one of the partite sets of $G_1(n, k)$.

Lemma 18. If $v \in R \setminus P$, then $||\{v, p\}, C|| \leq 6$ for every $C \in C$, with equality only if

(i)
$$|C| \in \{4, 6\}$$
 and $N_C(v) = N_C(p)$, or

(ii)
$$||p, C|| = |C| = 4.$$

Proof. Suppose $v \in R \setminus P$ and $||\{v, p\}, C|| \ge 6$ for some $C \in C$. If $||\{v, p\}, C|| \ge 7$, then either ||v, C|| = 4 or ||p, C|| = 4, so that $G[C] \cong K_4$ by Lemma 13. If ||v, C|| = 4, then ||p, C|| = 0, lest we extend P by adding a neighbor of p in C, and replace said neighbor in C with v, violating (O3). If ||p, C|| = 4, then $||v, C|| \le 2$, else there exists $c \in C$ such that $C - c + v \cong K_4$, and we can extend P by adding c, violating (O3). So, $||\{v, p\}, C|| \le 6$, and if equality holds, then either (ii) occurs, or ||v, C|| = ||p, C|| = 3. We may assume ||v, C|| = ||p, C|| = 3, so that $|C| \in \{4, 5, 6\}$ by Lemma 13.

By Lemma 15, $|C| \in \{4, 6\}$. Suppose |C| = 4 and ||v, C|| = ||p, C|| = 3. Note that $G[N_C(v) \cup \{v\}]$ forms a chorded 4-cycle with at least the same number of chords as C. If p is adjacent to the vertex in $V(C) \setminus N_G(v)$, we use that vertex to extend P, violating (O3). So (i) holds.

Finally, suppose |C| = 6. By Lemma 13, if v and p do not have the same neighborhood, they are adjacent to disjoint sets of vertices, and C + p and C + v both contain $K_{3,4}$. In this case, we extend P using any $c \in N_C(p)$, and replace C with a chorded cycle in C - c + v. This violates (O3), so (i) holds.

Lemma 19. For any $v \in \mathbb{R} \setminus P$, $||\{v, p\}, \mathbb{R}|| \ge 4$, so that $||v, \mathbb{R}|| \ge 2$. Moreover, $|P| \ge 3$.

Proof. Let $v \in R \setminus P$. By the maximality of P, $pv \notin E(G)$. Thus, by Lemma 18,

$$2(3k-1) \le d_G(v) + d_G(p) = ||\{v, p\}, C|| + ||\{v, p\}, R|| \le 6(k-1) + ||\{v, p\}, R||,$$

so $||\{v, p\}, R|| \ge 4$. Since $||p, R|| \le 2$, it follows that $||v, R|| \ge 2$. Then v and two of its neighbors form a path of length three in R, hence $|P| \ge 3$.

Lemma 20. For any maximal path P' in $R \setminus P$, label the (not necessarily distinct) endpoints v_1 and v_2 so that $||v_1, P|| \leq ||v_2, P||$. Then:

(a) $||v_2, P|| \le 2$, and if $v_1 \ne v_2$ then $||v_1, P|| \le 1$,

(b)
$$d_R(v_1) = 2$$
 (this implies $v_1 \in T \setminus V(P)$ so that $T \setminus V(P) \neq \emptyset$), and

(c) if $||v_2, P|| = 2$ and $||v_1, P|| = 1$, then $||P' - v_1 - v_2, P|| = 0$.

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Proof. Since R contains no chorded cycle, no vertex in $R \setminus P$ has three neighbors in P, so $||v_2, P|| \leq 2$. Lemma 17 then gives (a) and (c).

It remains to show (b). If $||v_1, P|| = 0$, then using Lemma 19 and the maximality of P', $d_R(v_1) = ||v_1, P'|| = 2$. If $v_1 = v_2$, then $||v_1, R|| = ||v_2, P|| = 2$. So suppose $||v_1, P|| = 1$ and $v_1 \neq v_2$. Since $||v_2, P|| \ge ||v_1, P|| = 1$, there exist $a_1, a_2 \in P$ (perhaps $a_1 = a_2$) such that $v_1a_1, v_2a_2 \in E(G)$. Then $v_1P'v_2a_2Pa_1v_1$ is a cycle. Since it has no chord, $||v_1, P'|| = 1$, so $||v_1, R|| = 2$ and $v_1 \in T$.

Lemma 21. $d_R(p) = d_R(p') = 2$. Additionally, for every $v \in T \setminus V(P)$ and every $C \in C$:

pvnhd pvnhdd pvnhdb pvnhdc

(a)
$$|C| \in \{4, 6\},\$$

(b) ||p, C|| = 3, and

(c)
$$N_{\mathcal{C}}(v) = N_{\mathcal{C}}(p).$$

Proof. By Lemma 20, $v \in T \setminus V(P)$ exists so that $d_R(v) = 2$. Lemma 19 implies $||\{v, p\}, R|| \ge 4$, and hence, $d_R(p) = 2$ and $||\{v, p\}, R|| = 4$. Since $vp \notin E(G)$, $||\{v, p\}, C|| \ge (6k - 2) - 4 = 6(k - 1)$. By Lemma 18, $||\{v, p\}, C|| = 6$ for all $C \in C$. If we can show that ||p, C|| = 3 for all $C \in C$, then we are done by Lemma 18.

If not, then there exists $C \in C$ such that ||p, C|| > 3, so ||p, C|| = 4 and $G[C] \cong K_4$ by Lemma 13. Thus, ||v, C|| = 2, and by Lemma 18, there exists $u \in N_C(p')$. Since ||p, P|| = 2, P + u forms a chorded cycle, so since C - u + v also forms chorded cycles, we have a contradiction. Thus, ||p, C|| = 3 as desired.

From Lemma 21 we immediately obtain the following.

mindeg Corollary 22. $d_G(p) = d_G(p') = 3k - 1$, and consequently, $d_G(v) \ge 3k - 1$ for all $v \in R \setminus P$.

Recall that \mathcal{P} is the set of vertices in P that are the endpoint of a path spanning V(P). Note Lemmas 18, 19, 20, and 21 apply to each $p^* \in \mathcal{P}$. Thus, $\mathcal{P} \subseteq T$, and furthermore, for all $p_1^*, p_2^* \in \mathcal{P}, N_{\mathcal{C}}(p_1^*) = N_{\mathcal{C}}(p_2^*)$.

Lemma 23. For every $C \in C$, $G[C] \cong K_{3,3}$.

Proof. If not, by Lemma 13 and Lemma 21, we may assume that there exists $C \in C$ with |C| = 4. Suppose $V(C) = \{c_1, c_2, c_3, c_4\}$. Let $v \in T \setminus V(P)$, which we know exists by Lemma 20. By Lemmas 13 and 21, we may assume that $N_C(p) = N_C(p') = N_C(v) = \{c_1, c_2, c_3\}$ and $c_2c_4 \in E(G)$. Since ||p, P|| = 2 by Lemma 21, $P + c_1$ and $C - c_1 + v$ contain chorded cycles, a contradiction.

For the remainder of this section, we will use the fact that for each $C \in C$, $G[C] \cong K_{3,3}$ and, that there exist $A \subseteq C$ such that A is a partite set of C and such that, for every $p^* \in \mathcal{P}$, $N_C(p^*) = A$, without mentioning Lemmas 21, and 23.

Lemma 24. For every $C \in C$, if $v \in R \setminus P$ has a neighbor in C, then $N_C(v) \subseteq N_C(p)$, unless $|N_C(v)| = 1$.

Proof. Fix $C \in C$, and let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be the partite sets of C such that $N_C(p) = N_C(p') = A$. Suppose on the contrary, there exists $v \in R \setminus P$ with $|N_C(v)| \ge 2$ such that, say $vb_3 \in E(G)$.

By Lemma 21, ||p, P|| = 2 so that $P + a_i$ contains a chorded cycle for each $i \in [3]$. If $vb_2 \in E(G)$, then $vb_3a_3b_1a_2b_2v$ is a cycle with chord a_2b_3 . However, $P + a_1$ also contains a chorded cycle, a contradiction.

So we may assume that $va_3 \in E(G)$. However, $vb_3a_2b_2a_3v$ is a 5-cycle with chord a_3b_3 contradicting (O1). Thus, $N_C(v) \subseteq A = N_C(p)$, as desired.

R-P=T Lemma 25. $R \setminus P$ is an independent set, and $V(R \setminus P) \subseteq T$.

Proof. Suppose $R \setminus P$ is not an independent set. Then there exists a maximal path P' in $R \setminus P$ with distinct endpoints v_1 and v_2 , labeled as in Lemma 20. Thus, $||v_2, P|| \leq 2$, and, hence, $d_R(v_2) \leq 4$. Since $pv_2 \notin E(G)$, Lemma 22 implies that $d_G(v_2) \geq 3k - 1 > 4$, which implies that there exists $C \in C$ such that v_2 has a neighbor $c \in C$.

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be the partite sets of C such that $N_C(p) = N_C(p') = A$. By Lemmas 20 and 21, $v_1 \in T \setminus V(P)$ and $N_C(v_1) = A$. We can assume $a_1 \neq c$, so that there exists a path W

|C|=6

nhdofv

in $C - a_1$ that contains a_2 and a_3 for which c is an endpoint. Since $||v_1, W|| \ge 2$ and v_2 is adjacent to an endpoint of W, $||\{v_1, v_2\}, W|| \ge 3$ and Lemma 17 implies there is a chorded cycle in $G[V(P') \cup V(C - a_1)]$. However, as ||p, P|| = 2, $P + a_1$ also contains a chorded cycle, a contradiction.

Let
$$\mathcal{S} := N_{\mathcal{C}}(p)$$
, and let $\mathcal{T} := ((\bigcup_{C \in \mathcal{C}} V(C)) \setminus \mathcal{S}) \cup T$.

bipar_sub Proposition 26. $G[S \cup T] \cong K_{3k-3,|T|}$, and no vertex in G has neighbors in both S and T.

Proof. By Lemma 23, C consists of k-1 copies of $K_{3,3}$. Lemmas 21 and 25 tell us that, for every $v \in R \setminus P$, $N_{\mathcal{C}}(v) = S$. Given $C \in C$, $a \in V(C) \cap \mathcal{T}$, and $v \in R \setminus P$, we can create a chorded cycle C' by swapping a and v in C. Note $G[C'] \cong K_{3,3}$, and we have not changed any vertices in P. Then replacing C with C' in C results in a collection of k-1 chorded cycles satisfying (O1) through (O3). Thus all the previous lemmas apply, and, in particular, Lemma 20 and Lemma 25 imply that $a \in T$. So by Lemma 21, and the fact that $N_C(a) = V(C) \cap S$, we conclude $N_C(a) = S$. Hence, every vertex in \mathcal{T} is adjacent to every vertex in S, and $G[S \cup \mathcal{T}]$ contains a copy of $K_{|S|,|\mathcal{T}|}$.

We claim $G[S \cup T]$ has no additional edges. Note $|\mathcal{T}| > 3(k-1)$ and |S| = 3(k-1). If there exists any edge with both endpoints in \mathcal{T} , or both endpoints in \mathcal{S} , then we find a set of k-1 chorded cycles, k-2 of which are 6-cycles, and one of which is a 4-cycle, violating (O1). So $G[S \cup \mathcal{T}] \cong K_{|S|,|\mathcal{T}|} \cong K_{3k-3,|\mathcal{T}|}$.

If any vertex of $V(G) \setminus (S \cup T)$ has neighbors in both S and T, then in a similar manner, we find k-1 disjoint chorded cycles, one of which is a 5-cycle and the rest of which are 6-cycles, again violating (O1).

Recall that q and q' were defined as the neighbors of p and p', respectively, on P. Since ||p, P|| = 2 by Lemma 21, there exists $w \in N_R(p) \setminus \{q\}$. As a consequence of Proposition 26, $w \neq p'$. Now the neighbor of w on pPw is the endpoint of a path that spans V(P). Thus, $|\mathcal{P}| \geq 3$.

|P|=3 Lemma 27.
$$|\mathcal{P}| = 3$$

Proof. Suppose $|\mathcal{P}| \geq 4$, with p_1, p_2, p_3, p_4 the first four members of \mathcal{P} along P. In particular, $p_1 = p$. Fix $C \in \mathcal{C}$, and let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be the particle sets of C such that $N_C(p_i) = A$ for each $i \in [4]$.

By Lemma 16, $N_C(q) \subseteq B$. So in particular, $q \neq p_2$. If q has a neighbor in C, say b_1 , then $qb_1a_1b_2a_2p_1q$ is a 6-cycle with chord p_1a_1 and $p_2Pp_4a_3p_2$ is a cycle with chord p_3a_3 , a contradiction.

So we may assume that for every $C \in C$, $N_C(q) = \emptyset$. That is, $||q, R|| = d_G(q)$. Since $||p_3, P|| = 2$ by Lemma 21, q is not adjacent to p_3 . Then since $d_G(p_3) = 3k - 1$ by Corollary 22, $d_G(q) \ge 3k - 1 \ge 5$. Since $||q, P|| \le 3$, q must be adjacent to two vertices $v_1, v_2 \in R \setminus P$. By Lemma 21, $N_C(v_1) = N_C(v_2) = A$. However, this yields the cycles $v_1qv_2a_2b_1a_1v_1$ and $p_2Pp_4a_3p_2$ with chords v_1a_2 and p_3a_3 , respectively, a contradiction.

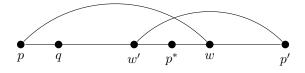


Figure 3: Setup for Lemma 28

pstar

K23

Lemma 28. $G[P] \cong K_{2,3}$

Proof. By Lemma 27, we may assume that $\mathcal{P} = \{p, p', p^*\}$. Recall that $\mathcal{P} \subseteq \mathcal{T}$, so Lemma 26 implies that \mathcal{P} is an independent set. Lemma 21 implies that $||p, \mathcal{P}|| = ||p', \mathcal{P}|| = ||p^*, \mathcal{P}|| = 2$, so there exist w and w' on \mathcal{P} such that $w \neq q$ and $w' \neq q'$ and $N_P(p) = \{q, w\}$ and $N_P(p') = \{q', w'\}$. Furthermore, since $|\mathcal{P}| = 3$, and both the neighbor of w on $p\mathcal{P}w$ and the neighbor of w' on $w'\mathcal{P}p'$ are in \mathcal{P} , we can conclude that $w \neq w'$ and $N_P(p^*) = \{w, w'\}$, i.e. $w\mathcal{P}w'$ is the path on three vertices wp^*w' .

Since G[P] does not contain a chorded cycle, $qq' \notin E$, so if w = q' and w' = q, then $G \cong K_{2,3}$. So if $G \ncong K_{2,3}$, then without loss of generality we can assume that $q \neq w'$ as in Figure 3. Thus, $qp', pw' \notin E(G)$ so, by Corollary 22, $d_G(q), d_G(w') \ge 3k - 1$.

Fix $C \in \mathcal{C}$ with particle sets $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ such that $N_C(p) = N_C(p') = N_C(p^*) = A$. By Lemma 16, $N_C(q) \subseteq B$ and $N_C(w') \subseteq B$.

Since $d_G(q) \ge 3k - 1$, $||q, C \cup R|| \ge (3k - 1) - 3(k - 2) = 5$. Also, $||q, P|| \le 3$. This holds for w' as well. Thus, both q and w' have two neighbors in $B \cup (R \setminus P)$. Let v_1 and v_2 be distinct vertices in $B \cup (R \setminus P)$ such that $v_1q, v_2w' \in E(G)$. We may assume that $v_2 \ne b_3$. Observe that $N_C(v_1) = N_C(v_2) = A$. Then the cycle $pqv_1a_1b_3a_3p$ has chord pa_1 , and the cycle $w'v_2a_2p'Pw'$ has chord a_2p^* , a contradiction.

Lemma 29. $G = G_1(n, k)$

Proof. By Lemma 28, let $\{p_1, p_2, p_3\}$ and $\{q_1, q_2\}$ denote the partite sets of G[P]. Recall that $\mathcal{P} \subseteq T$ so that $G[\mathcal{S} \cup \mathcal{T}]$ contains every vertex of G except for q_1 and q_2 .

By Lemmas 21 and 25 and Corollary 22, ||v, P|| = 2 for all $v \in R \setminus P$, and by Proposition 26, $N_R(v) = \{q_1, q_2\}$. Since \mathcal{T} is an independent set in G, for each $u \in \mathcal{T} \setminus T$, $||u, R|| \ge (3k-1) - 3(k-1) = 2$. Thus, $uq_1, uq_2 \in E(G)$, and so $N_G(q_i) \supseteq \mathcal{T}$ for $i \in [2]$. That is, $G \supseteq K_{|S|+2,|\mathcal{T}|} = K_{3k-1,|G|-3k+1} = G_1(n,k)$. Since adding any edge to $G_1(n,k)$ results in a graph with k disjoint chorded cycles, we conclude $G = G_1(n,k)$.

4 Suppose V(R) = V(P)

In this section, we assume V(P) = V(R). Since adding any edge to G results in k chorded cycles, by (O1) $|P| \ge 4$. If $|P| \ge 6$, we label $P = p_1q_1r_1\cdots r_2q_2p_2$. Note that, since G[R] has no chorded cycles, for every $v \in R$, $||r, R|| \le 4$. When |P| = 5, we let $P = p_1q_1r_2p_2$, and when |P| = 4, we let $P = p_1q_1q_2p_2$. We call an edge in $E(G[P]) \setminus E(P)$ a hop. If $Q = v_1 \cdots v_{|R|}$ is a spanning path of R, then we call an edge v_iv_j a hop (on Q) if |i - j| > 1.

Lemma 30. If $Q = v_1 \cdots v_{|R|}$ is a spanning path of R and $v_i v_j$ is a hop with i < j, then v_{i+1} and v_{i+2} cannot both be incident to hops, and similarly, v_{i-1} and v_{i-2} cannot both be incident to hops.

Proof. Suppose that, on the contrary, $v_{i+1}v_k$ and $v_{i+2}v_{k'}$ are both hops. Note that, if we consider only the hop v_iv_j and the hop $v_{i+1}v_k$, $v_jv_iv_{i+1}Qv_j$ is a chorded cycle if $i + 3 \le k \le j$, and $v_kQv_iv_jQv_{i+1}v_k$ is a chorded cycle if $k \le i-1$, so k > j. Repeating this argument but now only considering the hops $v_{i+1}v_k$ and $v_{i+2}v_{k'}$ gives us that k' > k, but then $v_iv_{i+1}v_{i+2}v_{k'}Qv_jv_i$ is a cycle with chord $v_{i+1}v_k$, a contradiction. By symmetry, the lemma holds.

pchords Lemma 31. For any $p \in \mathcal{P}$, $d_R(p) = 2$ unless R is a path.

Proof. Let $v_1 \cdots v_{|R|}$ be a spanning path in R, and let $p = v_1$. Assume $d_R(p) = 1$, and that R is not a path. Since R is not a path, hops exist. Let $v_i v_j$, i < j, be a hop such that for all $k, j < k \leq |R|$, v_k is not incident to a hop. Note, because $d_R(p) = 1$, that $i \neq 1$.

Let D be the cycle $v_j v_i v_{i+1} \cdots v_{j-1} v_j$. Since R contains no chorded cycles, v_j is incident to exactly one hop and v_{j-1} is incident to at most one hop. If v_{j-1} is not incident to a hop let $x = v_{j-1}$ and $y = v_j$, and if v_{j-1} is incident to exactly one hop, let $x = v_{j-2}$ and $y = v_{j-1}$. By Lemma 30, when v_{j-1} is incident to a hop, v_{j-2} is not incident to a hop, so in either case, $xy \in E(D)$, $d_R(x) + d_R(y) \leq 5$, and $px, py \notin E(G)$. Therefore,

$$2\|p,\mathcal{C}\| + \|\{x,y\},\mathcal{C}\| \ge 2(6k-2) - (2\|p,R\| + \|\{x,y\},R\|) > 12(k-1).$$

So there exists $C \in C$ such that $2||p, C|| + ||\{x, y\}, C|| \ge 13$. Thus, ||v, C|| = 4 for some $v \in \{p, x, y\}$, and by Lemma 13, $G[C] \cong K_4$. Further, $||\{x, y\}, C|| \ge 5$ so that there exists $c \in C$ such that $xc, yc \in E(G)$ and D + c contains a chorded cycle. Also $2||p, C|| \ge 5$, which implies $||p, C - c|| \ge 2$ so that C - c + p contains a chorded cycle, a contradiction.

long path

hops

Lemma 32. If $|R| \ge 6$, then there exists $F^+ \subseteq V(R)$ such that $|F^+| = 6$ and such that for every $C \in C$ and every pair of distinct vertices $u, u' \in F^+$, $||\{u, u'\}, C|| \ge 1$.

Proof. First we find $F^+ \subseteq V(R)$ such that $||F^+, R|| \leq 15$. If R is a path, this is trivial, so we assume R has at least one hop. By Lemmas 30 and 31, p_i is incident to a hop so that q_i and r_i cannot both be incident to hops. If $d_R(r_i) \leq 3$ for some $i \in [2]$, then since $d_R(q_i) \leq 3$ and $d_R(p_i) = 2$ by Lemma 31, $||\{p_i, q_i, r_i\}, R|| \leq 7$. If $d_R(r_i) = 4$, then $d_R(p_i) = d_R(q_i) = 2$, so that $||\{p_i, q_i, r_i\}, R|| \leq 8$. Therefore, $F^+ := \{p_1, q_1, r_1, r_2, q_2, p_2\}$ suffices when either $d_R(r_1) \leq 3$ or $d_R(r_2) \leq 3$. In this case, we let $r_1^* = r_1$.

When $d_R(r_1) = d_R(r_2) = 4$, $|R| \ge 7$, since R has no chorded cycles, and there exists a vertex u following r_1 on P with $d_R(u) \le 3$. Here, we let $F^+ := \{p_1, q_1, u, r_2, q_2, p_2\}$. and let $r_1^* = u$. Thus, in both cases, $F^+ = \{p_1, q_1, r_1^*, r_2, q_2, p_2\}$.

We claim that we can partition F^+ into three sets so that each set will consist of two nonadjacent vertices. Define $F_1 := \{p_1, q_1, r_1^*\}$ and $F_2 := \{p_2, q_2, r_2\}$, and let H be the subgraph of G on the vertex set F^+ containing precisely those edges of G with one endpoint in F_1 and the other in F_2 . Because R contains no chorded cycle, every vertex in F_2 has at most two neighbors in F_1 , and vice-versa. That is, $H \subseteq 3K_2$. Therefore we can label $F_1 = \{f_1, f_2, f_3\}$ so that f_1p_2, f_2q_2 , and f_3r_2 are all nonedges.

Therefore, $||F^+, \mathcal{C}|| \ge 3(6k-2) - 15 = 18(k-1) - 3$. Suppose there exists $C \in \mathcal{C}$ for which $||F^+, C|| \le 14$ so that there exists $C' \in \mathcal{C}$ such that $||F^+, C'|| \ge 19$. If we can find $v_1, v_2 \in F^+$ such that $||\{v_1, v_2\}, C'|| \le 6$, then $||F'-v_1-v_2, C'|| \ge 13$, contradicting Lemma 14. So for $F^+ = \{v_1, v_2, \dots, v_6\}$, $||\{v_i, v_{i+1}\}, C'|| \ge 7$ for $i \in \{1, 3, 5\}$. However this implies $||\{v_1, v_2, v_3, v_4\}, C'|| \ge 14$, a contradiction to Lemma 14.

Thus, $||F^+, C|| \ge 15$ for every $C \in C$. If there exists a pair of distinct vertices $u, u' \in F^+$ such that $||\{u, u'\}, C|| = 0$, then $||F^+ - u - u', C|| \ge 15$, again a violation of Lemma 14.

Lemma 33. There exists $F \subseteq V(R)$ such that $p_1, p_2 \in F$, |F| = 4 and

- (a) $||F,C|| \ge 12(k-1) 2$ if $R \cong K_{2,3}$, $||F,C|| \ge 12(k-1) + 2$ if R is a path, and $||F,C|| \ge 12(k-1)$ otherwise, and
- (b) if R is not a path, then for every $u \in F$, there exists a path Q in R u such that $F u \subseteq V(Q)$.

Proof. If R is a path or $R \cong K_{2,3}$, let $F := \{p_1, q_1, q_2, p_2\}$. When R is a path, ||F, R|| = 6, and $p_1q_2, p_2q_1 \notin E(G)$; when $R \cong K_{2,3}$, ||F, R|| = 10, and $p_1p_2, q_1q_2 \notin E(G)$. In both cases, (a) and (b) hold.

So we assume $R \not\cong K_{2,3}$ and R is not a path. By Lemma 31, for $i \in [2]$, $||p_i, P|| = 2$. Thus, p_i has a neighbor $w_i \in P - q_i$. Let t_i denote the neighbor of w_i on $w_i P p_i$. Observe that $t_i \in \mathcal{P}$, so by Lemma 31, $||t_i, P|| = 2$. Suppose $t_1 \neq t_2$, and, in this case, let $F := \{p_1, t_1, t_2, p_2\}$. Then $F \subseteq \mathcal{P}$, so (b) holds and $||F, R|| \leq 8$. If either $p_1 t_1, p_2 t_2 \notin E(G)$ or $p_1 t_2, p_2 t_1 \notin E(G)$, then (a) holds. Suppose (say) $p_1 t_1 \in E(G)$. Then $t_1 = q_1$, and $t_1 p_2 \notin E(G)$. Then $w_2 \notin \{p_1, t_1\}$, hence $t_2 \notin \{t_1, w_1\} = N_R(p_1)$, so also $p_1 t_2 \notin E(G)$. So in this case also, (a) holds.

So assume $t_1 = t_2$, which implies ||u, P|| = 2 for all $u \in V(P) - w_1 - w_2$, as otherwise R contains a chorded cycle. Also, when $t_1 = t_2$, we may assume that $q_1 \neq w_2$ since R is not isomorphic to $K_{2,3}$. In this case, let $F := \{p_1, q_1, t_1, p_2\}$ and note that $p_1t_1, q_1p_2 \notin E(G)$. Since $d_R(u) = 2$ for all $u \in F$, (a) holds. Since $t_1 = t_2, p_1w_1t_1w_2p_2$ is a path in $R - q_1$ containing $F - q_1$ and $F - q_1 \subseteq \mathcal{P}$, (b) holds.

Corollary 34. R is not a path.

Proof. Let $F \subseteq V(R)$ be as guaranteed in Lemma 33. If R is a path, then $||F, C|| \ge 12(k-1) + 2$, so that there exists $C \in C$ such that $||F, C|| \ge 13$, which violates Lemma 14. So R is not path.

struct2 Lemma 35. Let $F \subseteq V(R)$ be as guaranteed in Lemma 33. If ||F, C|| = 12 for any $C \in C$, then $G[C] \cong K_{3,3}$.

Proof. Let $F \subseteq V(R)$ be as guaranteed in Lemma 33 and let $C \in C$. Suppose that ||F, C|| = 12. By Lemmas 14 and 33, this is true for all $C \in C$, unless $R \cong K_{2,3}$. By Lemmas 13 and 15, $C \cong K_{3,3}$ unless |C| = 4, so assume |C| = 4. Note that for any $u \in F$ and $c \in C$, if C - c + u is a chorded cycle, then $||c, F - u|| \leq 2$, because there exists a path Q in R such that $F - u \subseteq V(Q)$ and G[Q + c] cannot contain a chorded cycle.

paths

Fdeg

struct1

R6toC

First assume that C is singly chorded, so we can label $V(C) = \{c_1, c_2, c_3, c_4\}$ such that $c_1c_2c_3c_4$ is a cycle and c_2c_4 is the chord. By Lemma 13, ||u, C|| = 3 for every $u \in F$, and $||c_i, F|| = 4$, for $i \in \{1, 3\}$. Recall that $p_1, p_2 \in F$ so that $C - c_1 + p_1$ and $P - p_1 + c_1$ both contain chorded cycles, a contradiction.

So for the remainder of the proof, we assume $G[C] \cong K_4$, with $V(C) = \{c_1, c_2, c_3, c_4\}$. Fix $u \in F$, and by Lemma 33, let Q be a path in R - u such that $F - u \subseteq V(Q)$. Suppose ||u, C|| = 3, so ||F - u, C|| = 9, and there exists $c \in C$ such that c is adjacent to all three vertices in F - u. This implies Q + c and C - c + uboth contain chorded cycles, a contradiction.

Now suppose ||u, C|| = 2 and $N_C(u) = \{c_1, c_2\}$. Then ||F - u, C|| = 10, and there exist two vertices in C adjacent to all three vertices in F - u. If c' is one of these two vertices and $c' \notin \{c_1, c_2\}$, then Q + c' and C - c' + u both contain chorded cycles, a contradiction. Therefore, every vertex in F is adjacent to both c_1 and c_2 . Since ||F, C|| = 12 and ||u, C|| = 2, there exists $v \in F - u$ such that ||v, C|| = 4. By Lemma 33, there exists a path Q' in R - v such that $F - v \subseteq V(Q')$, so that $C - c_1 + v$ and $Q' + c_1$ both contain chorded cycles, a contradiction.

So $||u, C|| \in \{0, 1, 4\}$, for every $u \in F$. Since ||F, C|| = 12, there exists $u' \in F$ such that ||u', C|| = 0 and ||u, C|| = 4 for every $u \in F - u'$. By Lemma 33, $p_1, p_2 \in F$, so we may assume $||p_1, C|| = 4$. Thus, for all $c \in C$, $C - c + p_1$ is a chorded cycle, and further $||c, P - p_1|| \leq 2$, else $P - p_1 + c$ contains a chorded cycle. Therefore, if $||R \setminus F, C|| > 0$, we can pick c such that $||c, P - p_1|| \geq 3$ so that $P - p_1 + c$ has a chorded cycle, a contradiction.

Thus $||R \setminus F, C|| = 0$. By Lemma 32, $|R| \leq 5$, as otherwise we can find $F^+ \subseteq V(R)$ with $|F^+| = 6$ so that for distinct $v, v' \in F^+ \setminus F$, $||\{v, v'\}, C|| \geq 1$, a contradiction. If |R| = 4, then u' has a neighbor $v \in F - u'$. Since R is not a path, by Lemma 31 $R \cong C_4$, so replacing C with C' := C - c + v in C gives a collection of k - 1 chorded cycles that satisfies (O1) - (O3), but R' := R - v + c has a path P' such that |P'| = |R'| and such that u' is an endpoint and such that ||u', R'|| = 1. This is a contradiction to Lemma 31.

So assume |R| = 5 so that $P = p_1q_1rq_2p_2$. By Lemma 31, either $p_1r, p_2r \in E(G)$, or $R \in \{C_5, K_{2,3}\}$. In each of these cases, we can assume that $F = \{p_1, q_1, q_2, p_2\}$, by the proof Lemma 33. Recall that $||p_1, C|| = 4$ and ||u', C|| = 0 for some $u' \in F$. Furthermore, since $||R \setminus F, C|| = 0$, ||r, C|| = 0.

Suppose $R \in \{C_5, K_{2,3}\}$. Let $F' := \{q_1, r, q_2, p_2\}$, so that $u' \in F'$, $||F', C|| \le 8$ and $||F', R|| \le 10$. Since $q_1q_2, rp_2 \notin E(G), ||F', C - C|| \ge 12(k-2) + 2$ so that $k \ge 3$ and $||F', C'|| \ge 13$ for some $C' \in C - C$, a contradiction to Lemma 14.

Thus $p_1r, p_2r \in E(G)$. Since three of the five vertices in R send four edges to C, there exists $i \in [2]$, such that at least two vertices in $\{r, q_i, p_i\}$ have four neighbors in C, and so have a common neighbor $c \in C$. This implies that $G[\{r, q_i, p_i, c\}]$ contains a chorded cycle. Furthermore, there exists $v \in \{p_{3-i}, q_{3-i}\}$ such that v has four neighbors in C, and so C - c + v contains a chorded cycle, a contradiction.

Thus, $|C| \neq 4$ and $G[C] \cong K_{3,3}$, as desired.

structure

Lemma 36. If $R \not\cong K_{2,3}$, then $G[C] \cong K_{3,3}$ for all $C \in C$. If $R \cong K_{2,3}$, then $G[C] \cong K_{3,3}$ for all but at most one $C \in C$, and for any such C, $G[C] \cong K_{1,1,2}$ and $G[V(R) \cup V(C)] \cong K_{1,4,4}$.

Proof. Let $F \subseteq V(R)$ be as guaranteed by Lemma 33. If R is not isomorphic to $K_{2,3}$, then $||F, C|| \ge 12(k-1)$. By Lemma 14, $||F, C|| \le 12$ for all $C \in C$ so that in fact, equality holds for all $C \in C$. Thus, by Lemma 35, $G[C] \cong K_{3,3}$ for all $C \in C$.

So assume $R \cong K_{2,3}$ with partite sets $A = \{p_1, p_2, p_3\}$ and $B = \{q_1, q_2\}$ with |A| = 3 and |B| = 2. Since A and B are independent, we have $||B, C|| \ge 6k - 8$ and

$$2\|A, \mathcal{C}\| = \sum_{a \in A} 2\|a, \mathcal{C}\| \ge 3(6k - 2) - 12 = 18k - 18k$$

so $||A, C|| \ge 9(k-1)$ and $||R, C|| \ge 15k - 17 = 15(k-1) - 2$. If $||R, C|| \ge 16$ for some $C \in C$, then there exists some $u \in R$ such that ||u, C|| = 4. By Lemma 14, $||R - u, C|| \le 12$ so that there exists $u' \in R - u$ such that $||u', C|| \le 3$. However, $||R - u', C|| \ge 13$, a contradiction to Lemma 14.

We therefore have that, for ever $C \in C$, $13 \leq ||R, C|| \leq 15$. Fix $C \in C$. At least two vertices in R have three neighbors each in C so that by Lemmas 13 and 15, |C| = 4 or $G[C] \cong K_{3,3}$. We claim that $G[C] \ncong K_4$.

Suppose on the contrary, $G[C] \cong K_4$. If $||p_i, C|| \ge 3$ for some $i \in [3]$, Lemma 14 implies that $||R, C|| \le 12$, a contradiction. So $||p_i, C|| \le 2$ for all $i \in [3]$. Hence $||B, C|| \ge 7$ so that for all $c \in C$ and $j \in [2]$, $C - c + q_j$ is a chorded cycle. As $||R, C|| \ge 13$, there exists $c \in C$ such that $||c, R|| \ge 4$. Without loss of generality, $N_R(c) \supseteq \{p_1, p_2, q_1\}$. However, $C - c + q_2$ and $p_1 c p_2 q_1 p_1$ each contain chorded cycles, a contradiction.

So for all $C \in C$, either |C| = 4 and C is singly chorded or $G[C] \cong K_{3,3}$. By Lemma 13, $||u, C|| \leq 3$ for all $u \in A$ and $C \in C$. Since $||A, C|| \geq 9(k-1)$, we deduce that ||A, C|| = 9 and so ||u, C|| = 3 for all $u \in A$ and $C \in C$.

Suppose |C| = 4 and C is singly chorded. We can label $V(C) = \{c_1, c_2, c_3, c_4\}$ such that $c_1c_2c_3c_4$ is a cycle and c_2c_4 is the chord. By Lemma 13, $uc_1, uc_3 \in E(G)$ for all $u \in A$. Since, $C - c_i + u$ is a chorded cycle for $i \in \{1, 3\}, R - u + c_i$ cannot contain a chorded cycle, which implies that $N_R(c_i) = A$. Hence, for every $v \in B$, $N_C(v) \subseteq \{c_2, c_4\}$, and since $||R, C|| \ge 13$, equality holds and $N_C(v) = \{c_2, c_4\}$ for every $v \in B$.

Fix $u \in A$. Without loss of generality, assume $N_C(u) = \{c_1, c_3, c_4\}$. Then $C - c_2 + u$ is a chorded cycle. If $u' \in A - u$ has $c_2 \in N_C(u)$, then $R - u + c_2$ contains a chorded cycle, a contradiction. Thus, for all $w \in A$, $N_C(w) = \{c_1, c_3, c_4\}$ so that $N_R(c_4) = V(R)$ and $G[R \cup C] \cong K_{4,4,1}$.

Recall that $||R, C|| \ge 15(k-1) - 2$ and $||R, C'|| \le 15$ for all $C' \in C$. Further, $||u, C'|| \le 3$ for all $u \in R$ and $C' \in C$. Since ||R, C|| = 13, ||R, C''|| = 15 for every $C'' \in C - C$. However, for any $u \in A$, $||u, C'|| \le 3$ so that F := R - u satisfies $||F, C''|| \ge 12$. Furthermore, F satisfies all the hypotheses of Lemmas 33 and 35, so that $G[C''] \cong K_{3,3}$ for all $C'' \in C - C$.

This completes the proof of the lemma.

pqchords

Lemma 37. For every $u \in R$ and $C \in C$, $||u, C|| \leq 3$. If P' is path that spans R, p is an endpoint of P' and q is adjacent to p on P', then $d_G(p) = 3k - 1$ and $d_G(q) \geq 3k - 1$. In particular, for every $C \in C$ ||p, C|| = 3 and $||q, C|| \geq 2$.

Proof. Let p and p' be the two endpoints of P', and let q and q' be the neighbors of p and p', respectively, on P'. By Lemmas 13 and 36, $||u, C|| \leq 3$ for all $u \in R$ and $C \in C$. Therefore, if $d_R(u) = 2$, then $d_G(u) \leq 3k-1$, so in particular, $d_G(p) \leq 3k-1$ and $d_G(p') \leq 3k-1$. If $pp' \notin E$, then $d_G(p') = d_G(p) = 3k-1$. Otherwise, $pp' \in E$ and p is not adjacent to q'. In this case, $d_R(q') = 2$ so that $d_G(p) = 3k-1$. Since $||u, C|| \leq 3$ for all $u \in R$ and $C \in C$, it follows that ||p, C|| = 3. By symmetry, this holds for p' as well.

Since $||q, R|| \leq 3$, if we can show that $d_G(q) \geq 3k - 1$, it follows that $||q, C|| \geq 2$ for all $C \in C$. So assume $d_G(q) \leq 3k - 2$. Now, $qp' \in E(G)$, as otherwise $d_G(q) \geq 3k - 1$. If |R| = 4, then by Lemma 31, R contains a chorded cycle. So |R| > 4, and as a result $qq' \notin E(G)$. Since $d_G(q) \leq 3k - 2$, we get $d_G(q') \geq 3k$, and furthermore, since $d_R(q') \leq 3$ and $||q', C|| \leq 3$ for all $C \in C$, we deduce that ||q', C|| = 3 and $d_R(q') = 3$. This implies $pq' \in E(G)$, as otherwise we get a chorded cycle in R. Furthermore, $d_G(q) = 3k - 2$ and $||q, R|| \leq 3$ so that $||q, C|| \geq 1$ for all $C \in C$.

Since $|R| \ge 5$, there exists $r' \notin \{p, p'\}$ a neighbor of q' on P'. Note that $r' \in \mathcal{P}$ so that by the above, $d_G(r') = 3k - 1$ and ||r', C|| = 3 for all $C \in \mathcal{C}$. If $|R| \ge 6$, then $r'q \notin E(G)$ and $d_G(q) \ge 3k - 1$, a contradiction. Hence, |R| = 5, and, furthermore, $R \cong K_{2,3}$ with partite sets $\{q, q'\}$ and $\{p, p', r'\}$. Observe that for all $u \in \{p, r', q', p'\}$ and $C \in \mathcal{C}$, ||u, C|| = 3.

If know fix $C \in C$, such that $||q, C|| \leq 2$, which must exist because d(q) = 3k - 2 and $d_R(q) = 3$. By Lemma 36, $G[C] \in \{K_{3,3}, K_{1,1,2}\}$. Furthermore, if $G[C] \cong K_{1,1,2}$, then $G[C \cup R] = K_{1,4,4}$, but this contradicts the fact that $||q', C \cup R|| = 6$. Hence, $C \cong K_{3,3}$ and let A and B denote its partite sets. By Lemmas 13 and 16, we may assume $N_C(p) = N_C(r') = N_C(p') = A$, $N_C(q') = B$, and $N_C(q) \subseteq B$. Since $||q, C|| \leq 2$, there exists $b \in B \setminus N_C(q)$. We can replace C with C - b + p' and replace P' with bq'P'p. Our new collection and path satisfy (O1)-(O3). However, b is an endpoint of our new path and by the above, $d_G(b) = 3k - 1$. Since $bq \notin E(G)$, $d_G(q) \geq 3k - 1$, a contradiction.

K32_K22 Lemma 38. R is either isomorphic to $K_{2,3}$ or $K_{2,2}$.

Proof. If |R| = 4, then Lemmas 31 implies that $R \cong K_{2,2}$, so assume $|R| \ge 5$ and R is not isomorphic to $K_{2,3}$. Let $P = u_1, \ldots, u_{|R|}, p := u_1, q := u_2, q' := u_{|R|-1}$ and $p' := u_{|R|}$. Let $C \in \mathcal{C}$. By Lemma 36, $G[C] \cong K_{3,3}$, so we let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be its partite sets. Recall that by Lemma 13, if ||u, C|| = 3 for any $u \in R$, then $N_C(u) \in \{A, B\}$.

First assume that R is Hamiltonian (that is, R contains a cycle of size |R|). Since every vertex in R is the endpoint of a path spanning R, by Lemma 37, ||u, C|| = 3 for every $C \in C$ and $u \in R$. By Lemma 16, we can assume that $N_C(u_i) = A$ if i is odd and $N(u_i) = B$ is i is even. Therefore, Lemma 16 implies that |R| is even, which further implies that $|R| \ge 6$. Then for any $a \in A$ and $b \in B$, $G[\{u_1, \ldots, u_4, a, b\}]$ and $C - a - b + u_5 + u_6$ contain chorded cycles, a contradiction.

So we can assume R is not Hamiltonian. Let pw be a hop on P so that $w \neq p'$. First assume $w \neq q'$. Without loss generality assume that $N_C(p') = A$. By Lemmas 16 and 37, $N_C(p) \cap N_C(q) = \emptyset$, and so there exists $cc' \in E(C)$ such that pcc'qPwp is a cycle with chord pq. By Lemmas 16 and 37, $|N_C(p') - c - c'| \geq 2$ and $|N_C(q') - c - c'| \geq 1$, so C - c - c' + p' + q' contains a chorded cycle, a contradiction.

Now we can assume that both pq' and qp' are edges. Since $R \neq K_{2,3}$, we have that $|R| \ge 6$. Let $r \neq p$ and $r' \neq p'$ be the neighbors of q and q', respectively, on P. Note that r and r' are endpoints of paths spanning R so that ||r, C|| = ||r', C|| = 3. By Lemmas 16 and 37, and because $pq', qp' \in E(G)$, we may assume that $N_C(p) = N_C(r) = N_C(r') = N_C(p') = A$ and $N_C(q) \cup N_C(q') \subseteq B$. In particular, we may assume $qb_1 \in E(G)$ so that $pa_1b_2a_2b_1qp$ is a cycle with chord pa_2 , and $rPp'a_3r$ is a cycle with chord a_3r' , a contradiction.

So |R| = 5 and $R \cong K_{2,3}$, as desired.

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Lemma 39. If $G[C] \cong K_{3,3}$ for every $C \in C$, then $G \cong G_1(n,k)$.

Proof. By Lemma 38, $R \in \{K_{2,2}, K_{2,3}\}$. So let $U_1, U_2 \subseteq V(R)$ be the partite sets of R such that $|U_1| \ge |U_2| = 2$, and let $u_1 \in U_1, V_2 := N_G(u_1)$, and $V_1 := V(G) \setminus V_2$. Since u_1 is the end of spanning path of R, Lemma 37 implies that $|V_2| = 3k - 1$. Since $|G| \le 6(k-1) + 5$, $|V_1| \le 3k$. We aim to show that $N_G(v) = V_2$ for all $v \in V_1$. This will imply that $G \cong G_1(n, k)$.

Fix $v \in V_1 - u_1$. Since $u_1 v \notin E(G)$, Lemma 37 implies that $d_G(v) \geq 3k - 1$. If $v \in U_1$, then v is the end of a spanning path of R, and by Lemmas 13, 16 and 37, $N_G(v) = N_G(u_1) = V_2$. So we may assume $v \in V_1 \setminus U_1$, and in particular, $v \in C$ for some $C \in C$.

Define $V'_1 := \{u \in V_1 : ||u, U_2|| \ge 1\}$, and suppose $v \in V'_1 \setminus U_1$. Recall that we are assuming $G[C] \cong K_{3,3}$ for all $C \in \mathcal{C}$ so that by Lemma 13, $G[C - v + u_1] \cong K_{3,3}$. Furthermore, v is an end of a path of length |R| in $R' := R - u_1 + v$. This new collection and path satisfy (O1)-(O3), so by Lemma 38, $R' \cong R$ and $N_G(v) = N_G(u_1) = V_2$.

Now suppose $v \in V_1 \setminus V'_1$. Since $d_G(v) \ge 3k - 1$ and v has at most 3(k - 1) neighbors in V_2 , v must have two neighbors in V_1 . By Lemmas 16 and 37, for every $u_2 \in U_2$, $d_G(u_2) \ge 3k - 1$ and $N_G(u_2) \subseteq V_1$, so that $|V'_1| \ge 3k - 1$. Since $|V_1| \le 3k$, v has a neighbor, say v', in V'_1 . However, by the above, $N_G(v') = V_2$, which contradicts the fact that vv' is an edge. Therefore, $V'_1 = V_1$ which finishes the proof of the lemma.

Lemma 40. Suppose there exists $C \in C$ with |C| = 4. Then $G \cong G_2(k)$.

Proof. By Lemmas 36 and 38, we can assume $R \cong K_{2,3}$, $G[C] \cong K_{1,1,2}$, and $G[R \cup C] \cong K_{1,4,4}$. Let A' and B' be the two partite sets of size four and $\{c\}$ be the partite set of size one in $G[R \cup C]$. By symmetry, we can assume that any $v \in A' \cup B'$ is an end of a spanning path in R or the end of a spanning path of $G[V(G) \setminus V(C')]$ for some collection C' of k-1 vertex disjoint cycles that satisfies (O1)-(O3), so, by Lemma 37, $d_G(v) = 3k-1$ and ||v, C-C|| = 3(k-2). By Lemma 36, for all $D \in C - C$, $G[D] \cong K_{3,3}$, and, with Lemma 16, we deduce that ||v, D|| = 3 and that we can label the partite sets of D as A_D and B_D so that for every $p \in A'$, $N_D(p) = B_D$ and for every $q \in B'$, $N_D(q) = A_D$. Therefore, there exists a partition $\{A, B, \{c\}\}$ of V(G) such that for every $p \in A'$, $N_G(p) = B + c$, for every $q \in B'$, $N_G(q) = A + c$, and |A| = |B| = 3k - 2.

If $u \in V(G) \setminus (A' \cup B')$, then there exists $D \in \mathcal{C}-C$, such that $u \in D$. Let $p \in A' \cap V(R)$, and $q \in B' \cap V(R)$ and label $\{w, w'\} = \{p, q\}$ so that $uw \notin E(G)$ and $uw' \in E(G)$. We have that $G[D - u + w] \cong K_{3,3}$ and $G[R - w + u] \cong K_{3,2}$, so there exists a collection \mathcal{C}' of k - 1 vertex disjoint cycles containing C that satisfies (O1)-(O3), and there exists a spanning path of of $G[V(G) \setminus V(\mathcal{C}')]$ such that u is an endpoint or u is the neighbor of an endpoint. Therefore, by Lemma 37, $d_G(u) \ge 3k - 1$, so, with Lemma 36, we have that $N_C(u) = (V(C) \setminus N_C(w')) + c$ and, for any $D' \in \mathcal{C}' - C$, by Lemma 16, $N_{D'}(u) = D' \setminus N_{D'}(w')$. Therefore, either $N_G(u) \supseteq B + c$ if $u \in A$ or $N_G(u) \supseteq A + c$ if $u \in B$. Hence, G contains $G_2(k)$ as a spanning subgraph. As $G_2(k)$ is edge-maximal with respect to not containing k disjoint chorded cycles, $G \cong G_2(k)$. Using Lemmas 36, 38, 39, and 40, we conclude $G \in \{G_1(n,k), G_2(k)\}$.

5 Concluding Remarks

remarks

Many variations on Theorems 1 and 5 have appeared, and suggest further extensions of Theorem 9. We present only a small selection below.

A result of Gould, Hirohata, and Horn [8] implies the following:

Theorem 41. Let G be a graph on $|G| \ge 6k$ vertices with $\delta(G) \ge 3k$. Then G contains k disjoint doubly chorded cycles.

While it is not clear that $|G| \ge 6k$ is necessary, it would be interesting to characterize the sharpness examples for this theorem; that is, if $|G| \ge 6k$ and $\delta(G) = 3k - 1$ but G does not contain k disjoint doubly chorded cycles, what does G look like? For more results on the existence of k disjoint multiply chorded cycles, see [9]

Additionally, rather than consider $\delta(G)$ or $\sigma_2(G)$, one may consider the neighborhood union, $\min\{|N(x) \cup N(y)| : xy \in E(\overline{G})\}$. See the following results.

Theorem 42 (Faudree-Gould, [6]). If G has $n \ge 3k$ vertices and $|N(x) \cup N(y)| \ge 3k$ for all nonadjacent pairs of vertices x, y, then G contains k disjoint cycles.

Theorem 43 (Gould-Hirohata-Horn, [8]). Let G be a graph on at least 4k vertices such that for any nonadjacent $x, y \in V(G)$, $|N(x) \cup N(y)| \ge 4k + 1$. Then G contains k disjoint chorded cycles.

Theorem 44 (Gould-Hirohata-Horn, [8]). Let G be a graph on n > 30k vertices such that for any nonadjacent $x, y \in V(G)$, $|N(x) \cup N(y)| \ge 2k + 1$. Then G contains k disjoint cycles.

Theorem 45 (Qiao, [13]). Let r, s be nonnegative integers, and let G be a graph on at least 3r + 4s vertices such that for any nonadjacent $x, y \in V(G)$, $|N(x) \cup N(y)| \ge 3r + 4s + 1$. Then G contains r + s disjoint cycles, s of them chorded.

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