# A refinement of theorems on vertex-disjoint chorded cycles 

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#### Abstract

In 1963, Corrádi and Hajnal settled a conjecture of Erdős by proving that, for all $k \geq 1$, any graph $G$ with $|G| \geq 3 k$ and minimum degree at least $2 k$ contains $k$ vertex-disjoint cycles. In 2008, Finkel proved that for all $k \geq 1$, any graph $G$ with $|G| \geq 4 k$ and minimum degree at least $3 k$ contains $k$ vertex-disjoint chorded cycles. Finkel's result was strengthened by Chiba, Fujita, Gao, and Li in 2010, who showed, among other results, that for all $k \geq 1$, any graph $G$ with $|G| \geq 4 k$ and minimum Ore-degree at least $6 k-1$ contains $k$ vertex-disjoint cycles. We refine this result, characterizing the graphs $G$ with $|G| \geq 4 k$ and minimum Ore-degree at least $6 k-2$ that do not have $k$ disjoint chorded cycles.


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## 1 Introduction

All graphs in this paper are simple, unless otherwise noted. Additionally, when referring to cycles in a graph, "disjoint" is always taken to mean "vertex-disjoint." For a graph $G$, we use $V(G)$ and $E(G)$ to denote the vertices and edges, respectively, and for a vertex $v$, we will use $v \in G$ to denote $v \in V(G)$. For a vertex $v \in G$, and for a subgraph $H$ of $G$ (where possibly $H=G$ ), the neighborhood of $v$ in $H$ is denoted by $N_{H}(v)$. The number of neighbors of $v$ in $H$ (i.e., $\left.\left|N_{H}(v)\right|\right)$ will be written by $d_{H}(v)$. Furthermore, we write $|G|$ for the order of a graph $G, \bar{G}$ for its complement, $\delta(G)$ for its minimum degree, and $\alpha(G)$ for its independence number.

The minimum Ore-degree of a non-complete graph $G$ is written $\sigma_{2}(G)$, and defined as

$$
\sigma_{2}(G):=\min \left\{d_{G}(x)+d_{G}(y): x y \in E(\bar{G})\right\}
$$

that is, $\sigma_{2}(G)$ is the minimum degree-sum of nonadjacent vertices. $K_{n}$ is the complete graph on $n$ vertices, and $K_{s_{1}, \ldots, s_{t}}$ is the complete $t$-partite graph with parts of size $s_{1}, \ldots, s_{t}$. For graphs $G$ and $H, G+H$ is the disjoint union of $G$ and $H$, and $G \vee H$ is the join of $G$ and $H$.

In 1963, Corrádi and Hajnal verified a conjecture of Erdős, proving the following.
Theorem 1 (Corrádi-Hajnal, [3]). Every graph $G$ on $|G| \geq 3 k$ vertices with $\delta(G) \geq 2 k$ contains $k$ disjoint cycles.

This result of Corrádi and Hajnal has been generalized in various ways. One such generalization is a strengthening by Enomoto and Wang, who independently proved the following.

Theorem 2 (Enomoto [5], Wang [14]). Every graph $G$ on $|G| \geq 3 k$ vertices with $\sigma_{2}(G) \geq 4 k-1$ contains $k$ disjoint cycles.

[^0]Both Theorems 1 and 2 are sharp, leading to the following natural question of Dirac.
DiracQ Question 3 (Dirac, [4]). Which $(2 k-1)$-connected multigraphs do not contain $k$ disjoint cycles?
Question 3 was answered in the case of simple graphs in [10, and then in multigraphs in [11. Indeed, [10] together with [12] answer a more general question for simple graphs, describing graphs with minimum Ore-degree at least $4 k-3$ with no $k$ disjoint cycles. To avoid going into too many technical details, we only provide part of this description below.

KKYT Theorem 4 ([10, [12]). Given an integer $k \geq 4$, let $G$ be a graph on $|G| \geq 3 k$ vertices with $\sigma_{2}(G) \geq 4 k-3$. Then $G$ contains $k$ disjoint cycles if and only if none of the following hold:

1. $\alpha(G) \geq|G|-2 k+1$.
2. $G=\left(K_{c}+K_{2 k-c}\right) \vee \overline{K_{k}}$ for some odd $c$
3. $G=\left(K_{1}+K_{2 k}\right) \vee \overline{K_{k-1}}$
4. $|G|=3 k$ and $\bar{G}$ is not $k$-colorable

In 2008 Finkel proved the following chorded-cycle analogue to Theorem 1 .
FinkelT Theorem 5 (Finkel, [7]). Every graph $G$ on $|G| \geq 4 k$ vertices with $\delta(G) \geq 3 k$ contains $k$ disjoint chorded cycles.

A stronger vertion of Theorem [5 was conjectured by Bialostocki, Finkel, and Gyárfás in [1], and proved by Chiba, Fujita, Gao, and Li in [2].

CFGLT Theorem 6 (Chiba-Fujita-Gao-Li, [2]). Let $r$ and $k$ be integers with $r+k \geq 1$. Every graph $G$ on $|G| \geq$ $3 r+4 k$ vertices with $\sigma_{2}(G) \geq 4 r+6 k-1$ contains a collection of $r+k$ disjoint cycles such that $k$ of these cycles are chorded.

In particular, the following corollary holds.
CFGLC
Corollary 7 (Chibia-Fujita-Gao-Li, [2]). Every graph $G$ on $|G| \geq 4 k$ vertices with $\sigma_{2}(G) \geq 6 k-1$ contains a collection of $k$ disjoint chorded cycles.

All hypotheses in Theorem 5 and Corollary 7 are sharp. First, since any chorded cycle contains at least four vertices, if $|G|<4 k$ then $G$ does not contain $k$ disjoint chorded cycles. Second, the conditions $\delta(G) \geq 3 k$ and $\sigma_{2}(G) \geq 6 k-1$ are best possible, as demonstrated by the two graphs below.

G1 Definition 8. For $n \geq 6 k-2$, define $G_{1}(n, k):=K_{3 k-1, n-3 k+1}$ (Figure 1a). For $k \geq 2$, define $G_{2}(k):=$ $K_{3 k-2,3 k-2,1}$ (Figure 1b).

For $n \geq 6 k-2,\left|G_{1}(n, k)\right|=n \geq 4 k$ and $\sigma_{2}\left(G_{1}(n, k)\right)=6 k-2$. Each chorded cycle in $G_{1}(n, k)$ uses at least three vertices from each part, so $G_{1}(n, k)$ does not contain $k$ disjoint chorded cycles. For $k \geq 2$, $\left|G_{2}(k)\right|=6 k-3 \geq 4 k$ and $\sigma_{2}\left(G_{2}(k)\right)=6 k-2$. Each chorded cycle in $G_{2}(k)$ uses three vertices from each of the big parts, or the dominating vertex and at least two vertices from a big part, so $G_{2}(k)$ does not contain $k$ chorded cycles.

We can now ask a question similar to Question 33 which graphs $G$ with $\sigma_{2}(G) \geq 6 k-2$ do not contain $k$ disjoint chorded cycles? Our main result is the following.
main Theorem 9. For $k \geq 2$, let $G$ be a graph with $n:=|G| \geq 4 k$ and $\sigma_{2}(G) \geq 6 k-2$. $G$ does not contain $k$ disjoint chorded cycles if and only if $G \in\left\{G_{1}(n, k), G_{2}(k)\right\}$.

The condition $k \geq 2$ in Theorem 9 is necessary, as subividing every edge of a graph results in a new graph with no chorded cycles. Thus, for $k=1$, we obtain the following characterization, which is analogous to the characterization of acyclic graphs as the graphs for which there exists at most one path between every pair of vertices.

(a) $G_{1}(n, k)$, shown for $k=2$ G2F

(b) $G_{2}(k)$, shown for $k=2$

Figure 1: Graphs $G_{1}(n, k)$ and $G_{2}(k)$ from Definition 8 ,

Proposition 10. A graph $G$ has no chorded cycle if and only if for all $u v \in E(G), G-u v$ has at most one path between $u$ and $v$.

Every graph $G$ with $\delta(G) \geq 3 k-1$ also satisfies $\sigma_{2}(G) \geq 6 k-2$. Therefore, Theorem 9 is a refinement of both Theorem 5 and Corollary 7 Two other immediate corollaries of Theorem 9 are listed here.
indep cor Corollary 11. For $k \geq 2$, let $G$ be a graph with $|G| \geq 4 k, \sigma_{2}(G) \geq 6 k-2$, and $\alpha(G) \leq n-3 k$. Then $G$ contains $k$ disjoint chorded cycles.

Every graph $G$ with $\sigma_{2}(G) \geq 6 k-2$ also satisfies $\alpha(G) \leq n-3 k+1$. So, requiring $\alpha(G) \leq n-3 k$ in Corollary 11 is equivalent to requiring the seemingly weaker condition $\alpha(G) \neq n-3 k+1$.

Corollary 12. For $k \geq 2$, let $G$ be a graph with $4 k \leq|G| \leq 6 k-4$ and $\sigma_{2}(G) \geq 6 k-2$. Then $G$ contains $k$ disjoint chorded cycles.

### 1.1 Outline

We present our result as follows. In Section 2 we detail the setup of our proof and present several important lemmas that will be used throughout our paper. In particular, we find and choose an 'optimal' collection of $k-1$ disjoint cycles, and use $R$ to denote the subgraph induced by the vertices outside our collection. Then, in Section 3, we consider the case when $R$ does not have a spanning path, and, in Section 4 we consider the case when $R$ has a spanning path. We conclude our paper in Section 5 with some remarks on further extensions.

## 2 Setup and Preliminaries

### 2.1 Notation

Let $G$ be a graph, and let $A, B \subseteq V(G)$, not necessarily disjoint. We define $\|A, B\|:=\sum_{a \in A}\left|N_{G}(a) \cap B\right|$. When $A=\{a\}$ or $A$ is the vertex set of some subgraph $\mathcal{A}$, we will often replace $A$ in the above notation with $a$ or $\mathcal{A}$, respectively. Additionally, if $\mathcal{L}$ is a collection of graphs, then $\|A, \mathcal{L}\|=\left\|A, \bigcup_{L \in \mathcal{L}} V(L)\right\|$. If $A$ is the vertex set of some subgraph $\mathcal{A}$, we will write $G[\mathcal{A}]$ for $G[A]$, the subgraph of $G$ induced by the vertices of $\mathcal{A}$. Furthermore, if $\mathcal{B}$ is a subgraph of $G$ with vertex set $B$, we will use $\mathcal{A} \backslash \mathcal{B}$ to denote $G[A \backslash B]$, and if $B=\left\{b_{1}, \ldots, b_{k}\right\}$ and $k$ is small, we will also use $\mathcal{A}-b_{1}-\cdots-b_{k}$. For a vertex $v$, we additionally write $\mathcal{A}+v$ for $G[A \cup\{v\}]$.

If $P=v_{1} \ldots v_{m}$ is a path, then for $1 \leq i \leq j \leq m, v_{i} P v_{j}$ is the path $v_{i} \cdots v_{j}$. An $n$-cycle is a cycle with $n$ vertices. A singly chorded cycle is a cycle with precisely one chord, and a doubly chorded cycle is a cycle with at least two chords.

### 2.2 Setup

We let $k \geq 2$ and consider a graph $G^{\prime}$ on $n$ vertices such that $n \geq 4 k$ and $\sigma_{2}\left(G^{\prime}\right)=6 k-2$, where $G^{\prime}$ does not contain $k$ disjoint chorded cycles. We then let $G$ be a graph with vertex set $V\left(G^{\prime}\right)$ such that $E\left(G^{\prime}\right) \subseteq E(G)$ and $G$ is "edge-maximal" in the sense that, for any $e \in E(\bar{G}), G+e$ does contain $k$ disjoint chorded cycles. We then prove that $G$ is $G_{1}(n, k)$ or $G_{2}(k)$, which implies that $G=G^{\prime}$, because any proper spanning subgraph of $G_{1}(n, k)$ or $G_{2}(k)$ has minimum Ore-degree less than $6 k-2$. Since we have already observed that $G_{1}(n, k)$ and $G_{2}(k)$ do not contain $k$ disjoint chorded cycles, this will prove Theorem 9 ,

Note that $G \not \approx K_{n}$, else $G$ contains $k$ disjoint chorded cycles. So there exists $e \in E(\bar{G})$, and by our edgemaximality condition, $G$ contains $k-1$ disjoint chorded cycles. Over all possible collections of $k-1$ disjoint chorded cycles in $G$, let $\mathcal{C}$ be such a collection which satisfies the following conditions when $R:=G \backslash \mathcal{C}$ :
(O1) the number of vertices in $\mathcal{C}$ is minimum,
(O2) subject to (O1), the total number of chords in the cycles of $\mathcal{C}$ is maximum, and
(O3) subject to (O1) and (O2), the length of the longest path in $R$ is maximum.
We use the convention that $P$ is a longest path in $R$. Since $G[P]$ may have several paths spanning $V(P)$ and the endpoints of such paths will behave in a similar manner, we let

$$
\mathcal{P}:=\{v \in V(P): v \text { is an endpoint of a path spanning } V(P)\}
$$

### 2.3 Preliminary Results

We begin with a number of observations about $G$ that follow directly from our setup. In the interest of readability, the observations in this paragraph will be used in the text without citation. Since $G$ does not contain $k$ disjoint chorded cycles, $R$ does not contain any chorded cycle, and for any $C \in \mathcal{C}, G[R \cup C]$ does not contain two disjoint chorded cycles. If $p$ is an endpoint of $P$ and has a neighbor in $R \backslash P$, we can extend $P$. Thus, $\|p, R\|=\|p, P\|$. If $\|p, P\| \geq 3$, then $G[P]$ contains a chorded cycle, so $\|p, R\| \leq 2$. Similarly, to avoid a chorded cycle in $R,\|q, P\| \leq 3$ and for any $v \in P,\|v, P\| \leq 4$. If $p$ has two neighbors in $P$, then $G[P]$ contains two distinct spanning paths.

An immediate corollary of (O1) is that, for any chorded cycle $C \in \mathcal{C}$, no vertex of $C$ is incident to two chords; otherwise, we could replace $C$ with a chorded cycle on fewer vertices. We will assume this fact in the proof of the following lemma.

Lemma 13. Let $v \in R$ and $C \in \mathcal{C}$.
(1) If $\|v, C\| \geq 4$, then $\|v, C\|=4=|C|$, and $G[C] \cong K_{4}$.
(2) If $\|v, C\|=3$, then $|C| \in\{4,5,6\}$. Moreover:
(a) if $|C|=4$, then $C$ has a chord incident to the non-neighbor of $v$ (see Figure 2a);
(b) if $|C|=5$, then $C$ is singly chorded, and the endpoints of the chord are disjoint from the neighbors of $v$ (see Figure 2b); and
(c) if $|C|=6$, then $C$ has three chords, with $G[C] \cong K_{3,3}$ and $G[C+v] \cong K_{3,4}$ (see Figure [2C).

Proof. If there exist vertices $c_{1}, c_{2} \in C$ that are adjacent along the cycle of $C$ such that $\left\|v, C-c_{1}-c_{2}\right\| \geq 3$, then $\left(C-c_{1}-c_{2}\right)+v$ contains a chorded cycle with strictly fewer vertices than $C$, contradicting (O1). This proves that if $\|v, C\|=3$, then $|C| \leq 6$. Similarly, if $\|v, C\| \geq 4$, then $|C|=4$ and $\|v, C\|=4$. If $\|v, C\|=4$ and $|C|=4$, then $v$ together with a triangle in $C$ give a doubly chorded 4-cycle, so by ( O 2 ), $G[C] \cong K_{4}$.

Suppose $\|v, C\|=3$. If $|C|=4$, then let $c \in C$ be the non-neighbor of $v$ in $C$. If $c$ is not incident to a chord, then $(C-c)+v$ gives a doubly chorded 4 -cycle, preferable to $C$ by ( O 2 ). This proves (目).

So $|C| \in\{5,6\}$. Since the vertices in $V(C) \backslash N_{G}(v)$ cannot be adjacent along the cycle $C, C-c+v$ contains a chorded cycle $C^{\prime}$ of the same length as $C$, for any $c \in V(C) \backslash N_{G}(v)$, If $c$ is not incident to a



(b) $|C|=5,\|v, C\|=36 \mathrm{v} 3$

(c) $|C|=6,\|v, C\|=3$

Figure 2: Lemma 13)(2)
chord, then $C^{\prime}$ has strictly more chords than $C$, violating (O2). So every vertex in $V(C) \backslash N_{G}(v)$ is incident to a chord.

If $|C|=6$, then $v$ is adjacent to every other vertex along the cycle, and every $c \in V(C) \backslash N_{G}(v)$ is incident to a chord. Since no vertex in $C$ is incident to two chords, (O1) implies (CC). If $|C|=5$, then (O1) implies that the only possible chord has the two non-neighbors of $v$ as its endpoints, which proves (b).

Lemma 14. Let $Q$ be a path in $R$ such that $|Q| \geq 4$ and let $C \in \mathcal{C}$. If $F \subseteq V(Q)$ such that $|F|=4$, then $\|F, C\| \leq 12$. Furthermore, if $G[C] \cong K_{4}$ and there exists an endpoint $v$ of $Q$ such that $\|v, C\| \geq 3$, then $\|Q, C\| \leq 12$ with $\|Q, C\|=12$ only if $\|v, C\|=4$.

Proof. Assume $\|F, C\| \geq 13$ for some $F \subseteq V(Q),|F|=4$, and let $u_{1}, u_{2}, u_{3}, u_{4}$ be the vertices of $F$ in the order they appear on the path $Q$. By Lemma 13, $G[C] \cong K_{4}$, so there exists $c \in C$ such that $\|c, F\| \geq 4$. Since $\left\|\left\{u_{1}, u_{4}\right\}, C\right\| \geq 5$, there exists $i \in\{1,4\}$ such that $\left\|u_{i}, C\right\| \geq 3$. So $Q-u_{i}+c$ and $C-c+u_{i}$ both contain chorded cycles, a contradiction.

To prove the second statement, suppose $G[C] \cong K_{4}$ and let $v$ be an endpoint of $Q$ such that $\|v, C\| \geq 3$. Note that for every $c \in C, C-c+v$ and $Q-v+c$ both contain chorded cycles if $\|c, Q-v\| \geq 3$. Thus, $\|Q, C\| \leq 12$, and furthermore, if $\|Q, C\|=12$, then $\|c, Q\|=3$ and $\|c, v\|=1$ for every $c \in C$.

Lemma 15. If $C \in \mathcal{C}$ and $\left\|v_{1}, C\right\|,\left\|v_{2}, C\right\| \geq 3$ for distinct $v_{1}, v_{2} \in R$, then $|C| \in\{4,6\}$.
Proof. If $C \notin\{4,6\}$, then $|C|=5$ and $N_{C}\left(v_{1}\right)=N_{C}\left(v_{2}\right)$, by Lemma 13. Furthemore, Lemma 13, implies that there are two adjacent vertices $c, c^{\prime} \in N_{C}\left(v_{1}\right)=N_{C}\left(v_{2}\right)$, but then $v_{1} c v_{2} c^{\prime} v_{1}$ is a chorded cycle contradicting (O1).

In the following sections, we will often show that every $C$ in $\mathcal{C}$ is a 6 -cycle. Furthermore, it will often be the case that there exists some $u \in R$ such that $\|u, C\|=3$ for every $C \in \mathcal{C}$. The following lemma will be useful in considering the neighbors of $u$ in $R$ and their adjacencies in $C$.
uvnhd Lemma 16. Let $C \in \mathcal{C}$ with $|C|=6$, and let $u, v \in R$ such that $u v \in E(G)$. If $\|u, C\|=3$ and $\|v, C\| \geq 1$, then $N_{C}(u) \cap N_{C}(v)=\emptyset$.

Proof. By Lemma 13 , we may assume that $A=\left\{a_{1}, a_{2}, a_{3}\right\}$ and $B=\left\{b_{1}, b_{2}, b_{3}\right\}$ are the partite sets of $G[C] \cong K_{3,3}$ with $N_{C}(u)=A$. Suppose on the contrary that $v a_{1} \in E(G)$. Then $u a_{2} b_{1} a_{1} v u$ is a 5 -cycle with chord $u a_{1}$. This contradicts (O1).

Lemma 17. Suppose $H$ is a graph with no chorded cycle. Let $U$ and $W$ be two disjoint paths in $H$ and let $u_{1}$ and $u_{2}$ be the endpoints of $U$. Then $\left\|\left\{u_{1}, u_{2}\right\}, W\right\| \leq 3$. If equality holds, then $u_{1} \neq u_{2}$ and for some $i \in[2],\left\|u_{i}, W\right\|=2$ and $\left\|u_{3-i}, W\right\|=1$, with the neighbor of $u_{3-i}$ strictly between the neighbors of $u_{i}$ on $W$; in addition, $\|U, W\|=3$.

Proof. Let $W=w_{1} w_{2} \ldots w_{t}$ for some $t \geq 1$. $\left\|u_{1}, W\right\| \leq 2$ and $\left\|u_{2}, W\right\| \leq 2$, as $H$ does not contain a chorded cycle. Thus, if $\left\|\left\{u_{1}, u_{2}\right\}, W\right\| \geq 3$, we may assume that $u_{1} \neq u_{2}$, and, without loss of generality,
that $\left\|u_{1}, W\right\|=2$ and $\left\|u_{2}, W\right\| \geq 1$. Suppose $u_{1} w_{i}, u_{1} w_{j} \in E(H)$ such that $i<j$, and let $u_{2} w_{\ell} \in E(H)$ for some $\ell$.

If $\ell \leq i$, then $w_{\ell} W w_{j} u_{1} U u_{2} w_{\ell}$ is a cycle with chord $u_{1} w_{i}$. If $\ell \geq j$, then $w_{i} W w_{\ell} u_{2} U u_{1} w_{i}$ is a cycle with chord $u_{1} w_{j}$. Thus, the neighbors of $u_{2}$ in $W$ are internal vertices of the path $w_{i} W w_{j}$. If $\left\|u_{2}, W\right\|=2$, then suppose $\ell$ is the largest index such that $u_{2} w_{\ell} \in E(H)$. However, $w_{i} W w_{\ell} u_{2} U u_{1} w_{i}$ is a cycle containing a chord incident to $u_{2}$. So $\left\|u_{2}, W\right\|=1$.

Now if $v$ is an internal vertex on $U$ such that $v w_{m} \in E(H)$, then by replacing $u_{2}$ with $v$, we deduce that $i \leq m \leq j$. If $m \leq \ell$, then $w_{i} W w_{\ell} u_{2} U u_{1} w_{i}$ is a cycle with chord $v w_{m}$, and if $m>\ell$, then $w_{\ell} W w_{j} u_{1} U u_{2} w_{j}$ is a cycle with chord $v w_{m}$. This proves the lemma.

## $3 \quad$ Suppose $V(R) \neq V(P)$.

In this section, we make the assumption that $V(R) \neq V(P)$. That is, there exists some vertex $v \in R \backslash P$. In addition, we will use the convention that $p$ and $p^{\prime}$ are the endpoints of $P$, and $q$ (resp. $q^{\prime}$ ) is the neighbor of $p$ (resp. $p^{\prime}$ ) on $P$. By the maximality of $P, v p \notin E(G)$ so that $d_{G}(v)+d_{G}(p) \geq 6 k-2$. Similarly for $v$ and $p^{\prime}$.

Our aim is to show that $G=G_{1}(n, k)$, which is a complete bipartite graph. To aid us, we define a set of vertices $T:=\left\{v \in R: d_{R}(v)=2\right\}$. We will show that $T$ is contained in one of the partite sets of $G_{1}(n, k)$.

Lemma 18. If $v \in R \backslash P$, then $\|\{v, p\}, C\| \leq 6$ for every $C \in \mathcal{C}$, with equality only if
(i) $|C| \in\{4,6\}$ and $N_{C}(v)=N_{C}(p)$, or
(ii) $\|p, C\|=|C|=4$.

Proof. Suppose $v \in R \backslash P$ and $\|\{v, p\}, C\| \geq 6$ for some $C \in \mathcal{C}$. If $\|\{v, p\}, C\| \geq 7$, then either $\|v, C\|=4$ or $\|p, C\|=4$, so that $G[C] \cong K_{4}$ by Lemma 13, If $\|v, C\|=4$, then $\|p, C\|=0$, lest we extend $P$ by adding a neighbor of $p$ in $C$, and replace said neighbor in $C$ with $v$, violating (O3). If $\|p, C\|=4$, then $\|v, C\| \leq 2$, else there exists $c \in C$ such that $C-c+v \cong K_{4}$, and we can extend $P$ by adding $c$, violating (O3). So, $\|\{v, p\}, C\| \leq 6$, and if equality holds, then either (ii) occurs, or $\|v, C\|=\|p, C\|=3$. We may assume $\|v, C\|=\|p, C\|=3$, so that $|C| \in\{4,5,6\}$ by Lemma 13 .

By Lemma 15, $|C| \in\{4,6\}$. Suppose $|C|=4$ and $\|v, C\|=\|p, C\|=3$. Note that $G\left[N_{C}(v) \cup\{v\}\right]$ forms a chorded 4-cycle with at least the same number of chords as $C$. If $p$ is adjacent to the vertex in $V(C) \backslash N_{G}(v)$, we use that vertex to extend $P$, violating (O3). So (i) holds.

Finally, suppose $|C|=6$. By Lemma 13, if $v$ and $p$ do not have the same neighborhood, they are adjacent to disjoint sets of vertices, and $C+p$ and $C+v$ both contain $K_{3,4}$. In this case, we extend $P$ using any $c \in N_{C}(p)$, and replace $C$ with a chorded cycle in $C-c+v$. This violates (O3), so (i) holds.
pbuds Lemma 19. For any $v \in R \backslash P,\|\{v, p\}, R\| \geq 4$, so that $\|v, R\| \geq 2$. Moreover, $|P| \geq 3$.
Proof. Let $v \in R \backslash P$. By the maximality of $P, p v \notin E(G)$. Thus, by Lemma 18 ,

$$
2(3 k-1) \leq d_{G}(v)+d_{G}(p)=\|\{v, p\}, \mathcal{C}\|+\|\{v, p\}, R\| \leq 6(k-1)+\|\{v, p\}, R\|,
$$

so $\|\{v, p\}, R\| \geq 4$. Since $\|p, R\| \leq 2$, it follows that $\|v, R\| \geq 2$. Then $v$ and two of its neighbors form a path of length three in $R$, hence $|P| \geq 3$.

R-Pv1v2 Lemma 20. For any maximal path $P^{\prime}$ in $R \backslash P$, label the (not necessarily distinct) endpoints $v_{1}$ and $v_{2}$ so that $\left\|v_{1}, P\right\| \leq\left\|v_{2}, P\right\|$. Then:
(a) $\left\|v_{2}, P\right\| \leq 2$, and if $v_{1} \neq v_{2}$ then $\left\|v_{1}, P\right\| \leq 1$,
comp T
(b) $d_{R}\left(v_{1}\right)=2$ (this implies $v_{1} \in T \backslash V(P)$ so that $T \backslash V(P) \neq \emptyset$ ), and
(c) if $\left\|v_{2}, P\right\|=2$ and $\left\|v_{1}, P\right\|=1$, then $\left\|P^{\prime}-v_{1}-v_{2}, P\right\|=0$.

Proof. Since $R$ contains no chorded cycle, no vertex in $R \backslash P$ has three neighbors in $P$, so $\left\|v_{2}, P\right\| \leq 2$. Lemma 17 then gives (a) and (c).

It remains to show (b). If $\left\|v_{1}, P\right\|=0$, then using Lemma 19 and the maximality of $P^{\prime}, d_{R}\left(v_{1}\right)=$ $\left\|v_{1}, P^{\prime}\right\|=2$. If $v_{1}=v_{2}$, then $\left\|v_{1}, R\right\|=\left\|v_{2}, P\right\|=2$. So suppose $\left\|v_{1}, P\right\|=1$ and $v_{1} \neq v_{2}$. Since $\left\|v_{2}, P\right\| \geq\left\|v_{1}, P\right\|=1$, there exist $a_{1}, a_{2} \in P$ (perhaps $a_{1}=a_{2}$ ) such that $v_{1} a_{1}, v_{2} a_{2} \in E(G)$. Then $v_{1} P^{\prime} v_{2} a_{2} P a_{1} v_{1}$ is a cycle. Since it has no chord, $\left\|v_{1}, P^{\prime}\right\|=1$, so $\left\|v_{1}, R\right\|=2$ and $v_{1} \in T$.

Lemma 21. $d_{R}(p)=d_{R}\left(p^{\prime}\right)=2$. Additionally, for every $v \in T \backslash V(P)$ and every $C \in \mathcal{C}$ :
(a) $|C| \in\{4,6\}$,
(b) $\|p, C\|=3$, and
(c) $N_{\mathcal{C}}(v)=N_{\mathcal{C}}(p)$.

Proof. By Lemma 20 $v \in T \backslash V(P)$ exists so that $d_{R}(v)=2$. Lemma 19 implies $\|\{v, p\}, R\| \geq 4$, and hence, $d_{R}(p)=2$ and $\|\{v, p\}, R\|=4$. Since $v p \notin E(G),\|\{v, p\}, \mathcal{C}\| \geq(6 k-2)-4=6(k-1)$. By Lemma 18 , $\|\{v, p\}, C\|=6$ for all $C \in \mathcal{C}$. If we can show that $\|p, C\|=3$ for all $C \in \mathcal{C}$, then we are done by Lemma 18 .

If not, then there exists $C \in \mathcal{C}$ such that $\|p, C\|>3$, so $\|p, C\|=4$ and $G[C] \cong K_{4}$ by Lemma 13, Thus, $\|v, C\|=2$, and by Lemma [18, there exists $u \in N_{C}\left(p^{\prime}\right)$. Since $\|p, P\|=2, P+u$ forms a chorded cycle, so since $C-u+v$ also forms chorded cycles, we have a contradiction. Thus, $\|p, C\|=3$ as desired.

From Lemma 21 we immediately obtain the following.
mindeg
Corollary 22. $d_{G}(p)=d_{G}\left(p^{\prime}\right)=3 k-1$, and consequently, $d_{G}(v) \geq 3 k-1$ for all $v \in R \backslash P$.
Recall that $\mathcal{P}$ is the set of vertices in $P$ that are the endpoint of a path spanning $V(P)$. Note Lemmas 18, 19, 20, and 21 apply to each $p^{*} \in \mathcal{P}$. Thus, $\mathcal{P} \subseteq T$, and furthermore, for all $p_{1}^{*}, p_{2}^{*} \in \mathcal{P}, N_{\mathcal{C}}\left(p_{1}^{*}\right)=N_{\mathcal{C}}\left(p_{2}^{*}\right)$.
$|\mathrm{C}|=6$ Lemma 23. For every $C \in \mathcal{C}, G[C] \cong K_{3,3}$.
Proof. If not, by Lemma 13 and Lemma 21, we may assume that there exists $C \in \mathcal{C}$ with $|C|=4$. Suppose $V(C)=\left\{c_{1}, c_{2}, c_{3}, c_{4}\right\}$. Let $v \in T \backslash V(P)$, which we know exists by Lemma 20. By Lemmas 13 and 21, we may assume that $N_{C}(p)=N_{C}\left(p^{\prime}\right)=N_{C}(v)=\left\{c_{1}, c_{2}, c_{3}\right\}$ and $c_{2} c_{4} \in E(G)$. Since $\|p, P\|=2$ by Lemma 21 $P+c_{1}$ and $C-c_{1}+v$ contain chorded cycles, a contradiction.

For the remainder of this section, we will use the fact that for each $C \in \mathcal{C}, G[C] \cong K_{3,3}$ and, that there exist $A \subseteq C$ such that $A$ is a partite set of $C$ and such that, for every $p^{*} \in \mathcal{P}, N_{C}\left(p^{*}\right)=A$, without mentioning Lemmas 21, and 23.

Lemma 24. For every $C \in \mathcal{C}$, if $v \in R \backslash P$ has a neighbor in $C$, then $N_{C}(v) \subseteq N_{C}(p)$, unless $\left|N_{C}(v)\right|=1$.
Proof. Fix $C \in \mathcal{C}$, and let $A=\left\{a_{1}, a_{2}, a_{3}\right\}$ and $B=\left\{b_{1}, b_{2}, b_{3}\right\}$ be the partite sets of $C$ such that $N_{C}(p)=$ $N_{C}\left(p^{\prime}\right)=A$. Suppose on the contrary, there exists $v \in R \backslash P$ with $\left|N_{C}(v)\right| \geq 2$ such that, say $v b_{3} \in E(G)$.

By Lemma 21, $\|p, P\|=2$ so that $P+a_{i}$ contains a chorded cycle for each $i \in[3]$. If $v b_{2} \in E(G)$, then $v b_{3} a_{3} b_{1} a_{2} b_{2} v$ is a cycle with chord $a_{2} b_{3}$. However, $P+a_{1}$ also contains a chorded cycle, a contradiction.

So we may assume that $v a_{3} \in E(G)$. However, $v b_{3} a_{2} b_{2} a_{3} v$ is a 5 -cycle with chord $a_{3} b_{3}$ contradicting (O1). Thus, $N_{C}(v) \subseteq A=N_{C}(p)$, as desired.
$\mathrm{R}-\mathrm{P}=\mathrm{T}$ Lemma 25. $R \backslash P$ is an independent set, and $V(R \backslash P) \subseteq T$.
Proof. Suppose $R \backslash P$ is not an independent set. Then there exists a maximal path $P^{\prime}$ in $R \backslash P$ with distinct endpoints $v_{1}$ and $v_{2}$, labeled as in Lemma 20. Thus, $\left\|v_{2}, P\right\| \leq 2$, and, hence, $d_{R}\left(v_{2}\right) \leq 4$. Since $p v_{2} \notin E(G)$, Lemma 22 implies that $d_{G}\left(v_{2}\right) \geq 3 k-1>4$, which implies that there exists $C \in \mathcal{C}$ such that $v_{2}$ has a neighbor $c \in C$.

Let $A=\left\{a_{1}, a_{2}, a_{3}\right\}$ and $B=\left\{b_{1}, b_{2}, b_{3}\right\}$ be the partite sets of $C$ such that $N_{C}(p)=N_{C}\left(p^{\prime}\right)=A$. By Lemmas 20 and 21, $v_{1} \in T \backslash V(P)$ and $N_{C}\left(v_{1}\right)=A$. We can assume $a_{1} \neq c$, so that there exists a path $W$
in $C-a_{1}$ that contains $a_{2}$ and $a_{3}$ for which $c$ is an endpoint. Since $\left\|v_{1}, W\right\| \geq 2$ and $v_{2}$ is adjacent to an endpoint of $W,\left\|\left\{v_{1}, v_{2}\right\}, W\right\| \geq 3$ and Lemma 17 implies there is a chorded cycle in $G\left[V\left(P^{\prime}\right) \cup V\left(C-a_{1}\right)\right]$. However, as $\|p, P\|=2, P+a_{1}$ also contains a chorded cycle, a contradiction.

$$
\text { Let } \mathcal{S}:=N_{\mathcal{C}}(p), \text { and let } \mathcal{T}:=\left(\left(\bigcup_{C \in \mathcal{C}} V(C)\right) \backslash \mathcal{S}\right) \cup T \text {. }
$$

bipar_sub Proposition 26. $G[\mathcal{S} \cup \mathcal{T}] \cong K_{3 k-3,|\mathcal{T}|}$, and no vertex in $G$ has neighbors in both $\mathcal{S}$ and $\mathcal{T}$.
Proof. By Lemma 23, $\mathcal{C}$ consists of $k-1$ copies of $K_{3,3}$. Lemmas 21 and 25 tell us that, for every $v \in R \backslash P$, $N_{\mathcal{C}}(v)=\mathcal{S}$. Given $C \in \mathcal{C}, a \in V(C) \cap \mathcal{T}$, and $v \in R \backslash P$, we can create a chorded cycle $C^{\prime}$ by swapping $a$ and $v$ in $C$. Note $G\left[C^{\prime}\right] \cong K_{3,3}$, and we have not changed any vertices in $P$. Then replacing $C$ with $C^{\prime}$ in $\mathcal{C}$ results in a collection of $k-1$ chorded cycles satisfying (O1) through (O3). Thus all the previous lemmas apply, and, in particular, Lemma 20 and Lemma 25 imply that $a \in T$. So by Lemma 21, and the fact that $N_{C}(a)=V(C) \cap \mathcal{S}$, we conclude $N_{\mathcal{C}}(a)=\mathcal{S}$. Hence, every vertex in $\mathcal{T}$ is adjacent to every vertex in $\mathcal{S}$, and $G[\mathcal{S} \cup \mathcal{T}]$ contains a copy of $K_{|\mathcal{S}|,|\mathcal{T}|}$.

We claim $G[\mathcal{S} \cup \mathcal{T}]$ has no additional edges. Note $|\mathcal{T}|>3(k-1)$ and $|\mathcal{S}|=3(k-1)$. If there exists any edge with both endpoints in $\mathcal{T}$, or both endpoints in $\mathcal{S}$, then we find a set of $k-1$ chorded cycles, $k-2$ of which are 6 -cycles, and one of which is a 4 -cycle, violating ( O 1$)$. So $G[\mathcal{S} \cup \mathcal{T}] \cong K_{|\mathcal{S}|,|\mathcal{T}|} \cong K_{3 k-3,|\mathcal{T}|}$.

If any vertex of $V(G) \backslash(\mathcal{S} \cup \mathcal{T})$ has neighbors in both $\mathcal{S}$ and $\mathcal{T}$, then in a similar manner, we find $k-1$ disjoint chorded cycles, one of which is a 5 -cycle and the rest of which are 6 -cycles, again violating (O1).

Recall that $q$ and $q^{\prime}$ were defined as the neighbors of $p$ and $p^{\prime}$, respectively, on $P$. Since $\|p, P\|=2$ by Lemma 21, there exists $w \in N_{R}(p) \backslash\{q\}$. As a consequence of Proposition 26, $w \neq p^{\prime}$. Now the neighbor of $w$ on $p P w$ is the endpoint of a path that spans $V(P)$. Thus, $|\mathcal{P}| \geq 3$.
$|\mathrm{P}|=3$ Lemma 27. $|\mathcal{P}|=3$
Proof. Suppose $|\mathcal{P}| \geq 4$, with $p_{1}, p_{2}, p_{3}, p_{4}$ the first four members of $\mathcal{P}$ along $P$. In particular, $p_{1}=p$. Fix $C \in \mathcal{C}$, and let $A=\left\{a_{1}, a_{2}, a_{3}\right\}$ and $B=\left\{b_{1}, b_{2}, b_{3}\right\}$ be the partite sets of $C$ such that $N_{C}\left(p_{i}\right)=A$ for each $i \in[4]$.

By Lemma 16, $N_{C}(q) \subseteq B$. So in particular, $q \neq p_{2}$. If $q$ has a neighbor in $C$, say $b_{1}$, then $q b_{1} a_{1} b_{2} a_{2} p_{1} q$ is a 6 -cycle with chord $p_{1} a_{1}$ and $p_{2} P p_{4} a_{3} p_{2}$ is a cycle with chord $p_{3} a_{3}$, a contradiction.

So we may assume that for every $C \in \mathcal{C}, N_{C}(q)=\emptyset$. That is, $\|q, R\|=d_{G}(q)$. Since $\left\|p_{3}, P\right\|=2$ by Lemma 21, $q$ is not adjacent to $p_{3}$. Then since $d_{G}\left(p_{3}\right)=3 k-1$ by Corollary 22, $d_{G}(q) \geq 3 k-1 \geq 5$. Since $\|q, P\| \leq 3, q$ must be adjacent to two vertices $v_{1}, v_{2} \in R \backslash P$. By Lemma 21, $N_{C}\left(v_{1}\right)=N_{C}\left(v_{2}\right)=A$. However, this yields the cycles $v_{1} q v_{2} a_{2} b_{1} a_{1} v_{1}$ and $p_{2} P p_{4} a_{3} p_{2}$ with chords $v_{1} a_{2}$ and $p_{3} a_{3}$, respectively, a contradiction.


Figure 3: Setup for Lemma 28

Lemma 28. $G[P] \cong K_{2,3}$
Proof. By Lemma 27, we may assume that $\mathcal{P}=\left\{p, p^{\prime}, p^{*}\right\}$. Recall that $\mathcal{P} \subseteq \mathcal{T}$, so Lemma 26 implies that $\mathcal{P}$ is an independent set. Lemma 21 implies that $\|p, P\|=\left\|p^{\prime}, P\right\|=\left\|p^{*}, P\right\|=2$, so there exist $w$ and $w^{\prime}$ on $P$ such that $w \neq q$ and $w^{\prime} \neq q^{\prime}$ and $N_{P}(p)=\{q, w\}$ and $N_{P}\left(p^{\prime}\right)=\left\{q^{\prime}, w^{\prime}\right\}$. Furthermore, since $|\mathcal{P}|=3$, and both the neighbor of $w$ on $p P w$ and the neighbor of $w^{\prime}$ on $w^{\prime} P p^{\prime}$ are in $\mathcal{P}$, we can conclude that $w \neq w^{\prime}$ and $N_{P}\left(p^{*}\right)=\left\{w, w^{\prime}\right\}$, i.e. $w P w^{\prime}$ is the path on three vertices $w p^{*} w^{\prime}$.

Since $G[P]$ does not contain a chorded cycle, $q q^{\prime} \notin E$, so if $w=q^{\prime}$ and $w^{\prime}=q$, then $G \cong K_{2,3}$. So if $G \not \neq K_{2,3}$, then without loss of generality we can assume that $q \neq w^{\prime}$ as in Figure 3. Thus, $q p^{\prime}, p w^{\prime} \notin E(G)$ so, by Corollary 22, $d_{G}(q), d_{G}\left(w^{\prime}\right) \geq 3 k-1$.

Fix $C \in \mathcal{C}$ with partite sets $A=\left\{a_{1}, a_{2}, a_{3}\right\}$ and $B=\left\{b_{1}, b_{2}, b_{3}\right\}$ such that $N_{C}(p)=N_{C}\left(p^{\prime}\right)=N_{C}\left(p^{*}\right)=$ $A$. By Lemma 16, $N_{C}(q) \subseteq B$ and $N_{C}\left(w^{\prime}\right) \subseteq B$.

Since $d_{G}(q) \geq 3 k-1,\|q, C \cup R\| \geq(3 k-1)-3(k-2)=5$. Also, $\|q, P\| \leq 3$. This holds for $w^{\prime}$ as well. Thus, both $q$ and $w^{\prime}$ have two neighbors in $B \cup(R \backslash P)$. Let $v_{1}$ and $v_{2}$ be distinct vertices in $B \cup(R \backslash P)$ such that $v_{1} q, v_{2} w^{\prime} \in E(G)$. We may assume that $v_{2} \neq b_{3}$. Observe that $N_{C}\left(v_{1}\right)=N_{C}\left(v_{2}\right)=A$. Then the cycle $p q v_{1} a_{1} b_{3} a_{3} p$ has chord $p a_{1}$, and the cycle $w^{\prime} v_{2} a_{2} p^{\prime} P w^{\prime}$ has chord $a_{2} p^{*}$, a contradiction.

Lemma 29. $G=G_{1}(n, k)$
Proof. By Lemma [28] let $\left\{p_{1}, p_{2}, p_{3}\right\}$ and $\left\{q_{1}, q_{2}\right\}$ denote the partite sets of $G[P]$. Recall that $\mathcal{P} \subseteq T$ so that $G[\mathcal{S} \cup \mathcal{T}]$ contains every vertex of $G$ except for $q_{1}$ and $q_{2}$.

By Lemmas 21 and 25] and Corollary 22, $\|v, P\|=2$ for all $v \in R \backslash P$, and by Proposition 26, $N_{R}(v)=$ $\left\{q_{1}, q_{2}\right\}$. Since $\mathcal{T}$ is an independent set in $G$, for each $u \in \mathcal{T} \backslash T,\|u, R\| \geq(3 k-1)-3(k-1)=2$. Thus, $u q_{1}, u q_{2} \in E(G)$, and so $N_{G}\left(q_{i}\right) \supseteq \mathcal{T}$ for $i \in[2]$. That is, $G \supseteq K_{|\mathcal{S}|+2,|\mathcal{T}|}=K_{3 k-1,|G|-3 k+1}=G_{1}(n, k)$. Since adding any edge to $G_{1}(n, k)$ results in a graph with $k$ disjoint chorded cycles, we conclude $G=G_{1}(n, k)$.

## $4 \quad$ Suppose $V(R)=V(P)$

In this section, we assume $V(P)=V(R)$. Since adding any edge to $G$ results in $k$ chorded cycles, by (O1) $|P| \geq 4$. If $|P| \geq 6$, we label $P=p_{1} q_{1} r_{1} \cdots r_{2} q_{2} p_{2}$. Note that, since $G[R]$ has no chorded cycles, for every $v \in R,\|r, R\| \leq 4$. When $|P|=5$, we let $P=p_{1} q_{1} r q_{2} p_{2}$, and when $|P|=4$, we let $P=p_{1} q_{1} q_{2} p_{2}$. We call an edge in $E(G[P]) \backslash E(P)$ a hop. If $Q=v_{1} \cdots v_{|R|}$ is a spanning path of $R$, then we call an edge $v_{i} v_{j}$ a hop (on $Q$ ) if $|i-j|>1$.
hops Lemma 30. If $Q=v_{1} \cdots v_{|R|}$ is a spanning path of $R$ and $v_{i} v_{j}$ is a hop with $i<j$, then $v_{i+1}$ and $v_{i+2}$ cannot both be incident to hops, and similarly, $v_{j-1}$ and $v_{j-2}$ cannot both be incident to hops.

Proof. Suppose that, on the contrary, $v_{i+1} v_{k}$ and $v_{i+2} v_{k^{\prime}}$ are both hops. Note that, if we consider only the hop $v_{i} v_{j}$ and the hop $v_{i+1} v_{k}, v_{j} v_{i} v_{i+1} Q v_{j}$ is a chorded cycle if $i+3 \leq k \leq j$, and $v_{k} Q v_{i} v_{j} Q v_{i+1} v_{k}$ is a chorded cycle if $k \leq i-1$, so $k>j$. Repeating this argument but now only considering the hops $v_{i+1} v_{k}$ and $v_{i+2} v_{k^{\prime}}$ gives us that $k^{\prime}>k$, but then $v_{i} v_{i+1} v_{i+2} v_{k^{\prime}} Q v_{j} v_{i}$ is a cycle with chord $v_{i+1} v_{k}$, a contradiction. By symmetry, the lemma holds.
pchords Lemma 31. For any $p \in \mathcal{P}, d_{R}(p)=2$ unless $R$ is a path.
Proof. Let $v_{1} \cdots v_{|R|}$ be a spanning path in $R$, and let $p=v_{1}$. Assume $d_{R}(p)=1$, and that $R$ is not a path. Since $R$ is not a path, hops exist. Let $v_{i} v_{j}, i<j$, be a hop such that for all $k, j<k \leq|R|, v_{k}$ is not incident to a hop. Note, because $d_{R}(p)=1$, that $i \neq 1$.

Let $D$ be the cycle $v_{j} v_{i} v_{i+1} \cdots v_{j-1} v_{j}$. Since $R$ contains no chorded cycles, $v_{j}$ is incident to exactly one hop and $v_{j-1}$ is incident to at most one hop. If $v_{j-1}$ is not incident to a hop let $x=v_{j-1}$ and $y=v_{j}$, and if $v_{j-1}$ is incident to exactly one hop, let $x=v_{j-2}$ and $y=v_{j-1}$. By Lemma 30 when $v_{j-1}$ is incident to a hop, $v_{j-2}$ is not incident to a hop, so in either case, $x y \in E(D), d_{R}(x)+d_{R}(y) \leq 5$, and $p x, p y \notin E(G)$. Therefore,

$$
2\|p, \mathcal{C}\|+\|\{x, y\}, \mathcal{C}\| \geq 2(6 k-2)-(2\|p, R\|+\|\{x, y\}, R\|)>12(k-1)
$$

So there exists $C \in \mathcal{C}$ such that $2\|p, C\|+\|\{x, y\}, C\| \geq 13$. Thus, $\|v, C\|=4$ for some $v \in\{p, x, y\}$, and by Lemma 13, $G[C] \cong K_{4}$. Further, $\|\{x, y\}, C\| \geq 5$ so that there exists $c \in C$ such that $x c, y c \in E(G)$ and $D+c$ contains a chorded cycle. Also $2\|p, C\| \geq 5$, which implies $\|p, C-c\| \geq 2$ so that $C-c+p$ contains a chorded cycle, a contradiction.

R6toC Lemma 32. If $|R| \geq 6$, then there exists $F^{+} \subseteq V(R)$ such that $\left|F^{+}\right|=6$ and such that for every $C \in \mathcal{C}$ and every pair of distinct vertices $u, u^{\prime} \in F^{+},\left\|\left\{u, u^{\prime}\right\}, C\right\| \geq 1$.

Proof. First we find $F^{+} \subseteq V(R)$ such that $\left\|F^{+}, R\right\| \leq 15$. If $R$ is a path, this is trivial, so we assume $R$ has at least one hop. By Lemmas 30 and 31, $p_{i}$ is incident to a hop so that $q_{i}$ and $r_{i}$ cannot both be incident to hops. If $d_{R}\left(r_{i}\right) \leq 3$ for some $i \in[2]$, then since $d_{R}\left(q_{i}\right) \leq 3$ and $d_{R}\left(p_{i}\right)=2$ by Lemma 31] $\left\|\left\{p_{i}, q_{i}, r_{i}\right\}, R\right\| \leq 7$. If $d_{R}\left(r_{i}\right)=4$, then $d_{R}\left(p_{i}\right)=d_{R}\left(q_{i}\right)=2$, so that $\left\|\left\{p_{i}, q_{i}, r_{i}\right\}, R\right\| \leq 8$. Therefore, $F^{+}:=\left\{p_{1}, q_{1}, r_{1}, r_{2}, q_{2}, p_{2}\right\}$ suffices when either $d_{R}\left(r_{1}\right) \leq 3$ or $d_{R}\left(r_{2}\right) \leq 3$. In this case, we let $r_{1}^{*}=r_{1}$.

When $d_{R}\left(r_{1}\right)=d_{R}\left(r_{2}\right)=4,|R| \geq 7$, since $R$ has no chorded cycles, and there exists a vertex $u$ following $r_{1}$ on $P$ with $d_{R}(u) \leq 3$. Here, we let $F^{+}:=\left\{p_{1}, q_{1}, u, r_{2}, q_{2}, p_{2}\right\}$. and let $r_{1}^{*}=u$. Thus, in both cases, $F^{+}=\left\{p_{1}, q_{1}, r_{1}^{*}, r_{2}, q_{2}, p_{2}\right\}$.

We claim that we can partition $F^{+}$into three sets so that each set will consist of two nonadjacent vertices. Define $F_{1}:=\left\{p_{1}, q_{1}, r_{1}^{*}\right\}$ and $F_{2}:=\left\{p_{2}, q_{2}, r_{2}\right\}$, and let $H$ be the subgraph of $G$ on the vertex set $F^{+}$containing precisely those edges of $G$ with one endpoint in $F_{1}$ and the other in $F_{2}$. Because $R$ contains no chorded cycle, every vertex in $F_{2}$ has at most two neighbors in $F_{1}$, and vice-versa. That is, $H \subseteq 3 K_{2}$. Therefore we can label $F_{1}=\left\{f_{1}, f_{2}, f_{3}\right\}$ so that $f_{1} p_{2}, f_{2} q_{2}$, and $f_{3} r_{2}$ are all nonedges.

Therefore, $\left\|F^{+}, \mathcal{C}\right\| \geq 3(6 k-2)-15=18(k-1)-3$. Suppose there exists $C \in \mathcal{C}$ for which $\left\|F^{+}, C\right\| \leq 14$ so that there exists $C^{\prime} \in \mathcal{C}$ such that $\left\|F^{+}, C^{\prime}\right\| \geq 19$. If we can find $v_{1}, v_{2} \in F^{+}$such that $\left\|\left\{v_{1}, v_{2}\right\}, C^{\prime}\right\| \leq 6$, then $\left\|F^{\prime}-v_{1}-v_{2}, C^{\prime}\right\| \geq 13$, contradicting Lemma 14. So for $F^{+}=\left\{v_{1}, v_{2}, \ldots, v_{6}\right\},\left\|\left\{v_{i}, v_{i+1}\right\}, C^{\prime}\right\| \geq 7$ for $i \in\{1,3,5\}$. However this implies $\left\|\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}, C^{\prime}\right\| \geq 14$, a contradiction to Lemma 14 .

Thus, $\left\|F^{+}, C\right\| \geq 15$ for every $C \in \mathcal{C}$. If there exists a pair of distinct vertices $u, u^{\prime} \in F^{+}$such that $\left\|\left\{u, u^{\prime}\right\}, C\right\|=0$, then $\left\|F^{+}-u-u^{\prime}, C\right\| \geq 15$, again a violation of Lemma 14 .

Lemma 33. There exists $F \subseteq V(R)$ such that $p_{1}, p_{2} \in F,|F|=4$ and
(a) $\|F, \mathcal{C}\| \geq 12(k-1)-2$ if $R \cong K_{2,3},\|F, \mathcal{C}\| \geq 12(k-1)+2$ if $R$ is a path, and $\|F, \mathcal{C}\| \geq 12(k-1)$ otherwise, and
(b) if $R$ is not a path, then for every $u \in F$, there exists a path $Q$ in $R-u$ such that $F-u \subseteq V(Q)$.

Proof. If $R$ is a path or $R \cong K_{2,3}$, let $F:=\left\{p_{1}, q_{1}, q_{2}, p_{2}\right\}$. When $R$ is a path, $\|F, R\|=6$, and $p_{1} q_{2}, p_{2} q_{1} \notin$ $E(G)$; when $R \cong K_{2,3},\|F, R\|=10$, and $p_{1} p_{2}, q_{1} q_{2} \notin E(G)$. In both cases, (a) and (b) hold.

So we assume $R \not \approx K_{2,3}$ and $R$ is not a path. By Lemma 31, for $i \in[2],\left\|p_{i}, P\right\|=2$. Thus, $p_{i}$ has a neighbor $w_{i} \in P-q_{i}$. Let $t_{i}$ denote the neighbor of $w_{i}$ on $w_{i} P p_{i}$. Observe that $t_{i} \in \mathcal{P}$, so by Lemma 31, $\left\|t_{i}, P\right\|=2$. Suppose $t_{1} \neq t_{2}$, and, in this case, let $F:=\left\{p_{1}, t_{1}, t_{2}, p_{2}\right\}$. Then $F \subseteq \mathcal{P}$, so (b) holds and $\|F, R\| \leq 8$. If either $p_{1} t_{1}, p_{2} t_{2} \notin E(G)$ or $p_{1} t_{2}, p_{2} t_{1} \notin E(G)$, then (a) holds. Suppose (say) $p_{1} t_{1} \in E(G)$. Then $t_{1}=q_{1}$, and $t_{1} p_{2} \notin E(G)$. Then $w_{2} \notin\left\{p_{1}, t_{1}\right\}$, hence $t_{2} \notin\left\{t_{1}, w_{1}\right\}=N_{R}\left(p_{1}\right)$, so also $p_{1} t_{2} \notin E(G)$. So in this case also, (a) holds.

So assume $t_{1}=t_{2}$, which implies $\|u, P\|=2$ for all $u \in V(P)-w_{1}-w_{2}$, as otherwise $R$ contains a chorded cycle. Also, when $t_{1}=t_{2}$, we may assume that $q_{1} \neq w_{2}$ since $R$ is not isomorphic to $K_{2,3}$. In this case, let $F:=\left\{p_{1}, q_{1}, t_{1}, p_{2}\right\}$ and note that $p_{1} t_{1}, q_{1} p_{2} \notin E(G)$. Since $d_{R}(u)=2$ for all $u \in F$, (a) holds. Since $t_{1}=t_{2}, p_{1} w_{1} t_{1} w_{2} p_{2}$ is a path in $R-q_{1}$ containing $F-q_{1}$ and $F-q_{1} \subseteq \mathcal{P}$, (b) holds.

Corollary 34. $R$ is not a path.
Proof. Let $F \subseteq V(R)$ be as guaranteed in Lemma 33. If $R$ is a path, then $\|F, \mathcal{C}\| \geq 12(k-1)+2$, so that there exists $C \in \mathcal{C}$ such that $\|F, C\| \geq 13$, which violates Lemma 14. So $R$ is not path.

Lemma 35. Let $F \subseteq V(R)$ be as guaranteed in Lemma 33. If $\|F, C\|=12$ for any $C \in \mathcal{C}$, then $G[C] \cong K_{3,3}$.
Proof. Let $F \subseteq V(R)$ be as guaranteed in Lemma 33 and let $C \in \mathcal{C}$. Suppose that $\|F, C\|=12$. By Lemmas 14 and 33, this is true for all $C \in \mathcal{C}$, unless $R \cong K_{2,3}$. By Lemmas 13 and 15, $C \cong K_{3,3}$ unless $|C|=4$, so assume $|C|=4$. Note that for any $u \in F$ and $c \in C$, if $C-c+u$ is a chorded cycle, then $\|c, F-u\| \leq 2$, because there exists a path $Q$ in $R$ such that $F-u \subseteq V(Q)$ and $G[Q+c]$ cannot contain a chorded cycle.

First assume that $C$ is singly chorded, so we can label $V(C)=\left\{c_{1}, c_{2}, c_{3}, c_{4}\right\}$ such that $c_{1} c_{2} c_{3} c_{4}$ is a cycle and $c_{2} c_{4}$ is the chord. By Lemma 13, $\|u, C\|=3$ for every $u \in F$, and $\left\|c_{i}, F\right\|=4$, for $i \in\{1,3\}$. Recall that $p_{1}, p_{2} \in F$ so that $C-c_{1}+p_{1}$ and $P-p_{1}+c_{1}$ both contain chorded cycles, a contradiction.

So for the remainder of the proof, we assume $G[C] \cong K_{4}$, with $V(C)=\left\{c_{1}, c_{2}, c_{3}, c_{4}\right\}$. Fix $u \in F$, and by Lemma 33, let $Q$ be a path in $R-u$ such that $F-u \subseteq V(Q)$. Suppose $\|u, C\|=3$, so $\|F-u, C\|=9$, and there exists $c \in C$ such that $c$ is adjacent to all three vertices in $F-u$. This implies $Q+c$ and $C-c+u$ both contain chorded cycles, a contradiction.

Now suppose $\|u, C\|=2$ and $N_{C}(u)=\left\{c_{1}, c_{2}\right\}$. Then $\|F-u, C\|=10$, and there exist two vertices in $C$ adjacent to all three vertices in $F-u$. If $c^{\prime}$ is one of these two vertices and $c^{\prime} \notin\left\{c_{1}, c_{2}\right\}$, then $Q+c^{\prime}$ and $C-c^{\prime}+u$ both contain chorded cycles, a contradiction. Therefore, every vertex in $F$ is adjacent to both $c_{1}$ and $c_{2}$. Since $\|F, C\|=12$ and $\|u, C\|=2$, there exists $v \in F-u$ such that $\|v, C\|=4$. By Lemma 33, there exists a path $Q^{\prime}$ in $R-v$ such that $F-v \subseteq V\left(Q^{\prime}\right)$, so that $C-c_{1}+v$ and $Q^{\prime}+c_{1}$ both contain chorded cycles, a contradiction.

So $\|u, C\| \in\{0,1,4\}$, for every $u \in F$. Since $\|F, C\|=12$, there exists $u^{\prime} \in F$ such that $\left\|u^{\prime}, C\right\|=0$ and $\|u, C\|=4$ for every $u \in F-u^{\prime}$. By Lemma 33] $p_{1}, p_{2} \in F$, so we may assume $\left\|p_{1}, C\right\|=4$. Thus, for all $c \in C, C-c+p_{1}$ is a chorded cycle, and further $\left\|c, P-p_{1}\right\| \leq 2$, else $P-p_{1}+c$ contains a chorded cycle. Therefore, if $\|R \backslash F, C\|>0$, we can pick $c$ such that $\left\|c, P-p_{1}\right\| \geq 3$ so that $P-p_{1}+c$ has a chorded cycle, a contradiction.

Thus $\|R \backslash F, C\|=0$. By Lemma 32, $|R| \leq 5$, as otherwise we can find $F^{+} \subseteq V(R)$ with $\left|F^{+}\right|=6$ so that for distinct $v, v^{\prime} \in F^{+} \backslash F,\left\|\left\{v, v^{\prime}\right\}, C\right\| \geq 1$, a contradiction. If $|R|=4$, then $u^{\prime}$ has a neighbor $v \in F-u^{\prime}$. Since $R$ is not a path, by Lemma $31 R \cong C_{4}$, so replacing $C$ with $C^{\prime}:=C-c+v$ in $\mathcal{C}$ gives a collection of $k-1$ chorded cycles that satisfies (O1) - (O3), but $R^{\prime}:=R-v+c$ has a path $P^{\prime}$ such that $\left|P^{\prime}\right|=\left|R^{\prime}\right|$ and such that $u^{\prime}$ is an endpoint and such that $\left\|u^{\prime}, R^{\prime}\right\|=1$. This is a contradiction to Lemma 31,

So assume $|R|=5$ so that $P=p_{1} q_{1} r q_{2} p_{2}$. By Lemma 31 either $p_{1} r, p_{2} r \in E(G)$, or $R \in\left\{C_{5}, K_{2,3}\right\}$. In each of these cases, we can assume that $F=\left\{p_{1}, q_{1}, q_{2}, p_{2}\right\}$, by the proof Lemma 33. Recall that $\left\|p_{1}, C\right\|=4$ and $\left\|u^{\prime}, C\right\|=0$ for some $u^{\prime} \in F$. Furthermore, since $\|R \backslash F, C\|=0,\|r, C\|=0$.

Suppose $R \in\left\{C_{5}, K_{2,3}\right\}$. Let $F^{\prime}:=\left\{q_{1}, r, q_{2}, p_{2}\right\}$, so that $u^{\prime} \in F^{\prime},\left\|F^{\prime}, C\right\| \leq 8$ and $\left\|F^{\prime}, R\right\| \leq 10$. Since $q_{1} q_{2}, r p_{2} \notin E(G),\left\|F^{\prime}, \mathcal{C}-C\right\| \geq 12(k-2)+2$ so that $k \geq 3$ and $\left\|F^{\prime}, C^{\prime}\right\| \geq 13$ for some $C^{\prime} \in \mathcal{C}-C$, a contradiction to Lemma 14.

Thus $p_{1} r, p_{2} r \in E(G)$. Since three of the five vertices in $R$ send four edges to $C$, there exists $i \in[2]$, such that at least two vertices in $\left\{r, q_{i}, p_{i}\right\}$ have four neighbors in $C$, and so have a common neighbor $c \in C$. This implies that $G\left[\left\{r, q_{i}, p_{i}, c\right\}\right]$ contains a chorded cycle. Furthermore, there exists $v \in\left\{p_{3-i}, q_{3-i}\right\}$ such that $v$ has four neighbors in $C$, and so $C-c+v$ contains a chorded cycle, a contradiction.

Thus, $|C| \neq 4$ and $G[C] \cong K_{3,3}$, as desired.
structure Lemma 36. If $R \nsubseteq K_{2,3}$, then $G[C] \cong K_{3,3}$ for all $C \in \mathcal{C}$. If $R \cong K_{2,3}$, then $G[C] \cong K_{3,3}$ for all but at most one $C \in \mathcal{C}$, and for any such $C, G[C] \cong K_{1,1,2}$ and $G[V(R) \cup V(C)] \cong K_{1,4,4}$.

Proof. Let $F \subseteq V(R)$ be as guaranteed by Lemma33, If $R$ is not isomorphic to $K_{2,3}$, then $\|F, \mathcal{C}\| \geq 12(k-1)$. By Lemma 14, $\|F, C\| \leq 12$ for all $C \in \mathcal{C}$ so that in fact, equality holds for all $C \in \mathcal{C}$. Thus, by Lemma 35, $G[C] \cong K_{3,3}$ for all $C \in \mathcal{C}$.

So assume $R \cong K_{2,3}$ with partite sets $A=\left\{p_{1}, p_{2}, p_{3}\right\}$ and $B=\left\{q_{1}, q_{2}\right\}$ with $|A|=3$ and $|B|=2$. Since $A$ and $B$ are independent, we have $\|B, \mathcal{C}\| \geq 6 k-8$ and

$$
2\|A, \mathcal{C}\|=\sum_{a \in A} 2\|a, \mathcal{C}\| \geq 3(6 k-2)-12=18 k-18
$$

so $\|A, \mathcal{C}\| \geq 9(k-1)$ and $\|R, \mathcal{C}\| \geq 15 k-17=15(k-1)-2$. If $\|R, C\| \geq 16$ for some $C \in \mathcal{C}$, then there exists some $u \in R$ such that $\|u, C\|=4$. By Lemma 14, $\|R-u, C\| \leq 12$ so that there exists $u^{\prime} \in R-u$ such that $\left\|u^{\prime}, C\right\| \leq 3$. However, $\left\|R-u^{\prime}, C\right\| \geq 13$, a contradiction to Lemma 14 .

We therefore have that, for ever $C \in \mathcal{C}, 13 \leq\|R, C\| \leq 15$. Fix $C \in \mathcal{C}$. At least two vertices in $R$ have three neighbors each in $C$ so that by Lemmas 13 and $15,|C|=4$ or $G[C] \cong K_{3,3}$. We claim that $G[C] \not \approx K_{4}$.

Suppose on the contrary, $G[C] \cong K_{4}$. If $\left\|p_{i}, C\right\| \geq 3$ for some $i \in[3]$, Lemma 14 implies that $\|R, C\| \leq 12$, a contradiction. So $\left\|p_{i}, C\right\| \leq 2$ for all $i \in[3]$. Hence $\|B, C\| \geq 7$ so that for all $c \in C$ and $j \in[2], C-c+q_{j}$ is a chorded cycle. As $\|R, C\| \geq 13$, there exists $c \in C$ such that $\|c, R\| \geq 4$. Without loss of generality, $N_{R}(c) \supseteq\left\{p_{1}, p_{2}, q_{1}\right\}$. However, $C-c+q_{2}$ and $p_{1} c p_{2} q_{1} p_{1}$ each contain chorded cycles, a contradiction.

So for all $C \in \mathcal{C}$, either $|C|=4$ and $C$ is singly chorded or $G[C] \cong K_{3,3}$. By Lemma $13,\|u, C\| \leq 3$ for all $u \in A$ and $C \in \mathcal{C}$. Since $\|A, \mathcal{C}\| \geq 9(k-1)$, we deduce that $\|A, C\|=9$ and so $\|u, C\|=3$ for all $u \in A$ and $C \in \mathcal{C}$.

Suppose $|C|=4$ and $C$ is singly chorded. We can label $V(C)=\left\{c_{1}, c_{2}, c_{3}, c_{4}\right\}$ such that $c_{1} c_{2} c_{3} c_{4}$ is a cycle and $c_{2} c_{4}$ is the chord. By Lemma 13, $u c_{1}, u c_{3} \in E(G)$ for all $u \in A$. Since, $C-c_{i}+u$ is a chorded cycle for $i \in\{1,3\}, R-u+c_{i}$ cannot contain a chorded cycle, which implies that $N_{R}\left(c_{i}\right)=A$. Hence, for every $v \in B, N_{C}(v) \subseteq\left\{c_{2}, c_{4}\right\}$, and since $\|R, C\| \geq 13$, equality holds and $N_{C}(v)=\left\{c_{2}, c_{4}\right\}$ for every $v \in B$.

Fix $u \in A$. Without loss of generality, assume $N_{C}(u)=\left\{c_{1}, c_{3}, c_{4}\right\}$. Then $C-c_{2}+u$ is a chorded cycle. If $u^{\prime} \in A-u$ has $c_{2} \in N_{C}(u)$, then $R-u+c_{2}$ contains a chorded cycle, a contradiction. Thus, for all $w \in A$, $N_{C}(w)=\left\{c_{1}, c_{3}, c_{4}\right\}$ so that $N_{R}\left(c_{4}\right)=V(R)$ and $G[R \cup C] \cong K_{4,4,1}$.

Recall that $\|R, \mathcal{C}\| \geq 15(k-1)-2$ and $\left\|R, C^{\prime}\right\| \leq 15$ for all $C^{\prime} \in \mathcal{C}$. Further, $\left\|u, C^{\prime}\right\| \leq 3$ for all $u \in R$ and $C^{\prime} \in \mathcal{C}$. Since $\|R, C\|=13,\left\|R, C^{\prime \prime}\right\|=15$ for every $C^{\prime \prime} \in \mathcal{C}-C$. However, for any $u \in A,\left\|u, C^{\prime}\right\| \leq 3$ so that $F:=R-u$ satisfies $\left\|F, C^{\prime \prime}\right\| \geq 12$. Furthermore, $F$ satisfies all the hypotheses of Lemmas 33 and 35. so that $G\left[C^{\prime \prime}\right] \cong K_{3,3}$ for all $C^{\prime \prime} \in \mathcal{C}-C$.

This completes the proof of the lemma.
pqchords Lemma 37. For every $u \in R$ and $C \in \mathcal{C},\|u, C\| \leq 3$. If $P^{\prime}$ is path that spans $R$, $p$ is an endpoint of $P^{\prime}$ and $q$ is adjacent to $p$ on $P^{\prime}$, then $d_{G}(p)=3 k-1$ and $d_{G}(q) \geq 3 k-1$. In particular, for every $C \in \mathcal{C}\|p, C\|=3$ and $\|q, C\| \geq 2$.

Proof. Let $p$ and $p^{\prime}$ be the two endpoints of $P^{\prime}$, and let $q$ and $q^{\prime}$ be the neighbors of $p$ and $p^{\prime}$, respectively, on $P^{\prime}$. By Lemmas 13 and 36, $\|u, C\| \leq 3$ for all $u \in R$ and $C \in \mathcal{C}$. Therefore, if $d_{R}(u)=2$, then $d_{G}(u) \leq 3 k-1$, so in particular, $d_{G}(p) \leq 3 k-1$ and $d_{G}\left(p^{\prime}\right) \leq 3 k-1$. If $p p^{\prime} \notin E$, then $d_{G}\left(p^{\prime}\right)=d_{G}(p)=3 k-1$. Otherwise, $p p^{\prime} \in E$ and $p$ is not adjacent to $q^{\prime}$. In this case, $d_{R}\left(q^{\prime}\right)=2$ so that $d_{G}(p)=3 k-1$. Since $\|u, C\| \leq 3$ for all $u \in R$ and $C \in \mathcal{C}$, it follows that $\|p, C\|=3$. By symmetry, this holds for $p^{\prime}$ as well.

Since $\|q, R\| \leq 3$, if we can show that $d_{G}(q) \geq 3 k-1$, it follows that $\|q, C\| \geq 2$ for all $C \in \mathcal{C}$. So assume $d_{G}(q) \leq 3 k-2$. Now, $q p^{\prime} \in E(G)$, as otherwise $d_{G}(q) \geq 3 k-1$. If $|R|=4$, then by Lemma 31, $R$ contains a chorded cycle. So $|R|>4$, and as a result $q q^{\prime} \notin E(G)$. Since $d_{G}(q) \leq 3 k-2$, we get $d_{G}\left(q^{\prime}\right) \geq 3 k$, and furthermore, since $d_{R}\left(q^{\prime}\right) \leq 3$ and $\left\|q^{\prime}, C\right\| \leq 3$ for all $C \in \mathcal{C}$, we deduce that $\left\|q^{\prime}, C\right\|=3$ and $d_{R}\left(q^{\prime}\right)=3$. This implies $p q^{\prime} \in E(G)$, as otherwise we get a chorded cycle in $R$. Furthermore, $d_{G}(q)=3 k-2$ and $\|q, R\| \leq 3$ so that $\|q, C\| \geq 1$ for all $C \in \mathcal{C}$.

Since $|R| \geq 5$, there exists $r^{\prime} \notin\left\{p, p^{\prime}\right\}$ a neighbor of $q^{\prime}$ on $P^{\prime}$. Note that $r^{\prime} \in \mathcal{P}$ so that by the above, $d_{G}\left(r^{\prime}\right)=3 k-1$ and $\left\|r^{\prime}, C\right\|=3$ for all $C \in \mathcal{C}$. If $|R| \geq 6$, then $r^{\prime} q \notin E(G)$ and $d_{G}(q) \geq 3 k-1$, a contradiction. Hence, $|R|=5$, and, furthermore, $R \cong K_{2,3}$ with partite sets $\left\{q, q^{\prime}\right\}$ and $\left\{p, p^{\prime}, r^{\prime}\right\}$. Observe that for all $u \in\left\{p, r^{\prime}, q^{\prime}, p^{\prime}\right\}$ and $C \in \mathcal{C},\|u, C\|=3$.

If know fix $C \in \mathcal{C}$, such that $\|q, C\| \leq 2$, which must exist because $d(q)=3 k-2$ and $d_{R}(q)=3$. By Lemma 36, $G[C] \in\left\{K_{3,3}, K_{1,1,2}\right\}$. Furthermore, if $G[C] \cong K_{1,1,2}$, then $G[C \cup R]=K_{1,4,4}$, but this contradicts the fact that $\left\|q^{\prime}, C \cup R\right\|=6$. Hence, $C \cong K_{3,3}$ and let $A$ and $B$ denote its partite sets. By Lemmas 13 and 16, we may assume $N_{C}(p)=N_{C}\left(r^{\prime}\right)=N_{C}\left(p^{\prime}\right)=A, N_{C}\left(q^{\prime}\right)=B$, and $N_{C}(q) \subseteq B$. Since $\|q, C\| \leq 2$, there exists $b \in B \backslash N_{C}(q)$. We can replace $C$ with $C-b+p^{\prime}$ and replace $P^{\prime}$ with $b q^{\prime} P^{\prime} p$. Our new collection and path satisfy (O1)-(O3). However, $b$ is an endpoint of our new path and by the above, $d_{G}(b)=3 k-1$. Since $b q \notin E(G), d_{G}(q) \geq 3 k-1$, a contradiction.

Lemma 38. $R$ is either isomorphic to $K_{2,3}$ or $K_{2,2}$.
Proof. If $|R|=4$, then Lemmas 31 implies that $R \cong K_{2,2}$, so assume $|R| \geq 5$ and $R$ is not isomorphic to $K_{2,3}$. Let $P=u_{1}, \ldots, u_{|R|}, p:=u_{1}, q:=u_{2}, q^{\prime}:=u_{|R|-1}$ and $p^{\prime}:=u_{|R|}$. Let $C \in \mathcal{C}$. By Lemma 36 , $G[C] \cong K_{3,3}$, so we let $A=\left\{a_{1}, a_{2}, a_{3}\right\}$ and $B=\left\{b_{1}, b_{2}, b_{3}\right\}$ be its partite sets. Recall that by Lemma 13, if $\|u, C\|=3$ for any $u \in R$, then $N_{C}(u) \in\{A, B\}$.

First assume that $R$ is Hamiltonian (that is, $R$ contains a cycle of size $|R|$ ). Since every vertex in $R$ is the endpoint of a path spanning $R$, by Lemma 37, $\|u, C\|=3$ for every $C \in \mathcal{C}$ and $u \in R$. By Lemma 16, we can assume that $N_{C}\left(u_{i}\right)=A$ if $i$ is odd and $N\left(u_{i}\right)=B$ is $i$ is even. Therefore, Lemma 16 implies that $|R|$ is even, which further implies that $|R| \geq 6$. Then for any $a \in A$ and $b \in B, G\left[\left\{u_{1}, \ldots, u_{4}, a, b\right\}\right]$ and $C-a-b+u_{5}+u_{6}$ contain chorded cycles, a contradiction.

So we can assume $R$ is not Hamiltonian. Let $p w$ be a hop on $P$ so that $w \neq p^{\prime}$. First assume $w \neq q^{\prime}$. Without loss generality assume that $N_{C}\left(p^{\prime}\right)=A$. By Lemmas 16 and 37, $N_{C}(p) \cap N_{C}(q)=\emptyset$, and so there exists $c c^{\prime} \in E(C)$ such that $p c c^{\prime} q P w p$ is a cycle with chord $p q$. By Lemmas 16 and 37] $\left|N_{C}\left(p^{\prime}\right)-c-c^{\prime}\right| \geq 2$ and $\left|N_{C}\left(q^{\prime}\right)-c-c^{\prime}\right| \geq 1$, so $C-c-c^{\prime}+p^{\prime}+q^{\prime}$ contains a chorded cycle, a contradiction.

Now we can assume that both $p q^{\prime}$ and $q p^{\prime}$ are edges. Since $R \neq K_{2,3}$, we have that $|R| \geq 6$. Let $r \neq p$ and $r^{\prime} \neq p^{\prime}$ be the neighbors of $q$ and $q^{\prime}$, respectively, on $P$. Note that $r$ and $r^{\prime}$ are endpoints of paths spanning $R$ so that $\|r, C\|=\left\|r^{\prime}, C\right\|=3$. By Lemmas 16 and 37 , and because $p q^{\prime}, q p^{\prime} \in E(G)$, we may assume that $N_{C}(p)=N_{C}(r)=N_{C}\left(r^{\prime}\right)=N_{C}\left(p^{\prime}\right)=A$ and $N_{C}(q) \cup N_{C}\left(q^{\prime}\right) \subseteq B$. In particular, we may assume $q b_{1} \in E(G)$ so that $p a_{1} b_{2} a_{2} b_{1} q p$ is a cycle with chord $p a_{2}$, and $r P p^{\prime} a_{3} r$ is a cycle with chord $a_{3} r^{\prime}$, a contradiction.

So $|R|=5$ and $R \cong K_{2,3}$, as desired.
6only Lemma 39. If $G[C] \cong K_{3,3}$ for every $C \in \mathcal{C}$, then $G \cong G_{1}(n, k)$.
Proof. By Lemma 38, $R \in\left\{K_{2,2}, K_{2,3}\right\}$. So let $U_{1}, U_{2} \subseteq V(R)$ be the partite sets of $R$ such that $\left|U_{1}\right| \geq$ $\left|U_{2}\right|=2$, and let $u_{1} \in U_{1}, V_{2}:=N_{G}\left(u_{1}\right)$, and $V_{1}:=V(G) \backslash V_{2}$. Since $u_{1}$ is the end of spanning path of $R$, Lemma 37 implies that $\left|V_{2}\right|=3 k-1$. Since $|G| \leq 6(k-1)+5,\left|V_{1}\right| \leq 3 k$. We aim to show that $N_{G}(v)=V_{2}$ for all $v \in V_{1}$. This will imply that $G \cong G_{1}(n, k)$.

Fix $v \in V_{1}-u_{1}$. Since $u_{1} v \notin E(G)$, Lemma 37 implies that $d_{G}(v) \geq 3 k-1$. If $v \in U_{1}$, then $v$ is the end of a spanning path of $R$, and by Lemmas 13 16 and 37, $N_{G}(v)=N_{G}\left(u_{1}\right)=V_{2}$. So we may assume $v \in V_{1} \backslash U_{1}$, and in particular, $v \in C$ for some $C \in \mathcal{C}$.

Define $V_{1}^{\prime}:=\left\{u \in V_{1}:\left\|u, U_{2}\right\| \geq 1\right\}$, and suppose $v \in V_{1}^{\prime} \backslash U_{1}$. Recall that we are assuming $G[C] \cong K_{3,3}$ for all $C \in \mathcal{C}$ so that by Lemma [13, $G\left[C-v+u_{1}\right] \cong K_{3,3}$. Furthermore, $v$ is an end of a path of length $|R|$ in $R^{\prime}:=R-u_{1}+v$. This new collection and path satisfy (O1)-(O3), so by Lemma 38, $R^{\prime} \cong R$ and $N_{G}(v)=N_{G}\left(u_{1}\right)=V_{2}$.

Now suppose $v \in V_{1} \backslash V_{1}^{\prime}$. Since $d_{G}(v) \geq 3 k-1$ and $v$ has at most $3(k-1)$ neighbors in $V_{2}, v$ must have two neighbors in $V_{1}$. By Lemmas 16 and 37, for every $u_{2} \in U_{2}, d_{G}\left(u_{2}\right) \geq 3 k-1$ and $N_{G}\left(u_{2}\right) \subseteq V_{1}$, so that $\left|V_{1}^{\prime}\right| \geq 3 k-1$. Since $\left|V_{1}\right| \leq 3 k, v$ has a neighbor, say $v^{\prime}$, in $V_{1}^{\prime}$. However, by the above, $N_{G}\left(v^{\prime}\right)=V_{2}$, which contradicts the fact that $v v^{\prime}$ is an edge. Therefore, $V_{1}^{\prime}=V_{1}$ which finishes the proof of the lemma.
one4 Lemma 40. Suppose there exists $C \in \mathcal{C}$ with $|C|=4$. Then $G \cong G_{2}(k)$.
Proof. By Lemmas 36 and 38, we can assume $R \cong K_{2,3}, G[C] \cong K_{1,1,2}$, and $G[R \cup C] \cong K_{1,4,4}$. Let $A^{\prime}$ and $B^{\prime}$ be the two partite sets of size four and $\{c\}$ be the partite set of size one in $G[R \cup C]$. By symmetry, we can assume that any $v \in A^{\prime} \cup B^{\prime}$ is an end of a spanning path in $R$ or the end of a spanning path of $G\left[V(G) \backslash V\left(\mathcal{C}^{\prime}\right)\right]$ for some collection $\mathcal{C}^{\prime}$ of $k-1$ vertex disjoint cycles that satisfies (O1)-(O3), so, by Lemma 37 $d_{G}(v)=3 k-1$ and $\|v, \mathcal{C}-C\|=3(k-2)$. By Lemma 36, for all $D \in \mathcal{C}-C, G[D] \cong K_{3,3}$, and, with Lemma 16. we deduce that $\|v, D\|=3$ and that we can label the partite sets of $D$ as $A_{D}$ and $B_{D}$ so that for every $p \in A^{\prime}, N_{D}(p)=B_{D}$ and for every $q \in B^{\prime}, N_{D}(q)=A_{D}$. Therefore, there exists a partition $\{A, B,\{c\}\}$ of $V(G)$ such that for every $p \in A^{\prime}, N_{G}(p)=B+c$, for every $q \in B^{\prime}, N_{G}(q)=A+c$, and $|A|=|B|=3 k-2$.

If $u \in V(G) \backslash\left(A^{\prime} \cup B^{\prime}\right)$, then there exists $D \in \mathcal{C}-C$, such that $u \in D$. Let $p \in A^{\prime} \cap V(R)$, and $q \in B^{\prime} \cap V(R)$ and label $\left\{w, w^{\prime}\right\}=\{p, q\}$ so that $u w \notin E(G)$ and $u w^{\prime} \in E(G)$. We have that $G[D-u+w] \cong K_{3,3}$ and $G[R-w+u] \cong K_{3,2}$, so there exists a collection $\mathcal{C}^{\prime}$ of $k-1$ vertex disjoint cycles containing $C$ that satisfies (O1)-(O3), and there exists a spanning path of of $G\left[V(G) \backslash V\left(\mathcal{C}^{\prime}\right)\right]$ such that $u$ is an endpoint or $u$ is the neighbor of an endpoint. Therefore, by Lemma 37, $d_{G}(u) \geq 3 k-1$, so, with Lemma 36, we have that $N_{C}(u)=\left(V(C) \backslash N_{C}\left(w^{\prime}\right)\right)+c$ and, for any $D^{\prime} \in \mathcal{C}^{\prime}-C$, by Lemma 16, $N_{D^{\prime}}(u)=D^{\prime} \backslash N_{D^{\prime}}\left(w^{\prime}\right)$. Therefore, either $N_{G}(u) \supseteq B+c$ if $u \in A$ or $N_{G}(u) \supseteq A+c$ if $u \in B$. Hence, $G$ contains $G_{2}(k)$ as a spanning subgraph. As $G_{2}(k)$ is edge-maximal with respect to not containing $k$ disjoint chorded cycles, $G \cong G_{2}(k)$.

Using Lemmas 36, 38, 39, and 40, we conclude $G \in\left\{G_{1}(n, k), G_{2}(k)\right\}$.

## 5 Concluding Remarks

Many variations on Theorems 1 and 5 have appeared, and suggest further extensions of Theorem 9 , We present only a small selection below.

A result of Gould, Hirohata, and Horn [8] implies the following:
Theorem 41. Let $G$ be a graph on $|G| \geq 6 k$ vertices with $\delta(G) \geq 3 k$. Then $G$ contains $k$ disjoint doubly chorded cycles.

While it is not clear that $|G| \geq 6 k$ is necessary, it would be interesting to characterize the sharpness examples for this theorem; that is, if $|G| \geq 6 k$ and $\delta(G)=3 k-1$ but $G$ does not contain $k$ disjoint doubly chorded cycles, what does $G$ look like? For more results on the existence of $k$ disjoint multiply chorded cycles, see [9]

Additionally, rather than consider $\delta(G)$ or $\sigma_{2}(G)$, one may consider the neighborhood union, $\min \{\mid N(x) \cup$ $N(y) \mid: x y \in E(\bar{G})\}$. See the following results.

Theorem 42 (Faudree-Gould, [6]). If $G$ has $n \geq 3 k$ vertices and $|N(x) \cup N(y)| \geq 3 k$ for all nonadjacent pairs of vertices $x, y$, then $G$ contains $k$ disjoint cycles.

Theorem 43 (Gould-Hirohata-Horn, 8]). Let $G$ be a graph on at least $4 k$ vertices such that for any nonadjacent $x, y \in V(G),|N(x) \cup N(y)| \geq 4 k+1$. Then $G$ contains $k$ disjoint chorded cycles

Theorem 44 (Gould-Hirohata-Horn, [8]). Let $G$ be a graph on $n>30 k$ vertices such that for any nonadjacent $x, y \in V(G),|N(x) \cup N(y)| \geq 2 k+1$. Then $G$ contains $k$ disjoint cycles.

Theorem 45 (Qiao, [13). Let $r, s$ be nonnegative integers, and let $G$ be a graph on at least $3 r+4 s$ vertices such that for any nonadjacent $x, y \in V(G),|N(x) \cup N(y)| \geq 3 r+4 s+1$. Then $G$ contains $r+s$ disjoint cycles, s of them chorded.

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