

# Coloring the power graph of a semigroup

Yaroslav Shitov

*National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa,  
Moscow 101000, Russia*

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## Abstract

Let  $G$  be a semigroup. The vertices of the power graph  $\mathcal{P}(G)$  are the elements of  $G$ , and two elements are adjacent if and only if one of them is a power of the other. We show that the chromatic number of  $\mathcal{P}(G)$  is at most countable, answering a recent question of Aalipour et al.

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## 1. Introduction

This note is devoted to the graph constructed in a special way from a given semigroup  $G$ . This graph is called the *power graph* of  $G$ , denoted by  $\mathcal{P}(G)$ , and its vertices are the elements of  $G$ . Elements  $g, h \in G$  are adjacent if and only if one of them is a power of the other, that is, if we have either  $g = h^k$  or  $h = g^k$  for some  $k \in \mathbb{N}$ . (Here and in what follows,  $\mathbb{N}$  denotes the set of positive integers.) This concept has attracted some attention in both the discrete mathematics and group theory, see [2, 3].

Let us consider a related directed graph  $D(G)$  on the set  $G$ . Assume  $D(G)$  contains an edge leading from  $x$  to  $y$  if and only if  $y$  is a power of  $x$ . Clearly, the outdegree of any vertex of  $D(G)$  is at most countable, and one gets the graph  $\mathcal{P}(G)$  by forgetting about the orientations of the edges of  $D(G)$ . A classical result by Fodor (see [4, 5]) shows that the chromatic number of  $\mathcal{P}(G)$  does not exceed any uncountable cardinal. Is this chromatic number always at most countable? This question was studied in [1] by Aalipour et al. They answer this question in the special cases which include groups

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*Email address:* yaroslav-shitov@yandex.ru (Yaroslav Shitov)

with finite exponent, free groups, and Abelian groups. However, the general version of this problem remained open even for groups, and it was posed in [1] as Question 42. This note gives an affirmative answer to this question. More than that, we prove that the chromatic number of  $\mathcal{P}(G)$  is countable even if  $G$  is an arbitrary power-associative magma. (Here, a *magma* is a set endowed with a binary operation. A magma is called *power-associative* if a sub-magma generated by a single element is associative.)

We proceed with the proof. The *order* of an element  $h \in G$  is the cardinality of the subsemigroup generated by  $h$ . An element  $h \in G$  is called *cyclic* if the subsemigroup generated by  $h$  is a finite group. In other words, an element  $h$  is cyclic if and only if the equality  $h = h^{n+1}$  holds for some positive integer  $n$ . If  $h$  is not cyclic but has a finite order, then the *pre-period* of  $h$  is defined as the largest  $p$  such that the element  $h^p$  occurs in the sequence  $h, h^2, h^3, \dots$  exactly once.

## 2. Coloring the elements of finite orders

The following claim allows us to split the set of all cyclic elements into a union of countably many independent sets.

**Claim 1.** *Fix a number  $n \in \mathbb{N}$ . The subgraph of  $\mathcal{P}(G)$  induced by the set of cyclic elements of order  $n$  is a union of cliques of size at most  $n$ .*

*Proof.* Denote this induced subgraph by  $P'$ . Let  $\sim$  be the relation on  $P'$  containing those pairs  $(x, y)$  such that  $x$  is a power of  $y$ ; this relation is clearly reflexive and transitive. Assuming that  $x \sim y$ , we get  $x = y^p$ , and we note that  $p$  is relatively prime to  $n$  because the orders of  $x, y$  are equal to  $n$ . So we get  $pq + p'n = 1$  for some  $p' \in \mathbb{Z}$ ,  $q \in \mathbb{N}$ , which shows that  $y = x^q$ . Therefore,  $\sim$  is an equivalence relation, and every equivalence class is a subset of the set of powers of some  $x \in P'$ .  $\square$

As we see from the proof, the sizes of the cliques as in Claim 1 are equal to  $\varphi(n)$ , where  $\varphi$  is the Euler's totient function. This result is similar to Theorem 15 in [1]. Now we are going to prove that the set of all non-cyclic elements of finite orders can be represented as a union of countably many independent sets. We need the following observation.

**Observation 2.** *If  $g \in G$  has a finite order  $n$  and pre-period  $p$ , then  $g^q$  is cyclic for all  $q > p$ .*

*Proof.* We have  $g^{n+1} = g^{p+1}$ , so that  $(g^q)^{n-p+1} = g^{p+1}g^{(n-p)q}g^{q-p-1} = g^q$ .  $\square$

**Claim 3.** *Let  $g, h \in G$  be distinct elements with finite orders. If  $g, h$  have the same pre-period  $p$ , then they are non-adjacent in  $\mathcal{P}(G)$ .*

*Proof.* Assume the result is not true. Then we have  $g = h^t$ , for some  $t > 1$ . (We omit the case  $h = g^t$ , which is considered similarly.) Observation 2 shows that  $g^p = h^{tp}$  is a cyclic element, which contradicts to the initial assumption that  $p$  is the pre-period of  $g$ .  $\square$

### 3. Coloring the elements of the infinite order

In the following claim, we assume  $m, n \in \mathbb{N}$ , and we denote by  $G(x, m, n)$  the set of all  $y \in G$  satisfying  $x^m = y^n$ .

**Claim 4.** *Let  $x \in G$  be an element of the infinite order. Then the set  $G(x, m, n)$  is independent in  $\mathcal{P}(G)$ .*

*Proof.* Assume that  $k \in \mathbb{N}$  and  $y, z \in G$  are such that  $x^m = y^n$ ,  $x^m = z^n$ ,  $y = z^k$ . Then we have  $x^{mk} = z^{kn} = y^n = x^m$ . Since  $x$  has the infinite order, we get  $k = 1$ , which implies  $y = z$  and completes the proof.  $\square$

In what follows, we denote by  $\pi \subset G$  the set of elements of finite orders and by  $\mathcal{P}_*(G)$  the graph obtained from  $\mathcal{P}(G)$  by removing the vertices in  $\pi$ .

**Claim 5.** *Let  $x \in G$  be an element of the infinite order. We define the set  $C(x) = \bigcup_{m,n \in \mathbb{N}} G(x, m, n)$ . Then  $C(x)$  is a connected component of  $\mathcal{P}_*(G)$ .*

*Proof.* If we have  $x^{m_1} = g^{n_1}$ ,  $x^{m_2} = h^{n_2}$  with  $m_1, m_2, n_1, n_2 \in \mathbb{N}$ , then both  $g$  and  $h$  are adjacent to  $x^{m_1 m_2} \in C(x)$ , which shows that  $C(x)$  is connected. Now assume that an element  $z \in \mathcal{P}_*(G)$  is adjacent to a vertex  $y$  in  $C(x)$ . Then we have  $x^m = y^n$  and  $y^p = z^q$  with positive integers  $m, n, p, q$  (and either  $p = 1$  or  $q = 1$ , but this fact is not relevant for our proof). We get  $x^{mp} = z^{nq}$ , which implies that  $z$  belongs to  $C(x)$  as well. In other words, the vertices in  $C(x)$  can be adjacent only to vertices in  $C(x)$ .  $\square$

Now we are ready to prove our main result, which states that  $G$  is a union of countably many independent subsets of  $\mathcal{P}(G)$ . Claim 1 shows that the subgraph of  $\mathcal{P}(G)$  induced by the cyclic elements can be covered by countably many independent sets. Claim 3 proves the same result for the subgraph induced by those elements that have finite orders but are not cyclic. These

claims together allow us to cover the set  $\pi$  by countably many independent sets of  $\mathcal{P}(G)$ .

We denote by  $\{C_\alpha\}$  the set of all connected components of the graph  $\mathcal{P}_*(G)$ , which is obtained from  $\mathcal{P}(G)$  by removing the vertices in  $\pi$ . We choose an element  $x_\alpha$  in every connected component  $C_\alpha$ , and we deduce from Claim 5 that  $C_\alpha = C(x_\alpha)$  for all indexes  $\alpha$ . Claim 4 shows that every  $C(x_\alpha)$  is the union of the independent sets  $G(x_\alpha, m, n)$  over all pairs of positive integers  $(m, n)$ . We see that  $G \setminus \pi$  is the union of the independent sets  $\cup_\alpha G(x_\alpha, m, n)$ , which completes the proof.

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