# The Roller-Coaster Conjecture Revisited

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#### Abstract

A graph is well-covered if all its maximal independent sets are of the same cardinality [25]. If G is a well-covered graph, has at least two vertices, and G - v is well-covered for every vertex v, then G is a 1-well-covered graph [26]. We call G a  $\lambda$ -quasi-regularizable graph if  $\lambda \cdot |S| \leq |N(S)|$  for every independent set S of G. The independence polynomial I(G; x) is the generating function of independent sets in a graph G [9].

The Roller-Coaster Conjecture [24], saying that for every permutation  $\sigma$  of the set  $\{\left\lceil \frac{\alpha}{2}\right\rceil, ..., \alpha\}$  there exists a well-covered graph G with independence number  $\alpha$  such that the coefficients  $(s_k)$  of I(G; x) satisfy

$$s_{\sigma(\lceil \frac{\alpha}{2} \rceil)} < s_{\sigma(\lceil \frac{\alpha}{2} \rceil + 1)} < \dots < s_{\sigma(\alpha)},$$

has been validated in [6].

In this paper we show that independence polynomials of  $\lambda$ -quasi-regularizable graphs are partially unimodal. More precisely, the coefficients of an upper part of I(G; x) are in non-increasing order. Based on this finding, we prove that the domain of the Roller-Coaster Conjecture can be shortened up to:

$$\left\{ \left\lceil \frac{\alpha}{2} \right\rceil, \left\lfloor \frac{\alpha}{2} \right\rfloor + 1, ..., \min\left\{ \alpha, \left\lceil \frac{n-1}{3} \right\rceil \right\} \right\}$$

for well-covered graphs, and up to

$$\left\{ \left\lceil \frac{2\alpha}{3} \right\rceil, \left\lceil \frac{2\alpha}{3} \right\rceil + 1, ..., \min\left\{ \alpha, \left\lceil \frac{n-1}{3} \right\rceil \right\} \right\}$$

for 1-well-covered graphs, where  $\alpha$  stands for the independence number, and n is the cardinality of the vertex set.

**Keywords**: independent set, well-covered graph, 1-well-covered graph, corona of graphs, independence polynomial.

### 1 Introduction

Throughout this paper G = (V, E) is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set  $V = V(G) \neq \emptyset$  and edge set E = E(G). If  $X \subset V$ , then G[X] is the subgraph of G induced by X. By G - W we mean the subgraph G[V-W], if  $W \subset V(G)$ . We also denote by G-F the subgraph of G obtained by deleting the edges of F, for  $F \subset E(G)$ , and we write shortly G-e, whenever  $F = \{e\}$ .

The neighborhood N(v) of  $v \in V(G)$  is the set  $\{w : w \in V(G) \text{ and } vw \in E(G)\}$ , while the closed neighborhood N[v] of v is the set  $N(v) \cup \{v\}$ . The neighborhood N(A)of  $A \subseteq V(G)$  is  $\{v \in V(G) : N(v) \cap A \neq \emptyset\}$ , and  $N[A] = N(A) \cup A$ .

 $C_n, K_n, P_n$  denote respectively, the cycle on  $n \ge 3$  vertices, the complete graph on  $n \ge 1$  vertices, and the path on  $n \ge 1$  vertices.

The disjoint union of the graphs  $G_1, G_2$  is the graph  $G_1 \cup G_2$  having the disjoint unions  $V(G_1) \cup V(G_2)$  and  $E(G_1) \cup E(G_2)$  as a vertex set and an edge set, respectively. In particular, nG denotes the disjoint union of n > 1 copies of the graph G.

An independent set in G is a set of pairwise non-adjacent vertices. An independent set of maximum size is a maximum independent set of G, and the independence number of G, denoted  $\alpha(G)$ , is the cardinality of a maximum independent set in G.

A graph is well-covered if all its maximal independent sets are of the same size [25]. If G is well-covered, without isolated vertices, and  $|V(G)| = 2\alpha(G)$ , then G is a very well-covered graph [7]. The only well-covered cycles are  $C_3$ ,  $C_4$ ,  $C_5$  and  $C_7$ , while  $C_4$ is the only very well-covered cycle. A well-covered graph (with at least two vertices) is 1-well-covered if the deletion of every vertex of the graph leaves a graph, which is wellcovered as well [26]. For instance,  $K_2$  is 1-well-covered, while  $P_4$  is very well-covered, but not 1-well-covered. Notice that  $C_7$  is well-covered but not 1-well-covered. The only 1-well-covered cycles are  $C_3$  and  $C_5$ . A graph G belongs to class  $W_2$  if every two disjoint independent sets in G are contained in two disjoint maximum independent sets [26, 27]. Clearly,  $W_1 \supseteq W_2$ , where  $W_1$  is the family of all well-covered graphs.

**Theorem 1.1** [26] Let G have no isolated vertices. Then G is 1-well-covered if and only if G belongs to the class  $W_2$ .

If G has an isolated vertex, then it is contained in all maximum independent sets, and hence G cannot be in class  $\mathbf{W}_2$ . However, a graph having isolated vertices may be 1-well-covered; e.g.,  $K_3 \cup K_1$ .

**Theorem 1.2** [21] Let G be a graph without isolated vertices. Then G is 1-well-covered if and only if for each non-maximum independent set A there are at least two disjoint independent sets  $B_1, B_2$  such that  $A \cup B_1, A \cup B_2$  are maximum independent sets in G.

Let  $s_k$  be the number of independent sets of size k in a graph G. The polynomial

$$I(G;x) = s_0 + s_1 x + s_2 x^2 + \dots + s_\alpha x^\alpha, \quad \alpha = \alpha(G),$$

is called the *independence polynomial* of G [9]. For a survey on independence polynomials of graphs see [14]. Closed formulae for I(G; x) of several families of graphs one can find in [19, 31], while some factorizations of independence polynomials for certain classes of graphs are given in [29]. A polynomial is called unimodal if the sequence  $(a_0, a_1, a_2, ..., a_n)$  of its coefficients is *unimodal*, i.e., if there exists an index  $k \in \{0, 1, ..., n\}$ , such that

$$a_0 \leq \dots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \dots \geq a_n.$$

In [1] it is proved that for every permutation  $\sigma$  of  $\{1, 2, ..., \alpha\}$  there is a graph G with  $\alpha(G) = \alpha$  such that the coefficients of I(G; x) satisfy  $s_{\sigma(1)} < s_{\sigma(2)} < ... < s_{\sigma(\alpha)}$ .

**Theorem 1.3** [18, 24] If G is a well-covered graph, then  $s_0 \leq s_1 \leq \cdots \leq s_{\lceil \frac{\alpha(G)}{2} \rceil}$ .

Several results concerning the independence polynomials of well-covered graphs are presented in [4, 11, 12, 13]. It is known that there exist well-covered graphs whose independence polynomials are not unimodal [16, 24].

**Conjecture 1.4 (Roller-Coaster Conjecture)** [24] For every permutation  $\sigma$  of the set  $\{\lceil \frac{\alpha}{2} \rceil, ..., \alpha\}$  there is a well-covered graph G with  $\alpha(G) = \alpha$  such that the coefficients of I(G; x) satisfy  $s_{\sigma(\lceil \frac{\alpha}{2} \rceil)} < s_{\sigma(\lceil \frac{\alpha}{2} \rceil + 1)} < \cdots < s_{\sigma(\alpha)}$ .

The Roller-Coaster Conjecture has been verified for well-covered graphs G having  $\alpha(G) \leq 7$  [24], and later for  $\alpha(G) \leq 11$  [23]. In the case of very well-covered graphs, the domain of the Roller-Coaster Conjecture can be shortened to  $\{\lceil \frac{\alpha}{2} \rceil, \lceil \frac{\alpha}{2} \rceil + 1, ..., \lceil \frac{2\alpha-1}{3} \rceil\}$ , where  $\alpha$  stands for the independence number [15]. Recently, the Roller-Coaster Conjecture was validated in [6].

In this paper we show that the domain of the Roller-Coaster Conjecture can be shortened to:

- $\left\{ \left\lceil \frac{\alpha}{2} \right\rceil, \left| \frac{\alpha}{2} \right| + 1, ..., \min \left\{ \alpha, \left\lceil \frac{n-1}{3} \right\rceil \right\} \right\}$  for well-covered graphs of order *n*;
- $\left\{ \left\lceil \frac{2\alpha}{3} \right\rceil, \left\lceil \frac{2\alpha}{3} \right\rceil + 1, ..., \min \left\{ \alpha, \left\lceil \frac{n-1}{3} \right\rceil \right\} \right\}$  for 1-well-covered graphs of order n.

Actually, min  $\left\{\alpha, \left\lceil \frac{n-1}{3} \right\rceil\right\} < \alpha$  only for  $n \leq 3\alpha - 2$ . It means that one may formulate an overhauled Roller-Coaster Conjecture as follows.

**Conjecture 1.5** Let  $\alpha \geq 2$  and  $n \geq 4$  be integers satisfying  $2\alpha \leq n \leq 3\alpha - 2$ . Then for every permutation  $\sigma$  of the set  $\{\left\lceil \frac{\alpha}{2} \right\rceil, \left\lceil \frac{\alpha}{2} \right\rceil + 1, ..., \left\lceil \frac{n-1}{3} \right\rceil\}$  there exists a well-covered graph G with  $\alpha(G) = \alpha$  and |V(G)| = n such that the coefficients of I(G; x) satisfy  $s_{\sigma(\left\lceil \frac{\alpha}{2} \right\rceil)} \leq s_{\sigma(\left\lceil \frac{\alpha}{2} \right\rceil + 1)} \leq \cdots \leq s_{\sigma(\left\lceil \frac{n-1}{2} \right\rceil)}$ .

### 2 Results

We call G a  $\lambda$ -quasi-regularizable graph if  $\lambda > 0$  and  $\lambda \cdot |S| \leq |N(S)|$  is true for every independent set S of G. If  $\lambda = 1$ , then G is a quasi-regularizable graph [3].

For a graph G and  $1 \leq k < \alpha(G)$ , let  $\Omega_k(G) = \{W : |W| = k, W \text{ is independent} in G\}$  and  $H_k(G) = (\Omega_k(G), \Omega_{k+1}(G), Y)$  be the bipartite graph with bipartition  $\{\Omega_k(G), \Omega_{k+1}(G)\}$  and such that  $WU \in Y$  if and only if  $W \subset U$ . It is clear that  $|\Omega_k| = s_k$ .

**Theorem 2.1** If G is a  $\lambda$ -quasi-regularizable graph of order n, then the following assertions are true:

(i) 
$$(k+1) \cdot s_{k+1} \leq (n-(\lambda+1)k) \cdot s_k, 0 \leq k < \alpha(G);$$
  
(ii)  $s_r \geq s_{r+1} \geq \cdots \geq s_{\alpha(G)}, \text{ for } r = \left\lceil \frac{n-1}{\lambda+2} \right\rceil.$ 

**Proof.** Every  $U \in \Omega_{k+1}(G)$  has k+1 subsets in  $\Omega_k(G)$ , which means that the degree of every vertex U in  $H_k(G)$  is equal to k+1. Consequently, we obtain

$$|Y| = (k+1) \cdot |\Omega_{k+1}(G)| = (k+1) \cdot s_{k+1}$$

Every  $W \in \Omega_k(G)$  may be extended to some  $U \in \Omega_{k+1}(G)$  by means of a vertex belonging to V(G) - N[W]. Since G is a  $\lambda$ -quasi-regularizable, we have

$$|N[W]| = |W \cup N(W)| \ge (\lambda + 1) \cdot |W|,$$

and hence,

$$(k+1) \cdot s_{k+1} \le (n - (\lambda + 1)k) \cdot s_k.$$

Therefore, we get

$$s_{k+1} \le \frac{n - (\lambda + 1) k}{k+1} \cdot s_k,$$

which implies  $s_{k+1} \leq s_k$  for every k satisfying

$$\frac{n-(\lambda+1)\cdot k}{k+1} \leq 1 \Leftrightarrow k \geq \left\lceil \frac{n-1}{\lambda+2} \right\rceil,$$

as claimed.  $\blacksquare$ 

In particular, for  $\lambda = 1$ , we deduce the following.

**Corollary 2.2** Let G be a quasi-regularizable graph of order  $n \ge 2$  with  $\alpha(G) = \alpha$ . Then (i)  $(k+1) \cdot s_{k+1} \le (n-2k) \cdot s_k, 1 \le k < \alpha$ ; (ii)  $s_{\lceil \frac{n-1}{2} \rceil} \ge s_{\lceil \frac{n-1}{2} \rceil + 1} \ge \cdots \ge s_{\alpha}$ .

Taking into account Theorem 1.3, Corollary 2.2, and the fact that every well-covered graph is quasi-regularizable [3], we arrive at the following.

**Corollary 2.3** Let G be a well-covered graph of order  $n \ge 2$  with  $\alpha(G) = \alpha$ . Then (i)  $(\alpha - k) \cdot s_k \le (k+1) \cdot s_{k+1} \le (n-2k) \cdot s_k, 1 \le k < \alpha;$ (ii)  $s_0 \le s_1 \le \cdots \le s_{\lceil \frac{\alpha}{2} \rceil}$  and  $s_{\lceil \frac{n-1}{3} \rceil} \ge s_{\lceil \frac{n-1}{3} \rceil+1} \ge \cdots \ge s_{\alpha}.$ 

Combining Theorem 1.3 and Corollary 2.3, we infer that for well-covered graphs, the domain of the Roller-Coaster Conjecture can be shortened to  $\{\left\lceil \frac{\alpha}{2} \right\rceil, \left\lceil \frac{\alpha}{2} \right\rceil + 1, ..., \left\lceil \frac{n-1}{3} \right\rceil\}$ , whenever  $2 \le \alpha$  and  $4 \le n \le 3\alpha - 2$ .

Since each very well-covered graph is of order twice its independence number, we obtain the following.

**Corollary 2.4** [15] If G is a very well-covered graph of order  $n \ge 2$  with  $\alpha(G) = \alpha$ , then  $s_0 \le s_1 \le \cdots \le s_{\lceil \frac{\alpha}{2} \rceil}$  and  $s_{\lceil \frac{2\alpha-1}{3} \rceil} \ge s_{\lceil \frac{2\alpha-1}{3} \rceil+1} \ge \cdots \ge s_{\alpha}$ . Clearly,  $nK_2$  is 1-well-covered for  $n \ge 1$ , and has exactly  $2\alpha(G)$  vertices, while each graph  $G \in \{C_5 \cup nK_2, C_3 \cup nK_2\}, n \ge 1$ , is 1-well-covered and has exactly  $2\alpha(G) + 1$  vertices. One can show that  $C_3$  and  $C_5$  are the only connected 1-well-covered graphs with exactly  $2\alpha(G) + 1$  vertices [21].

**Proposition 2.5** [21] If a connected graph  $G \neq K_2$  is 1-well-covered, then:

(i) G has at least  $2\alpha(G) + 1$  vertices;

(ii) |A| < |N(A)| for every independent set A.

Proposition 2.5(i) implies that  $K_2$  is the unique very well-covered connected graph and also 1-well-covered. In addition,  $I(K_2; x) = 1 + 2x$  is unimodal.

**Theorem 2.6** If G is a connected 1-well-covered graph, |V(G)| = n > 2, and  $\alpha(G) = \alpha$ , then the following assertions are true:

(i)  $2(\alpha - k) \cdot s_k \leq (k+1) \cdot s_{k+1}, 1 \leq k < \alpha;$ (ii)  $s_0 \leq s_1 \leq \cdots \leq s_{\lceil \frac{2\alpha}{3} \rceil};$ (iii)  $(k+1) \cdot s_{k+1} < (n-2k) \cdot s_k, 1 \leq k < \alpha;$ (iv)  $s_{\lceil \frac{n-1}{3} \rceil} > s_{\lceil \frac{n-1}{3} \rceil + 1} > \cdots > s_{\alpha}.$ 

**Proof.** (i) According to Proposition 2.5(i), we have that  $2\alpha \cdot s_0 = 2\alpha \leq s_1 = |V(G)|$ .

Every  $U \in \Omega_{k+1}(G)$  has k+1 subsets in  $\Omega_k(G)$ , which means that the degree of every vertex U in H is equal to k+1. Consequently,  $|Y| = (k+1) \cdot |\Omega_{k+1}(G)| = (k+1) \cdot s_{k+1}$ . On the other hand, by Theorem 1.2, every  $W \in \Omega_k(G)$  can be extended by two disjoint independent sets  $B_1, B_2$  such that  $W_i \cup B_1, W_i \cup B_2$  are maximum independent sets in G. In other words, the degree of every vertex  $W \in \Omega_k(G)$  is at least  $2(\alpha - k)$ .

In conclusion, we obtain  $2(\alpha - k) \cdot s_k \leq (k+1) \cdot s_{k+1}$ , and this implies (i).

(ii) According Part (i), we have

$$s_k \le \frac{k+1}{2\left(\alpha - k\right)} \cdot s_{k+1}$$

which ensures that  $s_k \leq s_{k+1}$  for every k satisfying  $\frac{k+1}{2(\alpha-k)} \leq 1$ , i.e., for  $k \leq \frac{2\alpha-1}{3}$ , at least. In other words, the monotone part of the sequence of coefficients goes up to  $k+1 \leq \lfloor \frac{2\alpha+2}{3} \rfloor = \lceil \frac{2\alpha}{3} \rceil$ .

(*iii*) and (*iv*) By Proposition 2.5(*ii*), |A| < |N(A)| for every independent set A. To get the result, one has just to follow the lines of the proof of Theorem 2.1 changing " $\leq$ " for "<", when needed.

In other words, for 1-well-covered graphs, the domain of the Roller-Coaster Conjecture can be shortened to  $\{\left\lceil \frac{2\alpha}{3}\right\rceil, \left\lceil \frac{2\alpha}{3}\right\rceil + 1, ..., \left\lceil \frac{n-1}{3}\right\rceil\}$ , whenever  $n \leq 3\alpha - 2$ .

Let  $\mathcal{H} = \{H_v : v \in V(G)\}$  be a family of graphs indexed by the vertex set of a graph G. The corona  $G \circ \mathcal{H}$  of G and  $\mathcal{H}$  is the disjoint union of G and  $H_v, v \in V(G)$ , with additional edges joining each vertex  $v \in V(G)$  to all the vertices of  $H_v$ . If  $H_v = H$  for every  $v \in V(G)$ , then we denote  $G \circ H$  instead of  $G \circ \mathcal{H}$  [8].

**Theorem 2.7** [21] Let G be an arbitrary graph and  $\mathcal{H} = \{H_v : v \in V(G)\}$  be a family of non-empty graphs. Then  $G \circ \mathcal{H}$  is 1-well-covered if and only if each  $H_v \in \mathcal{H}$  is a complete graph of order two at least, for every non-isolated vertex v, while for each isolated vertex u, its corresponding  $H_u$  may be any complete graph.

It is easy to see that  $H \circ K_1$  is very well-covered for every graph H, and some properties of  $I(H \circ K_1; x)$  are presented in [18]. Several findings concerning the palindromicity of  $I(H \circ Y; x)$  are proved in [17, ?, 30].

**Theorem 2.8** [10] 
$$I(H \circ Y; x) = (I(Y; x))^n \bullet I(H; \frac{x}{I(Y; x)})$$
, where  $n = |V(H)|$ .

Theorem 2.8 allows finding closed formulae for  $I(H \circ Y; x)$ , once such formulae are known for both I(H;x) and I(Y;x); for instance, one can obtain closed formulae for  $I(H \circ K_p; x)$ , where  $H \in \{P_n, C_n, K_{1,n}\}$  [2, 9, 18].

**Theorem 2.9** Let H be a connected graph. If  $G = H \circ K_2$  and  $\alpha(G) = \alpha$ , then the following assertions are true:

- (i) G is a 1-well-covered graph;
- (ii) G is a 2-quasi-regularizable graph of order  $n = 3\alpha$ ;
- (iii)  $2(\alpha k) \cdot s_k \leq (k+1) \cdot s_{k+1} \leq 3(\alpha k) \cdot s_k, 1 \leq k < \alpha;$ (iv)  $s_0 \leq s_1 \leq \cdots \leq s_{\lceil \frac{2\alpha}{3} \rceil}$  and  $s_{\lceil \frac{3\alpha-1}{4} \rceil} \geq \cdots \geq s_{\alpha-1} \geq s_{\alpha};$ (v) if  $\alpha \geq 3$ , then  $s_{\alpha-3} \cdot s_{\alpha-1} \leq s_{\alpha-2}^2$  and  $s_{\alpha-2} \cdot s_{\alpha} \leq s_{\alpha-1}^2;$

- (vi) if  $\alpha \leq 17$ , then I(G; x) is unimodal.

**Proof.** (i) It follows from Theorem 2.7.

(ii) Let  $S = S_1 \cup S_2$  be an independent set in G, where  $S_1 \subseteq V(H)$ , while  $S_2 \subseteq V(H)$ V(G) - V(H). Then  $2|S_1| = |N_G(S_1) - V(H)| \le |N_G(S_1)|$ , because every vertex of  $S_1$ has exactly two neighbors in V(G) - V(H), and  $2|S_2| = |N_G(S_2)|$ , since each vertex from  $S_2$  has exactly two neighbors in G. Hence, we get that:

$$2|S| = 2|S_1| + 2|S_2| \le |N_G(S_1) - V(H)| + |N_G(S_2)| \le |N_G(S)|,$$

i.e., G is 2-quasi-regularizable. Clearly,  $\alpha = |V(H)|$ . Thus  $n = 3\alpha$ .

(*iii*) It follows from Theorem 2.6(*i*), Theorem 2.1(*i*), and the fact that  $n = 3\alpha$ .

(*iv*) By Theorem 2.1,  $s_{\lceil \frac{3\alpha-1}{4}\rceil} \geq \cdots \geq s_{\alpha-1} \geq s_{\alpha}$ , because G is 2-quasi-regularizable. According to Theorem 2.6 (*iii*), the polynomial  $I(G \circ K_2; x)$  satisfies  $s_0 \leq s_1 \leq \cdots \leq s_{\lceil \frac{2\alpha}{3} \rceil}$ .

(v) Let us specialize the inequality  $2(\alpha - k) \cdot s_k \leq (k+1) \cdot s_{k+1}$  at  $k = \alpha - 3$  and the inequality  $(k+1) \cdot s_{k+1} \leq 3(\alpha-k) \cdot s_k$  at  $k = \alpha - 2$ . It implies

$$s_{\alpha-3} \cdot s_{\alpha-1} \le \frac{(\alpha-2)}{(\alpha-1)} \cdot s_{\alpha-2}^2 \le s_{\alpha-2}^2$$

When we substitute  $k = \alpha - 2$  and  $k = \alpha - 1$  in the same manner, we obtain

$$s_{\alpha-2} \cdot s_{\alpha} \le \frac{3\left(\alpha-1\right)}{4\alpha} \cdot s_{\alpha-1}^2 \le s_{\alpha-1}^2.$$

(vi) By part (iv), the sequence of coefficients of I(G;x) is non-decreasing up to  $\frac{2\alpha}{3}$ and non-increasing starting from  $\left\lceil \frac{3\alpha-1}{4} \right\rceil$ . In addition, the constraint  $\alpha \leq 17$  ensures that  $\left\lceil \frac{3\alpha - 1}{4} \right\rceil - \left\lceil \frac{2\alpha}{3} \right\rceil \leq 1$ .

In other words, if G can be represented as  $H \circ K_2$ , then G is 1-well-covered and the domain of the Roller-Coaster Conjecture can be shortened to  $\{\left\lceil \frac{2\alpha}{3}\right\rceil, \left\lceil \frac{2\alpha}{3}\right\rceil + 1, ..., \left\lceil \frac{3\alpha-1}{4}\right\rceil\}$ .

It is known that:

- each polynomial with positive coefficients that has only real roots is unimodal;
- there exist graphs whose independence polynomials have all the roots real (for example,  $K_{1,3}$ -free graphs [5],  $P_n \circ K_1$  for any  $n \ge 1$  [18]);
- $I(H \circ K_p; x)$  has only real roots if and only if the same is true for I(H; x) [18, 22].

Hence, using Theorem 2.7, we get the following.

**Corollary 2.10** If I(H; x) has only real roots and  $p \ge 2$ , then every graph

 $G \in \{H \circ K_p, (H \circ K_p) \circ K_p, ((H \circ K_p) \circ K_p) \circ K_p, ...\}$ 

is 1-well-covered and its I(G; x) is unimodal, as having all its roots real.

## **3** Conclusions and future work

In this paper we proved that for 1-well-covered graphs the "chaotic interval"  $\left(\left\lceil \frac{\alpha}{2} \right\rceil, \left\lceil \frac{\alpha}{2} \right\rceil + 1, ..., \alpha\right)$  involved in Roller-Coaster Conjecture can be shortened to  $\left\{ \left\lceil \frac{2\alpha}{3} \right\rceil, \left\lceil \frac{2\alpha}{3} \right\rceil + 1, ..., \alpha \right\}$ . Based on this finding, we propose a Roller-Coaster Conjecture for 1-well-covered graphs as follows.

**Conjecture 3.1** For every permutation  $\sigma$  of the set  $\{\lfloor \frac{2\alpha}{3} \rfloor, \lfloor \frac{2\alpha}{3} \rfloor + 1, ..., \alpha\}$  there exists a 1-well-covered graph G with  $\alpha(G) = \alpha$  and |V(G)| = n such that the coefficients of I(G; x) satisfy  $s_{\sigma(\lfloor \frac{2\alpha}{3} \rfloor)} < s_{\sigma(\lfloor \frac{2\alpha}{3} \rfloor + 1)} < \cdots < s_{\sigma(\alpha)}$ .

We incline to think that Conjecture 3.1 can be validated using a technique similar to one presented in [6]. The only obstacle we see now is in constructing a 1-well-covered graph G such that for every given positive integer k each  $S \in \Omega_{k+1}(G)$  is included in exactly two maximum independent sets.

**Problem 3.2** Characterize 1-well-covered graphs whose independence polynomials are unimodal.

The nature and location of the roots of I(G; x) for a well-covered graph G were first analyzed in [4]. It is worth mentioning that there are 1-well-covered graphs whose independence polynomials have non-real roots; e.g.,  $I(K_{1,3} \circ K_2; x) = 1 + 12x + 51x^2 + 93x^3 + 62x^4$ . Taking into account Corollary 2.10, we propose the following.

**Problem 3.3** Characterize 1-well-covered graphs whose independence polynomials have all the roots real.

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