

Coloring Graphs to Produce Properly Colored Walks

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Abstract

For a connected graph, we define the proper-walk connection number as the minimum number of colors needed to color the edges of a graph so that there is a walk between every pair of vertices without two consecutive edges having the same color. We show that the proper-walk connection number is at most three for all cyclic graphs, and at most two for bridgeless graphs. We also characterize the bipartite graphs that have proper-walk connection number equal to two, and show that this characterization also holds for the analogous problem where one is restricted to properly colored paths.

1 Introduction

We consider the problem of coloring the edges of a graph so that it is possible to get between every pair of vertices without two consecutive edges having the same color. Obviously, this can be achieved by giving every edge a different color, and indeed by any proper coloring of the edges. So the real question is what is the minimum number of colors one needs.

Borožan et al. [2] introduced this problem for paths. In particular, for a connected graph G , they defined the *proper connection number* as the minimum number of colors that one needs so that there is a properly colored path between every two vertices. For example, they showed that the parameter is at most 3 for any block. Also, if a graph has a Hamiltonian path, then the parameter is at most 2 [1], and thus almost surely this holds for a random graph [3]. For a recent survey, see [5].

We consider here the analogous concept for walks. For a connected graph G , we define the *proper-walk connection number* $pW(G)$ as the minimum number

of colors if one is allowed any properly colored walk. For symmetry, we will use $pP(G)$ to denote the proper connection number. Trivially, $pW(G) \leq pP(G)$.

We proceed as follows. In Section 2 we show that for any connected cyclic graph the proper-walk connection number is at most three, and in Section 3 we characterize the bipartite graphs that have proper-walk and proper connection numbers two. Thereafter, we show in Section 4 that the parameter is two for any graph with two disjoint odd cycles and in Section 5 that the parameter is two for any bridgeless graph. In Section 6 we provide some thoughts on the general case. We conclude with a comment about the directed version and some thoughts for future work.

2 An Upper Bound

It is immediate that a properly colored walk cannot use the same edge twice in succession. It follows that, in a tree, every properly colored walk is a path. As observed in [1], for the property in trees, one needs the edges of the tree to be properly colored, and thus:

Observation 1 *If T is a tree with maximum degree Δ , then $pW(T) = pP(T) = \Delta$.*

We present next a general upper bound on the proper-walk connection number of cyclic graphs.

Theorem 1 *Let G be a connected graph that is not a tree. Then $pW(G) \leq 3$.*

PROOF. We may assume that G is unicyclic (else take suitable spanning subgraph). Consider the cycle C . Take any proper coloring of the cycle C . For every vertex v of the cycle, it is incident with two colors in the cycle; so let all other edges incident with v have the third color. Color the remaining edges so that for every vertex w not on the cycle, the path J_w from w to the closest vertex of C is properly colored.

There is a properly colored walk between every pair u and v of vertices. For example, if both u and v are off the cycle, then use J_u to get to the cycle, go

around the cycle to the vertex closest to v , and then use J_v in reverse to get to v .
QED

Figure 1 gives an example of a graph G where $pW(G) = 3$. (For a proof of this, see Theorem 7.)

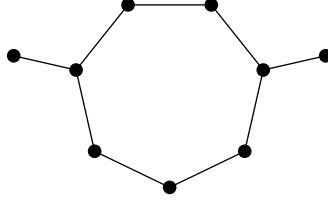


Figure 1. A graph G such that $pW(G) = 3$

Note that the complete graph has $pW(G) = pP(G) = 1$, while noncomplete graphs have $pW(G) \geq 2$. So the big question is: for which graphs is $pW(G) = 2$?

3 Bipartite Graphs

We next determine which bipartite graphs G have $pW(G) = 2$.

For graph G , define $M(G)$ as the spanning subgraph that results if one removes all the bridges of G . Note that each component of $M(G)$ is either an isolated vertex or is 2-edge-connected.

Theorem 2 *Let G be a connected bipartite graph with order at least 3. Then $pW(G) = 2$ if and only if every component of $M(G)$ is incident with at most two bridges.*

PROOF. (1) Assume that every component of $M(G)$ is incident with at most two bridges. We will color the edges of G with two colors.

We first color the edges of $M(G)$. Let H be a nontrivial component of $M(G)$. Then H is 2-edge connected. By Robbins' Theorem [7], such a graph has a strongly connected orientation, say $\vec{H}$. (That is, an orientation such that one can get from every vertex to every other vertex respecting the orientation.) Give the vertices of the subgraph H their bipartite coloring; then color each arc of $\vec{H}$

by the color of the head. It follows that all directed walks in the orientation $\vec{H}$ alternate colors. And within the undirected H , each pair of vertices is joined by a properly colored walk that starts with any designated color or that ends with any designated color, by either following the arcs or going against the arcs. Do this process for all nontrivial components H in $M(G)$.

We next color the bridges of G . We will color them such that two bridges incident with the same component H of $M(G)$ have the same color if and only if their ends in H are in different partite sets. This can be achieved by considering the graph F obtained from G by contracting each (nontrivial) component H of $M(G)$ to a single vertex c_H . Note that F is acyclic, has maximum degree at most 2, and is connected, so that F is a path. Each edge in F corresponds to a bridge in G ; for each bridge e of G , let e' be the corresponding edge in F .

We color F as follows. Start at a leaf-edge and color it arbitrarily. For subsequent edges, suppose that edge e' is colored and we need to color adjacent edge f' . Say edges e' and f' have common end c_H in F . Then let v_e be the end of e in H and similarly with v_f . If v_e and v_f are in the same partite set of G , then give edges e and f different colors; and if v_e and v_f are in different partite sets of G , then give edges e' and f' the same color. Finally, transfer the coloring of F to G ; that is, give each bridge e of G the color of its corresponding edge e' in F .

We claim the above coloring has the desired property; that is, there is a properly colored walk between every two vertices u and v of G .

Every path from u to v uses the same set of bridges in the same order. So consider two consecutive of these bridges, say b_1 and b_2 . Then there is a component of $M(G)$, say H , to which they are both incident. By the way we colored the bridges, if b_1 and b_2 have the same color, say red, then their ends in H are in different partite sets and so every path between them finishes with the same color it starts with. By above there is a path between those two ends starting and finishing with a blue edge. On the other hand, if b_1 and b_2 have different colors, say the former is red and the latter is blue, then these ends are in the same partite set, and so every path between them finishes with color different to its start. By above there is a path between these ends starting with a blue edge (and necessarily ending with a red edge). Thus we can piece together the

bridges with suitable paths in $M(G)$ to obtain the alternating u - v -walk

(2) Assume that G has a suitable 2-coloring. Since G is bipartite, every closed walk has the same parity. So assume a properly colored walk enters a subgraph H of $M(G)$ along bridge b_1 to vertex v_1 and exits H along bridge b_2 from vertex v_2 (with $v_1 = v_2$ allowed). Then b_1 and b_2 must have color determined by the parity of the distance between v_1 and v_2 . That is, bridges b_1 and b_2 have the same color if and only if v_1 and v_2 are in different partite sets in G .

So suppose there are three bridges b_1, b_2, b_3 incident with (not necessarily distinct) vertices v_1, v_2, v_3 of H . Without loss of generality, v_1 and v_2 are in the same partite set X . Thus b_1 and b_2 need different colors. But then if v_3 is in X , the bridge b_3 needs a color different from both b_1 and b_2 ; and if v_3 is in the other partite set, then b_3 needs to be the same as both b_1 and b_2 ; in each case an impossibility. QED

It turns out that the above characterization also holds for the proper connection number. For, in a bipartite graph, all closed walks have even length. Thus, if the edges are 2-colored, then there is a properly colored walk between two vertices if and only if there is a properly colored path between them. That is:

Theorem 3 *Let G be a connected bipartite graph of order at least 3. Then $pP(G) = 2$ if and only if every component in $M(G)$ is incident with at most two bridges.*

It was known that $pP(G) = 2$ for bridgeless bipartite graphs [2].

4 Disjoint Odd Cycles

We now consider the general problem of which graphs G have $pW(G) = 2$.

Theorem 4 *If a connected noncomplete graph G has two edge-disjoint odd cycles, then $pW(G) = 2$.*

PROOF. Since the graph is noncomplete, we need at least two colors.

Let C_1 and C_2 be edge-disjoint odd cycles. If they are also vertex-disjoint, let P be a shortest path joining them; say P starts with vertex u_1 in C_1 and ends at u_2 in C_2 . If the cycles have a vertex in common, then let $u_1 = u_2$ be such a vertex. Let H be the subgraph consisting of C_1 , C_2 , and P if needed.

Now, color the two edges of C_1 incident with u_1 red; then color the remaining edges of C_1 alternating red and blue so that u_1 is the only vertex not incident with an edge of each color. Further, if P exists, color the edges of P alternating colors so that the edge incident with u_1 is blue. Now, if P has even length or the cycles had a vertex in common, color the two edges of C_2 incident with u_2 blue; then color the remaining edges of C_2 alternating red and blue so that u_2 is the only vertex not incident with an edge of each color. On the other hand, if P has odd length, then proceed similarly, except that the two edges of C_2 incident with u_2 are colored red.

We claim that this coloring has the property that between every pair u and u' of (not necessarily distinct) vertices in H , there is a properly colored walk that starts and finishes with any prescribed colors. To see this, first note that every vertex of H is incident with at least one edge of each color. Thus one can start walking from u with any prescribed color. Then one can extend this alternating walk indefinitely such that eventually one traverses P in both directions (if it exists), and goes around both C_1 and C_2 in both directions. Using this, one can arrive at vertex u' having just traversed any designated incident edge.

Now consider the vertices not in H . By choosing a spanning subgraph if needed, one may assume that for each vertex v not in H there is a unique path J_v from v to H . Color the remaining edges such that each J_v is properly colored. See Figure 2.

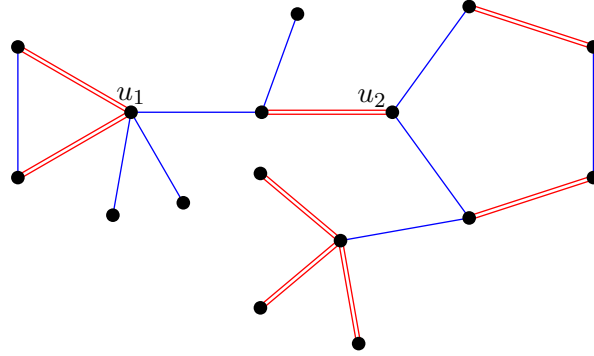


Figure 2. *A coloring of a graph with two disjoint odd cycles*

We claim that the coloring has the desired property. To get between any two vertices v and w in G , use the alternating path J_v to get to H if necessary, go around H in the appropriate direction, and then use the alternating path J_w in reverse if needed. QED

Our focus is on simple graphs, but we consider in passing what happens if the graph has loops. It is immediate from the above that if the graph has two loops then $pW(G) = 2$, as one can treat the loops as odd cycles. But actually, $pW(G) = 2$ for any graph with a loop. For, one can color the loop blue say, the edges incident with the loop red, and then alternate colors away from the loop. There is a properly colored walk between every pair of vertices by going via the loop.

5 Bridgeless Graphs

In this section, we show that $pW(G) \leq 2$ for all connected graphs G without bridges.

5.1 Preliminaries

We will need the following simple observation.

Observation 2 *Let P be an induced path. (That is, the subgraph induced by the vertices of P is a path.) If there is an odd cycle that shares at least one edge with the path P , then there exists a nontrivial path S that is internally disjoint from P and creates an odd cycle with P .*

PROOF. Let C be any odd cycle that shares an edge with P . Consider the vertices of $C \cap P$. Since P is induced, there must be at least one vertex in C not on P . Since C and P share an edge, there are at least two vertices in $C \cap P$. Now, partition the edges of C not in P into segments, where the ends of a segment are in P and internal vertices of each segment are not in P . If every segment creates an even cycle with P , then the result is bipartite, a contradiction. So some segment creates an odd cycle with P , as required. QED

We will also need the following result.

Lemma 1 *Let P be a path in a graph G from vertex u to vertex v such that for every vertex w not on P there are two internally disjoint paths from w to P ending at different vertices. Then one can orient G such that:*

- (a) P is oriented from u to v ;
- (b) one can get from u to every other vertex w by a directed walk; and
- (c) for all vertices w_1 and w_2 there is a directed walk between them in at least one direction.

PROOF. We will create a spanning oriented subgraph $\vec{H}$ such that for each vertex w not on P : there exist distinct vertices q_w and r_w on P , with q_w nearer to v , such that there is a directed walk from q_w to w and a directed walk from w to r_w .

Start with $\vec{H}$ as the path P oriented from u to v . We will grow $\vec{H}$ to contain all the vertices. Let w be any vertex not on P . Since P contains all the cut-vertices of G , there are two internally disjoint paths from w to P . Say these paths end at vertices h_1 and h_2 , where h_1 is nearer to v . Add all the vertices of both these paths to H , and orient the path between w and h_1 towards w while orienting the path between w and h_2 away from w . For all newly added vertices, h_1 is the q -vertex and h_2 the r -vertex.

If there is still a vertex not in H , let w' be such a vertex. Take the two internally disjoint paths from w' to P and cut each when it reaches a vertex that is already in H . Say we have internally disjoint paths L_1 and L_2 from w' to vertices k_1 and k_2 . For convenience, if vertex k is on P then we define $r_k = q_k = k$. By reordering k_1 and k_2 if necessary, it follows that we may assume q_{k_1} is strictly nearer to v than r_{k_2} .

Add all the vertices of both paths L_1 and L_2 to H . Then orient L_1 towards w' and orient L_2 away from w' . For all newly added vertices, q_{k_1} is the q -vertex and r_{k_2} the r -vertex. Repeat this procedure until H contains all the vertices.

We claim this orientation $\vec{H}$ has the desired three properties. The first property was explicitly satisfied. For a directed walk from u to w , go along P to q_w and then along the walk to w . Further, without loss of generality, we may assume that r_{w_2} is not farther from v than r_{w_1} ; this means that q_{w_2} is nearer to v than r_{w_1} . So one can get from w_1 to w_2 by going to r_{w_1} , going along P to q_{w_2} , and thence to w_2 . QED

5.2 Main Result

We define a ***theta-graph*** as a graph that is formed by taking a cycle C of even length (called the outer cycle) and a path P (called the inverter) and identifying the ends of the path P with two vertices u and v of the cycle C such that the result is nonbipartite. See Figure 3 for an example.

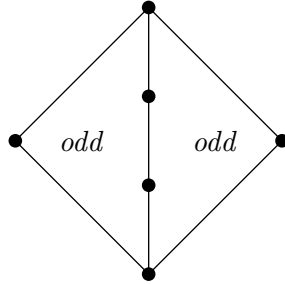


Figure 3. A *theta-graph*

Theorem 5 *There does not exist a 2-connected graph G such that $pW(G) = 3$.*

PROOF. Suppose block G has $pW(G) = 3$. We saw above (Theorem 2) that G cannot be bipartite. Also, we saw (Theorem 4) that G does not contain two edge-disjoint odd cycles.

Consider some odd cycle of the graph G . If it is a hamilton cycle, then it is easily seen that $pW(G) = 2$. So assume there is a vertex not on this cycle. By 2-connectedness, we can find two disjoint paths from this vertex to the cycle, ending at vertices u and v say. That is, we have three internally disjoint u - v paths such that the result is not bipartite. Two of these paths have the same parity; choose them to be the outer cycle, and the other path to be the inverter. That is, the result is a theta-graph.

Out of all theta-subgraphs,

choose the theta-subgraph where the inverter P is as short as possible.

Let C be the outer cycle of the chosen theta-graph.

Claim 1 *The graph $G - C$ is bipartite.*

PROOF. Suppose there is an odd cycle in $G - C$. Since there are not two edge-disjoint odd cycles in G , that odd cycle must share an edge with (the interior of) P . Then by Observation 2, there is segment S in $G - (C \cup P)$ that joins two vertices of the interior of P but is otherwise disjoint from P and creates an odd cycle with P . This segment S combined with P and either half of C provides a theta-graph with a shorter inverter, which contradicts our choice of theta-subgraph. QED

Let P' be the path P minus u and v . Partition the vertices not in the theta-graph into two sets: let A be those vertices that can reach the outer cycle C without going through P' , and let B be those that cannot.

Now, color the graph G as follows. Color the theta-graph such that the outer cycle C is properly colored, as is the inverter P . Without loss of generality,

assume that C is drawn so that every properly colored walk leaving the inverter proceeds clockwise on the outer cycle.

For each vertex w of A , retain one path J_w to C that does not intersect P' . Color the edge of J_w incident with C such that one can go across that edge and proceed counter-clockwise around the outer cycle. Color the remaining edges of the path J_w so that it is properly colored.

Finally, consider the set B . By Claim 1, the graph H induced by $P' \cup B$ is bipartite. Since the graph G is 2-connected, there are two internally disjoint paths from every vertex $w \in B$ to the theta-graph. By the definition of B , these paths must meet the theta-graph on P' . Thus we can apply Lemma 1 to H and P' to obtain an orientation $\vec{H}$ with the properties listed in that lemma. Give each vertex of H its bipartite coloring; then color each arc of $\vec{H}$ by the color of its head. As we used in the proof of Theorem 2, in such a coloring every walk that respects the orientation automatically alternates colors.

See Figure 4 for an example, where the vertices of A are drawn outside the outer cycle and the vertices of B are drawn inside the outer cycle.

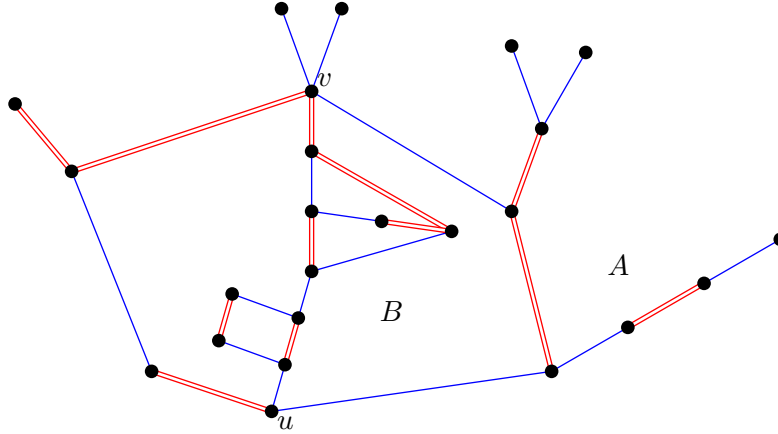


Figure 4. *Coloring of theta-graph and a spanning subgraph*

We claim that the resultant coloring of G has a properly colored walk between any pair of vertices. For example, to get from a vertex w_1 of A to another vertex w_2 of A , follow J_{w_1} , go counter-clockwise around the outer cycle, over the inverter, clockwise around the outer cycle, and then use J_{w_2} in reverse. To get from a vertex w_1 of A to a vertex w_3 of B , follow J_{w_1} , go counter-clockwise

around the outer cycle to u , and then over the inverter to the first vertex of P' , and then use the directed walk in $\vec{H}$ to w_3 . And, to get between vertices w_3 and w_4 of B , use the directed walk in $\vec{H}$. QED

From the above result, the question of bridgeless graphs is easily resolved:

Theorem 6 *If G is a connected bridgeless graph, then $pW(G) \leq 2$.*

PROOF. Assume G is bridgeless but not 2-connected. Consider the blocks of G . If any two of these are nonbipartite, then there are two edge-disjoint odd cycles, and the result follows from Theorem 4. If all the blocks are bipartite, then the result follows from Theorem 2. So assume that exactly one block, say H , is not bipartite.

By the above theorem, that block H can be colored with two colors to have a properly colored walk between every pair of vertices in H . Color all remaining blocks properly, as in Theorem 2. We claim the resultant coloring has the desired property. To find a properly colored walk between vertices u and v , let u' be the vertex of H nearest to u and v' the vertex of H nearest to v . Then find the properly colored walk between u' and v' . This can be extended to a properly colored walk between u and v , since there is a walk from u to u' ending with any prescribed color, and a walk from v' to v starting with any desired color. QED

6 Unicyclic Graphs

It is unclear what happens in general in graphs with bridges. The precise placement of bridges seems to matter. For example, consider the collection $\mathcal{G}$ of graphs formed by taking an odd cycle and adding feet to some of the vertices of the cycle. (By adding a foot we mean adding a new vertex and joining it to exactly one vertex of the cycle.)

Theorem 7 *Let G be a graph of $\mathcal{G}$. Then $pW(G) = 2$ if and only if there are three consecutive vertices u, v, w on the cycle such that u is adjacent to at most one foot, w is adjacent to at most one foot, and all vertices other than u, v, w are incident with no feet.*

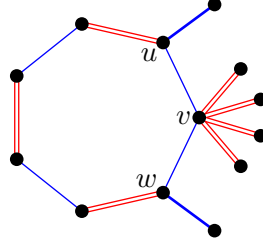


Figure 5. A graph in $\mathcal{G}$ with $pW = 2$

PROOF. (1) We first prove that the conditions are necessary for the graph to have $pW = 2$. That is, assume the graph has a 2-coloring such that every vertex can reach every other vertex by a properly colored walk.

Case A: Assume the odd cycle has length at least 5.

Call a vertex of the cycle a break if the two cycle edges incident with it have the same color. The number of breaks has the same parity as the number of vertices; that is, there is an odd number of breaks. Suppose there are at least three breaks, say vertices v_1 , v_2 , and v_3 . These divide up the cycle into three paths, at least one of which must have more than one edge, say the v_1 – v_2 path. Then there is no alternating walk between v_3 and an interior vertex of that path, a contradiction. That is, there is exactly one break.

So let v be the unique vertex on the cycle incident with two edges of the same color, with neighbors u and w on the cycle. Suppose there is a foot attached to a vertex x that is neither u , v , nor w . Then the two edges of the cycle incident with x have different colors, and so any walk from the foot can proceed in only one direction around the cycle, and gets stopped at v without reaching all the vertices. Thus, all feet must be attached to one of u , v , or w .

Consider a foot incident with u . In order for it to reach all vertices, the edge incident with it must have the same color as the uv edge. It follows that the foot is unique, since otherwise the two feet would not be able to reach each other.

Case B: Assume the odd cycle is a triangle.

If the triangle has exactly one break, then by the same argument as Case A, the other two vertices of the triangle can be incident with at most one foot each. Further, if the triangle is monochromatic, then it is easy to see that each vertex of the cycle is incident with at most one foot.

(2) We second prove that the conditions are sufficient. Color the cycle such that v is incident with two edges of the same color and every other vertex sees both colors. Color the leaf incident with u and/or w with the same color as the uv edge; color all leaves incident with v with the other color. It is easily checked that this coloring has the desired property. QED

7 Directed Graphs

For a strongly connected digraph, one can define the proper-walk connection number as in the undirected case. This idea was recently introduced for paths by Magnant et al. [6]. They showed that:

Theorem 8 [6] *Let D be a strongly-connected digraph. Then $pP(D) \leq 3$.*

This is sharp, even for the proper-walk case, since an odd cycle needs three colors; that is, $pW(D) = pP(D) = 3$ if D is an odd cycle.

We note that the two parameters can be different. That is, there are digraphs with $pW(D) = 2$ and $pP(D) = 3$. For example, take two disjoint directed triangles and identify one vertex of each. See Figure 6.

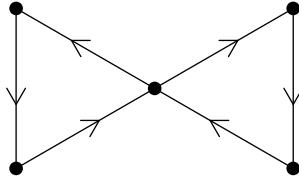


Figure 6. A graph where $pW(D) = 2$ and $pP(D) = 3$

Another direction is to add loops. If one adds loops at all vertices, then one needs only two colors (color all original arcs one color and all loops a second color).

8 Conclusion

We proved that every connected graph has proper-walk connection number at most three, and showed that it is two for some families. One natural open

problem is the complexity of recognizing which graphs have the parameter 2. Is there a polynomial-time algorithm, or is it NP-hard? Note that it is easy to check using a breadth-first-search whether a given coloring has a properly colored walk between two vertices.

Other directions of interest include the question where some of the edges of the graph are already colored. For example, Kézdy and Wang [4] asked when one could complete a 2-coloring such that there is an alternating path between two specified vertices. One could also insist on stronger properties; for example, that every pair of vertices is in a properly colored cycle, or closed walk.

Acknowledgements

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