# On the game total domination number* 

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#### Abstract

The total domination game is a two-person competitive optimization game, where the players, Dominator and Staller, alternately select vertices of an isolate-free graph $G$. Each vertex chosen must strictly increase the number of vertices totally dominated. This process eventually produces a total dominating set of $G$. Dominator wishes to minimize the number of vertices chosen in the game, while Staller wishes to maximize it. The game total domination number of $G, \gamma_{\mathrm{tg}}(G)$, is the number of vertices chosen when Dominator starts the game and both players play optimally.

Recently, Henning, Klavžar, and Rall proved that $\gamma_{\operatorname{tg}}(G) \leq \frac{4}{5} n$ holds for every graph $G$ which is given on $n$ vertices such that every component of it is of order at least 3; they also conjectured that the sharp upper bound would be $\frac{3}{4} n$. Here, we prove that $\gamma_{\mathrm{tg}}(G) \leq \frac{11}{14} n$ holds for every $G$ which contains no isolated vertices or isolated edges.


Keywords: Dominating set, total dominating set, total domination game, open neighborhood hypergraph, transversal game.

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## 1 Introduction

Total domination game is a two-person competitive optimization game based on the notion of total domination. We study the corresponding graph invariant $\gamma_{\mathrm{tg}}(G)$, called game total domination number. Our main contribution is a general upper bound $\frac{11}{14} n$ on $\gamma_{\operatorname{tg}}(G)$ that holds for every graph $G$ of order $n$ not containing isolated vertices or isolated edges. In the proof, we will consider the so-called 'open neighborhood hypergraph' $\mathcal{H}$ instead of $G$, and assign weights to the vertices and edges of $\mathcal{H}$. Then, we analyze a greedy strategy of the 'fast' player, called Dominator.

[^0]
### 1.1 Basic terminology

For a graph $G$ and a vertex $v \in V(G)$, the open neighborhood of $v$ is $N_{G}(v)=\{u: u v \in$ $E(G)\}$, and its closed neighborhood is $N_{G}[v]=N_{G}(v) \cup\{v\}$. If $S \subseteq V(G), N_{G}(S)=$ $\bigcup_{v \in S} N_{G}(v)$ and $N_{G}[S]=\bigcup_{v \in S} N_{G}[v]$. We say that a vertex $v$ totally dominates $u$ if $u \in N_{G}(v)$, while $v$ dominates $u$ if $u \in N_{G}[v]$. A set $D$ of vertices is a total dominating set and a dominating set in $G$ if $N_{G}[D]=V(G)$ and $N_{G}(D)=V(G)$ holds respectively. Equivalently, $D$ is a total dominating set if each vertex has a neighbor in $D$, and $D$ is a dominating set if each vertex which is not in $D$ has a neighbor in $D$. The invariant total domination number $\gamma_{t}(G)$ and domination number $\gamma(G)$ is the minimum size of a total dominating set and that of a dominating set in $G$, respectively.

The notion of total domination game was introduced recently by Henning, Klavžar, and Rall [10]. It is played on an isolate-free graph $G$ by two players, namely Dominator and Staller, who alternately select vertices of $G$. A move (a selection) is legal if the chosen vertex totally dominates at least one vertex which is not totally dominated by the set of vertices previously selected. The game is over when the set $D$ of chosen vertices becomes a total dominating set in $G$. Dominator wishes to finish the game as soon as possible, while Staller wishes to delay the end of the game. The game total domination number, $\gamma_{\mathrm{tg}}(G)$, of $G$ is the number of vertices chosen when Dominator starts the game and both players play optimally ${ }^{11}$

A hypergraph $\mathcal{H}$ is a set (multi)system over the vertex set $V(\mathcal{H})$. The edge set $E(\mathcal{H})$ of $\mathcal{H}$ contains nonempty subsets of $V(\mathcal{H})$. An edge $e \in E(\mathcal{H})$ is a $k$-edge if $|e|=k$. $\mathcal{H}$ is a linear hypergraph, if for any two different edges $e_{1}, e_{2} \in E(\mathcal{H}),\left|e_{1} \cap e_{2}\right| \leq 1$ holds. In particular, there are no multiple edges of size greater than 1 in a linear hypergraph. The degree $d_{\mathcal{H}}(v)$ of a vertex $v \in V(\mathcal{H})$ is the number of edges incident to $v$, and the maximum degree $\Delta(\mathcal{H})$ equals $\max \left\{d_{\mathcal{H}}(v): v \in V(\mathcal{H})\right\}$. A vertex cover (also called transversal) in $\mathcal{H}$ is a set $T$ of vertices which contains at least one vertex from each edge. Remark that unlike the usual terminology, here we allow also the presence of multiple edges and 1-edges in hypergraphs.

Transversal game on hypergraphs was introduced recently in [6] and further studied in [7]. Its definition is analogous to that of total domination game. Two players, namely Dominator ${ }^{2}$ and Staller, alternately choose vertices of a hypergraph $\mathcal{H}$. A move is legal if the vertex chosen covers at least one edge which has not been covered in the game so far. The game is over when every edge of $\mathcal{H}$ is covered. Dominator wishes to end the game as soon as possible, while Staller wishes to delay the end of the game. Assuming that Dominator starts the game on $\mathcal{H}$, and also that both players play optimally, the length of the game is uniquelly determined. It is called the game transversal number of $\mathcal{H}$ and denoted by $\tau_{g}(\mathcal{H})$.

[^1]Given an isolate-free graph $G$, its open neighborhood hypergraph $\operatorname{ONH}(G)$ is the hypergraph with vertex set $V(G)$ and edge set

$$
E(O N H(G))=\left\{N_{G}(v): v \in V(G)\right\} .
$$

It is easy to see (and also observed earlier) that a vertex set $T$ is a total dominating set in $G$ if and only if it is a vertex cover in $O N H(G)$. Similarly, a sequence of moves defines a legal total domination game on $G$, if and only if it is a legal transversal game on $\operatorname{ONH}(G)$. Consequently, $\gamma_{\mathrm{tg}}(G)=\tau_{g}(O N H(G))$ holds for every isolate-free graph $G$.

### 1.2 Results

In the introductory paper [10], among other basic results, the sharp bounds $\gamma(G) \leq \gamma_{\mathrm{tg}}(G) \leq$ $3 \gamma(G)-2$ are proved. The exact value of $\gamma_{\operatorname{tg}}(G)$ for paths and cycles were established in [8]. Our present subject is strongly connected to the following ' $\frac{3}{4}$-Game Total Domination Conjecture', posed by Henning, Klavžar, and Rall in [11].

Conjecture 1. If $G$ is a graph on $n$ vertices in which every component contains at least three vertices, then $\gamma_{\mathrm{tg}}(G) \leq \frac{3}{4} n$.

Note that the restriction given on the size of the components is necessary, because otherwise the upper bound on $\gamma_{\mathrm{tg}}(G)$ could not be better than $n$. We also remark that if the conjecture is true then it is sharp. Tight examples given in [11] are the graphs each component of which is a path of length 4 or 8 .

The following results related to Conjecture 2 have been proved so far. In each of them, it is assumed that $G$ is a graph of order $n$ in which every component contains at least three vertices.

- $\gamma_{\mathrm{tg}}(G) \leq \frac{4}{5} n$. 11 ]
- If $\delta(G) \geq 2$, then $\gamma_{t g}(G)<\frac{8}{11} n<\frac{3}{4} n$. [6]
- If $\operatorname{deg}(u)+\operatorname{deg}(v) \geq 4$ for every edge $u v \in E(G)$, and no two degree- 1 vertices are at distance 4, then $\gamma_{t g}(G) \leq \frac{3}{4} n$. [12]

In this paper our main contribution is a new general upper bound on the game total domination number that improves the earlier bound $4 n / 5$.

Theorem 1. If $G$ is a graph of order $n$ in which every component contains at least three vertices, then

$$
\gamma_{\mathrm{tg}}(G) \leq \frac{11}{14} n .
$$

This theorem will be proved in Section[2. In Section 3, we make some concluding remarks on the Staller-start version of the total domination game.

## 2 Proof of the upper bound $11 n / 14$

In this section we prove our main result, namely Theorem [1. Given a graph $G$ which does not contain isolated vertices and isolated edges, we construct its open neighborhood hypergraph $\mathcal{H}_{0}=O N H(G)$. Then, the total domination game on $G$ will be represented by the transversal game on the hypergraph $\operatorname{ONH}(G)$ where the same sequence of vertices is played. Since our aim is to give a general upper bound on $G$, degree-1 vertices in $G$ and the corresponding edges of size one in $O N H(G)$ are not excluded. Throughout the proof, we denote by $j^{*}$ the number of turns in the game. Let $m_{k}$ be the vertex chosen in the $k$ th turn $\left(1 \leq k \leq j^{*}\right)$. We set $D_{0}=\emptyset$ and define $D_{i}=\left\{m_{k}: 1 \leq k \leq i\right\}$ for $1 \leq i \leq j^{*}$.

### 2.1 Residual hypergraph and special vertices

During the transversal game, the edges which are already covered and the vertices which are not incident with any uncovered edges do not influence the continuation of the game. Hence, we delete them and obtain the residual hypergraph $\mathcal{H}_{i}$. It is defined formally as

$$
E\left(\mathcal{H}_{i}\right)=\left\{e \in E\left(\mathcal{H}_{0}\right): e \cap D_{i}=\emptyset\right\} \quad \text { and } \quad V\left(\mathcal{H}_{i}\right)=\bigcup_{e \in E\left(\mathcal{H}_{i}\right)} e,
$$

where $\mathcal{H}_{0}$ denotes $O N H(G)$. Note that $\mathcal{H}_{j^{*}}$ is the empty hypergraph.
Roughly, we would like to say that a vertex $v$ and the corresponding edge $e_{v}=N_{G}(v)$ is special in $\operatorname{ONH}(G)=\mathcal{H}_{0}$ if $d_{G}(v)=d_{\mathcal{H}_{0}}(v)=1$ and consequently, $e_{v}$ is a 1-edge in the hypergraph. But we do not need more than one special vertex inside any edge of $\mathcal{H}_{0}$. So, the definition will be the following. Consider all the edges of $\mathcal{H}_{0}$ that contains at least one degree-1 vertex and (arbitrarily) fix exactly one degree-1 vertex from each such edge. These vertices will be referred to as special vertices and the set of the special vertices will be denoted by $S$. If $v \in S$, then the corresponding edge $e_{v}$ is called special edge. The definitions imply the following simple statements.

Observation 2. Let $G$ be a graph which contains no isolated vertices and isolated edges, $\mathcal{H}_{0}$ be its open neighborhood hypergraph, and $S$ be a fixed set of special vertices in $\mathcal{H}_{0}$. Let $\mathcal{H}_{i}$ be the residual hypergraph obtained in a transversal game on $\mathcal{H}_{0}(0 \leq i)$.
(i) The number of special vertices equals the number of special edges in $\mathcal{H}_{0}$.
(ii) Any edge of $\mathcal{H}_{i}$ contains at most one special vertex.
(iii) Any vertex in $\mathcal{H}_{i}$ is incident with at most one special edge.
(iv) No special edge contains a special vertex in $\mathcal{H}_{i}$.

Proof. The definitions immediately imply that the statements $(i)-(i i i)$ are valid for $\mathcal{H}_{0}$. Moreover, if $\mathcal{H}_{0}$ satisfies (ii) and (iii), these remain valid for every later residual hypergraph $\mathcal{H}_{i}$. Concerning (iv), we observe that a special edge containing a special vertex in $\mathcal{H}_{0}$ would
correspond to a degree- 1 vertex in $G$ the neighbor of which is also of degree 1. This contradicts the exclusion of $P_{2}$-components from $G$. Since $\mathcal{H}_{0}$ satisfies $(i v)$, every residual hypergraph $\mathcal{H}_{i}(i \geq 1)$ satisfies it as well.

We emphasize that an edge or a vertex is special in $\mathcal{H}_{i}$, if it was special in $\mathcal{H}_{0}$ and it is still present in the residual hypergraph $\mathcal{H}_{i}$.

### 2.2 Weights and phases

In a residual hypergraph $\mathcal{H}_{i}$, a component will be called Type- $X$ component, if it corresponds to an isolated edge which contains at least two non-special vertices. The number of Type-X components in $\mathcal{H}_{i}$ is denoted by $x_{i}$. Moreover, $n_{i}^{h}$ and $n_{i}^{s}$ denote the number of non-special and special vertices present in $\mathcal{H}_{i}$, while $e_{i}^{h}$ and $e_{i}^{s}$ stand for the number of non-special and special edges present in $\mathcal{H}_{i}$, respectively. We define the following function on the residual hypergraphs

$$
f\left(\mathcal{H}_{i}\right)=13 n_{i}^{h}+7 n_{i}^{s}+9 e_{i}^{h}+15 e_{i}^{s}-7 x_{i}
$$

In an equivalent formulation, we may say that the following weights $f(v)$ and $f(e)$ are assigned to every vertex $v$ and edge $e$.

|  | Non-special | Special |
| :---: | :---: | :---: |
| Vertex $v$ | $f(v)=13$ | $f(v)=7$ |
| Edge $e$ | $f(e)=9$ | $f(e)=15$ |

Then, the weight of the residual hypergraph is

$$
f\left(\mathcal{H}_{i}\right)=\sum_{v \in V\left(\mathcal{H}_{i}\right)} f(v)+\sum_{e \in E\left(\mathcal{H}_{i}\right)} f(e)-7 x_{i} .
$$

Since every $v \in V\left(\mathcal{H}_{0}\right)$ and the corresponding edge $e_{v}$ satisfies $f(v)+f\left(e_{v}\right)=22$, we have $f\left(\mathcal{H}_{0}\right) \leq 22 n$ and $f\left(\mathcal{H}_{j^{*}}\right)=0$. In the $i$ th turn of the game, $1 \leq i \leq j^{*}$, the decrease in the weight is $d_{i}=f\left(\mathcal{H}_{i-1}\right)-f\left(\mathcal{H}_{i}\right)$. We will suppose that Dominator follows a greedy strategy in the transversal game; that is, for every odd $i$, he plays a vertex in $\mathcal{H}_{i-1}$ which results in the possible maximum decrease $d_{i}$. Our aim is to prove that, under this greedy strategy,

$$
\frac{\sum_{i=1}^{j^{*}} d_{i}}{j^{*}} \geq 28
$$

is always valid for the average decrease in a turn, independently of Staller's strategy.
To analyze the game, we split it into four phases. Let $\left[j^{*}\right]$ denote $\left\{1, \ldots, j^{*}\right\}$ and define the following sets

$$
\begin{aligned}
\mathcal{P}^{1} & =\left\{i \in\left[j^{*}\right]: \forall \ell\left((\ell \text { is odd and } \ell \leq i) \rightarrow d_{\ell} \geq 40\right)\right\} \\
\mathcal{P}^{2} & =\left\{i \in\left[j^{*}\right]: \forall \ell\left((\ell \text { is odd and } \ell \leq i) \rightarrow d_{\ell} \geq 38\right)\right\} \backslash \mathcal{P}^{1} \\
\mathcal{P}^{3} & =\left\{i \in\left[j^{*}\right]: \Delta\left(\mathcal{H}_{i-1}\right) \geq 2\right\} \backslash\left(\mathcal{P}^{1} \cup \mathcal{P}^{2}\right) \\
\mathcal{P}^{4} & =\left\{i \in\left[j^{*}\right]: \Delta\left(\mathcal{H}_{i-1}\right)=1\right\} \backslash\left(\mathcal{P}^{1} \cup \mathcal{P}^{2}\right)
\end{aligned}
$$

By definition, each $\mathcal{P}^{k}$ (if not empty) contains consecutive integers. Moreover, $\left\{\mathcal{P}^{1}, \mathcal{P}^{2}, \mathcal{P}^{3}, \mathcal{P}^{4}\right\}$ gives a partition of $\left[j^{*}\right]$. We say that the $i$ th turn of the game belongs to Phase $k$ if $i \in \mathcal{P}^{k}$. To simplify the later formulas, we also define $a_{k}=\min \left(\mathcal{P}^{k}\right)$ and $b_{k}=\max \left(\mathcal{P}^{k}\right)$ if $\mathcal{P}^{k}$ is not empty. If $\mathcal{P}^{k}$ is empty, we define $b_{k}=b_{k-1}(1 \leq k \leq 4)$ artificially that can be done recursively if we set $b_{0}=0$.

### 2.3 Phase 1

At the beginning of this subsection, we prove two general lemmas that remain valid throughout the game. The first of them gives a lower bound on the decrease of the weights, if a vertex from an isolated edge is played.

Lemma 3. If $i \in\left[j^{*}\right], m_{i}=v$ and $v$ belongs to an isolated edge $e$ in $\mathcal{H}_{i-1}$, then $d_{i} \geq 28$. In particular, $d_{i} \geq 28$, if $v$ is from a Type- $X$ component.

Proof. Since $e$ is an isolated edge, after the move $m_{i}=v$, the edge $e$ and all vertices from it will be deleted. If $e$ is a 1-edge, $e$ is special and, by Observation 2(iv), $v$ is not special. Thus, $d_{i} \geq 15+13=28$. If $e$ is a 2-edge and contains a special vertex, the other vertex is not special and hence, $d_{i} \geq 9+7+13=29$. In the remaining cases $e$ contains at least two non-special vertices; that is, $e$ is from a Type- X component. This means $x_{i}=x_{i-1}-1$, and the decrease is $d_{i} \geq 9+2 \cdot 13-7=28$.

Next, we prove a lower bound on the decrease $d_{i}$ in the weight. It is true regardless of that the next player is Dominator or Staller.

Lemma 4. For every $i \in\left[j^{*}\right], d_{i} \geq 16$.
Proof. In the $i$ th turn, at least one new edge is covered and deleted from the hypergraph $\mathcal{H}_{i-1}$; and at least one vertex (the one which was played) is deleted. Hence, if $x_{i} \geq x_{i-1}$, then $d_{i} \geq 9+7=16$. If $x_{i}<x_{i-1}$, then a vertex from a Type-X component was played. By Lemma 3, we have $d_{i} \geq 28$ that completes the proof.

Now we are ready to prove that the average decrease in a turn is at least 28 in Phase 1.
Lemma 5. If $\mathcal{P}^{1} \neq \emptyset$,

$$
\frac{\sum_{i=1}^{b_{1}} d_{i}}{b_{1}} \geq 28
$$

Proof. If $i$ is odd and $1 \leq i \leq b_{1}$, the definition of $\mathcal{P}^{1}$ ensures that $d_{i} \geq 40$. By Lemma (4, we have $d_{i}+d_{i+1} \geq 40+16=2 \cdot 28$ that implies the statement. Remark that if the game finishes with Dominator's turn in Phase 1 (i.e., $b_{1}$ is odd and equal to $j^{*}$ ), then the last decrease $d_{b_{1}}$ is at least 40 , and $\sum_{i=1}^{b_{1}} d_{i} \geq \frac{b_{1}-1}{2} \cdot 56+40>28 b_{1}$. Hence, the lemma is valid for this special case as well.

We may prove some properties which are true for each residual hypergraph after the end of Phase 1.

Lemma 6. For every $i \geq b_{1}$, the residual hypergraph $\mathcal{H}_{i}$ satisfies the following properties.
(i) $\Delta\left(\mathcal{H}_{i}\right) \leq 2$.
(ii) $\mathcal{H}_{i}$ is a linear hypergraph.
(iii) If $v$ is special vertex and $u$ is a neighbor of $v$ in $\mathcal{H}_{i}$, then $u$ is not contained in any special edges.
(iv) If $d_{\mathcal{H}_{i}}(v)=2$, then $\left|S \cap N_{\mathcal{H}_{i}}(v)\right| \leq 1$.

Proof. (i) Assume for a contradiction that there exists a vertex $v \in V\left(\mathcal{H}_{b_{1}}\right)$ which is incident with at least three edges. Then Dominator may play $v$ in the next turn and $\mathcal{H}_{b_{1}+1}$ is obtained from $\mathcal{H}_{b_{1}}$ by deleting $v$ which is not special (and maybe, some further vertices), and at least three edges. Clearly, $x_{b_{1}+1} \geq x_{b_{1}}$. Thus, we have $d_{b_{1}+1} \geq 13+3 \cdot 9=40$ that would imply $b_{1}+1 \in \mathcal{P}_{1}$ that is a contradiction. Hence, $\Delta\left(\mathcal{H}_{i}\right) \leq 2$ holds for $i=b_{1}$ and for every larger index.
(ii) Now, assume that there exist two edges, say $e_{1}$ and $e_{2}$, in $\mathcal{H}_{b_{1}}$ such that $\left|e_{1} \cap e_{2}\right| \geq 2$. By $(i)$, every vertex from $e_{1} \cap e_{2}$ is of degree 2 . Playing a common vertex $v$ of $e_{1}$ and $e_{2}$, at least two non-special vertices and two edges will be deleted. Since the number of Type-X components is not decreased, $d_{b_{1}+1} \geq 2 \cdot 13+2 \cdot 9=44$ that is a contradiction, again. This proves (ii) for $i=b_{1}$ and implies the linearity for every later residual hypergraph.
(iii) Assume for a contradiction that $v \in S, u \in N_{\mathcal{H}_{b_{1}}}(v)$ and $e=\{u\}$ is a special edge. By Observation $2(i i), u$ is not a special vertex. If Dominator plays $u$, then $u, v, e$, and the edge incident with $v$ will be deleted. Since $x_{b_{1}+1} \geq x_{b_{1}}$, we have $d_{b_{1}+1} \geq 13+7+15+9=44>40$, a contradiction. As new special vertices and edges cannot arise during the game, (iii) holds for every $i \geq b_{1}$.
(iv) Consider first $\mathcal{H}_{b_{1}}$. Suppose that a vertex $v$ is incident with two edges $e_{1}, e_{2}$ and that $N_{\mathcal{H}_{b_{1}}}(v)$ contains two special vertices $u_{1}$ and $u_{2}$. If Dominator plays $v$ in the next turn, the vertices $v, u_{1}, u_{2}$ and the edges $e_{1}, e_{2}$ will be deleted from $\mathcal{H}_{b_{1}}$ and $x_{b_{1}+1} \geq x_{b_{1}}$. This would yield $d_{b_{1}} \geq 13+2 \cdot 7+2 \cdot 9=45>40$ that is a contradiction. This proves the statement for $i=b_{1}$ from which (iv) follows for every $i \geq b_{1}$.

### 2.4 Phase 2

In Phase 2, every residual hypergraph satisfies the properties $(i)-(i v)$ from Lemma 6, and $d_{i} \geq 38$ for every odd $i$. We will prove that the average decrease is at least 28 over the turns in Phase 2.

Lemma 7. If $\mathcal{P}^{2} \neq \emptyset$,

$$
\frac{\sum_{i=a_{2}}^{b_{2}} d_{i}}{\left|\mathcal{P}^{2}\right|} \geq 28 .
$$

Proof. Suppose that Staller plays a vertex $v$ in the $i$ th turn, $a_{2}<i \leq b_{2}$. By definition of $\mathcal{P}^{2}, d_{i-1} \geq 38$ holds. We have the following cases concerning the move $m_{i}=v$.

- If $v$ is from an isolated edge, then, by Lemma 3, $d_{i} \geq 28$. This gives $d_{i-1}+d_{i} \geq$ $38+28>2 \cdot 28$.
- If $v$ is a non-special vertex and it is not from an isolated edge (i.e., $x_{i} \geq x_{i-1}$ ), then $d_{i} \geq 13+9=22$ and again, we have $d_{i-1}+d_{i} \geq 38+22>2 \cdot 28$.
- In the third case, $v$ is a special vertex and it is contained in a non-isolated edge in $\mathcal{H}_{i}$. Then, $v$ has a degree- 2 neighbor, say $u$. Let $e$ be the edge containing both $v$ and $u$, and let $e^{\prime}$ be the other edge incident with $u$. By Lemma $6(i v), e^{\prime}$ is not special. This implies $\left|e^{\prime}\right| \geq 2$. Moreover, by Lemma 6 (iii), $e^{\prime}$ does not contain any special edges. Consequently, $e^{\prime}$ contains at least two non-special vertices. If $e^{\prime}$ becomes isolated in $\mathcal{H}_{i+1}$, then it will be of a Type-X edge. Therefore, Staller's move results in a decrease of $d_{i} \geq 7+9+7=23$, since $v$ and $e$ are deleted and $x_{i+1} \geq x_{i}+1$. Hence, we have, $d_{i-1}+d_{i} \geq 38+23>2 \cdot 28$. In the other case, $e^{\prime}$ is not isolated and contains a degree- 2 vertex $u^{\prime}$ in $\mathcal{H}_{i+1}$. Then, Dominator may choose $u^{\prime}$ in the $(i+1)$ st turn. This means that two non-special vertices, namely $u, u^{\prime}$, and two edges are deleted from the residual hypergraph. Hence, $d_{i+1} \geq 2 \cdot 13+2 \cdot 9=44$. If the game is finished with the $(i+1)$ st turn, then $d_{i-1}+d_{i}+d_{i+1} \geq 38+16+44>3 \cdot 28$. In the other case, $\mathcal{H}_{i+1}$ is not empty, and we have $d_{i-1}+d_{i}+d_{i+1}+d_{i+2} \geq 38+16+44+16>4 \cdot 28$.
Lemma 8. For every $i \geq b_{2}$, every special vertex present in $\mathcal{H}_{i}$ is contained in an isolated edge.

Proof. Consider $\mathcal{H}_{b_{2}}$ and assume that the special vertex $v$ has a degree- 2 neighbor $u$. If Dominator plays $u$ in the next turn, then $v, u$ and two edges are deleted, moreover $x_{b_{2}+1} \geq x_{b_{2}}$. This gives $d_{b_{2}+1} \geq 7+13+2 \cdot 9=38$ that contradicts the definition of $\mathcal{P}^{3}$. Thus, the lemma is valid with $i=b_{2}$ and in turn, it implies the statement for every later residual hypergraph.
Lemma 9. For every $i \geq b_{2}+1, d_{i} \geq 22$ holds.

Proof. After the end of Phase 2, in every turn, either an isolated edge is deleted that gives $d_{i} \geq 28$ by Lemma 3, or a vertex $v$ is played which does not belong to an isolated edge. In the latter case, by Lemma 8 , $v$ cannot be special and the move results in a decrease $d_{i} \geq 13+9=22$.

### 2.5 Phase 3 and 4

If Phase 3 is not empty, it starts with the turn $a_{3}$. By the definition of the phases, Dominator's greedy strategy gives $d_{a_{3}}<38$, while $d_{i} \geq 38$ for every odd $i$ smaller than $a_{3}$. Moreover, $\Delta\left(\mathcal{H}_{k}\right)=2$ holds for each $a_{3}-1 \leq k \leq b_{3}-1$. Therefore, in Phase 3, Dominator can always play a vertex of degree 2 that results in a decrease of at least $13+2 \cdot 9=31$ in the weight of the residual hypergraph.

Lemma 10. If $\mathcal{P}^{3} \neq \emptyset$,

$$
\frac{\sum_{i=a_{3}}^{b_{3}} d_{i}}{\left|\mathcal{P}^{3}\right|} \geq 28
$$

Proof. Consider an odd $i$ with $a_{3} \leq i \leq b_{3}$. First suppose that there exists a degree- 2 vertex $v$ which has a degree- 1 neighbor $u$ in $\mathcal{H}_{i-1}$. Remark that by Lemma \& both $v$ and $u$ are non-special vertices. Then, Dominator may play $v$, and this move results in $d_{i} \geq 2 \cdot 13+2 \cdot 9=44$. Since by Lemma 9, $d_{i+1} \geq 22$, we have $d_{i}+d_{i+1} \geq 44+22>2 \cdot 28$ in this case.

Second, suppose that every component of $\mathcal{H}_{i-1}$ which is not an isolated edge is 2-regular. If Dominator may play a vertex such that a new isolated edge arises, then $d_{i} \geq 13+2$. $9+7=38$ and $d_{i}+d_{i+1} \geq 38+22>2 \cdot 28$ follows. Also, if there exists a special edge on the degree- 2 vertex $v$, then the choice of $v$ gives $d_{i} \geq 13+15+9=37$ and in turn, $d_{i}+d_{i+1} \geq 37+22>2 \cdot 28$. In the remaining case, Dominator plays a degree- 2 vertex $v$ which has at least two neighbors, say $u_{1}$ and $u_{2}$. Further, as new isolated edges do not arise, $u_{1}$ and $u_{2}$ become degree- 1 vertices in a component the maximum degree of which is two. This results in $d_{i} \geq 31$. If Staller's move creates a new isolated edge, $d_{i+1} \geq 13+9+7=29$. If after Staller's turn both $u_{1}$ and $u_{2}$ are deleted from the residual graph, $d_{i+1} \geq 2 \cdot 13+9=35$. In both cases $d_{i}+d_{i+1} \geq 31+29>2 \cdot 28$. So, it is enough to consider the case when $d_{i} \geq 31$, $d_{i+1} \geq 22$, and at least one of $u_{1}$ and $u_{2}$, say $u_{1}$, is a degree- 1 vertex contained in a nonisolated edge of $\mathcal{H}_{i+1}$. Thus, $u_{1}$ has a neighbor $w$ of degree 2 in $\mathcal{H}_{i+1}$. If Dominator selects $w$ in the next turn, $u_{1}, w$ and two edges will be deleted. Hence, $d_{i+2} \geq 2 \cdot 13+2 \cdot 9=44$. If the game is finished with this turn, $d_{i}+d_{i+1}+d_{i+2}>3 \cdot 28$ holds. If the game continues with the $(i+3)$ rd turn,

$$
d_{i}+d_{i+1}+d_{i+2}+d_{i+3} \geq 31+22+44+22=119>4 \cdot 28
$$

follows. This finishes the proof of the lemma.
Lemma 11. If $\mathcal{P}^{4} \neq \emptyset$,

$$
\frac{\sum_{i=a_{4}}^{b_{4}} d_{i}}{\left|\mathcal{P}^{4}\right|} \geq 28
$$

Proof. In Phase 4, by definition, we have only isolated edges and by Lemma3, $d_{i} \geq 28$ follows for every $i \geq a_{4}$ in the game. This proves the lemma.

By Lemma [5, Lemma 7, Lemma 10, and Lemma 11, we have that

$$
\frac{\sum_{i=1}^{j^{*}} d_{i}}{j^{*}}=\frac{f\left(\mathcal{H}_{0}\right)-f\left(\mathcal{H}_{j^{*}}\right)}{j^{*}} \geq 28
$$

where $j^{*}$ denotes the length of the game when Dominator follows a greedy strategy based on the function $f$, and Staller plays optimally, according to her goal. Consequently,

$$
\gamma_{\mathrm{tg}}(G)=\tau_{g}\left(\mathcal{H}_{0}\right) \leq j^{*} \leq \frac{f\left(\mathcal{H}_{0}\right)}{28} \leq \frac{11}{14} n
$$

follows, which proves Theorem 1 .

## 3 Concluding remarks

Analogously to the game total domination number $\gamma_{\mathrm{tg}}(G)$ (resp., to the game transversal number $\tau_{g}(\mathcal{H})$ ), the Staller-start game total domination number, $\gamma_{\mathrm{tg}}^{\prime}(G)$ (resp., the Stallerstart game transversal number $\tau_{g}(\mathcal{H})$ )is the length of the game if Staller starts and both players play optimally. It was proved already in the introductory paper [10] that for any $\operatorname{graph} G,\left|\gamma_{\mathrm{tg}}(G)-\gamma_{\mathrm{tg}}^{\prime}(G)\right| \leq 1$.

In [11], the authors also posed a conjecture on the Staller-start version of the total domination game.

Conjecture 2. If $G$ is a graph on $n$ vertices in which every component contains at least three vertices, then $\gamma_{\mathrm{tg}}^{\prime}(G) \leq \frac{3 n+1}{4}$.

In the same paper, they proved the upper bound $\frac{4 n+2}{5}$. Our proof given for Theorem 1 can be easily extended with a Preliminary Phase which contains the first move $m_{0}$ taken by Staller. In this short part, $f(O N H(G))$ is decreased by $d_{0} \geq 16$ and then, from the next turn, the determination of the phases and the proof is just the same as it was in Section 2 , Consequently, we have the upper bound $\frac{22 n+12}{28}=\frac{11 n+6}{14}$.

Proposition 12. If $G$ is a graph on $n$ vertices in which every component contains at least three vertices, then $\gamma_{\mathrm{tg}}^{\prime}(G) \leq \frac{11 n+6}{14}$.

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[^1]:    ${ }^{1}$ Remark that the total domination game is an analogous version of the domination game, introduced by Brešar, Klavžar, and Rall in 2010 [2], where the choice of $v$ is legal if it dominates at least one new vertex; that is, if $N[v] \backslash N[D] \neq \emptyset$. The corresponding invariant is the game domination number, $\gamma_{g}(G)$. For the exact definitions and results on the domination game see [2, 13, 1,
    ${ }^{2}$ The 'fast' player is called Edge-hitter in [6] and [7]. To have the two players with the same names in the total domination and in the transversal game, we prefer to call him Dominator instead of Edge-hitter, here.

