Odd induced subgraphs in graphs with treewidth at most two *

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Abstract

A long-standing conjecture asserts that there exists a constant c > 0 such that every graph of order n without isolated vertices contains an induced subgraph of order at least cn with all degrees odd. Scott (1992) proved that every graph G has an induced subgraph of order at least $|V(G)|/(2\chi(G))$ with all degrees odd, where $\chi(G)$ is the chromatic number of G, this implies the conjecture for graphs with bounded chromatic number. But the factor $1/(2\chi(G))$ seems to be not best possible, for example, Radcliffe and Scott (1995) proved $c = \frac{2}{3}$ for trees, Berman, Wang and Wargo (1997) showed that $c = \frac{2}{5}$ for graphs with maximum degree 3, so it is interesting to determine the exact value of cfor special family of graphs. In this paper, we further confirm the conjecture for graphs with treewidth at most 2 with $c = \frac{2}{5}$, and the bound is best possible.

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1 Introduction

Gallai [5] proved that for every graph G, the vertex set V(G) can be partitioned into two sets, each of which induces a subgraph with all degrees even. This implies that every graph of order n contains an induced subgraph of order at least $\lceil \frac{n}{2} \rceil$ with all degrees even, and this is best possible by considering paths. This motivates us to consider the problem that how large we can find an induced subgraph with all degrees odd. We call a graph with all degrees odd an *odd graph*. Let f(G) denote the maximum order of an odd induced subgraph in a graph G. The following longstanding conjecture was cited by Caro in [2] as "part of the graph theory folklore" and the origin is unclear.

Conjecture 1. There exists a constant c > 0 such that for every graph G without isolated vertices, $f(G) \ge c|V(G)|$.

The "without isolated vertices" constraint is natural because an odd graph does not contain isolated vertices. Many results related to Conjecture 1 have been obtained in literatures. In particular, Caro [2] proved that $f(G) \ge (1-o(1))\sqrt{|V(G)|/6}$, laterly, Scott [7] improved the lower bound to $\frac{c|V(G)|}{\log |V(G)|}$ for some c > 0, in the same paper, Scott also proved that every graph G has an odd induced subgraph of order at least $|V(G)|/(2\chi(G))$, where $\chi(G)$ is the chromatic number of G, this implies the conjecture for graphs with bounded chromatic number. But the factor $1/(2\chi(G))$ seems to be not best possible, for example, Radcliffe and Scott [6] confirmed the conjecture for trees (graphs with treewidth one) with $c = \frac{2}{3}$ and Berman, Wang and Wargo [1] proved the conjecture for graphs with maximum degree 3 with $c = \frac{2}{5}$. In this paper, we further confirm Conjecture 1 for graphs with treewidth at most 2 with $c = \frac{2}{5}$, and the value of c is best possible.

A tree decomposition of a graph G is a tree T, where

- (1) Each vertex i of T is labeled by a subset B_i of vertices of G.
- (2) Each edge of G is in a subgraph induced by at least one of the B_i ,

(3) For every three vertices i, j, k in T with j lying on the path from i to k in T, $B_i \cap B_k \subseteq B_j$.

The *tree-width* tw(G) of G is the minimum integer p such that there exists a tree decomposition of G with all subsets of cardinality at most p + 1. Tree-decomposition is one of the most general and effective techniques for designing efficient algorithms, and a tree-like structure allows us to solve certain difficult problems. It is well-known that a connected graph has treewidth one if and only if it is tree. In terms of treewidth, the result of Radcliffe and Scott [6] can be restated as follows.

Theorem 2. [6] For any connected graph T with tw(T) = 1, $f(T) \ge 2\lfloor \frac{|V(T)|+1}{3} \rfloor$.

The following theorem is our main result.

Theorem 3. For every graph G with $tw(G) \le 2$ and without isolated vertices, $f(G) \ge \frac{2}{5}|V(G)|$.

The lower bound is sharp by considering the graph of which each component is a cycle of length 5. We remark that graph with treewidth at most two is also known as K_4 -minor-free graph, see Proposition 1 in section 3. Some upper and lower bounds on graphs with small treewidth are also discussed in the last section.

In this paper, standard notation follows from [3]. In particular, for a graph Gand a set $S \subseteq V(G)$, let G[S] be the subgraph induced by S and let $N_G(S)$ be the union of neighbors of vertices in S, for a vertex $u \in V(G)$, let $N_G^1(u) = \{x \mid x \in$ $N_G(u)$ and $d_G(x) = 1\}$ and $N_G^2(u) = \{x \mid x \in N_G(u) \text{ and } d_G(x) = 2\}$, and denote $N_G^2(u, v) = N_G^2(u) \cap N_G^2(v)$. A vertex of degree k is called a k-vertex. Define $S_G(u) =$ $\{x \mid x \in N_G(u) \text{ with } d_G(x) \geq 3 \text{ or there exists a vertex } z \in N_G^2(u, x)\}$. Let $D_G(u) =$ $|S_G(u)|$. For two sets S, T, we use $S \setminus T$ denote $S - (S \cap T)$.

The rest of the paper is arranged as follows. In section 2, we establish structural properties of minimum counterexample of Theorem 3. Then the proof of Theorem 3 is presented in Section 3, and in the last section, we give some discussions.

2 Properties of minimal counterexample

Let G be a minimum counterexample of Theorem 3 with respect to the order of G. The main idea of the proof is as the following. We first pick some set $V_0 \subset V(G)$ so that $G' = G - V_0$ has no isolated vertex, by the minimality of G, G' has an odd induced subgraph H' with $|V(H')|/|V(G')| \ge 2/5$. We will find a set $S_0 \subset V_0$ with $|S_0| \ge \frac{2}{5}|V_0|$ such that $S_0 \cup V(H')$ induces an odd induced subgraph H of G. We should be careful to remain the parity of the degrees of the vertices in $N_G(S_0) \cap V(H')$ and $S_0 \cap N_G(V(H'))$. Here we allow $V(G') = \emptyset$.

Lemma 4. Let u be a vertex of G with $D_G(u) = 1$ and let $S_G(u) = \{v\}$. Then $N_G^1(u) \cup N_G^2(u, v) = \emptyset$.

Proof. Suppose to the contrary that G has a vertex u with $D_G(u) = 1$ and $N_G^1(u) \cup N_G^2(u, v) \neq \emptyset$. Let $t_1 = |N_G^1(u)|$ and $t_2 = |N_G^2(u, v)|$. Then $t_1 + t_2 > 0$. Case 1. $|N_G^1(v)| \leq 1$.

Set $V_0 = N_G^1(u) \cup N_G^2(u, v) \cup \{u, v\} \cup N_G^1(v)$ and $G' = G - V_0$. Then G' has no isolated vertex, so, by the minimality of G, G' has an odd induced subgraph H' with $|V(H')| \geq \frac{2}{5}|V(G')|$. Now let $S_0 = V_0 \setminus (N_G^1(v) \cup \{v\})$. Then $G[S_0] \cong K_{1,t_1+t_2}$, and so $G[S_0]$ contains an odd induced subgraph $K = K_{1,t}$ with $t = t_1 + t_2$ if $t_1 + t_2$ is odd or $t = t_1 + t_2 - 1$ if $t_1 + t_2$ is even. So $(t+1)/|V_0| \geq (t+1)/(t_1 + t_2 + 3) \geq 2/5$. Furthermore, we have $N_G(V(K)) \cap V(H') = \emptyset$ and $V(K) \cap N_G(V(H')) = \emptyset$. Hence $H = H' \cup K$ is an odd induced subgraph of G with $|V(H)| \geq \frac{2}{5}|V(G)|$, a contradiction. **Case 2.** $|N_G^1(v)| \geq 2$.

Choose a vertex $x \in N_G^1(v)$ and set $V_0 = N_G^1(u) \cup N_G^2(u, v) \cup \{u, x\}$ and $G' = G - V_0$. Then G' has no isolated vertex and so, by the minimality of G, G' has an odd induced subgraph H' with $|V(H')|/|V(G')| \ge 2/5$.

Claim 1. v must be in V(H').

Suppose to the contrary that $v \notin V(H')$. Set $S_0 = V_0 \setminus \{x\}$, then $G[S_0] \cong K_{1,t_1+t_2}$, and so $G[S_0]$ contains an odd induced subgraph $K = K_{1,t}$ with $t = t_1+t_2$ or $t = t_1+t_2-1$ 1 with respect to the parity of t_1+t_2 . Note that $(t+1)/|V_0| = (t+1)/(t_1+t_2+2) > 2/5$, $N_G(V(K)) \cap V(H') = \emptyset$ and $V(K) \cap N_G(V(H')) = \emptyset$. Therefore, $H = K \cup H'$ is an odd induced subgraph of G with |V(H)|/|V(G)| > 2/5, a contradiction. The claim is true.

Now suppose $v \in V(H')$.

Claim 2. We have $t_2 \leq t_1$.

Suppose to the contrary that $t_2 \ge t_1 + 1$. Set $S_0 = N_G^2(u, v) \cup \{x\}$ if t_2 is odd or $S_0 = N_G^2(u, v)$ if t_2 is even, then $S_0 \cup V(H')$ still induces an odd subgraph H of G with $|V(H)| = |S_0| + |V(H')| \ge \frac{2}{5}|V_0| + \frac{2}{5}|V(G')| = \frac{2}{5}|V(G)|$, a contradiction, where the second inequality holds since $|S_0|/|V_0| \ge |S_0|/(t_1 + t_2 + 2) \ge |S_0|/(2t_2 + 1) \ge 2/5$. Hence the claim holds.

Now suppose $t_2 \leq t_1$ and let T_1 (resp. T_2) be a maximum subset of odd (resp. even) order in $N_G^1(u)$. Set $S_0 = T_1 \cup \{u\}$ if $uv \notin E(G)$ or $S_0 = T_2 \cup \{u, x\}$ if $uv \in E(G)$. In both cases, $|S_0|/|V_0| = |S_0|/(t_1 + t_2 + 2) \geq |S_0|/(2t_1 + 2) \geq 2/5$ unless $t_1 = t_2 = 2$ and $uv \notin E(G)$. Then $S_0 \cup V(H')$ induces an odd subgraph H of G with $|V(H)| = |S_0| + |V(H')| \geq 2/5|V_0| + 2/5|V(G')| = 2/5|V(G)|$, a contradiction. For $t_1 = t_2 = 2$ and $uv \notin E(G)$, reset $V_0 = N_G^1(u) \cup N_G^2(u, v) \cup \{u\} = N_G(u) \cup \{u\}$ and let $G' = G - V_0$, then G' has no isolated vertex and so, by the minimality of G, G' has an odd induced subgraph H' with $|V(H')|/|V(G')| \ge 2/5$. Let $N_G^1(u) = \{a, b\}$ and set $S_0 = \{a, u\}$. Then $S_0 \cup V(H')$ induces an odd subgraph H of G with $|V(H)| \ge 2 + \frac{2}{5}|V(G')| = \frac{2}{5}|V(G)|$, a contradiction again.

This completes the proof of the lemma.

Lemma 5. Let u be a vertex of G with $D_G(u) = 2$ and let $S_G(u) = \{v, w\}$. Then $N_G^1(u) \cup N_G^2(u, v) \cup N_G^2(u, w) = \emptyset$.

Proof. Suppose to the contrary that G has a vertex u with $S_G(u) = \{v, w\}$ and $N_G^1(u) \cup N_G^2(u, v) \cup N_G^2(u, w) \neq \emptyset$. Let $t_1 = |N_G^1(u)|, t_2 = |N_G^2(u, v)|$ and $t_3 = |N_G^2(u, w)|$. Then $t_1 + t_2 + t_3 > 0$. Let $\bar{N}_G^2(v, w) = N_G^2(v, w) \setminus \{u\}$.

Claim 3. If $N_G^1(v) = \emptyset$ then $N_G^1(w) \cup \bar{N}_G^2(v, w) \neq \emptyset$; symmetrically, if $N_G^1(w) = \emptyset$ then $N_G^1(v) \cup \bar{N}_G^2(v, w) \neq \emptyset$.

We only prove the first statement, the second one can be proved similarly. Suppose to the contrary that $N_G^1(w) \cup \bar{N}_G^2(v, w) = \emptyset$. Set $V_0 = N_G(u) \cup \{u, v, w\}$ and $G' = G - V_0$. Then G' has no isolated vertex and so, by the minimality of G, G' has an odd induced subgraph H' with $|V(H')| \ge \frac{2}{5}|V(G')|$. Let S be a maximum subset of $N_G^1(u) \cup N_G^2(u, v) \cup N_G^2(u, w)$ so that s = |S| is odd. Then $S_0 = S \cup \{u\}$ induces an odd subgraph $K \cong K_{1,s}$ of $G[V_0]$, furthermore $|S_0|/|V_0| \ge (s+1)/(t_1+t_2+t_3+3) \ge 2/5$. Note that $N_G(S_0) \cap V(H') = \emptyset$ and $S_0 \cap N_G(V(H')) = \emptyset$. Therefore, $H = K \cup H'$ is an odd induced subgraph of G with $|V(H)| \ge \frac{2}{5}|V_0| + \frac{2}{5}|V(G')| = \frac{2}{5}|V(G)|$, a contradiction. The claim is true.

Case 1. $N_G^1(v) = \emptyset$.

Subcase 1.1. $|N_G(w) \setminus (N_G^2(u, w) \cup \{u, v\})| \le 1$.

Note that $|N_G(w) \setminus (N_G^2(u, w) \cup \{u, v\})| \leq 1$ implies that $|N_G^1(w) \cup \overline{N}_G^2(v, w)| \leq 1$. By Claim 3, $|N_G^1(w) \cup \overline{N}_G^2(v, w)| = 1$ and so $N_G(w) \setminus (N_G^2(u, w) \cup \{u, v\}) = N_G^1(w) \cup \overline{N}_G^2(v, w)$. Let $N_G^1(w) \cup \overline{N}_G^2(v, w) = \{x\}$ and set $V_0 = N_G(u) \cup \{u, v, w, x\}$ and $G' = G - V_0$. Then G' has no isolated vertex and so, by the minimality of G, G' has an odd induced subgraph H' with $|V(H')| \geq \frac{2}{5}|V(G')|$. Let S be a maximum subset of $N_G^1(u) \cup N_G^2(u, v) \cup N_G^2(u, w)$ so that s = |S| is odd. Then $S_0 = S \cup \{u\}$ induces an odd subgraph $K \cong K_{1,s}$ of $G[V_0]$, furthermore $|S_0|/|V_0| \geq (s+1)/(t_1+t_2+t_3+4) \geq 2/5$ unless $t_1+t_2+t_3=2$. Note that $N_G(S_0) \cap V(H') = \emptyset$ and $S_0 \cap N_G(V(H')) = \emptyset$. Hence $H = K \cup H'$ is an odd induced subgraph of G with $|V(H)| \ge \frac{2}{5}|V_0| + \frac{2}{5}|V(G')| = \frac{2}{5}|V(G)|$ provided that $t_1 + t_2 + t_3 \ne 2$, a contradiction.

For $t_1 + t_2 + t_3 = 2$, notice that $E_G(w, V(G')) = \emptyset$ because $N_G(w) \setminus (N_G^2(u, w) \cup$ $\{u,v\}$ = $N_G^1(w) \cup \bar{N}_G^2(v,w)$. If $E_G(v,V(G')) = \emptyset$ then G is a graph of order six, it can be easily checked that G cannot be a counterexample. If $t_3 = 2$ then $S_0 =$ $N_G^2(u,w) \cup \{w,x\}$ induces an odd subgraph $K \cong K_{1,3}$, and therefore $H = K \cup H'$ is an odd induced subgraph of G with $|V(H)| \ge 4 + \frac{2}{5}|V(G')| > \frac{2}{5}|V(G)|$, a contradiction. Hence suppose $E_G(v, V(G')) \neq \emptyset$ and $t_3 < 2$. Reset $V_0 = (N_G(u) \cup \{u, w, x\}) \setminus \{v\}$ and $G' = G - V_0$. Then G' has no isolated vertex and so, by the minimality of G, G' has an odd induced subgraph L' with $|V(L')| \geq \frac{2}{5}|V(G')|$. If $v \notin V(L')$ or $vw, vx \notin E(G)$, then $\{w, x\} \cup V(L')$ induces an odd subgraph H of G with $|V(H)| \ge 2 + \frac{2}{5}|V(G')| \ge 2$ $\frac{2}{5}|V(G)|$, a contradiction. So suppose $v \in V(L')$ and $vw \in E(G)$ or $vx \in E(G)$. If $N_G(v) \cap V_0$ has two nonadjacent vertices, say $\{a, b\}$, then $\{a, b\} \cup V(L')$ induces an odd subgraph of G with order at least $\frac{2}{5}|V(G)|$, a contradiction. This implies that $N_G^2(u,v) = \emptyset$ (i.e. $t_2 = 0$), $vx \notin E(G)$ (i.e. $x \in N_G^1(w)$) and $vw, uw \in E(G)$ (otherwise, it is easy to choose two nonadjacent vertices from $N_G^2(u, v) \cup \{u, w, x\}$). As $t_1 + t_2 + t_3 = 2$, $t_2 = 0$, and $t_3 < 2$, we have $t_1 > 0$. Choose $a \in N^1_G(u)$, then $\{a, u, w, x\} \cup V(L')$ induces an odd subgraph H of G with $|V(H)| \ge 4 + \frac{2}{5}|V(G')| > 1$ $\frac{2}{5}|V(G)|$, a contradiction.

Subcase 1.2. $|N_G(w) \setminus (N_G^2(u, w) \cup \{u, v\})| \ge 2.$

Choose $x \in N_G^1(w) \cup \bar{N}_G^2(v, w)$ (this can be done because $N_G^1(w) \cup \bar{N}_G^2(v, w) \neq \emptyset$ by Claim 3) and set $V_0 = (N_G(u) \cup \{u, v, x\}) \setminus \{w\}$ and $G' = G - V_0$. Then G' has no isolated vertex and so, by the minimality of G, G' has an odd induced subgraph H' with $|V(H')| \geq \frac{2}{5}|V(G')|$.

Claim 4. $w \in V(H')$.

If $w \notin V(H')$, choose a maximum subset S of $N_G(u) \setminus \{v, w\}$ so that s = |S|is odd, then $S_0 = S \cup \{u\}$ induces an odd subgraph $K \cong K_{1,s}$ of $G[V_0]$ such that $|S_0|/|V_0| = (s+1)/(t_1 + t_2 + t_3 + 3) \ge 2/5$. Clearly, $N_G(S_0) \cap V(H_0) = \emptyset$ and $S_0 \cap N_G(V(H_0)) = \emptyset$. Hence $H = K \cup H'$ is an odd induced subgraph of G with $|V(H)| \ge \frac{2}{5}|V(G)|$, a contradiction. The claim holds.

Claim 5. $t_3 \leq t_1 + t_2$.

If $t_3 \ge t_1 + t_2 + 1$, choose a maximum subset S_0 of $N_G^2(u, w) \cup \{x\}$ so that $|S_0|$ is even, then $|S_0|/|V_0| = |S_0|/(t_1 + t_2 + t_3 + 3) \ge |S_0|/(2t_3 + 2) \ge 2/5$ unless $t_3 = 2$

and $t_1 + t_2 = 1$. Therefore, $S_0 \cup V(H')$ induces an odd subgraph H of G with $|V(H)| \geq \frac{2}{5}|V(G)|$ unless $t_3 = 2$ and $t_1 + t_2 = 1$. For $t_3 = 2$ and $t_1 + t_2 = 1$, reset $V_0 = (N_G(u) \cup \{u, v\}) \setminus \{w\}$ and $G' = G - V_0$, then, again by the minimality of G, G' has an odd induced subgraph L' with $|V(L')| \geq \frac{2}{5}|V(G')|$. If $w \in V(L')$, set $S_0 = N_G^2(u, w)$, and if $w \notin V(L')$, set $S_0 = \{u, y\}$, where y is a vertex in $N_G^2(u, w)$. In both cases, $S_0 \cup V(L')$ induces an odd subgraph H of G with $|V(H)| \geq \frac{2}{5}|V(G)|$. Therefore, we always obtain a contradiction and so the claim follows.

Now let S be a maximum subset of $N_G^1(u) \cup N_G^2(u, v)$ so that s = |S| is even if $uw \in E(G)$, and s = |S| is odd if $uw \notin E(G)$. Set $S_0 = S \cup \{u, x\}$ if $uw \in E(G)$ and $S_0 = S \cup \{u\}$ if $uw \notin E(G)$. Clearly, $S_0 \cup V(H')$ induces an odd subgraph H of G and furthermore, for $uw \in E(G)$, $|S_0|/|V_0| \ge (s+2)/(t_1+t_2+t_3+3) \ge (t_1+t_2+1)/(2t_1+2t_2+3) \ge 2/5$; and for $uw \notin E(G)$, $|S_0|/|V_0| = (s+1)/(t_1+t_2+t_3+3) \ge 2/5$ unless $t_1 + t_2 = 2$, $t_3 = 1$ or $t_1 + t_2 = 2$, $t_3 = 2$ or $t_1 + t_2 = t_3 = 4$. Therefore, but some exceptions, H is an odd induced subgraph with $|V(H)| \ge \frac{2}{5}|V(G)|$, a contradiction. Note that all the exceptions occur under the assumption $uw \notin E(G)$. In the following of the case, we show that each of the three exceptions cannot occur in the minimal counterexample G as well.

For $t_1 + t_2 = 2$ and $t_3 = 1$, reset $V_0 = N_G(u) \cup \{u, v\}$ and let $G' = G - V_0$, then, by the minimality of G, G' has an odd induced subgraph L' with $|V(L')| \ge \frac{2}{5}|V(G')|$. Choose a vertex $a \in N_G^1(u) \cup N_G^2(u, v)$, then $S_0 = \{u, a\}$ induces an odd subgraph $K \cong K_{1,1}$ of $G[V_0]$. As $N_G(S_0) \cap V(L') = \emptyset$ and $S_0 \cap N_G(V(L')) = \emptyset$, $H = K \cup L'$ is an odd induced subgraph of G with $|V(H)| \ge \frac{2}{5}|V(G)|$, a contradiction.

For $t_1 + t_2 = t_3 = 2$. If $|N_G^1(w) \cup \bar{N}_G^2(v, w)| \leq 2$, reset $V_0 = N_G(u) \cup N_G^1(w) \cup \bar{N}_G^2(v, w) \cup \{u, v, w, x\}$, then $G' = G - V_0$ has no isolated vertex and so, by the minimality of G, G' has an odd induced subgraph L' with $|V(L')| \geq \frac{2}{5}|V(G')|$. Let S be a subset of $N_G(u) \setminus \{v\}$ with s = |S| = 3 (this can be done because $|N_G(u)| \geq t_1 + t_2 + t_3 = 4$). Then $S_0 = S \cup \{u\}$ induces an odd subgraph $K \cong K_{1,3}$ of $G[V_0]$ and therefore $H = K \cup L'$ is an odd induced subgraph of G with $|V(H)| \geq \frac{2}{5}|V(G)|$, a contradiction. Now suppose $|N_G^1(w) \cup \bar{N}_G^2(v, w)| \geq 3$. Choose a vertex $y \in N_G^1(w) \cup \bar{N}_G^2(v, w)$ with $y \neq x$. Reset $V_0 = N_G(u) \cup \{u, v, x, y\}$ and $G' = G - V_0$, then, by the minimality of G, G' has an odd induced subgraph L' with $|V(L')| \geq \frac{2}{5}|V(G')|$. Let $S_0 = N_G^2(u, w) \cup \{x, y\}$ if $w \in V(L')$, and let $S_0 = S \cup \{u\}$ if $w \notin V(L')$, where S is a maximum subset of $N_G(u) \setminus \{v\}$ with s = |S| = 3. Clearly, $S_0 \cup V(L')$ induces an odd subgraph H of G with $|V(H)| > \frac{2}{5}|V(G)|$, a contradiction.

For $t_1 + t_2 = t_3 = 4$, reset $V_0 = N_G(u) \cup \{u, v\}$ and $G' = G - V_0$, then G'

has no isolated vertices and so, by the minimality of G, G' has an odd induced subgraph L' with $|V(L')| \geq \frac{2}{5}|V(G')|$. Let $S_0 = N_G^2(u, w)$ if $w \in V(L')$, or let $S_0 = N_G^1(u) \cup N_G^2(u, v) \cup \{u\}$ if $w \notin V(L')$. Then $|S_0|/|V_0| \geq 2/5$ and $S_0 \cup V(L')$ induces an odd subgraph H of G with $|V(H)| \geq \frac{2}{5}|V(G)|$, a contradiction.

This proves Case 1. By symmetry, we may also assume $N_G^1(w) \neq \emptyset$ to verify the following remaining case.

Case 2. $N_G^1(v) \neq \emptyset$.

Choose $x \in N_G^1(v)$ and $y \in N_G^1(w)$, set $V_0 = (N_G(u) \cup \{u, x, y\}) \setminus \{v, w\}$ and $G' = G - V_0$.

Claim 6. G' has no isolated vertex.

Suppose to the contrary that G' has isolated vertices. Then v or w must be an isolated vertex of G'. Without loss of generality, assume v is an isolated vertex of G'. Then $D_G(v) = 1$. But $N_G^1(v) \neq \emptyset$, this is a contradiction to Lemma 4.

Hence G' has no isolated vertex and so, by the minimality of G, G' has an odd induced subgraph H' with $|V(H')| \geq \frac{2}{5}|V(G')|$.

Claim 7. H' contains at least one of $\{v, w\}$.

Suppose to the contrary that H' contains none of $\{v, w\}$. Let S be a maximum subset of $N_G(u) \setminus \{v, w\}$ so that s = |S| is odd. Then $S_0 = S \cup \{u\}$ induces an odd subgraph $K \cong K_{1,s}$ with $|S_0|/|V_0| = (s+1)/(t_1 + t_2 + t_3 + 3) \ge 2/5$. Note that $N_G(S_0) \cap V(H') = \emptyset$ and $S_0 \cap N_G(V(H')) = \emptyset$. Thus $H = K \cup H'$ is an odd induced subgraph of G with $|V(H)| \ge \frac{2}{5}|V(G)|$, a contradiction.

Claim 8. If $w \in V(H')$ then $t_3 \leq t_1 + t_2$. Symmetrically, if $v \in V(H')$ then $t_2 \leq t_1 + t_3$.

We show that $t_3 \leq t_1 + t_2$ when $w \in V(H')$. Suppose to the contrary that $t_3 \geq t_1 + t_2 + 1$. Let S_0 be a maximum subset of $N_G^2(u, w) \cup \{y\}$ such that $|S_0|$ is even. Then $S_0 \cup V(H')$ induces an odd subgraph H of G with $|V(H)| = |S_0| + |V(H')| \geq \frac{2}{5}|V_0| + \frac{2}{5}|V(G')| = \frac{2}{5}|V(G)|$ unless $t_3 = 2$ and $t_1 + t_2 = 1$.

For $t_3 = 2$ and $t_1 + t_2 = 1$, reset $V_0 = (N_G(u) \cup \{u, x\}) \setminus \{v, w\}$ and $G' = G - V_0$, then G' has no isolated vertex and so G' has an odd induced subgraph L' with $|V(L')| \geq \frac{2}{5}|V(G')|$ by the minimality of G. If $w \in V(L')$, let $S_0 = N_G^2(u, w)$, then $S_0 \cup V(L')$ induces an odd subgraph H with $|V(H)| \geq \frac{2}{5}|V(G)|$. Now suppose $w \notin V(L')$. If $v \notin V(L')$, choose a vertex z from $N_G^1(u) \cup N_G^2(u, v)$, then $\{u, z\} \cup V(L')$ induces an odd subgraph H of G with $|V(H)| \ge \frac{2}{5}|V(G)|$. Hence $v \in V(L')$, choose a vertex $z \in N_G^2(u, v) \cup \{u\}$ which is adjacent to v, then $\{x, z\} \cup V(L')$ induces an odd subgraph H of G with $|V(H)| \ge \frac{2}{5}|V(G)|$. In all cases we get contradictions and so the claim follows.

We shall show that certain special cases cannot occur in the minimal counterexample G, which would be helpful to eliminate exception values in later discussion.

Claim 9. If $uw \notin E(G)$ then none of the following occurs in the minimal counterexample G.

- (a) $t_1 + t_2 = 2$ and $t_3 = 1$;
- (b) $t_1 + t_2 = t_3 = p$, p = 2 or 4.

For $t_1 + t_2 = 2$ and $t_3 = 1$, reset $V_0 = (N_G(u) \cup \{u, x\}) \setminus \{w, v\}$ and $G' = G - V_0$, then G' has no isolated vertex and so, by the minimality of G, G' has an odd induced subgraph L' with $|V(L')| \ge \frac{2}{5}|V(G')|$. If $v \in V(L')$, choose a vertex $z \in N_G^2(u, v) \cup \{u\}$ which is adjacent to v, note that $uw \notin E(G)$, then $\{x, z\} \cup V(L')$ induces an odd subgraph H of G with $|V(H)| \ge \frac{2}{5}|V(G)|$, a contradiction. Hence $v \notin V(L')$, choose a vertex z from $N_G^1(u) \cup N_G^2(u, v)$, then $\{u, z\} \cup V(L')$ induces an odd subgraph H of G with $|V(H)| \ge \frac{2}{5}|V(G)|$, a contradiction.

For $t_1 + t_2 = t_3 = p$, p = 2, 4, reset $V_0 = (N_G(u) \cup \{u\}) \setminus \{v, w\}$ and $G' = G - V_0$, then G' has no isolated vertex and so, by the minimality of G, G' has an odd induced subgraph L' with $|V(L')| \ge \frac{2}{5}|V(G')|$. If $w \in V(L')$, note that $|N_G^2(u, w)| = t_3 = p$ is even, $N_G^2(u, w) \cup V(L')$ induces an odd subgraph H of G with $|V(H)| \ge \frac{2}{5}|V(G)|$, a contradiction. So suppose $w \notin V(L')$. If $uv \notin E(G)$ or $v \notin V(L')$, choose a subset S of $N_G^2(u, w)$ so that |S| = p - 1, then $S \cup \{u\} \cup V(L')$ induces an odd subgraph H of G with $|V(H)| \ge \frac{2}{5}|V(G)|$, a contradiction. Hence $uv \in E(G)$ and $v \in V(L')$. If $x \in V(L')$, then $N_G^2(u, w) \cup \{u\} \cup (V(L') \setminus \{x\})$ induces an odd subgraph H with $|V(H)| = p + 1 + |V(L')| - 1 \ge \frac{2}{5}|V(G)|$, a contradiction. Hence $x \notin V(L')$. Then $N_G^2(u, w) \cup \{u\} \cup V(L') \cup \{x\}$ induces an odd subgraph H of G with $|V(H)| \ge \frac{2}{5}|V(G)|$, a contradiction. This proves the claim.

By Claim 7, we may assume, without loss of generality, $w \in V(H')$. Hence, by Claim 8, $t_3 \leq t_1 + t_2$. Now we divide the discussion into two subcases below.

Subcase 2.1. $v \notin V(H')$.

Let S be a maximum subset of $N_G^1(u) \cup N_G^2(u, v)$ such that s = |S| is odd if $uw \notin E(G)$ or s = |S| is even if $uw \in E(G)$. Set $S_0 = S \cup \{u\}$ if $uw \notin E(G)$ or

 $S_0 = S \cup \{u, y\}$ if $uw \in E(G)$. Note that $s = t_1 + t_2$ or $t_1 + t_2 - 1$ depending on the parity of $t_1 + t_2$ and $|S_0| = s + 1$ or s + 2 depending on $uw \notin E(G)$ or $uw \in E(G)$. Notice that $t_3 \leq t_1 + t_2$, we have $|S_0|/|V_0| = |S_0|/(t_1 + t_2 + t_3 + 3) \geq 2/5$ unless $uw \notin E(G)$ and $t_1 + t_2 = 2$, $t_3 = 1$, or $t_1 + t_2 = t_3 = 2$, or $t_1 + t_2 = t_3 = 4$. Therefore $S_0 \cup V(H')$ induces an odd subgraph H of G with $|V(H)| \geq \frac{2}{5}|V(G)|$ but three exceptions. However, none of the exceptions occur in G by Claim 9. This yields a contradiction and verifies Subcase 2.1.

Subcase 2.2. $v \in V(H')$.

By Claim 8, we have $t_3 \leq t_1 + t_2$ and $t_2 \leq t_1 + t_3$. Furthermore, we have the following claim.

Claim 10. We have $t_2 + t_3 \le t_1$.

Suppose to the contrary that $t_2 + t_3 \ge t_1 + 1$. Let S_v be a maximum subset of $N_G^2(u, v) \cup \{x\}$ such that $|S_v|$ is even, let S_w be a maximum subset of $N_G^2(u, w) \cup \{y\}$ such that $|S_w|$ is even, and set $S_0 = S_u \cup S_v$. Then $S_0 \cup V(H')$ induces an odd subgraph of G. By checking the parity of t_2 and t_3 with certain calculation, we have $|S_0|/|V_0| \ge \frac{2}{5}$ unless $t_1 = 1$, $t_2 + t_3 = 2$ and t_i , i = 2, 3, is even. But this exception cannot occur because $t_3 \le t_1 + t_2$ and $t_2 \le t_1 + t_3$, a contradiction. Hence the claim holds.

Now, we choose a set S_0 according to the following rules:

(i) If $uv \in E(G)$, $uw \in E(G)$, let $S_0 = S_u \cup \{u, x, y\}$, where S_u is the maximum subset of $N_G^1(u)$ with size odd;

(ii) If $uv \in E(G)$, $uw \notin E(G)$, let $S_0 = S_u \cup \{u, x\}$, where S_u is the maximum subset of $N^1_G(u)$ with size even;

(iii) If $uv \notin E(G)$, $uw \in E(G)$, let $S_0 = S_u \cup \{u, y\}$, where S_u is the maximum subset of $N_G^1(u)$ with size even;

(iv) If $uv \notin E(G)$, $uw \notin E(G)$, let $S_0 = S_u \cup \{u\}$, where S_u is the maximum subset of $N_G^1(u)$ with size odd.

Then $S_0 \cup V(H')$ induces an odd subgraph of G by definition. It remains to compute $|S_0|/|V_0|$.

If t_1 is odd, we have $|S_0|/|V_0| \ge (t_1 + 1)/(t_1 + t_2 + t_3 + 3) \ge 2/5$ by Claim 10 in each of the cases (i)-(iv). If t_1 is even, it follows from Claim 10 that $|S_0|/|V_0| \ge (t_1 + 2)/(t_1 + t_2 + t_3 + 3) \ge 2/5$ in each of the cases (i)-(iii), and in the case (iv), $|S_0|/|V_0| = t_1/(t_1 + t_2 + t_3 + 3) \ge 2/5$ unless $t_1 = 2$, $t_2 + t_3 = 2$ or $t_1 = 4$, $t_2 + t_3 = 4$. Therefore, $S_0 \cup V(H')$ induces an odd subgraph H of G with $|V(H)| \ge \frac{2}{5}|V(G)|$ unless $t_1 = t_2 + t_3 = 2$ or $t_1 = t_2 + t_3 = 4$.

For $t_1 = t_2 + t_3 = p$, p = 2, 4, reset $V_0 = N_G(u) \cup \{u\}$ and $G' = G - V_0$, then G' has no isolated vertex and so, by the minimality of G, G' has an odd induced subgraph L' with $|V(L')| \ge \frac{2}{5}|V(G')|$. Choose a subset S of $N_G^1(u)$ so that |S| = p - 1, then $S \cup \{u\} \cup V(L')$ induces an odd subgraph H of G with $|V(H)| \ge \frac{2}{5}|V(G)|$, a contradiction.

The proof of the lemma is completed.

The following three structural properties of the minimum counterexample G are direct consequence of Lemmas 4 and 5.

Corollary 6. Let V_1 be the set of all 1-vertices in G and let $P = N_G(V_1)$. Suppose $G_1 = G - V_1$, then $d_{G_1}(x) \ge 3$ for any $x \in P$.

Proof. Suppose to the contrary that there is a vertex $x \in P$ with $d_{G_1}(x) \leq 2$. If $d_{G_1}(x) = 0$ then G is isomorphic to a star, which cannot be a counterexample. Hence $0 < d_{G_1}(x) \leq 2$. This implies that $0 < D_G(x) \leq 2$. But $|N_G^1(x)| \geq 1$, this is a contradiction to Lemmas 4 or 5.

Corollary 7. G has no adjacent 2-vertices.

Proof. Suppose to the contrary that G has two adjacent 2-vertices u, v. Then $D_G(u) \leq 2$. Let $v_1 = N_G(v) \setminus \{u\}$. Then $v \in N_G^2(u, v_1)$, which is a contradiction to Lemmas 4 or 5.

Corollary 8. G has no vertex u with $d_G(u) \ge 3$ so that $D_G(u) \le 2$.

Proof. Suppose to the contrary that G has a vertex u with $d_G(u) \ge 3$ and $D_G(u) \le 2$. By Lemmas 4 and 5, u has no neighbor of degree at most 2 since G cannot be isomorphic to a star. This implies $D_G(u) \ge d_G(u) \ge 3$, a contradiction.

3 Proof of Theorem 3

Before giving the proof, we need some definition and structural properties of graphs with treewidth at most 2. A graph G contains a graph H as a *minor* if H can be obtained from a subgraph of G by contracting edges, and G is called H-minor free if G does not have H as a minor. It is well known that

Proposition 1. [Proposition 12.4.2, [3]] A graph has treewidth at most 2 if and only if it is K_4 -minor free.

For K_4 -minor free graphs, Lih, Wang, and Zhu ([4]) gave a powerful structural property of them.

Lemma 9. [Lemma 2, [4]] If G is a K_4 -minor free graph, then one of the following holds:

- (a) $\delta(G) \leq 1;$
- (b) there exist two adjacent 2-vertices;
- (c) there exists a vertex u with $d_G(u) \ge 3$ such that $D_G(u) \le 2$.

Proof of Theorem 3. Let G be a minimum counterexample with respect to the order of G. By the minimality of G, G must be connected. Let V_1 be the set of all 1vertices in G and $P = N_G(V_1)$. Let $G_1 = G - V_1$. By Corollaries 6 and 7, $\delta(G_1) \ge 2$ and G_1 has no adjacent 2-vertices. Clearly, $tw(G_1) \le 2$ and hence G_1 is K_4 -minor free. By Lemma 9, G_1 has a vertex u with $d_{G_1}(u) \ge 3$ and $D_{G_1}(u) \le 2$. Clearly, $d_G(u) = d_{G_1}(u) + |N_G^1(u)|$ and the adding of the vertices of $N_G^1(u)$ to G_1 does not increases the value of $D_{G_1}(u)$. So $D_G(u) = D_{G_1}(u) \le 2$, this is a contradiction to Corollary 8. The proof of Theorem 3 is completed.

4 Concluding remarks

Let

 $\mathcal{G}_k = \{ G: tw(G) \le k \text{ and } G \text{ contains no isolated vertex} \},\$

and $c_k = \min_{G \in \mathcal{G}_k} \frac{f(G)}{|V(G)|}$. Since each graph in \mathcal{G}_k has chromatic number at most k+1, Scott's result [7] implies $c_k \geq \frac{1}{2k+2}$. The follow graphs H_k in Figure 1 gives an upper bound $c_k \leq \frac{2}{k+3}$ for k = 1, 2, 3, 4. Note that the graph H_4 is found by Caro [2], which is the smallest known ratio of $\frac{f(G)}{|V(G)|}$ for all graphs G. As we have known, Theorem 2 of Radcliffe and Scott [6] and the upper bound of c_k implies $c_1 = 1/2$, and in this paper, we show that $c_2 = 2/5$ (Theorem 3). As a far more step, we want ask the question: what is the exact value c_k for graphs in \mathcal{G}_k . It is plausible that $c_3 = \frac{1}{3}$ and $c_4 = \frac{2}{7}$.

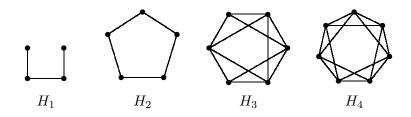


Figure 1: Graphs H_k with treewidth k and $\frac{f(H_k)}{|V(H_k)|} = \frac{2}{k+3}$ for k = 1, 2, 3, 4.

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